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ALGEBRAIC K-THEORY OF THE VARIETIES $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$, $\mathrm{E}_6/\mathrm{F}_4$ AND THEIR TWISTED FORMS

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Abstract. Let SL_{2n}, Sp_{2n}, $\mathrm{E}_6 = G^{sc}(\mathrm{E}_6)$, $\mathrm{F}_4 = G(\mathrm{F}_4)$ be simply connected split algebraic groups over an arbitrary field F. Algebraic K-theory of the affine homogeneous varieties $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ and $\mathrm{E}_6/\mathrm{F}_4$ is computed. Moreover, explicit elements that generate $K_*(\mathrm{SL}_{2n}/\mathrm{Sp}_{2n})$ and $K_*(\mathrm{E}_6/\mathrm{F}_4)$ as $K_*(F)$-algebras are provided. Also, K-theory is computed for some twisted forms of these varieties.

Introduction

Algebraic K-theory is already known for some classes of algebraic varieties. For the first time, it was computed for Severi–Brauer varieties by D. Quillen [9] and for smooth projective quadrics by R. Swan [12]. Then M. Levine [4] computed the K-theory of split semisimple simply connected algebraic groups. I. Panin [2] generalized this computation to all semisimple simply connected algebraic groups and computed the K-theory of flag varieties (see [8]). Later A. Ananyevskiy [3] computed the K-theory of homogeneous varieties G/H, where $H \subset G$ are connected reductive algebraic groups of the same rank. In all these cases, K-theory turned out to be isomorphic to a sum of K-theories for some central semisimple algebras.

We provide a computation of K-theory for the affine homogeneous varieties $\mathrm{SL}_{2n} / \mathrm{Sp}_{2n}$ and $\mathrm{E}_6 / \mathrm{F}_4$. The computation is based on using the Merkurjev spectral sequence for the equivariant K-theory (see [1]). The key point that allows us to accomplish the computation is the following fact: for the chosen varieties G/H, there is an epimorphism $i^*\colon R(G) \to R(H)$ on the rings of representations, and its kernel is generated by explicit elements. Here a big difference can be seen with the case of G/H where G and H have the same rank. In that case, $R(H)$ is a free $R(G)$-module (see [11] and [3, Theorem 2]).

The following theorem is proved.

Theorem. *There are isomorphisms of graded $K_*(F)$-modules*:

$$K_*(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n}) \simeq K_*(F) \otimes \Lambda(\mathbb{Z}^{n-1}),$$
$$K_*(\mathrm{E}_6 / \mathrm{F}_4) \simeq K_*(F) \otimes \Lambda(\mathbb{Z}^2),$$

where $\Lambda(\mathbb{Z}^m)$ is an exterior algebra considered with the natural grading.

Moreover, we provide $t_1, \dots, t_{n-1} \in K_1(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ and $s_1, s_2 \in K_1(\mathrm{E}_6 / \mathrm{F}_4)$ that are multiplicative generators of the $K_*(F)$-algebras $K_*(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ and $K_*(\mathrm{E}_6 / \mathrm{F}_4)$, respectively. These elements are constructed much as those in the topological K-theory of these varieties (see [7]). The proof is based on M. Levine's computation [4, Theorem 2.1] of multiplicative generators for $K_*(\mathrm{SL}_{2n})$ and $K_*(\mathrm{E}_6)$ as algebras over $K_*(F)$.

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Explicitly constructed isomorphisms in the split case allow us to compute K-theory for some twisted forms of these varieties by using Panin's splitting principle [2].

Theorem. *Assume that* $\mathrm{char}(F) \neq 2$. *Let* $\gamma\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to (\mathrm{Sp}_{2n}/\mu_2)(F^{\mathrm{sep}})$ *be a* 1*-cocycle, let* $A = \mathrm{End}(V)$, *where* V *is a* $2n$*-dimensional vector space over* F, *and let* τ *be the standard symplectic involution on* A. *Denote* B_i *the central simple algebra* A_γ *for* i *odd, and* F *for* i *even* $(1 \leq i \leq n-1)$ *and put* $B_I = B_{i_1} \otimes \cdots \otimes B_{i_q}$ *for every* $I = \{i_1 < \cdots < i_q\} \subseteq \{1, \ldots, n-1\}$. *Then the following graded* $K_*(F)$*-modules are isomorphic*:

$$K_*(\mathrm{SL}_{1,A_\gamma} / \mathrm{Sp}(A_\gamma, \tau_\gamma)) \simeq \bigoplus_{I \subseteq \{1,\ldots,n-1\}} K_{*-|I|}(B_I).$$

Let $\delta\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{F}_4(F^{\mathrm{sep}})$ *be a* 1*-cocycle. Then the following graded* $K_*(F)$*-modules are isomorphic*:

$$K_*((\mathrm{E}_6 / \mathrm{F}_4)_\delta) \simeq \bigoplus_{I \subseteq \{1,2\}} K_{*-|I|}(F).$$

In §1 we construct multiplicative generators of K-theory and introduce some notation. In §2 we study the Merkurjev spectral sequence, which is used in §3 to compute the K-theories of the varieties in question as graded modules over the K-theory of a ground field. In §4 we compute the multiplicative generators of the K-theory and state the answer in the split case. In §5 we state the problem for twisted forms of the varieties. Then in §6 we describe how to twist the multiplicative generators with a cocycle. Finally, in §7 we show how Panin's splitting principle helps to reduce the problem to the split case, which is already solved.

§1. Construction of generators for $K_1(G/H)$

1.1. Representation rings of SL_{2n} and Sp_{2n}, E_6, and F_4.

Definition 1. Let G be an algebraic group over a field F. The *representation ring* $R(G)$ of a group G is the Grothendieck group of the category $\mathrm{Rep}_F(G)$ with multiplication induced by the tensor product of representations.

Suppose we have a subgroup $i\colon H \hookrightarrow G$. Then restriction of representations induces a homomorphism $i^*\colon R(G) \to R(H)$.

SL_{2n} *and* Sp_{2n}. Denote the vector representation by V. Then for the representation rings of the groups SL_{2n} and Sp_{2n} we have

$$R(\mathrm{SL}_{2n}) = \mathbb{Z}[V, \Lambda^2 V, \ldots, \Lambda^{2n-1} V], \quad R(\mathrm{Sp}_{2n}) = \mathbb{Z}[V, \Lambda^2 V, \ldots, \Lambda^n V].$$

The representations $\Lambda^k V$ and $\Lambda^{2n-k} V$ of the group SL_{2n} become isomorphic after restriction to Sp_{2n} for every $k = 1, \ldots, n-1$. The homomorphism $i^*\colon R(\mathrm{SL}_{2n}) \to R(\mathrm{Sp}_{2n})$ is surjective. The ideal $\mathrm{Ker}\, i^*$ is generated by the elements $\Lambda^k V - \Lambda^{2n-k} V$, where $k = 1, \ldots, n-1$.

E_6 *and* F_4. Let ρ and $\rho^\vee$ be the 27-dimensional fundamental representations of E_6, and let ρ' be the 26-dimensional fundamental representation of F_4. Then for the representation rings of the groups E_6 and F_4 we have

$$R(\mathrm{E}_6) = \mathbb{Z}[\rho, \rho^\vee, \Lambda^2\rho, \Lambda^2\rho^\vee, \Lambda^3\rho, \mathrm{Ad}_{\mathrm{E}_6}], \quad R(\mathrm{F}_4) = \mathbb{Z}[\rho', \Lambda^2\rho', \Lambda^3\rho', \mathrm{Ad}_{\mathrm{F}_4}],$$

and $\Lambda^3\rho \simeq \Lambda^3\rho^\vee$. The representations ρ and $\rho^\vee$ become isomorphic after restriction to F_4. It is known that $i^*(\rho) = i^*(\rho^\vee) = \rho' + 1$; $i^*(\mathrm{Ad}_{\mathrm{E}_6}) = \rho' + \mathrm{Ad}_{\mathrm{F}_4}$, see [5, p. 298]. Hence, the homomorphism i^* is surjective. The ideal $\mathrm{Ker}\, i^*$ is generated by the elements $\rho - \rho^\vee$ and $\Lambda^2\rho - \Lambda^2\rho^\vee$.

1.2. Construction. Suppose we have an affine homogeneous variety G/H. Assume that there are two nonisomorphic representations of the group G that are isomorphic when restricted to the subgroup H. In other words, there are homomorphisms $\phi, \psi\colon G \to \mathrm{GL}_k(F)$ and a matrix $\alpha \in \mathrm{GL}_k(F)$ such that $\phi(h) = \alpha^{-1}\psi(h)\alpha$ for all $h \in H$.

Using these data, we construct a well-defined map χ from G/H to $\mathrm{GL}_k(F)$: $[g] \mapsto \phi(g)\alpha^{-1}\psi(g)^{-1}\alpha$. We identify $\mathrm{Mor}_F(G/H, \mathrm{GL}_k)$ with $\mathrm{GL}_k(F[G/H])$ and consider the composition

$$\mathrm{GL}_k(F[G/H]) \hookrightarrow \mathrm{GL}(F[G/H]) \twoheadrightarrow K_1(F[G/H]) = K_1(G/H).$$

This way, the map χ gives us an element in $K_1(G/H)$. It is denoted by $\beta(\phi - \psi)$ and is defined by the following formula:

$$\beta(\phi - \psi) = \big[\, [g] \mapsto \phi(g)\alpha^{-1}\psi(g)^{-1}\alpha \big] \in K_1(G/H); \quad [g] \in G/H.$$

1.3. Application. Here we are going to exhibit certain elements of $K_1(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ and $K_1(\mathrm{E}_6 / \mathrm{F}_4)$, and later we shall show that they are multiplicative generators of the $K_*(F)$-algebras $K_*(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ and $K_*(\mathrm{E}_6 / \mathrm{F}_4)$. These varieties are affine as quotients of reductive groups by reductive subgroups (see [10]), so we can apply the construction described in Subsection 1.2.

For the group SL_{2n}, consider the vector representation V and its exterior powers $\Lambda^k V$. For every $1 \le k \le n-1$ the representations $\Lambda^k V$ and $\Lambda^{2n-k} V$ are isomorphic when restricted to Sp_{2n} (see 1.1). The corresponding elements of $K_1(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ are defined as follows:

$$t_k = \beta(\Lambda^k V - \Lambda^{2n-k} V), \quad 1 \le k \le n-1.$$

For the group E_6, consider the fundamental representations ρ and $\rho^\vee$, which are isomorphic when restricted to F_4 (see 1.1). Here are the desired elements of $K_1(\mathrm{E}_6 / \mathrm{F}_4)$:

$$s_1 = \beta(\rho - \rho^\vee); \quad s_2 = \beta(\Lambda^2\rho - \Lambda^2\rho^\vee).$$

1.4. Notation. We introduce some notation that will be used later.

- G/H (or X) stands for both varieties $\mathrm{SL}_{2n} / \mathrm{Sp}_{2n}$ and $\mathrm{E}_6 / \mathrm{F}_4$;
- $\rho_1, \dots, \rho_l$ are fundamental representations of the group G;
- $\{(\rho_{i_1}, \rho_{i_2})\}_{i=1}^m$ are pairs of fundamental representations of G that are isomorphic when restricted to H ($m = n-1$ or $m = 2$);
- $\widehat{\rho_i} = \rho_{i_1} - \rho_{i_2}$ are elements of $R(G)$ that generate $\mathrm{Ker}\, i^*$ (see Subsection 1.1).

§2. Merkurjev spectral sequence

The Merkurjev spectral sequence allows us to express the K-theory of a variety X equipped with an action of an algebraic group G in terms of the G-equivariant K-theory of X (see [1, 6]).

Definition 2. Let X be a variety equipped with an action of an algebraic group G. The *G-equivariant K-theory* of X is the K-theory of the category of G-equivariant vector bundles over X. It is denoted by $K_*(G; X)$.

For computing $K_*(G/H)$ as a $K_*(F)$-module, we need the following theorem by Merkurjev [1, Theorem 4.3].

Theorem (Merkurjev)**.** *Let G be a split reductive group such that $\pi_1(G)$ is torsion free, and let X be a G-scheme. Then there is a spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K_q(G; X)) \Longrightarrow K_{p+q}(X).$$

Since both groups $G = \mathrm{SL}_{2n}$ and $G = \mathrm{E}_6$ are simple and simply connected, their fundamental groups are trivial, see [1, Corollary 1.3]. Applying this theorem to the case of the variety G/H, on which the group G acts by left translation, we get the following spectral sequence:

$$E^2_{p,q} = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, K_q(G; G/H)) \Longrightarrow K_{p+q}(G/H).$$

Let us compute the terms of its second page.

2.1. Computation of $E^2_{p,q}$.

Lemma 1. *$K_i(G; G/H) \simeq R(H) \otimes K_i(F)$ as $R(H)$-modules.*

Proof. This statement was proved in [3, Lemma 9]. The proof is based on the fact that the categories $\mathrm{Vect}^G(G/H)$ and $\mathrm{Rep}(H)$ are equivalent, see [1, Proposition 2.10]. □

Therefore, we need to compute the second page of the following spectral sequence:

$$E^2_{p,q} = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, R(H) \otimes K_q(F)) \Longrightarrow K_{p+q}(G/H).$$

First, we treat the case where $q = 0$.

Computation of $\mathrm{Tor}_p^{R(G)}(\mathbb{Z}, R(H))$. We start with observing that $\mathbb{Z}$ can be viewed as an $R(G)$-module via the dimension homomorphism $R(G) \to \mathbb{Z}$, and $R(H)$ becomes an $R(G)$-module via the homomorphism $i^*\colon R(G) \to R(H)$. Recall that for the two varieties G/H under consideration the homomorphism i^* is surjective (see 1.1).

Note that the sequence $(\widehat{\rho_1}, \dots, \widehat{\rho_m})$ is regular in $R(G)$. Hence, we can write the corresponding Koszul resolution $\mathrm{K}_\bullet \to R(H)$, consisting of free $R(G)$-modules:

$$\Lambda^m(R(G)^m) \xhookrightarrow{d_m} \dots \longrightarrow \Lambda^2(R(G)^m) \xrightarrow{d_2} R(G)^m \xrightarrow{d_1} R(G) \xrightarrow{i^*} R(H)$$

Let e_i generate $R(G)^m$ as a free $R(G)$-module ($i = 1 \dots m$); then the differentials are defined in the following way: $d_1\colon e_i \mapsto \widehat{\rho_i}$; $\quad d_2\colon e_i \wedge e_j \mapsto \widehat{\rho_i} \cdot e_j - \widehat{\rho_j} \cdot e_i$; etc.

Consider the isomorphism $R(H) \otimes_{R(G)} \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$: $\quad \phi \otimes n \mapsto \dim(\phi) \cdot n$. We multiply the resolution $\mathrm{K}_\bullet$ termwise by $\mathbb{Z}$ and apply this isomorphism:

$$- \otimes_{R(G)} \mathbb{Z}\colon \qquad \Lambda^m(\mathbb{Z}^m) \to \dots \to \Lambda^2(\mathbb{Z}^m) \to \mathbb{Z}^m \to \mathbb{Z} \to \mathbb{Z} \to 0$$

All the differentials will become zero because $\dim(\rho_{i_1}) = \dim(\rho_{i_2})$, and so $\dim(\widehat{\rho_i}) = 0$ for every i.

As a result, we get the formula

$$\mathrm{Tor}_p^{R(G)}(\mathbb{Z}, R(H)) = \mathrm{H}_p(\mathrm{K}_\bullet \otimes_{R(G)} \mathbb{Z}) = \Lambda^p(\mathbb{Z}^m). \tag{1}$$

Final presentation of $E^2_{p,q}$. To finish the computation of $E^2_{p,q}$, we need the following lemma.

Lemma 2. $\mathrm{Tor}_p^{R(G)}(\mathbb{Z}, R(H) \otimes K_i(F)) = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, R(H)) \otimes K_i(F)$ *for every* $i \geq 0$.

Proof. Because of the associativity of the tensor product, we have

$$(\mathbb{Z} \otimes_{R(G)} R(H)) \otimes K_i(F) = \mathbb{Z} \otimes_{R(G)} (R(H) \otimes K_i(F)).$$

This implies the existence of two spectral sequences that converge to the triple Tor:

$$\widetilde{E}_2^{p,q} = \mathrm{Tor}_p^{\mathbb{Z}}(\mathrm{Tor}_q^{R(G)}(\mathbb{Z}, R(H)), K_i(F)) \Longrightarrow \mathrm{Tor}_{p+q}(\mathbb{Z}, R(H), K_i(F)),$$
$$\widehat{E}_2^{p,q} = \mathrm{Tor}_p^{R(G)}(\mathbb{Z}, \mathrm{Tor}_q^{\mathbb{Z}}(R(H), K_i(F))) \Longrightarrow \mathrm{Tor}_{p+q}(\mathbb{Z}, R(H), K_i(F)).$$

Observe that $\widetilde{E}_2^{p,q} = 0$ for $p \neq 0$, because $\mathrm{Tor}_q^{R(G)}(\mathbb{Z}, R(H))$ is a free $\mathbb{Z}$-module (see (1)); $\widehat{E}_2^{p,q} = 0$ for $q \neq 0$ because $R(H)$ is a free $\mathbb{Z}$-module. Therefore, both spectral

sequences degenerate on the second page, and $\widetilde{E}_2^{0,p} = \widehat{E}_2^{p,0}$, which is indeed the claim of the lemma. □

Lemma 1, formula (1), and Lemma 2 imply that the Merkurjev spectral sequence for the varieties $G/H = \mathrm{SL}_{2n} / \mathrm{Sp}_{2n}$ and $G/H = \mathrm{E}_6 / \mathrm{F}_4$ looks like this:

$$E^2_{p,q} = \Lambda^p(\mathbb{Z}^m) \otimes K_q(F) \Longrightarrow K_{p+q}(G/H),$$

where $m = rk(G) - rk(H)$. The spectral sequence is first-quadrant, its differential $d^2_{p,q}$ acts from $E^2_{p,q}$ to $E^2_{p-2,q+1}$.

2.2. Degeneration of $E^*_{p,q}$. The Merkurjev spectral sequence is a special case of the Levine spectral sequence, see [1, 3.1]. There is a multiplicative structure on the zero row of the second page of the Levine spectral sequence, which is denoted by $\smile_2$, see [4, §1]. To check that the spectral sequence $E^*_{p,q}$ degenerates we need the following technical lemma.

Lemma 3. *The multiplicative structure $\smile_2$ on $\bigoplus_p E^2_{p,0}$ coincides with the natural product on $\bigoplus_p \Lambda^p(\mathbb{Z}^m)$.*

Proof. The following statement is true (see [4, Example 1.1]): let R be a local ring, $\mathfrak{m}$ its maximal ideal, $x_1, \dots, x_n$ a regular sequence in $\mathfrak{m}$, and B an ideal in R. Then the multiplicative structure $\smile_2$ on $\bigoplus_p \mathrm{Tor}^R_p(R/(x_1, \dots, x_n), R/B)$ coincides with the natural product.

Under the conditions of the lemma, we show that on $\bigoplus_p \Lambda^p(\mathbb{Z}^m) = \bigoplus_p \mathrm{Tor}^{R(G)}_p(\mathbb{Z}, R(H))$ the two products coincide. Let us reduce this case to the proposition stated above.

Observe that the sequence $(\rho_1, \dots, \rho_l)$ is regular in $R(G)$ and $\mathbb{Z} = R(G)/(\rho_1, \dots, \rho_l)$. Recall that for our varieties we have $R(H) = R(G)/I$, where $I = \mathrm{Ker}\, i^*$. Thus

$$\bigoplus_p E^2_{p,0} = \bigoplus_p \mathrm{Tor}^{R(G)}_p(R(G)/(\rho_1, \dots, \rho_l), R(G)/I).$$

A product on $\bigoplus_p \Lambda^p(\mathbb{Z}^m)$ admits natural extension by applying the localization homomorphism $\bigoplus_p \Lambda^p(\mathbb{Z}^m) \hookrightarrow \bigoplus_p \Lambda^p(\mathbb{Q}^m)$. Passing to localization allows us to consider the graded ring $\bigoplus_p \mathrm{Tor}^R_p(R/\mathfrak{a}, R/J)$ in which the ideal $\mathfrak{a}$ is already maximal. Using the identity

$$\mathrm{Tor}^R_p(R/\mathfrak{a}, R/J) = \mathrm{Tor}^{R_\mathfrak{a}}_p(R_\mathfrak{a}/(\mathfrak{a} \cdot R_\mathfrak{a}), R_\mathfrak{a}/J_\mathfrak{a}),$$

we reduce the statement to the case of a local ring R. □

Consider the edge homomorphisms $g_i \colon K_i(G/H) \to E^2_{i,0} = \Lambda^i(\mathbb{Z}^m)$. Since the differentials $d^r_{1,0}$ are zero for every $r \geq 2$, we see that $E^\infty_{1,0} = E^2_{1,0}$, hence g_1 is surjective. The edge homomorphism is multiplicative with respect to the product $\smile_2$, see [4, Proposition 1.3], i.e., $g_i(a) \smile_2 g_j(b) = g_{i+j}(a \cup b)$. From Lemma 3 it follows that the edge homomorphism is multiplicative with respect to the natural product on $\Lambda(\mathbb{Z}^m)$. The algebra $\Lambda(\mathbb{Z}^m)$ is generated by the component $\Lambda^1(\mathbb{Z}^m)$, so that the surjectivity of g_1 implies that the g_i are surjective for every i. By the surjectivity of the homomorphisms g_i, we have $E^2_{i,0} = E^\infty_{i,0}$. Therefore, all the differentials $d^r_{p,0}$ are zero for every $r \geq 2$.

The Levine spectral sequence is a module over $K_*(F)$, see [4, Lemma 1.2]. Since $E^2_{p,q} = E^2_{p,0} \otimes K_q(F)$ and $d^2_{p,0} = 0$, all the differentials on the second page are zero. Since $d^r_{p,0} = 0$ and $E^r_{p,q}$ is a $K_*(F)$-module for every $r \geq 2$, we see that the differentials are zero on the higher pages also. As a result, we conclude that the spectral sequence degenerates at the second page.

Corollary 1. *There is a filtration on $K_*(G/H)$ whose successive quotients are $K_*(F)$, $K_*(F)^m$, $\Lambda^2(K_*(F)^m), \dots, \Lambda^m(K_*(F)^m)$.*

Proof. Since $E^{\infty}_{p,q} = E^2_{p,q} = \Lambda^p(\mathbb{Z}^m) \otimes K_q(F) \Longrightarrow K_{p+q}(G/H)$, there is a filtration on each $K_i(G/H)$ with the successive quotients $K_i(F)$, $K_{i-1}(F)^m$, $\Lambda^2(K_{i-2}(F)^m)$, $\ldots$, $\Lambda^i(\mathbb{Z}^m)$. These filtrations give a general filtration on $K_*(G/H)$ with the desired successive quotients. □

Corollary 2. *$K_*(G/H)$ is a free $K_*(F)$-module of rank 2^m.*

Proof. Consider the filtration on $K_*(G/H)$ defined in Corollary 1. Since all the successive quotients are free $K_*(F)$-modules of finite rank, the short exact sequences ending with those modules are split. This means that we have an isomorphism of $K_*(F)$-modules (which may fail to respect the graded structures):

$$K_*(G/H) \simeq K_*(F) \oplus K_*(F)^m \oplus \Lambda^2(K_*(F)^m) \oplus \cdots \oplus \Lambda^m(K_*(F)^m).$$

□

2.3. Application of $E^*_{p,q}$. We get some information about $K_*(G/H)$ using the spectral sequence considered above.

Lemma 4. *$K_1(G/H) \simeq K_1(F) \oplus \mathbb{Z}^m$. In particular, for reduced K-theory we have $\widetilde{K}_1(G/H) \simeq \mathbb{Z}^m$.*

Proof. The filtration on $K_1(G/H)$ implies the existence of a short exact sequence

$$0 \longrightarrow K_1(F) \longrightarrow K_1(G/H) \longrightarrow \mathbb{Z}^m \longrightarrow 0.$$

It splits via the homomorphism $j^*\colon K_1(G/H) \to K_1(F)$ induced by the inclusion $j\colon pt \hookrightarrow G/H$. □

We introduce some notation: A is the graded ring $K_*(F)$; $A^+ = \bigoplus_{i>0} A_i$ $(A/A^+ = \mathbb{Z})$; B is the graded A-module $K_*(G/H)$. The quotient module $B/(A^+ \cdot B)$ has the structure of a $\mathbb{Z}$-module.

Lemma 5. *There is an isomorphism of Abelian groups $B/(A^+ \cdot B) \simeq \Lambda(\mathbb{Z}^m)$.*

Proof. For every $p > 0$, there is a filtration on $K_p(G/H)$ of length $p+1$ such that $K_p(G/H)^{(p)} = K_p(F)$. Taking the quotient by $K_p(G/H)^{(p)}$, we get a filtration of length p on the quotient group, the first term of which is again zero in $B/(A^+ \cdot B)$. Iterating this process we see that the homomorphism $B \to B/(A^+ \cdot B)$ sends the free summand $K_p(G/H)$ to $\Lambda^p(\mathbb{Z}^m) = E^2_{p,0}$. □

§3. Computation of $K_*(G/H)$ as a graded $K_*(F)$-module

We view the exterior algebra $\Lambda(\mathbb{Z}^m)$ as an Abelian group with the natural grading: $\Lambda(\mathbb{Z}^m)_i = \Lambda^i(\mathbb{Z}^m)$.

Proposition 1. *There is an isomorphism of graded $K_*(F)$-modules:*

$$K_*(G/H) \simeq K_*(F) \otimes_{\mathbb{Z}} \Lambda(\mathbb{Z}^m).$$

Proof. Let S be a graded ring, and let $S^+ = \bigoplus_{i>0} S_i$. For a graded S-module P, we shall denote by $\overline{P}$ the S/S^+-module $P/(S^+ \cdot P)$.

As earlier, we shall write A for $K_*(F)$. For graded A-modules, we denote $B = K_*(G/H)$, $C = K_*(F) \otimes \Lambda(\mathbb{Z}^m)$, and let $j\colon A \hookrightarrow B$ be the canonical inclusion.

Consider the following homomorphism of graded A-modules:

$$\phi = j \otimes \Lambda(\mathrm{id})\colon C \to B,$$

where $\Lambda(\mathrm{id}) = \mathrm{id}\colon \mathbb{Z}^m \to \mathbb{Z}^m \subset B_1$ (see Lemma 4), and $\Lambda(\mathrm{id})(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \Lambda(\mathrm{id})(e_{i_1}) \cup \cdots \cup \Lambda(\mathrm{id})(e_{i_r}) \in B_r$.

We show that ϕ is an isomorphism. For that, we use the graded version of the Nakayama lemma.

Lemma (Graded Nakayama Lemma)**.** *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, and let $R^+ = \bigoplus_{i>0} R_i$. Let M be a graded R-module such that $M_j = 0$ for $j << 0$. Then $R^+ \cdot M = M$ implies $M = 0$.*

First, we check that ϕ is an epimorphism.

Lemma 6. *The homomorphism ϕ is surjective.*

Proof. Observe that $\overline{C} = \mathbb{Z} \otimes \Lambda(\mathbb{Z}^m) \simeq \Lambda(\mathbb{Z}^m)$. From Lemma 5 it follows that also $\overline{B} \simeq \Lambda(\mathbb{Z}^m)$. The induced homomorphism of $\mathbb{Z}$-modules $\overline{\phi}\colon \overline{C} \to \overline{B}$ maps $\mathbb{Z}^m$ to $\mathbb{Z}^m$ isomorphically. Thus, $\overline{\phi}$ is an isomorphism, implying that $\overline{\operatorname{Coker}\phi} = 0$. Then $\operatorname{Coker}\phi = 0$ by the graded Nakayama lemma. □

Corollary 2 and Lemma 6 show that the homomorphism ϕ is a graded epimorphism of free finitely-generated $K_*(F)$-modules of the same rank. This implies that $\overline{\operatorname{Ker}\phi} = 0$, and so $\operatorname{Ker}\phi = 0$ by the graded Nakayma lemma. Thus, ϕ is an isomorphism of graded $K_*(F)$-modules. □

§4. Computation of generators of $K_*(G/H)$ as a $K_*(F)$-module

4.1. Computation of generators of $\widetilde{K}_1(G/H)$. From Lemma 4 it follows that for the reduced K-theory we have $\widetilde{K}_1(G/H) \simeq \mathbb{Z}^m$. To get the final answer, we only need to find m generating elements for $\widetilde{K}_1(G/H)$. First, we consider $\widetilde{K}_1(G)$. In [4, Theorem 2.1 and Corollary 2.2], M. Levine proved that for $G = \mathrm{SL}_{2n}$ and $G = \mathrm{E}_6$ there is an isomorphism $\widetilde{K}_1(G) \simeq \mathbb{Z}^l$, where $l = rk(G)$; moreover, $\widetilde{K}_1(G)$ is generated by the elements $[\rho_1], \dots, [\rho_l] \in K_1(G)$.

Recall that in Subsection 1.2 for each pair of representations (ρ_{i_1}, ρ_{i_2}) we constructed an element $\beta(\rho_{i_1} - \rho_{i_2}) \in K_1(G/H)$.

Proposition 2. *$\widetilde{K}_1(G/H)$ is generated by the elements $u_i = \beta(\rho_{i_1} - \rho_{i_2})$, $1 \le i \le m$.*

Proof. Let $\mathbb{Z}^m$ be generated by elements $e_1, \dots, e_m$ as a free Abelian group. Consider the following diagram of Abelian groups and their homomorphisms:

$$\begin{tikzcd} \mathbb{Z}^m \arrow[r, "\psi"] & \widetilde{K}_1(G/H) \simeq \mathbb{Z}^m \arrow[dl, "p^*"] \\ \widetilde{K}_1(G) \simeq \mathbb{Z}^l \arrow[u, "\chi"] & \end{tikzcd}$$

The homomorphisms are defined in the following way:

1) $\psi\colon e_i \mapsto u_i$;

2) $p^*\colon \widetilde{K}_1(G/H) \to \widetilde{K}_1(G)$ is induced by the projection $p\colon G \to G/H$;

3) $\chi\colon \widetilde{K}_1(G) \to \mathbb{Z}^m$ is defined on generators: $[\rho_k] \mapsto \begin{cases} e_i & \text{if } k = i_1, \\ 0 & \text{otherwise.} \end{cases}$

We show that ψ is an isomorphism. Observe that

$$p^*(u_i) = p^*([\rho_{i_1} \cdot \alpha_i^{-1} \cdot \rho_{i_2}^{-1} \cdot \alpha_i]) = [\rho_{i_1}] + [\alpha_i^{-1}] + [\rho_{i_2}^{-1}] + [\alpha_i] = [\rho_{i_1}] - [\rho_{i_2}].$$

Therefore,

$$(\chi \circ p^* \circ \psi)(e_i) = (\chi \circ p^*)(u_i) = \chi([\rho_{i_1}] - [\rho_{i_2}]) = e_i.$$

Hence, $\chi \circ p^* \circ \psi = \mathrm{id}$, i.e., ψ is injective. Note that $\psi\colon \mathbb{Z}^m \to \mathbb{Z}^m$ has a left inverse $\chi \circ p^*$. This implies that $\operatorname{Coker}\psi = 0$ and so ψ is surjective. □

4.2. Final result. To study later the K-theory of twisted forms of varieties, now we formulate the result obtained above in functorial terms.

Let X be a variety, and let $\xi = (x_1, \dots, x_m)$ be a set of elements in $K_1(X)$. For every subset of indices $I = \{i_1 < \dots < i_q\} \subseteq \{1, \dots, m\}$, we denote $x_I = x_{i_1} \cup \dots \cup x_{i_q} \in K_{|I|}(X)$, where $|I|$ is the cardinality of I. For $I = \varnothing$, define $x_\varnothing = 1 \in K_0(F)$.

Consider the following homomorphisms of graded $K_*(F)$-modules:

$$\Theta_{I,\xi}\colon K_{*-|I|}(F) \to K_*(X); \quad \alpha \mapsto x_I \cup \alpha.$$

We define the following homomorphism Θ_ξ:

$$\Theta_\xi = \sum_I \Theta_{I,\xi}\colon \bigoplus_I K_{*-|I|}(F) \to K_*(X),$$

where I runs through all the subsets of a set $\{1, \dots, m\}$.

The final result follows from Propositions 1 and 2.

Theorem 1. *Let $t = (t_1, \dots, t_{n-1})$ and $s = (s_1, s_2)$ be the collections of elements in $K_1(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})$ and $K_1(\mathrm{E}_6 / \mathrm{F}_4)$, respectively, defined in Subsection* 1.3. *Then the homomorphisms Θ_t and Θ_s of graded $K_*(F)$-modules are isomorphisms*:

$$\Theta_t\colon \bigoplus_{I \subseteq \{1,\dots,n-1\}} K_{*-|I|}(F) \xrightarrow{\sim} K_*(\mathrm{SL}_{2n} / \mathrm{Sp}_{2n});$$

$$\Theta_s\colon \bigoplus_{I \subseteq \{1,2\}} K_{*-|I|}(F) \xrightarrow{\sim} K_*(\mathrm{E}_6 / \mathrm{F}_4).$$

§5. Twisted forms and central simple algebras

From now on we assume that $\mathrm{char}(F) \neq 2$. As earlier, we denote the varieties $\mathrm{SL}_{2n} / \mathrm{Sp}_{2n}$ and $\mathrm{E}_6 / \mathrm{F}_4$ by G/H or X. We denote the center of an algebraic group G by $Z(G)$.

Consider the action of the group H on the variety G/H by left translation. This action can be extended to $\overline{H} = H/Z(H)$ (in the first case $\overline{H} = \mathrm{Sp}_{2n} / \mu_2$, and in the second case $\overline{H} = \mathrm{F}_4$). We fix a 1-cocycle $\gamma\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \overline{H}(F^{\mathrm{sep}})$. Since there is an action of $\overline{H}$ on G/H, we can consider a twisted form X_γ of the variety X. The rest of the paper is devoted to the computation of $K_*(X_\gamma)$.

5.1. Twisting of central simple algebras.

Definition 3. For an algebraic group G, we introduce a notation for the *group of characters of the center*: $\mathrm{Ch}(G) = \mathrm{Hom}(Z(G), \mathbb{G}_m)$.

Definition 4. A representation $\sigma\colon G \to \mathrm{GL}(V)$ of an algebraic group G is said to be Ch-*homogeneous* if there is a character $\chi \in \mathrm{Ch}(G)$ such that $\sigma(z)v = \chi(z) \cdot v$ for every $z \in Z(G)$, $v \in V$. In particular, the irreducible representations are Ch-homogeneous.

Let $\sigma\colon H \to \mathrm{GL}(V)$ be a Ch-homogeneous representation of the group H, and let $A = \mathrm{End}_F(V)$ be a central simple algebra (we write $\mathrm{End}(V)$ for $\mathrm{End}_F(V)$). Consider the action of H on A by conjugation: $(h, \alpha) \mapsto \sigma(h)\alpha\sigma(h)^{-1}$. It induces an action of $\overline{H}$ on the algebra A. Given the action of $\overline{H}$ on the algebra A and a cocycle $\gamma\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \overline{H}(F^{\mathrm{sep}})$, we can use the Tits construction to build a twisted algebra A^σ_γ (see [13, §3] or [2, 8.6]).

Remark 1. Let V be a $2n$-dimensional vector space over F, let $A = \mathrm{End}(V)$, and let τ be an involution on A corresponding to the standard antisymmetric form. Consider

the action of $\overline{\mathrm{Sp}}_{2n}$ on SL_{2n} and Sp_{2n} by conjugation. Then for the twisted forms of $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ we have:

$$(\mathrm{SL}_{2n}/\mathrm{Sp}_{2n})_\gamma \simeq (\mathrm{SL}_{2n})_\gamma/(\mathrm{Sp}_{2n})_\gamma = (\mathrm{SL}_{1,A})_\gamma/\mathrm{Sp}(A,\tau)_\gamma = \mathrm{SL}_{1,A_\gamma}/\mathrm{Sp}(A_\gamma,\tau_\gamma).$$

5.2. Computation of central simple algebras. Let $\sigma\colon H \to \mathrm{GL}(V)$ be a Ch-homogeneous representation of a group H, and let $A = \mathrm{End}(V)$. The class of the algebra A_γ^σ in the Brauer group $\mathrm{Br}(F)$ depends only on the character $\chi \in \mathrm{Ch}(H)$ representing the action of $Z(H)$ on V under σ, see [2, 8.7]. We shall compute the equivalence classes of the algebras A_γ^σ in the Brauer group for fundamental representations σ of the groups $H = \mathrm{Sp}_{2n}$ and $H = \mathrm{F}_4$.

The center of the group Sp_{2n} is equal to μ_2, so $\mathrm{Ch}(\mathrm{Sp}_{2n}) = \mathbb{Z}/2\mathbb{Z}$. Under the vector representation V, the center acts by the character $\overline{1} \in \mathbb{Z}/2\mathbb{Z}$. Under the representation $\Lambda^r V$, the center acts by the character $\overline{r} \in \mathbb{Z}/2\mathbb{Z}$. Hence, in $\mathrm{Br}(F)$ we have the following equivalences for $A_{i,\gamma} = \mathrm{End}(\Lambda^i V)_\gamma^{\Lambda^i V}$:

$$A_{i,\gamma} \sim A_\gamma \ \text{ if } \ i \ \text{ is odd}; \quad A_{i,\gamma} \sim F \ \text{ if } \ i \ \text{ is even},$$

where $A = \mathrm{End}(V)$, V is a $2n$-dimensional vector space, $i = 1, \dots, n$.

Since the center of the group F_4 is trivial, the group of characters is also trivial. Therefore, for all four algebras $A_{i,\gamma} = \mathrm{End}(V_i)_\gamma^{\sigma_i}$ corresponding to the fundamental representations σ_i of the group F_4 we have $A_{i,\gamma} \sim F$.

§6. Construction of certain elements in K_1

Definition 5. Let B be a central simple F-algebra. For an affine variety Y, we put $B[Y] = B \otimes_F F[Y]$. Then K_1 with coefficients in B is defined as follows:

$$K_1(Y,B) = K_1(B[Y]) = \mathrm{GL}(B[Y])/E(B[Y]).$$

General construction. Suppose there is a morphism $f \in \mathrm{Mor}_F(Y, \mathrm{GL}_{1,B})$. We identify $\mathrm{Mor}_F(Y, \mathrm{GL}_{1,B})$ with $\mathrm{GL}_1(B[Y])$ (see [2, §9]), and consider the composition

$$\mathrm{GL}_1(B[Y]) \hookrightarrow \mathrm{GL}(B[Y]) \twoheadrightarrow K_1(B[Y]) \xrightarrow{\sim} K_1(B^{op}[Y])$$

Thus, to the morphism f we can assign an element $[f] \in K_1(Y, B^{op})$.

Application. Suppose that the representations (ρ_{i_1}, ρ_{i_2}) of the group G (see Subsection 1.4) act on a vector space V_i. Then each pair gives rise to the map $\widetilde{\rho}_i\colon G/H \to \mathrm{GL}(V_i)$ described in Subsection 1.2:

$$\widetilde{\rho}_i\colon [g] \mapsto \rho_{i_1}(g)\alpha_i^{-1}\rho_{i_2}(g)^{-1}\alpha_i,$$

where the α_i satisfy $\rho_{i_1}(h) = \alpha_i^{-1}\rho_{i_2}(h)\alpha_i$ for every $h \in H$.

Consider the action of $\overline{H}$ on $\mathrm{GL}(V_i)$: $(h,a) \mapsto \rho_{i_1}(h)a\rho_{i_1}(h)^{-1}$. Then the $\widetilde{\rho}_i$ are $\overline{H}$-equivariant morphisms (with respect to the action of $\overline{H}$ on G/H by left translation).

Denote $A_i = \mathrm{End}(V_i)$. Observe that the $\rho_{i_1}\colon H \to \mathrm{GL}(V_i)$ are Ch-homogeneous. Therefore, we can twist $\mathrm{GL}(V_i) = \mathrm{GL}_{1,A_i}$ with a 1-cocycle $\gamma\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \bar{H}(F^{\mathrm{sep}})$ (see Subsection 5.1). We shall write $A_{i,\gamma}$ for $A_{i,\gamma}^{\rho_{i_1}}$. Furthermore, we can twist $X = G/H$ and $\widetilde{\rho}_i$ with this cocycle (because of the $\overline{H}$-equivariance of the morphisms $\widetilde{\rho}_i$). We get the following objects:

$$(G/H)_\gamma; \quad \mathrm{GL}(V_i)_\gamma = GL_{1,A_{i,\gamma}}; \quad \widetilde{\rho}_{i,\gamma}\colon (G/H)_\gamma \to \mathrm{GL}_{1,A_{i,\gamma}},$$

where $\widetilde{\rho}_{i,\gamma} \in \mathrm{Mor}_F(X_\gamma, \mathrm{GL}_{1,A_{i,\gamma}})$. With the morphism $\widetilde{\rho}_{i,\gamma}$ we associate the element $[\widetilde{\rho}_{i,\gamma}] \in K_1(X_\gamma, A_{i,\gamma}^{op})$ as it was described in the general construction. After fixing a cocycle γ, the corresponding elements of the K-theory will be denoted by $[\widetilde{t}_1], \dots, [\widetilde{t}_{n-1}]$ in the case of the variety $(\mathrm{SL}_{2n}/\mathrm{Sp}_{2n})_\gamma$ and by $[\widetilde{s}_1], [\widetilde{s}_2]$ in the case of the variety $(\mathrm{E}_6/\mathrm{F}_4)_\gamma$.

Recall that from Subsection 5.2 we know the equivalence classes of the algebras $A_{i,\gamma}$ in the Brauer group. Since $K_1(Y, F^{op}) = K_1(Y)$ for every variety Y, we have

$$\begin{aligned}
&[\widetilde{t}_i] \in K_1((\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})_\gamma, A_\gamma^{op}) \ \text{ if } \ i \ \text{ is odd},\\
&[\widetilde{t}_i] \in K_1((\mathrm{SL}_{2n} / \mathrm{Sp}_{2n})_\gamma) \quad \text{if } \ i \ \text{ even};\\
&[\widetilde{s}_1], [\widetilde{s}_2] \in K_1((\mathrm{E}_6 / \mathrm{F}_4)_\gamma),
\end{aligned}$$

where $0 \le i \le n-1$, V is a $2n$-dimensional F-vector space, $A = \mathrm{End}(V)$.

§7. Computation of K-theory for twisted forms

Let Y be an affine variety, $B_1, \dots, B_m$ central simple F-algebras, and $\xi = (x_1, \dots, x_m)$ a set of elements such that $x_i \in K_1(Y, B_i^{op})$. For every subset $I = \{i_1 < \dots < i_q\} \subseteq \{1, \dots, m\}$ we denote $x_I = x_{i_1} \cup \dots \cup x_{i_q} \in K_{|I|}(Y, B_{i_1}^{op} \otimes \dots \otimes B_{i_q}^{op})$.

Define $B_I = B_{i_1} \otimes \dots \otimes B_{i_q}$ and consider the following homomorphism of graded $K_*(F)$-modules:

$$\Theta_{I,\xi} \colon K_{*-|I|}(B_I) \to K_*(Y); \quad \alpha \mapsto x_I \cup_{B_I} \alpha.$$

The homomorphism Θ_ξ is introduced as follows:

$$\Theta_\xi = \sum_I \Theta_{I,\xi} \colon \bigoplus_I K_{*-|I|}(B_I) \to K_*(Y),$$

where I runs through all subsets of the set $\{1, \dots, m\}$.

For the variety $X_\gamma = (G/H)_\gamma$ we take central simple algebras B_i equal to $A_{i,\gamma} = \mathrm{End}(V_i)_\gamma^{\rho_{i_1}}$, $i = 1, \dots, m$, where V_i is the vector space where the representations ρ_{i_1} and ρ_{i_2} of the group G act. We consider the set of elements $\widetilde{\rho} = ([\widetilde{\rho}_1], \dots, [\widetilde{\rho}_m])$, where $[\widetilde{\rho}_i] \in K_1(X_\gamma, A_{i,\gamma}^{op})$ (see §6). This way, we can define the homomorphism

$$\Theta_{\widetilde{\rho}} \colon \bigoplus_{I \subseteq \{1,\dots,m\}} K_{*-|I|}(B_I) \to K_*((G/H)_\gamma).$$

Panin's splitting principle tells us (see [2, Theorem 1.1]) that in order to prove that the homomorphism $\Theta_{\widetilde{\rho}}$ is an isomorphism, it suffices to check the following property.

Proposition 3. *Let $F \subset E$ be any field extension such that the cocycle γ_E is a coboundary. Then the homomorphism $\Theta_{\widetilde{\rho}}$ after extension of scalars up to the field E becomes an isomorphism*:

$$\Theta_{\widetilde{\rho}}(E) \colon \bigoplus_{I \subseteq \{1,\dots,m\}} K_{*-|I|}(B_I \otimes_F E) \xrightarrow{\sim} K_*((G/H)_\gamma \times \mathrm{Spec}\, E).$$

Proof. Since γ_E is trivial, all the twistings trivialize over the field E:

$$\begin{aligned}
&(G/H)_\gamma \times \mathrm{Spec}\, E \simeq (G/H)_E,\\
&\qquad A_{i,\gamma} \otimes E \simeq A_i \otimes E \sim E \ \ (\text{equivalence in } \mathrm{Br}(E)),\\
&\qquad [\widetilde{\rho}_i] \otimes E = t_{i,E} \ \text{ if } \ G/H = \mathrm{SL}_{2n} / \mathrm{Sp}_{2n},\\
&\qquad [\widetilde{\rho}_i] \otimes E = s_{i,E} \ \text{ if } \ G/H = \mathrm{E}_6 / \mathrm{F}_4,
\end{aligned}$$

where the t_i and s_i are defined as in Theorem 1. We see that the homomorphism $\Theta_{\widetilde{\rho}}(E)$ in the case of every variety G/H under consideration coincides with the corresponding isomorphism from Theorem 1 after scalar extension up to the field E. □

Therefore, for the varieties $\mathrm{SL}_{2n,\gamma} / \mathrm{Sp}_{2n,\gamma}$, as well as for the varieties $(\mathrm{E}_6 / \mathrm{F}_4)_\gamma$, the homomorphism $\Theta_{\widetilde{\rho}}$ is an isomorphism. This implies the final result.

Theorem 2. *Assume* $\mathrm{char}(F) \neq 2$. *Let* $\gamma\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to (\mathrm{Sp}_{2n}/\mu_2)(F^{\mathrm{sep}})$ *be a* 1*-cocycle. Let* $\widetilde{t} = ([\widetilde{t}_1], \dots, [\widetilde{t}_{n-1}])$ *be the set of elements defined in* §6*, let* $A = \mathrm{End}(V)$*, where* V *is a* $2n$*-dimensional vector space over* F*, and let* τ *be the standard symplectic involution on* A*. Let* B_i *denote the central simple algebra* A_γ *for* i *odd, and* F *for* i *even* $(1 \leq i \leq n-1)$*. Then the homomorphism* $\Theta_{\widetilde{t}}$ *is an isomorphism of graded* $K_*(F)$*-modules*:

$$\Theta_{\widetilde{t}}\colon \bigoplus_{I \subseteq \{1,\dots,n-1\}} K_{*-|I|}(B_I) \xrightarrow{\sim} K_*((\mathrm{SL}_{2n}/\mathrm{Sp}_{2n})_\gamma) = K_*(\mathrm{SL}_{1,A_\gamma}/\mathrm{Sp}(A_\gamma, \tau_\gamma)).$$

Let $\delta\colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{F}_4(F^{\mathrm{sep}})$ *be a* 1*-cocycle. Let* $\widetilde{s} = ([\widetilde{s}_1], [\widetilde{s}_2])$ *be the set of elements defined in* §6*. Then the homomorphism* $\Theta_{\widetilde{s}}$ *is an isomorphism of graded* $K_*(F)$*-modules*:

$$\Theta_{\widetilde{s}}\colon \bigoplus_{I \subseteq \{1,2\}} K_{*-|I|}(F) \xrightarrow{\sim} K_*((\mathrm{E}_6/\mathrm{F}_4)_\delta).$$

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