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SUBRING SUBGROUPS OF SYMPLECTIC GROUPS IN CHARACTERISTIC 2

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Abstract. In 2012, the second author obtained a description of the lattice of subgroups of a Chevalley group $G(\Phi, A)$ that contain the elementary subgroup $E(\Phi, K)$ over a subring $K \subseteq A$ provided $\Phi = B_n$, C_n, or F_4, $n \geq 2$, and 2 is invertible in K. It turned out that this lattice is a disjoint union of "sandwiches" parametrized by the subrings R such that $K \subseteq R \subseteq A$. In the present paper, a similar result is proved in the case where $\Phi = C_n$, $n \geq 3$, and $2 = 0$ in K. In this setting, more sandwiches are needed, namely those parametrized by the form rings (R, Λ) such that $K \subseteq \Lambda \subseteq R \subseteq A$. The result generalizes Ya. N. Nuzhin's theorem of 2013 concerning the root systems $\Phi = B_n$, C_n, $n \geq 3$, where the same description of the subgroup lattice is obtained, but under the condition that A and K are fields such that A is algebraic over K.

Introduction

Throughout this paper K, R, and A will denote commutative rings. Let $G = \mathrm{G}_P(\Phi, \_)$ denote a Chevalley–Demazure group scheme with a reduced irreducible root system Φ and weight lattice P. If the weight lattice P is not important, we leave it out of the notation. Let $E(A) = \mathrm{E}_P(\Phi, A)$ denote the elementary subgroup of $G(A)$, i.e., the subgroup generated by all elementary root unipotent elements $x_\alpha(t)$, $\alpha \in \Phi$, $t \in A$. Let K be a subring of A. We study the lattice $\mathcal{L} = L\big(E(K), G(A)\big)$ of subgroups of $G(A)$ that contain $E(K)$.

The standard description of $\mathcal{L}$ is called a *sandwich classification theorem.* It states that for each $H \in \mathcal{L}$ there exists a unique subring R between K and A such that H lies between $E(R)$ and its normalizer $N_A(R)$ in $G(A)$. The lattice $L\big(E(R), N_A(R)\big)$ of all subgroups of $G(A)$ that lie between $E(R)$ and $N_A(R)$ is called a *standard sandwich.* Thus, the sandwich classification theorem holds if and only if $\mathcal{L}$ is the disjoint union of all standard sandwiches. In [17], the second author proved the sandwich classification theorem in the case where Φ is doubly laced and 2 is invertible in K.

In this paper we consider the symplectic case of rank $n \geq 3$ *with* $2 = 0$ *in* A*, in particular, we always assume that* $\Phi = C_n$. In this situation we show that the sandwich classification theorem, as formulated above, fails. Let R be a subring of A containing K. Recall (see [5]) that an additive subgroup Λ of R is called a (symplectic) *form parameter* if it contains $2R$ and is closed under multiplication by ξ^2 for all $\xi \in R$. If $2 = 0$, the set $R^2 = \{\xi^2 \mid \xi \in R\}$ is a subring of R, and a form parameter Λ in R is an R^2-submodule of R. Let $\mathrm{Ep}_{2n}(R, \Lambda)$ denote the subgroup of $\mathrm{Sp}_{2n}(R)$ generated by all root unipotents $x_\alpha(\mu)$ and $x_\beta(\lambda)$, where α is a short root and $\mu \in R$, and β is a long root and $\lambda \in \Lambda$. Suppose now that $\Lambda \supseteq K$. Then, clearly, $\mathrm{Ep}_{2n}(R, \Lambda) \geq \mathrm{Ep}_{2n}(K)$. But

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one can check that if $\Lambda \neq R$, then $\mathrm{Ep}_{2n}(R, \Lambda)$ is not contained in any standard sandwich $L\big(\mathrm{Ep}_{2n}(R'), N_A(R')\big)$ such that $K \subseteq R'$. This shows that $\mathcal{L}$ cannot be a union of standard sandwiches, unless we enlarge our standard sandwiches or introduce additional ones. It turns out that the latter approach is the correct one. It is similar to that followed by the first author in [3] and by E. Abe and K. Suzuki in [2, 1] in order to provide enough sandwiches to classify the subgroups of $\mathrm{Sp}_{2n}(R)$ that are normalized by $\mathrm{Ep}_{2n}(R)$.

Under the assumption that A and K are fields and A is algebraic over K, the sandwich classification theorem was obtained by Ya. N. Nuzhin in [14, 15]. In [15] he considered the case of doubly laced root systems in characteristic 2 and G_2 in characteristics 2 and 3. In all these cases sandwiches were parametrized by pairs; each pair consisted of a subring and an additive subgroup with certain properties.

Let F be a field of characteristic 2. Then there are injections

$$\mathrm{Sp}_{2n}(F) \rightarrowtail \mathrm{SO}_{2n+1}(F) \rightarrowtail \mathrm{Sp}_{2n}(F),$$

which become isomorphisms if F is perfect, see [16, Theorem 28 and Example (a) after the theorem, and the Remark before Theorem 29]. Note that in this case the Chevalley groups $\mathrm{G}_P(C_n, F)$ and $\mathrm{G}_P(B_n, F)$ do not depend on the weight lattice P ([16], Exercise after Corollary 5 to Theorem 4$'$). Therefore, the lattice $L\big(\mathrm{E}(\Phi, K), \mathrm{G}(\Phi, F)\big)$ embeds into the lattice $L\big(\mathrm{Ep}_{2n}(\mathbb{F}_2), \mathrm{Sp}_{2n}(F)\big)$ for $\Phi = B_n, C_n$ and a subring K of F, and its description follows from the main result of the present paper.

Actually, the injections mentioned above were constructed in [16] at the level of Steinberg groups. Since over a field $K_2(\Phi, F)$ is generated by Steinberg symbols, they induce the injections of Chevalley groups. Over a ring A, there is no appropriate description of K_2 available, and the above arguments cannot be used. Moreover, in the ring case the group $\mathrm{G}_P(\Phi, A)$ depends on P. For these reasons, the adjoint group of type C_n and the groups of type B_n over rings will be considered in a subsequent article.

Now we state our main result. Following [5], we call a pair (R, Λ) consisting of a ring R and a form parameter Λ in R a *form ring.*

Theorem 1. *Let A denote a (commutative) ring such that $2 = 0$ in A. Let K be a subring of A. If H is a subgroup of $\mathrm{Sp}_{2n}(A)$ containing $\mathrm{Ep}_{2n}(K)$, $n \geq 3$, then there is a unique form ring (R, Λ) such that $K \subseteq \Lambda \subseteq R \subseteq A$ and*

$$\mathrm{Ep}_{2n}(R, \Lambda) \leq H \leq N_A(R, \Lambda),$$

where $N_A(R, \Lambda)$ is the normalizer of $\mathrm{Ep}_{2n}(R, \Lambda)$ in $\mathrm{Sp}_{2n}(A)$.

Let $\mathcal{L}(R, \Lambda) = L\big(\mathrm{Ep}_{2n}(R, \Lambda), N_A(R, \Lambda)\big)$ denote the lattice of all subgroups H of $\mathrm{Sp}_{2n}(A)$ such that $\mathrm{Ep}_{2n}(R, \Lambda) \leq H \leq N_A(R, \Lambda)$. From now on, $\mathcal{L}(R, \Lambda)$ is what we shall mean by a *standard sandwich.* In view of Theorem 1, it is natural to study the lattice structure of $\mathcal{L}(R, \Lambda)$. By definition, $\mathrm{Ep}_{2n}(R, \Lambda)$ is normal in $N_A(R, \Lambda)$. Therefore, the lattice structure of $\mathcal{L}(R, \Lambda)$ is the same as that of the quotient group $N_A(R, \Lambda)/\mathrm{Ep}_{2n}(R, \Lambda)$. The lattice $\mathcal{L}(R, \Lambda)$ contains an important subgroup $\mathrm{Sp}_{2n}(R, \Lambda)$, which is called a *Bak symplectic group*, whose definition will be recalled in §1. A group is said to be *quasi-nilpotent* if it is a direct limit of nilpotent subgroups, i.e., the group has a directed system of nilpotent subgroups whose colimit is the group itself.

In the following result we use the notion of the Bass–Serre dimension of a ring, introduced by the first author in [6, §4]. Recall that the Krull (or combinatorial) dimension of a topological space is the supremum of the lengths of the proper chains of nonempty closed irreducible subsets. The *Bass–Serre dimension* $\operatorname{BS-dim} R$ of a ring R is the smallest integer d such that the maximal spectrum $\operatorname{Max} R$ is a union of a finite number of irreducible Noetherian subspaces of Krull dimension not greater than d. If there is no such integer, then $\operatorname{BS-dim} R = \infty$.

Theorem 2. *Let A denote a (commutative) ring, and let R be a subring of A and (R, Λ) a form ring. Suppose $n \geq 2$. Then*

1. $\mathrm{Sp}_{2n}(R, \Lambda) = \mathrm{Sp}_{2n}(R) \cap N_A(R, \Lambda)$.
2. $\mathrm{Sp}_{2n}(R, \Lambda)$ *is normal in* $N_A(R, \Lambda)$.
3. $N_A(R, \Lambda)/\mathrm{Sp}_{2n}(R, \Lambda)$ *is Abelian.*
4. $\mathrm{Sp}_{2n}(R, \Lambda)/\mathrm{Ep}_{2n}(R, \Lambda)$ *is quasi-nilpotent.*
5. *Let R_0 denotes the subring of R generated by all elements ξ^2 such that $\xi \in R$. If the Bass–Serre dimension of R_0 is finite, then* $\mathrm{Sp}_{2n}(R, \Lambda)/\mathrm{Ep}_{2n}(R, \Lambda)$ *is nilpotent.*

In particular, the sandwich quotient group $N_A(R, \Lambda)/\mathrm{Ep}_{2n}(R, \Lambda)$ is quasi-nilpotent by Abelian or nilpotent by Abelian if $\text{BS-dim}\, R_0 < \infty$.

Although Theorem 1 is proved under the assumptions $2 = 0$ and $n \geq 3$, we do not invoke these assumptions until the proof of that theorem in §6. In particular, Theorem 2 is proved without these assumptions. Of course, we always assume that $n \geq 2$.

Notation. Let H be a group. For two elements $x, y \in H$ we write $[x, y] = xyx^{-1}y^{-1}$ for their commutator and $x^y = y^{-1}xy$ for the y-conjugate of x. For subgroups $X, Y \leq H$, we let X^Y denote the normal closure of X in the subgroup generated by X and Y, while $[X, Y]$ stands for the mixed commutator group generated by X and Y. By definition, it is the group generated by all commutators $[x, y]$ such that $x \in X$ and $y \in Y$. The commutator subgroup $[X, X]$ of X will be also denoted by $D(X)$, and we set $D^k(X) = \big[D^{k-1}(X), D^{k-1}(X)\big]$. Recall that a group X is said to be *perfect* if $D(X) = X$.

The identity matrix is denoted by e, as well as the identity element of a Chevalley group. We denote by e_{ij} the matrix with 1 in position (ij) and zeros elsewhere. The entries of a matrix g are denoted by g_{ij}. For the entries of the inverse matrix we use the abbreviation $(g^{-1})_{ij} = g'_{ij}$. The transpose of a matrix g is denoted by $g^{\top}$, so that $(g^{\top})_{ij} = g_{ji}$.

§1. The symplectic group

The symplectic group $\mathrm{Sp}_{2n}(R)$, its elementary root unipotent elements, and its elementary subgroup $\mathrm{Ep}_{2n}(R)$ will be recalled below. The groups $\mathrm{Sp}_{2n}(R, \Lambda)$ and their elementary subgroups $\mathrm{Ep}_{2n}(R, \Lambda)$ will be defined in §2.

Since the groups $\mathrm{Sp}_{2n}(R, \Lambda)$ are not in general algebraic (in fact, $\mathrm{Sp}_{2n}(R, \Lambda)$ is algebraic if and only if $\Lambda = R$ or $\Lambda = \{0\}$), it is convenient to work with the standard matrix representation of $\mathrm{Sp}_{2n}(R)$. On the other hand, we want to use the notions of a parabolic subgroup, unipotent radical, etc. from the theory of algebraic groups. Thus, to simplify the exposition, we below define these notions directly in terms of the matrix representation we use.

Following Bourbaki [9], we view the root system C_n and its set of fundamental roots Π in the following way. Let V_n denote the n-dimensional Euclidean space with the orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. Let

$$C_n = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_k \mid 1 \leq k \leq n\},$$
$$\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \text{ where } i = 1, \ldots, n-1,\ \alpha_n = 2\varepsilon_n\}.$$

The elements $\pm\varepsilon_i \pm \varepsilon_j$ are called *short roots* and the elements $\pm 2\varepsilon_k$ are *long roots.*

From the perspective of algebraic groups, we want the standard Borel subgroup of $\mathrm{Sp}_{2n}(R)$ to be the group of all upper triangular matrices in $\mathrm{Sp}_{2n}(R)$. Accordingly, we fix the following matrix description of $\mathrm{Sp}_{2n}(R)$. Let J denote the $(n \times n)$-matrix with 1 at each antidiagonal position and zeroes elsewhere. Let $F = \left(\begin{smallmatrix} 0 & J \\ -J & 0 \end{smallmatrix}\right)$. Then, the group

$\mathrm{Sp}_{2n}(R)$ is the subgroup of $\mathrm{GL}_{2n}(R)$ consisting of all matrices that preserve the bilinear form whose matrix is F. In other words,

$$\mathrm{Sp}_{2n}(R) = \{g \in \mathrm{GL}_{2n}(R) \mid g^{\mathsf{T}} F g = F\}.$$

Let $I = (1, \dots, n, -n, \dots, -1)$ denote the linearly ordered set whose linear ordering is obtained by reading from left to right. We enumerate the rows and columns of the matrices of $\mathrm{GL}_{2n}(R)$ by indices from I. Thus the position of a coordinate of a matrix in $\mathrm{GL}_{2n}(R)$ is denoted by a pair $(i, j) \in I \times I$.

In the language of algebraic groups, the set of all diagonal matrices in $\mathrm{Sp}_{2n}(R)$ is a *maximal torus*. The following matrices are *elementary root unipotent elements* of $\mathrm{Sp}_{2n}(R)$ with respect to the above torus:

$$\begin{aligned}
x_{\varepsilon_i-\varepsilon_j}(\xi) &= T_{ij}(\xi) = T_{-j\,-i}(-\xi) = e + \xi e_{ij} - \xi e_{-j\,-i},\\
x_{\varepsilon_i+\varepsilon_j}(\xi) &= T_{i\,-j}(\xi) = T_{j\,-i}(\xi) = e + \xi e_{i\,-j} + \xi e_{j\,-i},\\
x_{-\varepsilon_i-\varepsilon_j}(\xi) &= T_{-i\,j}(\xi) = T_{-j\,i}(\xi) = e + \xi e_{-i\,j} + \xi e_{-j\,i},\\
x_{2\varepsilon_k}(\xi) &= T_{k\,-k}(\xi) = e + \xi e_{k\,-k},\\
x_{-2\varepsilon_k}(\xi) &= T_{-k\,k}(\xi) = e + \xi e_{-k\,k},\\
&\text{where } \xi \in R,\ \ 1 \le i, j, k \le n,\ \ i \ne j.
\end{aligned}$$

Note that the subscripts (i, j), (i, j), $(i, -j)$, $(j, -i)$, $(-j, -i)$, $(k, -k)$, and $(-k, k)$ at the T above all belong to the set $\widetilde{C}_n = I \times I \setminus \{(k, k) \mid k \in I\}$ of nondiagonal positions of a matrix from $\mathrm{Sp}_{2n}(R)$ and exhaust $\widetilde{C}_n$.

There is a surjective map $p : \widetilde{C}_n \to C_n$ defined by the following rule:

$$\begin{aligned}
p(i, j) &= p(-j, -i) = \varepsilon_i - \varepsilon_j;\\
p(i, -j) &= p(j, -i) = \varepsilon_i + \varepsilon_j;\\
p(-i, j) &= p(-j, i) = -\varepsilon_i - \varepsilon_j;\\
p(k, -k) &= 2\varepsilon_k;\\
p(-k, k) &= -2\varepsilon_k;\\
&\text{where } \xi \in R,\ \ 1 \le i, j, k \le n,\ \ i \ne j.
\end{aligned}$$

With this notation, the correspondence between elementary symplectic transvections $T_{ij}(\xi)$ and elementary root elements $x_\alpha(\xi)$ looks like this

$$T_{ij}(\xi) = x_{p(i,j)}(-\operatorname{sgn}(ij)\xi) \ \text{ for all } \ i \ne j \in I.$$

Note that p maps symmetric (with respect to the antidiagonal) positions to the same root. Therefore, for $(i, j) \in \widetilde{C}_n$ and $\xi \in R$ we have $T_{ij}(\xi) = T_{-j\,-i}(-\operatorname{sgn}(ij)\xi)$.

The root subgroup scheme X_α is defined by $X_\alpha(R) = \{x_\alpha(\xi) \mid \xi \in R\}$. The scheme X_α is naturally isomorphic to $\mathbb{G}_a$, i.e., $x_\alpha(\xi)x_\alpha(\mu) = x_\alpha(\xi + \mu)$ for all $\xi, \mu \in R$. The following commutator formulas are well known in matrix language, cf. [8, §3]. They are special cases of the Chevalley commutator formula in the algebraic group theory:

$$\begin{aligned}
&[x_\alpha(\lambda), x_\beta(\mu)] = x_{\alpha+\beta}(\pm\lambda\mu) \ \text{ if } \ \alpha + \beta \in \Phi,\ \widehat{\alpha\beta} = 2\pi/3;\\
&[x_\alpha(\lambda), x_\beta(\mu)] = x_{\alpha+\beta}(\pm 2\lambda\mu) \ \text{ if } \ \alpha + \beta \in \Phi,\ \widehat{\alpha\beta} = \pi/2;\\
&[x_\alpha(\lambda), x_\beta(\mu)] = x_{\alpha+\beta}(\pm\lambda\mu)x_{\alpha+2\beta}(\pm\lambda\mu^2) \ \text{ if } \ \alpha + \beta, \alpha + 2\beta \in \Phi,\ \widehat{\alpha\beta} = 3\pi/4;\\
&[x_\alpha(\lambda), x_\beta(\mu)] = e \ \text{ if } \ \alpha + \beta \notin \Phi \cup \{0\}.
\end{aligned}$$

In our proofs we make frequent use of the *parabolic subgroup* P_1 of Sp_{2n}. In the matrix language as above, it is defined as follows:

$$P_1(R) = \{g \in \mathrm{Sp}_{2n}(R) \mid g_{i1} = g_{-1\,-i} = 0 \quad \forall i \ne 1\}.$$

The above definition of $\mathrm{Sp}_{2n}(R)$ shows that $g_{11} = g_{-1\,-1}^{-1}$ for any matrix $g \in P_1(R)$. The *unipotent radical* U_1 of P_1 is the subgroup generated by all root subgroups T_{1i} such that $i \neq 1$. The *Levi subgroup* $L_1(R)$ of $P_1(R)$ consists of all $g \in P_1(R)$ such that $g_{1i} = g_{-i\,-1} = 0$ for all $i \neq 1$. As a group scheme it is isomorphic to $\mathbb{G}_m \times \mathrm{Sp}_{2n-2}$.

§2. Bak symplectic groups

The *Bak symplectic group* $\mathrm{Sp}_{2n}(R, \Lambda)$ is the particular case of the Bak general unitary group where the involution is trivial and the symmetry $\lambda = -1$. The main references for the definition and the structure of the general unitary group is the book [5] and the paper [8] by N. Vavilov and the first author. In this section we recall the definitions and simple properties to be used in the sequel.

Let R be a commutative ring. An additive subgroup Λ of R is called a *symplectic form parameter* in R if it contains $2R$ and is closed under multiplication by squares, i.e., $\mu^2\lambda \in \Lambda$ for all $\mu \in R$ and $\lambda \in \Lambda$. Define $\mathrm{Sp}_{2n}(R, \Lambda)$ as the subgroup of $\mathrm{Sp}_{2n}(R)$ consisting of all matrices preserving the quadratic form that takes values in R/Λ and is determined by the matrix $\left(\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right)$.

Let $*$ denote the involution on the matrix ring $\mathrm{M}_n(R)$ given by the formula $a^* = Ja^{\mathsf{T}}J$. Note that this is the reflection of a matrix with respect to the antidiagonal. Define

$$\mathrm{M}_n(R, \Lambda) = \{a \in \mathrm{M}_n(R) \mid a = a^*,\ a_{n-k+1\,k} \in \Lambda \text{ for all } k = 1, \dots, n\}.$$

Write a matrix of degree $2n$ in the block form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b, c, d \in \mathrm{M}_n(R)$. From [8, Lemma 2.2] and its proof it follows that under the notation above we have the following formula.

Lemma 2.1. $\mathrm{Sp}_{2n}(R, \Lambda) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a^*d - c^*b = e \ \ \textit{and} \ \ c^*a, d^*b \in \mathrm{M}_n(R, \Lambda)\right\}$.

It is easy to check that $\mathrm{M}_n(R, \Lambda)$ is a form parameter in the ring $\mathrm{M}_n(R)$ with involution $*$, corresponding to the symmetry $\lambda = -1$. The minimal form parameter in $\mathrm{M}_n(R)$ with the same involution and symmetry is $\mathrm{M}_n(R, 2R)$; it is denoted by $\mathrm{Min}_n(R)$. Let $\bar{\mathrm{M}}_n(R, \Lambda)$ denote the additive group $\mathrm{M}_n(R, \Lambda)/\mathrm{Min}_n(R)$.

Lemma 2.2. *The group* $\bar{\mathrm{M}}_n(R, \Lambda)$ *has a natural structure of a left* $M_n(R)$*-module under the operation* $a \circ \bar{b} = aba^* \bmod \mathrm{Min}_n(R)$*, where* $a \in \mathrm{M}_n(R)$*,* $\bar{b} \in \bar{\mathrm{M}}_n(R, \Lambda)$*, and* b *is a preimage of* $\bar{b}$ *in* $\mathrm{M}_n(R, \Lambda)$*.*

By abuse of notation, for $a \in \mathrm{M}_n(R)$ and $b \in \mathrm{M}_n(R, \Lambda)$ we shall write $a \circ b$ instead of $a \circ (b + \mathrm{Min}_n(R))$.

It is easy to check that an elementary root unipotent element $x_\alpha(\xi) \in \mathrm{Sp}_{2n}(R)$ belongs to $\mathrm{Sp}_{2n}(R, \Lambda)$ if and only if α is a short root or $\xi \in \Lambda$. Denote by $\mathrm{Ep}_{2n}(R, \Lambda)$ the group generated by all such elements:

$$\mathrm{Ep}_{2n}(R, \Lambda) = \langle x_\alpha(\xi) \mid \alpha \in C_n^{\text{short}} \,\&\, \xi \in R \vee \alpha \in C_n^{\text{long}} \,\&\, \xi \in \Lambda\rangle.$$

§3. Subgroups generated by elementary root unipotents

In this section we assume that $n \geq 3$. Let H be a subgroup of $\mathrm{Sp}_{2n}(A)$ containing $\mathrm{Ep}_{2n}(K)$. The following lemma shows that we can uncouple a short root element from a long one inside H.

Lemma 3.1. *Let* $\alpha, \beta \in C_n$ *be a short and a long root, respectively, such that* $\alpha + \beta$ *is not a root. Let* $g = x_\alpha(\mu)x_\beta(\lambda)$*, where* $\lambda, \mu \in A$*. If* $\mathrm{Ep}_{2n}(K)^g \leq H$ *(e.g.,* $g \in H$*), then each factor of* g *belongs to* H*.*

Proof. Since $n \geq 3$, there exists a short root γ such that $\gamma + \alpha$ is a short root and $\gamma + \beta$ is not a root. Then, $X_\beta(A)$ commutes with $X_\alpha(A)$ and $X_\gamma(A)$, whence $[g, x_\gamma(1)] = x_{\alpha+\gamma}(\pm\mu) \in H$. Conjugating this element by an appropriate element of the Weyl group over K and taking the inverse if necessary, we get $x_\alpha(\mu) \in H$. It follows that $\mathrm{Ep}_{2n}(K)^{x_\beta(\lambda)} \leq H$.

Now, take a short root δ such that $\beta + \delta$ is a root. Then,

$$[x_\delta(1), x_\beta(\lambda)] = x_{\delta+\beta}(\pm\lambda)x_{2\delta+\beta}(\pm\lambda) \in H.$$

Notice that the root $\delta + \beta$ is short whereas $2\delta + \beta$ is long. As in the first paragraph of the proof, we conclude that $x_{\delta+\beta}(\pm\lambda) \in H$, whence $x_{2\delta+\beta}(\pm\lambda) \in H$. Again, using the action of the Weyl group and taking inverse if necessary, we see that $x_\beta(\lambda) \in H$. $\square$

Put $P_\alpha(H) = \{t \in A \,|\, x_\alpha(t) \in H\}$. Since the Weyl group acts transitively on the set of roots of the same length, it is easily seen that $P_\alpha(H) = P_\beta(H)$ if $|\alpha| = |\beta|$. Let $R = R_H = P_\alpha(H)$ for any short root α, and let $\Lambda = \Lambda_H = P_\beta(H)$ for any long root β.

Lemma 3.2. *With the above notation, (R, Λ) is a form ring and $K \subseteq \Lambda \subseteq R \subseteq A$.*

Proof. Clearly, $P_\alpha(H)$ is an additive subgroup of A. Since $n \geq 3$, there are two short roots α, α' such that $\alpha + \alpha'$ is also short. The commutator formula

$$[x_\alpha(\lambda), x_{\alpha'}(\mu)] = x_{\alpha+\alpha'}(\pm\lambda\mu)$$

shows that R is a ring.

Now, let α, α' be short roots that are orthogonal in V_n and $\beta = \alpha + \alpha' \in \Phi$ is a long root. Then

$$[x_\alpha(\mu), x_{\alpha'}(1)] = x_\beta(\pm 2\mu).$$

If $\mu \in R$, then this element belongs to H. This proves that $2R \subseteq \Lambda$. Finally, we show that Λ is closed under multiplication by squares in R. Let

$$g = [x_\beta(\lambda), x_{-\alpha}(\mu)] = x_{\alpha'}(\pm\lambda\mu)x_{\beta-2\alpha}(\pm\lambda\mu^2).$$

If $\lambda \in \Lambda$ and $\mu = 1$, then $g \in H$, and Lemma 3.1 shows that $x_{\alpha'}(\pm\lambda) \in H$. Therefore, $\Lambda \subseteq R$. On the other hand, if $\mu \in R$ and $\lambda \in \Lambda$, then g also lies in H, and by Lemma 3.1 we have $x_{\beta-2\alpha}(\pm\lambda\mu^2) \in H$. This shows that Λ is stable under multiplication by the squares of elements of R. $\square$

If H is a subgroup of $\mathrm{Sp}_{2n}(A)$ containing $\mathrm{Ep}_{2n}(K)$, then (R_H, Λ_H) is called the *form ring associated with H.*

§4. The normalizer

Let (R, Λ) be a form subring of a ring A such that $1 \in \Lambda$. In this section we develop properties of the normalizer $N_A(R, \Lambda)$ of the group $\mathrm{Ep}_{2n}(R, \Lambda)$ in $\mathrm{Sp}_{2n}(A)$ and prove Theorem 2. Here we do not assume that $n \geq 3$. By a result of Bak and Vavilov [7, Theorem 1.1] we know that $\mathrm{Ep}_{2n}(R, \Lambda)$ is normal in $\mathrm{Sp}_{2n}(R, \Lambda)$, so that $\mathrm{Sp}_{2n}(R, \Lambda) \leq N_A(R, \Lambda)$. First, we show that the quotient $N_A(R, \Lambda)/\,\mathrm{Sp}_{2n}(R, \Lambda)$ is Abelian.

Lemma 4.1. *Let $g \in \mathrm{Sp}_{2n}(A)$. If $\mathrm{Ep}_{2n}(R, \Lambda)^g \leq \mathrm{GL}_{2n}(R)$ then $g_{ij}g_{kl} \in R$ for all $i, j, k, l \in I$.*

Proof. To begin, we express a matrix unit e_{jk} as a linear combination $\sum \xi_m a^{(m)}$ for some $\xi_m \in R$ and $a^{(m)} \in \mathrm{Ep}_{2n}(R, \Lambda)$. Note that the set of such linear combinations is closed under multiplication. Suppose $j \neq k$ and let $i \neq \pm j, \pm k$. Then $e_{jk} = (T_{ji}(1) - e)(T_{ik}(1) - e)$ and $e_{jj} = e_{jk}e_{kj}$. It follows that if $\mathrm{Ep}_{2n}(R, \Lambda)^g \leq \mathrm{GL}_{2n}(R)$, then $g^{-1}e_{jk}g \in \mathrm{M}_n(R)$. Thus $g'_{ij}g_{kl} \in R$ for all $i, j, k, l \in I$ (recall that $g'_{ij} = (g^{-1})_{ij}$). The conclusion of the lemma follows, because the entries of g^{-1} coincide with the entries of g up to sign and a permutation. □

The next proposition describes the normalizer in the case of $A = R$.

Proposition 4.2. $N_R(R,\Lambda) = \mathrm{Sp}_{2n}(R,\Lambda)$.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in N_R(R,\Lambda)$, where $a,b,c,d \in \mathrm{M}_n(R)$. Since $g \in \mathrm{Sp}_{2n}(R)$, we know that $d^*a - b^*c = e$. We must prove that c^*a, $d^*b \in M_n(R,\Lambda)$. Since $h = \left(\begin{smallmatrix} e & J \\ 0 & e \end{smallmatrix}\right) \in \mathrm{Ep}_{2n}(R,\Lambda)$, we have

$$f = ghg^{-1} = \begin{pmatrix} e - aJc^* & aJa^* \\ -cJc^* & e + cJa^* \end{pmatrix} \in \mathrm{Ep}_{2n}(R,\Lambda).$$

It follows that the matrices $-(cJc^*)^*(e-aJc^*)$ and $(e+cJa^*)^*(aJa^*)$ belong to $M_n(R,\Lambda)$. Since cJc^* and aJa^* are automatically in $M_n(R,\Lambda)$, we see that cJc^*aJc^*, $aJc^*aJa^* \in M_n(R,\Lambda)$. Modulo $\mathrm{Min}_n(R)$ we can write $(cJ) \circ (c^*a)$, $(aJ) \circ (c^*a) \in \overline{M}_n(R,\Lambda)$. By Lemma 2.2, $\overline{M}_n(R,\Lambda)$ is an $M_n(R)$ module, therefore

$$\begin{aligned} c^*a + \mathrm{Min}_n(R) &= \big(J(d^*a - b^*c)J\big) \circ (c^*a) \\ &= (Jd^*) \circ \big((aJ) \circ (c^*a)\big) - (Jb^*) \circ \big((cJ) \circ (c^*a)\big) \in \overline{M}_n(R,\Lambda). \end{aligned}$$

The proof that $d^*b \in M_n(R,\Lambda)$ is essentially the same. □

Corollary 4.3. $[N_A(R,\Lambda), N_A(R,\Lambda)] \le \mathrm{Sp}_{2n}(R,\Lambda)$.

Proof. If $g,h \in N_A(R,\Lambda)$ and $f = [g,h]$, then, by Lemma 4.1, for all indices p,q we have

$$f_{pq} = \sum_{i,j,k=1}^{n} g'_{pi} h'_{ij} g_{jk} h_{kq} = \sum_{i,j,k=1}^{n} (g'_{pi} g_{jk})(h'_{ij} h_{kq}) \in R.$$

Therefore,

$$f \in \mathrm{Sp}_{2n}(R) \cap N_A(R,\Lambda) = N_R(R,\Lambda) = \mathrm{Sp}_{2n}(R,\Lambda).$$ □

The first three items of Theorem 2 are already proved. Our next goal is to show that the main theorem of [12] implies that $\mathrm{Sp}_{2n}(R,\Lambda)/\mathrm{Ep}_{2n}(R,\Lambda)$ is nilpotent, provided that R_0 has finite Bass–Serre dimension. This will imply the rest claims of Theorem 2. Recall that R_0 denotes the subring of R generated by all elements ξ^2 such that $\xi \in R$.

Lemma 4.4. *If R_0 is semilocal and $1 \in \Lambda$, then* $\mathrm{Sp}_{2n}(R,\Lambda) = \mathrm{Ep}_{2n}(R,\Lambda)$.

Proof. Since R is integral over R_0, it is a direct limit of R_0-subalgebras $R' \subseteq R$ such that R' is module finite and integral over R_0. By the first theorem of Cohen–Seidenberg, each R' is semilocal. Thus, by [4, Lemma 4], $\mathrm{Sp}_{2n}(R', \Lambda \cap R') = \mathrm{Ep}_{2n}(R', \Lambda \cap R')$ for each R'. Since Sp_{2n} and Ep_{2n} commute with direct limits, it follows that $\mathrm{Sp}_{2n}(R,\Lambda) = \mathrm{Ep}_{2n}(R,\Lambda)$. □

Lemma 4.5. *If R is a finitely generated $\mathbb{Z}$-algebra, then so is R_0.*

Proof. Since $2\xi = (\xi+1)^2 - \xi^2 - 1$, we have $2R \subseteq R_0$. Let S be a finite set of generators for R as a $\mathbb{Z}$-algebra, which includes 1. Set $S_0 = \big\{2\prod_{s\in S'} s \mid S' \subseteq S\big\}$ and denote by R_1 the $\mathbb{Z}$-subalgebra generated by $S^2 \cup S_0$. Clearly, $2R \subseteq R_1 \subseteq R_0$, $2R$ is generated as an R_1-ideal by S_0, and $R_1/2R = R_0/2R$ because the map $R \to R_0/2R$, $\xi \mapsto \xi^2 + 2R$, is a ring epimorphism. Thus $S^2 \cup S_0$ generates R_0 as a $\mathbb{Z}$-algebra. □

Proof of Theorem 2. Since $N_R(R,\Lambda) = N_A(R,\Lambda) \cap \mathrm{Sp}_{2n}(R)$, the first statement follows from Proposition 4.2, the second is a particular case of [7, Theorem 1.1], and the third coincides with Corollary 4.3.

Recall that $\mathrm{Sp}^0_{2n}(R,\Lambda)=\bigcap_\varphi \operatorname{Ker}\varphi$, where φ ranges over all group homomorphisms $\mathrm{Sp}_{2n}(R,\Lambda)\to\mathrm{Sp}_{2n}(\widetilde{R},\widetilde{\Lambda})/\mathrm{Ep}_{2n}(\widetilde{R},\widetilde{\Lambda})$ induced by form ring morphisms $(R,\Lambda)\to(\widetilde{R},\widetilde{\Lambda})$ such that $\widetilde{R}_0$ is semilocal. By Lemma 4.4, $\mathrm{Sp}^0_{2n}(R,\Lambda)=\mathrm{Sp}_{2n}(R,\Lambda)$.

Since R is integral over R_0, it is a direct limit of R_0-subalgebras $R'\subseteq R$ such that R' is module finite over R_0. By [12, Theorem 3.10], the group $\mathrm{Sp}^0_{2n}(R',\Lambda\cap R')/\mathrm{Ep}_{2n}(R',\Lambda\cap R')$ is nilpotent, provided that R_0 has finite Bass–Serre dimension. Since Sp_{2n} and Ep_{2n} commute with direct limits, this proves (5).

Any commutative ring is a direct limit of finitely generated $\mathbb{Z}$-algebras. Let $R=\operatorname{inj\,lim}R^{(i)}$, where i ranges over some index set I and each $R^{(i)}$ is a finitely generated $\mathbb{Z}$-algebra. Then $R_0=\operatorname{inj\,lim}R_0^{(i)}$ and $\mathrm{Sp}_{2n}(R,\Lambda\cap R)/\mathrm{Ep}_{2n}(R,\Lambda)=\operatorname{inj\,lim}Sp(R^{(i)},\Lambda\cap R^{(i)})/\mathrm{Ep}_{2n}(R^{(i)},\Lambda\cap R^{(i)})$. By Lemma 4.5, each $R_0^{(i)}$ is a finitely generated $\mathbb{Z}$-algebra. Therefore, each $R_0^{(i)}$ has finite Krull dimension and, hence, finite Bass–Serre dimension. By (5) each group $Sp(R^{(i)},\Lambda\cap R^{(i)})/\mathrm{Ep}_{2n}(R^{(i)},\Lambda\cap R^{(i)})$ is nilpotent, which implies item (4). ☐

The following statement is crucial for the proof of the main theorem. For Noetherian rings it is an almost immediate consequence of Theorem 2.

Lemma 4.6. *Let $g\in\mathrm{Sp}_{2n}(A)$, $n\ge 3$. If $\mathrm{Ep}_{2n}(R,\Lambda)^g\le N_A(R,\Lambda)$, then $g\in N_A(R,\Lambda)$. Moreover, $\mathrm{Ep}_{2n}(R,\Lambda)$ is a characteristic subgroup of $N_A(R,\Lambda)$.*

Proof. Let θ be either an automorphism of $N_A(R,\Lambda)$ or an automorphism of $\mathrm{Sp}_{2n}(A)$ such that $\mathrm{Ep}_{2n}(R,\Lambda)^\theta\le N_A(R,\Lambda)$ (we denote by h^θ the image of an element $h\in N_A(R)$ under the action of θ). Since $n\ge 3$, the Chevalley commutator formula (see §1) implies that the group $\mathrm{Ep}_{2n}(R,\Lambda)$ is perfect. By Corollary 4.3, we have

$$\begin{aligned}\mathrm{Ep}_{2n}(R,\Lambda)^\theta&=[\mathrm{Ep}_{2n}(R,\Lambda),\mathrm{Ep}_{2n}(R,\Lambda)]^\theta\\&\le[N_A(R,\Lambda),N_A(R,\Lambda)]\le\mathrm{Sp}_{2n}(R,\Lambda).\end{aligned}$$

We shall prove that $h^\theta\in\mathrm{Ep}_{2n}(R,\Lambda)$ for any $h\in\mathrm{Ep}_{2n}(R,\Lambda)$. Write h as a product of elementary root unipotents $x_{\alpha_1}(s_1)\dots x_{\alpha_m}(s_m)$. Let R' denote the $\mathbb{Z}$-subalgebra of R generated by all s_i's, and let Λ' denote the form parameter of R' generated by those s_j for which α_j is a long root. Clearly $h\in\mathrm{Ep}_{2n}(R',\Lambda')$ and $\mathrm{Ep}_{2n}(R',\Lambda')$ is a finitely generated group. Let R'' denote the R'-algebra generated by all entries of the matrices y^θ, where y ranges over all generators of $\mathrm{Ep}_{2n}(R',\Lambda')$. Let $\Lambda''=\Lambda\cap R''$. The inclusion $\mathrm{Ep}_{2n}(R,\Lambda)^\theta\le\mathrm{Sp}_{2n}(R,\Lambda)$ shows that $R''\subseteq R$. Note that $\mathrm{Ep}_{2n}(R',\Lambda')^\theta\le\mathrm{Sp}_{2n}(R'',\Lambda'')$ by the choice of (R'',Λ'').

Since R'' is a finitely generated R'-algebra, it is a finitely generated $\mathbb{Z}$-algebra. By Lemma 4.5, R''_0 is a finitely generated $\mathbb{Z}$-algebra. Therefore, it has finite Krull dimension and, hence, finite Bass–Serre dimension. Thus, by Theorem 2(5), the kth commutator subgroup $D^k\,\mathrm{Sp}_{2n}(R'',\Lambda'')$ is equal to $\mathrm{Ep}_{2n}(R'',\Lambda'')$ for some positive integer k. Now, since $\mathrm{Ep}_{2n}(R',\Lambda')$ is perfect, it is equal to $D^k\,\mathrm{Ep}_{2n}(R',\Lambda')$. It follows that

$$\mathrm{Ep}_{2n}(R',\Lambda')^\theta=D^k\,\mathrm{Ep}_{2n}(R',\Lambda')^\theta\le D^k\,\mathrm{Sp}_{2n}(R'',\Lambda'')=\mathrm{Ep}_{2n}(R'',\Lambda'').$$

In particular, $h^\theta\in\mathrm{Ep}_{2n}(R'',\Lambda'')\le\mathrm{Ep}_{2n}(R,\Lambda)$. Thus, $\mathrm{Ep}_{2n}(R,\Lambda)$ is invariant under θ.

If θ is an automorphism of $N_A(R)$, this implies that $\mathrm{Ep}_{2n}(R,\Lambda)$ is a characteristic subgroup of $N_A(R)$. If θ is an inner automorphism defined by $g\in G(A)$, then the statement we have proved is the first assertion of the lemma. ☐

The following straightforward corollary shows that the normalizers of all subgroups of the sandwich $L\big(\mathrm{Ep}_{2n}(R,\Lambda),N_A(R,\Lambda)\big)$ lie in the sandwich.

Corollary 4.7. *For any $H\le N_A(R,\Lambda)$ containing $\mathrm{Ep}_{2n}(R,\Lambda)$, its normalizer is contained in $N_A(R,\Lambda)$. In particular, the group $N_A(R,\Lambda)$ is self-normalizing.*

§5. Inside a parabolic subgroup

Let $I = \{1, \dots, n, -n, \dots, -1\}$ denotes the linearly ordered set defined in Section 1. Thus, for example, the product $\prod_{j=2}^{-1} T_{1j}(\mu_j)$ means by definition the product

$$T_{12}(\mu_2) \dots T_{1n}(\mu_n) T_{1\,-n}(\mu_{-n}) \dots T_{1\,-1}(\mu_{-1}).$$

Recall that $U_1(A)$ denotes the unipotent radical of the parabolic subgroup $P_1(A)$ of $\mathrm{Sp}_{2n}(A)$, see the end of Section 1.

Let H be a subgroup of $\mathrm{Sp}_{2n}(A)$ normalized by $\mathrm{Ep}_{2n}(K)$. Denote by (R, Λ) the form ring associated with H.

Lemma 5.1. *If $g \in U_1(A)$ and $\mathrm{Ep}_{2n}(K)^g \le H$, then $g \in \mathrm{Ep}_{2n}(R, \Lambda)$.*

Proof. Let $g = \prod_{j=2}^{-1} T_{1j}(\mu_j)$. We have to prove that $\mu_j \in R$ for any $j \ne -1$ and $\mu_{-1} \in \Lambda$. Let k be the smallest element of I such that $\mu_k \ne 0$. We proceed by downward induction on k. If $k = -1$, then g consists of a single factor and the result holds by the definition of a form ring associated with H. If $k = -2$, then the result follows from Lemma 3.1.

Now, let $k < -2$. Denote by l the successor of k in I and let $\widetilde{U}$ be the subgroup of $U_1(A)$ generated by $T_{1j}(\xi)$ for all $j > l$ and $\xi \in A$. The Chevalley commutator formula expressed in terms of the matrices $T_{ij}(\xi)$ (see Section 1) or simple matrix computations show that $\widetilde{U}$ is normalized by $T_{kl}(\zeta)$ for all $\zeta \in A$ and by $U_1(A)$. Since $g \in T_{1k}(\mu_k)\widetilde{U}$, we have $[T_{kl}(-1), g] \in T_{1l}(\mu_k)\widetilde{U}$. On the other hand, $[T_{kl}(-1), g] \in H$ by the condition of the lemma. By the induction hypothesis, $\mu_k \in R$. Therefore, the matrix $T_{1k}(-\mu_k)g$ satisfies the conditions of the lemma. Again by the induction hypothesis, we see that $T_{1k}(-\mu_k)g \in \mathrm{Ep}_{2n}(R, \Lambda)$. Thus, $g \in \mathrm{Ep}_{2n}(R, \Lambda)$. □

It is known that in a Chevalley group the unipotent radicals of two opposite standard parabolic subgroups span the elementary group (see, e. g., [18, Lemma 2.1]). The next lemma shows that this also holds for the unipotent radicals of P_1 and its opposite in $\mathrm{Sp}_{2n}(R, \Lambda)$.

Lemma 5.2. *The set*

$$\{T_{1i}(\mu),\, T_{i1}(\mu),\, T_{1\,-1}(\lambda),\, T_{-1\,1}(\lambda) \mid i \ne \pm 1,\, \mu \in R,\, \lambda \in \Lambda\}$$

generates the elementary group $\mathrm{Ep}_{2n}(R, \Lambda)$.

Proof. If $i \ne \pm j, \pm 1$ then $T_{ij}(\mu) = [T_{i1}(\mu), T_{1j}(1)]$. On the other hand, for $\lambda \in \Lambda$ we have $T_{-i\,i}(\lambda) = [T_{-1\,1}(\lambda), T_{1i}(1)]T_{-1\,i}(-\lambda)$. □

Lemma 5.3. *Let H be a subgroup of $\mathrm{Sp}_{2n}(A)$ containing $\mathrm{Ep}_{2n}(K)$, and let (R, Λ) be the form ring associated with H. Suppose that $g \in H$ commutes with a long root subgroup $X_\gamma(K)$. Then $g \in N_A(R, \Lambda)$.*

Proof. Without loss of generality we may assume that $\gamma = 2\varepsilon_1$ is the maximal root. Then g belongs to the standard parabolic subgroup P_1 corresponding to the simple root $\alpha_1 = \varepsilon_1 - \varepsilon_2$, in other words $g_{i1} = g_{-1\,i} = g_{-1\,1} = 0$ for all $i \ne \pm 1$. Moreover, $g_{11} = g_{-1\,-1} = g'_{11} = g'_{-1\,-1} = \mu$, where $\mu^2 = 1$. Then $g = ab$ for some $b \in U_1(A)$ and $a \in L_1(A)$ (a and b are not necessarily in H). For any $d \in U_1(K)$ the element d^g belongs to $H \cap U_1(A)$. By Lemma 5.1, $d^g \in \mathrm{Ep}_{2n}(R, \Lambda)$. If $d = T_{1i}(1)$, then the above inclusion implies that $\mu g_{ij} \in R$ for all $i \ne 1$ and all $j \in I$. It follows that $T_{1i}(1)^a$ and $T_{i1}(1)^a$ belong to $\mathrm{Ep}_{2n}(R, \Lambda)$. On the other hand, a commutes with the root subgroups $X_{\pm\gamma}(K)$. Applying the previous lemma, we conclude that a normalizes $\mathrm{Ep}_{2n}(R, \Lambda)$.

Now, we have $\mathrm{Ep}_{2n}(K)^b \le \mathrm{Ep}_{2n}(R, \Lambda)^g \le H$ and $b \in \mathrm{Ep}_{2n}(R, \Lambda)$ by Lemma 5.1. Thus, $g \in N_A(R, \Lambda)$, as required. □

§6. Proof of Theorem 1

In this section we assume that $2 = 0$ in K. Let G be a Chevalley group with a not simply laced root system, e.g., $G = \mathrm{Sp}_{2n}$. Recall that in this case a short root unipotent element h is called a *small unipotent element*, see [11]. The terminology reflects the fact that the conjugacy class of h is small. In our setting a small unipotent element is conjugate to $T_{ij}(\mu)$, where $i \neq \pm j$ and $\mu \in R$.

The following lemma was obtained over a field by Golubchik and Mikhalev in [10], by Gordeev in [11], and by Nesterov and Stepanov in [13].

Lemma 6.1. *Let $\Phi = B_n, C_n, F_4$, and let R be a ring such that $2 = 0$. Let α be a long root and $g \in G(\Phi, R)$. If $h \in G(R)$ is a small unipotent element, then $X_\alpha(R)^{h^g}$ commutes with $X_\alpha(R)$.*

Proof. The identity with constants $[X_\alpha(R)^{h^g}, X_\alpha(R)] = \{1\}$ is inherited by subrings and quotient rings. Any commutative ring with $2 = 0$ is a quotient of a polynomial ring over $\mathbb{F}_2$, which is a subring of a field of characteristic 2. $\square$

The next lemma is the last ingredient of the proof of Theorem 1. It follows from the normal structure of the general unitary group $\mathrm{GU}_{2n}(R, \Lambda)$, obtained by Bak and Vavilov in the middle of 1990s. Since this result has not been published yet, we give a proof of a simple special case.

Lemma 6.2. *A normal subgroup N of $\mathrm{Ep}_{2n}(R, \Lambda)$ containing a root element $T_{ij}(1)$ coincides with $\mathrm{Ep}_{2n}(R, \Lambda)$.*

Proof. First, suppose that $i \neq \pm j$. Take $k \neq \pm i, \pm j$. Then $[T_{ij}(1), T_{jk}(\xi)] = T_{ik}(\mu) \in N$. Since the Weyl group acts transitively on the set of all short roots, we have $T_{lm}(\mu) \in N$ for all $l \neq \pm m$ and $\mu \in R$. Next,

$$T_{l\,-m}(-\lambda)[T_{lm}(1), T_{m\,-m}(\lambda)] = T_{l\,-l}(\lambda) \in N$$

for all $\lambda \in \Lambda$ and $l \in I$.

Now, let $i = m$, $j = -m$, and $l \neq \pm m$. Put $\lambda = 1$ in the commutator identity above. Then, $T_{l\,-m}(1)T_{l\,-l}(1) \in N$. By the transitivity of the Weyl group we know that $T_{l\,-l}(1) \in N$; therefore, $T_{l\,-m}(1) \in N$. By the first paragraph of the proof, N contains all the generators of $\mathrm{Ep}_{2n}(R, \Lambda)$. $\square$

Now we are ready to prove Theorem 1. The idea of the proof is the same as in the proof of the main result of [17].

Proof of Theorem 1. Let (R, Λ) be the form subring associated with H. Put $y = T_{12}(1)$ and $x = T_{1\,-1}(\lambda)$, where $\lambda \in \Lambda$. Take arbitrary elements $h \in H$ and $a, b \in \mathrm{Ep}_{2n}(R, \Lambda)$ and consider the element $c = x^{y^{ahb}} \in H$. By Lemma 6.1, this element commutes with a long root subgroup, and by Lemma 5.3, $c \in N_A(R, \Lambda)$. Rewrite c in the form

$$c = \big(h^{-1}(a^{-1}y^{-1}a)h(bxb^{-1})h^{-1}(a^{-1}ya)h\big)^b.$$

Since $b \in \mathrm{Ep}_{2n}(R, \Lambda)$, the element bcb^{-1} is in $N_A(R, \Lambda)$. Fix a and let b and λ vary. The subgroup generated by bxb^{-1} is normal in $\mathrm{Ep}_{2n}(R, \Lambda)$. By Lemma 6.2, it must coincide with $\mathrm{Ep}_{2n}(R, \Lambda)$. Thus, $\mathrm{Ep}_{2n}(R, \Lambda)^{h^{-1}(a^{-1}y^{-1}a)h} \leq N_A(R, \Lambda)$ and by Lemma 4.6, we have $(a^{-1}y^{-1}a)^h \in N_A(R, \Lambda)$.

Again, as a ranges over $\mathrm{Ep}_{2n}(R, \Lambda)$, the elements of the form $a^{-1}y^{-1}a$ generate a normal subgroup in $\mathrm{Ep}_{2n}(R, \Lambda)$. The minimal normal subgroup of $\mathrm{Ep}_{2n}(R, \Lambda)$ containing y must be equal to $\mathrm{Ep}_{2n}(R, \Lambda)$, by Lemma 6.2. Therefore, $\mathrm{Ep}_{2n}(R, \Lambda)^h \leq N_A(R, \Lambda)$. By Lemma 4.6, we have $h \in N_A(R, \Lambda)$, which completes the proof. $\square$

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