SYMMETRIZATION OF BOUNDED REMAINDER SETS

V. G. ZHURAVLEV

Abstract. A new method for constructing exchanged toric developments is suggested. This method is based on symmetrization of embedded or induced toric developments and is the main tool for obtaining bounded remainder sets in arbitrary dimension.

INTRODUCTION

1. Bounded remainder sets. Let \( S_\alpha(x) \equiv x + \alpha \mod \mathbb{Z}^D \) be the translation of a torus \( T^D = \mathbb{R}^D / \mathbb{Z}^D \) of dimension \( D \) by a vector \( \alpha \in \mathbb{R}^D \). For any subset \( \mathcal{X} \) in \( T^D \), consider the distribution function

\[
\rho_\mathcal{X}(i, x_0) = \#\{j; S_\alpha^j(x_0) \in \mathcal{X}, \ 0 \leq j < i\},
\]

where \( x_0 \in T^D \) is the starting point of the orbit \( \text{Orb}(x_0, S_\alpha) \) with respect to the translation \( S_\alpha \), i.e., \( \rho_\mathcal{X}(i, x_0) \) is the number of the first \( i \) points in \( \text{Orb}(x_0, S_\alpha) \) that lie in \( \mathcal{X} \subset T^D \). Suppose that the ratio

\[
\frac{\rho_\mathcal{X}(i, x_0)}{i} = \mu(\mathcal{X}) + o(1)
\]

as \( i \to +\infty \) varies around some value \( \mu(\mathcal{X}) \) independent of \( i \). Under this condition, we may consider the deviation function

\[
\delta_\mathcal{X}(i, x_0) = \rho_\mathcal{X}(i, x_0) - i\mu(\mathcal{X})
\]

of the distribution (0.1) with respect to the expected value \( i\mu(\mathcal{X}) \).

Then \( \mathcal{X} \) is called a bounded remainder set or simply a BR-set if the inequality

\[
|\delta_\mathcal{X}(i, x_0)| \leq c_{\mathcal{X}, x_0}
\]

is fulfilled for an arbitrary starting point \( x_0 \in T^D \) and for all \( i = 0, 1, 2, \ldots \). Here \( c_{\mathcal{X}, x_0} \) is a constant that depends only on the set \( \mathcal{X} \) and on the choice of a starting point \( x_0 \).

2. Examples of BR-sets. For the first time, bounded remainder sets were discovered by Hecke [5]. Those were the intervals \( \mathcal{X} \subset [0, 1) \) of length \( |\mathcal{X}| = a\alpha + b \) with integral parameters \( a, b \in \mathbb{Z} \).

In the two-dimensional case, Szusz [13] constructed a family of parallelograms \( \mathcal{X} \subset \mathbb{T}^2 \) for which estimate (0.3) holds true. Analyzing Szusz’s construction, Liardet [8] found a reduction of BR-sets \( \mathcal{X} \subset \mathbb{T}^D \) to similar sets of smaller dimension \( \mathcal{X}' \subset \mathbb{T}^{D-1} \). This was an important step, because it allowed to construct BR-sets in arbitrary dimension \( D \). For example, Szusz’s parallelograms are obtained by lifting Hecke’s intervals. Another approach to constructing such sets \( \mathcal{X} \) was found by Rauzy [11] and Ferenczi [3]. They related the property of \( \mathcal{X} \) being a BR-set to the properties of the corresponding

2010 Mathematics Subject Classification. Primary 52C17.
Key words and phrases. Bounded remainder sets, toric developments, symmetrization.
Supported by RSF (project no. 14-11-00433).
induced map $S_\alpha \mathcal{X}$ — the map of the first return to $\mathcal{X}$, or the Poincaré map. The idea of Rauzy–Ferenczi was realized in [17], where the general inequality
\begin{equation}
(0.4) \quad |\delta \chi(i, x_0)| < 2
\end{equation}
was proved for all sets $X \subset T^2$ from the sequence of nested self-similar fractal Rauzy sets $R^{(1)} \supset \cdots \supset R^{(m)} \supset \cdots$ tending to $\{0\}$.

A global approach to finding $BR$-sets was suggested in [22], where instead of sets $X \subset T^D$ by themselves, complete partitions of the tori $T^D = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_D$ into $BR$-sets $X = X_k$ were considered. The main idea was to define a lift of the torus $T^D$ to the covering space $\mathbb{R}^D$:
\begin{equation}
(0.5) \quad \begin{array}{ccc}
\mathbb{R}^D & \xrightarrow{S_\alpha} & \mathbb{R}^D \\
\pi & & \pi \\
T^D & \xrightarrow{S_\alpha} & T^D
\end{array}
\end{equation}
After that, a rotation $S_\alpha$ of the torus corresponds to an exchange transformation $S_\alpha$ of some subsets $X_0, X_1, \ldots, X_D$ in $\mathbb{R}^D$, i.e., a translation of each of these sets by some separate vector. The union $T^D = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_D$ in $\mathbb{R}^D$ corresponds to a development of the torus $T^D$. Such developments $T^D$ were constructed in [23] by using stretching of $D$-dimensional unit cubes.

In [24], we used (0.5), to construct simplest multidimensional $BR$-sets $X = P^D$ that are $D$-dimensional polyhedra: parallelepipeds or convex parallelohedra with $\sharp V(P^D) = 2D + 1 - 2$ vertices. In dimensions $D = 1$ and 2 these are, respectively, sets containing Hecke intervals, and hexagons with opposite sides parallel; and for $D = 3, 4$ these are Voronoi parallelohedra [16], e.g., Fedorov’s rhombic dodecahedron [30]. In [24], the inequality
\begin{equation}
(0.6) \quad |\delta \chi(i, x_0)| < Dh
\end{equation}
was proved for these $BR$-polyhedra $X = P^D$; (0.6) is a multidimensional analog of Hecke’s theorem [5]. Here, the set $X \subset T^D$ is obtained by a reduction $X \mod \mathbb{Z}^D$, and deviations $\delta \chi(i, x_0)$ are considered for the translation $S_\beta$ of the torus by a vector $\beta = \frac{1}{h}(\alpha + l)$, where $h$ is a natural number, and $l$ is an arbitrary vector in the cubic lattice $\mathbb{Z}^D$.

The next advancement was more formal: topological and algebraic considerations were used. It turned out that the parallelohedras $X = P^D$ described above admit small deformations such that the property of being $BR$-sets is retained [27]. Moreover, in [23] we defined a certain noncommutative and nonassociative product $\otimes_k$ that allowed us to turn parallelohedra $P^{D_1}$ and $P^{D_2}$ under exchange transformations into a new parallelohedron $P^D = P^{D_1} \otimes_k P^{D_2}$ of dimension $D = D_1 + D_2$, with exchange transformation producing a partition into $BR$-sets.

As a result we got canonical developments $T$ of the tori $T^D$ that consist of parallelohedra and their deformations.

3. Symmetrization technique. Besides canonical developments described in Subsection 2, there are examples of much more general developments that occur as attractors of $(D + 1)$-translational transformations of the space $\mathbb{R}^D$, see [26, 27].

In the present paper we introduce a new technique for constructing $BR$-sets, based on embedding toric developments $T \subset \mathbb{R}^D$ with exchange transformations into the torus $T^D$ followed by their symmetrizations. As examples of such developments we mention the induced subsets $T = T^{(m)}$ of level $m = 0, 1, 2, \ldots$ in one-dimensional Fibonacci partitions [18], the two-dimensional Rauzy partitions [10, 17], and their analogs [12]. The symmetrization technique under consideration is general, because changing the level

\begin{enumerate}
\item[$\bullet$] Finitely generated groups.
\item[$\bullet$] Poincaré map.
\item[$\bullet$] Symmetrization technique.
\end{enumerate}
m of the induced developments $T^{(m)}$ we can obtain developments $\tilde{T}^{(m)} \subset \mathbb{R}^D$ of the torus $\mathbb{T}^D$ that are good approximations of more complicated exchange transformation developments, as well as arbitrary $BR$-sets.

Theorem 6.1 shows that the described developments possess partitions

$$\tilde{T}^{(m)} = \tilde{T}_0^{(m)} \cup \tilde{T}_1^{(m)} \cup \cdots \cup \tilde{T}_D^{(m)}$$

into portions $\tilde{T}_k^{(m)} \subset \mathbb{R}^D$ such that under the reduction $X_k = \tilde{T}_k^{(m)} \mod \mathbb{Z}^D$ onto a torus $\mathbb{T}^D$ each of them induces a bounded remainder sets $X_k \subset \mathbb{T}^D$ with deviation function (0.2) satisfying

$$|\delta_{X_k}(i, x_0)| \leq c_k h.$$  

Here the constants $c_k = c_{\tilde{T}_k}$ do not depend on the parameter $h$ of the translation vector $\beta$ in (0.6) and on the iterations step $i$. They are determined exclusively by the size of the toric development $\tilde{T}^{(m)}$ of the torus $\mathbb{T}^D$. Comparing the bounds in (0.6) and in (0.7), note that the constants $c_k$ in (0.7) play the role of the dimension $D$ in (0.6).

4. Historical account. Breaking the style, we say that Kronecker’s stern shadow arises: a mathematical entity does not exist unless it is constructed. A congenial thought seized Cantor’s imagination so much that he was forced to escape it in the asylum [29]. “Mathematics and schizophrenia”, a hot topic, attracts not the writers among mathematicians, but clients solely.

Let us follow Hecke as an example of tranquility and good sense. The incentives of Hecke [5] that led to the appearance of $BR$-sets were Dirichlet series with fractional parts as coefficients, and analytic continuation of these series. That branch of number theory was developed in [28]. Meanwhile, the general path for the development of the theory of $BR$-sets forced its way through a porism, i.e., in a completely unexpected manner. The $BR$-sets themselves, and subsets of integers that they generate — sequences of iteration indices of the translation $S_\alpha$ — possess many properties important for applications: algebraic (quasirings, see [19]); arithmetical (number systems, see [19]) and expansions in multidimensional continuous fractions by the Jacobi–Perron algorithm (see [4]); Diophantine (equations on $\circ$-multiplication of Matiyasevich–Knuth, see [20, 21]); geometrical (quasilattices, see [19]); combinatorial (complexity theory, balanced words, see [25]), which are multidimensional analogs of Sturmian words (see [9]); dynamical (Poincaré maps and renormalizations (see [2]); and a lot of other properties (see, for example, [1, 6, 7, 14]).

§1. Embedding developments into tori

1.1. Exchanged developments of the torus. Let

$$L = \mathbb{Z}[l_1, \ldots, l_D]$$

be a complete lattice in $\mathbb{R}^D$ with basis $l_1, \ldots, l_D$, i.e., the vectors $l_1, \ldots, l_D$ are linearly independent over the field $\mathbb{R}$ of real numbers, and let $T$ be some subset in $\mathbb{R}^D$. We say that $T$ is a development of the torus $\mathbb{T}^D_L = \mathbb{R}^D/L$ if the map

$$T \rightarrow \mathbb{T}^D_L: \quad x \mapsto x \mod L$$

is bijective. We call $T$ an exchanged developments if we are given a partition

$$T = T_0 \cup T_1 \cup \cdots \cup T_D$$

and an exchange transformation

$$T \xrightarrow{S'} T: S'(x) = x + v_{\text{col}}(x)$$
by vectors $v_0, v_1, \ldots, v_D$ in $\mathbb{R}^D$ that are related to the chosen basis of $L$ by the formulas
\begin{equation}
 l_k = v_k - v_0 \quad \text{for} \quad k = 1, \ldots, D.
\end{equation}
In (1.3), col$(x) = k$ denotes the color of a point $x \in T_k$ for $k = 0, 1, \ldots, D$.
If we substitute
\begin{equation}
v_0 = \alpha',
\end{equation}
relations (1.4) and (1.5) imply the congruences
\[ v_k \equiv \alpha' \mod L \]
for all $k = 0, 1, \ldots, D$. Thus, the exchange transformation (1.3) is equivalent to the translation
\begin{equation}
T_L^D \xrightarrow{S'} \mathbb{T}_L^D: S'(x) \equiv x + \alpha' \mod L,
\end{equation}
by the vector $\alpha' \mod L$ occurring in (1.5).

Remark 1.1. The role of elementary exchanged developments $T$ of the torus $\mathbb{T}^D$ can be played by the exchange polyhedra $P$ constructed in [25], which may be either convex or nonconvex.

Besides the torus $T_L^D$, we need another torus $\mathbb{T}^D = \mathbb{R}^D / \mathbb{Z}^D$, where $\mathbb{Z}^D = \mathbb{Z}[e_1, \ldots, e_D]$ is a cubic lattice in $\mathbb{R}^D$ with the orthonormal basis $e_1 = (1, \ldots, 0), \ldots, e_D = (0, \ldots, 1)$. We introduce a translation $S = S_\alpha$ of the torus $\mathbb{T}^D$ by a vector $\alpha = (\alpha_1, \ldots, \alpha_D)$ by setting
\begin{equation}
T^D \xrightarrow{S'} \mathbb{T}^D: x \mapsto S(x) \equiv x + \alpha \mod \mathbb{Z}^D,
\end{equation}
under the assumption that the vector $\alpha$ has coordinates $0 < \alpha_k < 1$ for $k = 1, \ldots, D$. In what follows, the torus $\mathbb{T}^D$ will be used as an enveloping space for embeddings of different tori $T_L^D$ with different lattices $L$.

1.2. Developments embedded into torus.

Definition 1.1. We say that an exchanged development $T$ as in (1.2) embeds
\begin{equation}
T \xrightarrow{\text{emb}} \mathbb{T}^D
\end{equation}
into a torus $\mathbb{T}^D$ with respect to a translation $S = S_\alpha$ if the following conditions are satisfied.
1. The subset $T \subset \mathbb{R}^D$ is $\mathbb{Z}^D$-distinguishable, i.e., for any $x, y$ in $T$ such that $x \equiv y \mod \mathbb{Z}^D$ we have $x = y$. Thus, the map
\begin{equation}
T \longrightarrow T \mod \mathbb{Z}^D: x \mapsto x \mod \mathbb{Z}^D
\end{equation}
is a bijection; hence, the map (1.9), allows us to identify $T$ with a subset
\begin{equation}
T \subset \mathbb{T}^D
\end{equation}
of the torus $\mathbb{T}^D$.
2. The exchange vectors (1.3) are of the form
\begin{equation}
v_k \equiv n_k \alpha \mod \mathbb{Z}^D
\end{equation}
for all $k = 0, 1, \ldots, D$ and some coefficients $n_k = 1, 2, 3, \ldots$.
3. Let
\begin{equation}
\text{Orb}^*(T_k) = \{S^j(T_k); j = 1, \ldots, n_k - 1\}
\end{equation}
denote the orbit of the subset $T_k \subset T$. Using (1.10), we may assume that $\text{Orb}'_k \subset \mathbb{T}^D$. Then, by definition, the orbits in (1.12) satisfy the condition
\begin{equation}
\text{Orb}'(T_k) \cap T = \emptyset
\end{equation}
for $k = 0, 1, \ldots, D$.

In order to state the next result, we also need to define, in addition to (1.12), the \textit{complete} orbits
\begin{equation}
\text{Orb}(T_k) = \{ S^j(T_k); j = 0, 1, \ldots, n_k - 1 \}.
\end{equation}

Moreover, we shall say that the translation vector $\alpha = (\alpha_1, \ldots, \alpha_D)$ (see (1.7)) is \textit{irrational} whenever
\begin{equation}
\text{the numbers } 1, \alpha_1, \ldots, \alpha_D \text{ are linearly independent over } \mathbb{Z}.
\end{equation}

**Theorem 1.1.** Suppose that a development $T$ is embedded (see (1.8)) into the torus $\mathbb{T}^D$, that $T$ has an interior point, and that the translation vector $\alpha$ in (1.7) is irrational. Then the following holds true.

1. The sets of the complete orbits $\text{Orb}(T_k)$ are disjoint, i.e.,
\begin{equation}
S^{j_1}(T_{k_1}) \cap S^{j_2}(T_{k_2}) \neq \emptyset
\end{equation}
only if $j_1 = j_2$ and $k_1 = k_2$.
2. There is a partition
\begin{equation}
T = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_D
\end{equation}
of the torus $\mathbb{T}^D$ such that
\begin{equation}
T_k = T_k \sqcup S^1(T_k) \sqcup \cdots \sqcup S^{n_k - 1}(T_k)
\end{equation}
is the orbit partition, consisting of the sets that are in the orbit $\text{Orb}(T_k)$.

**Proof.** Suppose that (1.16) is true; we may assume that $j_1 \geq j_2$. Then (1.16) implies
\begin{equation}
S^{j_1 - j_2}(T_{k_1}) \cap S^0(T_{k_2}) = S^{j_1 - j_2}(T_{k_1}) \cap T_{k_2} \neq \emptyset
\end{equation}
whence
\begin{equation}
S^j(T_{k_1}) \cap T \neq \emptyset,
\end{equation}
where $j = j_1 - j_2$. By assumption, we have $0 \leq j \leq j_1 \leq n_k - 1$. Together with Definition 1.1 and formula (1.13), this implies that $j = 0$, i.e., $j_1 = j_2$. Therefore, property (1.18) implies that $T_{k_1} \cap T_{k_2} \neq \emptyset$, so that $k_1 = k_2$, and the first assertion (1.16) is proved.

In order to prove the second assertion of Theorem 1.1, we need the next lemma.

**Lemma 1.1.** The set $\mathcal{T} \subset \mathbb{T}^D$ defined in (1.17) is closed,
\begin{equation}
S : \mathcal{T} \longrightarrow \mathcal{T},
\end{equation}
under the translation $S = S_\alpha$.

**Proof of the lemma.** Choose any set $S^j(T_k)$ in the orbit $\text{Orb}(T_k)$ with degree $0 \leq j < n_k - 1$. Then
\begin{equation}
S(S^j(T_k)) = S^{j+1}(T_k) \subset \text{Orb}(T_k),
\end{equation}
because $j \leq n_k - 1$. If we choose $S^{n_k - 1}(T_k)$ in $\text{Orb}(T_k)$, we have
\begin{equation}
S(S^{n_k - 1}(T_k)) = S^{n_k}(T_k) \equiv T_k + n_k \alpha \mod \mathbb{Z}^D \equiv T_k + v_k \mod \mathbb{Z}^D.
\end{equation}
Since $T_k + v_k \subset T$ by Definition (1.3), relation (1.21) shows that
\begin{equation}
S(S^{n_k - 1}(T_k)) \subset T \mod \mathbb{Z}^D.
\end{equation}
Inclusions (1.20) and (1.22) imply (1.19). The lemma is proved. \[\square\]
We proceed to the proof of the second part of Theorem 1. By assumption, the set $T$ has an interior point $x^* \in T$. This means that there exists a ball $B_\varepsilon(x^*) \subset T$ of radius $\varepsilon > 0$ centered at $x^*$. By Definition 1.1, the set $T \subset \mathbb{R}^D$ is $\mathbb{Z}^D$-distinguishable; therefore, its subset $B_\varepsilon(x^*)$ is also $\mathbb{Z}^D$-distinguishable. Hence, we may assume that the ball $B_\varepsilon(x^*)$ is contained in the torus $T^D$.

Let $x$ be an arbitrary point on the torus $T^D$. By the theorem’s assumptions, the translation $S = S_\alpha$ of the torus is defined for an irrational vector $\alpha$. By the Weyl theorem [15], the orbit of any point under the translation $S$ by an irrational vector $(1.15)$ is everywhere dense on the torus $T^D$. Therefore, there exists $j = 0, 1, 2, \ldots$ such that the image $S^{-j}(x)$ of a point $x \in T^D$ lies in the ball $B_\varepsilon(x^*)$.

Now we have a point $x_0$ such that $x_0 = S^{-j}(x) \in B_\varepsilon(x^*)$, and, therefore, due to the inclusions $B_\varepsilon(x^*) \subset T \subset T$, the point

$$x_0 = S^{-j}(x) \in T$$

lies in the partition $(1.17)$. But Lemma 1.1 implies that the partition $T$ is closed under the translation $S$ of the torus $T^D$. Then by (1.23), the point $S^j(x_0) = x$ also lies in the partition $T$. Since $x$ was arbitrary, the second part of Theorem 1.1 is proved. \hfill \Box

§2. Rauzy Developments

2.1. Rauzy partitions. We already mentioned the one-dimensional Fibonacci partitions (see [13]) in the Introduction. Here we consider another well-known class of two-dimensional partitions.

The Rauzy fractal $R$ was constructed in [10]. It is a connected subset of $\mathbb{R}^2$ that has a fractal boundary $\partial R$ and is a development of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

The Rauzy partition $R^{(m)}$ of level $m$ is defined by the following recurrence [17]:

1) for the starting levels, the Rauzy partitions

$$R^{(-3)} = R, \quad R^{(-2)} = R, \quad R^{(-1)} = R$$

coincide with the Rauzy fractal $R$;

2) for the next levels $m = 0, 1, 2, \ldots$, the partitions $R^{(m)}$ are defined by the recurrent relation

$$R^{(m)} = B R^{(m-1)} \sqcup (B^2 R^{(m-2)} + z) \sqcup (B^3 R^{(m-3)} + z + Bz),$$

where

$$z = \left( \begin{array}{c} \zeta - 1 \\ \zeta^2 \end{array} \right), \quad B = \left( \begin{array}{cc} -\zeta & -\zeta \\ 1 - \zeta^2 & -\zeta^2 \end{array} \right),$$

and $\zeta = \theta^{-1}$, where $\theta$ denotes the Pisot number that is the real root of the equation

$$x^3 - x^2 - x - 1 = 0.$$ 

The matrix $B$ has a similar property, see [10]:

$$B^m = B^{m-1} + B^{m-2} + B^{m-3}.$$ 

The first nontrivial Rauzy partition $R^{(0)}$ has zero level $m = 0$, and

$$R^{(0)} = R^{(0)}_0 \sqcup R^{(0)}_1 \sqcup R^{(0)}_2,$$

where

$$R^{(0)}_0 = B R, \quad R^{(0)}_1 = B^2 R + z, \quad R^{(0)}_2 = B^3 R + z + Bz.$$
2.2. Induced sets. For an arbitrary level $m = 0, 1, 2, \ldots$, the induced sets $T^{(m)}$ are generated by the affine map with matrix $B$ as in (2.3) by the following rule:

\[(2.6) \quad T^{(m)} = T^{(m)}_0 \cup T^{(m)}_1 \cup T^{(m)}_2,\]

where

\[(2.7) \quad T^{(m)}_0 = B^m R_0^{(0)}, \quad T^{(m)}_1 = B^m R_1^{(0)}, \quad T^{(m)}_2 = B^m R_2^{(0)}.\]

Definitions (2.6) and (2.7) imply that for the zeroth level the induced set $T^{(0)}$ coincides with the Rauzy partition $R^{(0)}$.

The induced set $T^{(0)} = T^{(0)}_0 \cup T^{(0)}_1 \cup T^{(0)}_2$ is a toric development $T^2$, which admits the exchange transformation

\[(2.8) \quad T^{(0)} \xrightarrow{S'} T^{(0)} : S'(x) = x + v_{\text{col}(x)}\]

by the vectors

\[(2.9) \quad v_0 = \alpha - e_1, \quad v_1 = \alpha - e_2, \quad v_2 = \alpha,\]

where the column

\[(2.10) \quad \alpha = \begin{pmatrix} \zeta \\ \zeta^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\]

is from now on viewed as a translation vector of the torus $T^2$.

2.3. Orbit partitions. The results of [17] show that the Rauzy partitions $R^{(m)}$ defined by the recurrent relation (2.1), (2.2) are orbit partitions

\[(2.11) \quad T^{(m)} = T^{(m)}_0 \cup T^{(m)}_1 \cup T^{(m)}_2,\]

generated by the induced sets $T^{(m)}$ occurring in (2.6), (2.7). In (2.11), the components $T^{(m)}_k$ are the orbits

\[(2.12) \quad T^{(m)}_k = T^{(m)}_k \cup S^1(T^{(m)}_k) \cup \cdots \cup S^{n_k-1}(T^{(m)}_k)\]

of the domains $T^{(m)}_k$ of the partition $T^{(m)} = T^{(m)}_0 \cup T^{(m)}_1 \cup T^{(m)}_2$ under the translation $\mathbb{S} = \mathbb{S}_\alpha$ of the torus $T^2$ by the vector (2.10). The orbits in (2.12) have lengths

\[(2.13) \quad n_0 = t_m, \quad n_1 = t_{m-1} + t_{m-2}, \quad n_2 = t_{m-1},\]

respectively. Here the $t_m$'s are the Tribonacci numbers defined by the recurrent relation

\[(2.14) \quad t_m = t_{m-1} + t_{m-2} + t_{m-3}\]

with initial conditions $t_0 = 1$, $t_1 = 2$, $t_2 = 4$. It should be noted that the recurrent relation (2.2) for the Rauzy partitions $R^{(m)}$ is a geometric and combinatorial interpretation of relation (2.14) for the Tribonacci numbers $t_m$.

Using Definition (1.8), we see that the induced sets $T^{(m)}$ of all levels $m = 0, 1, 2, \ldots$, which are exchanged developments, embed

\[(2.15) \quad T^{(m)} \hookrightarrow T^2\]

into the torus $T^2$. This follows immediately from the existence of the partitions (2.6), (2.11), (2.12), and from the congruences

\[(2.16) \quad B^m v_k \equiv n_k \alpha \mod \mathbb{Z}^2\]

for $k = 0, 1, 2$. Relations (2.16) can be proved by induction on the basis of the property (2.4) of the matrix $B$. 

SYMMETRIZATION OF BOUNDED REMAINDER SETS 497
§3. Matrices and exchange transformations

3.1. Admissible bases of a cubic lattice. Suppose that a matrix $U$ lies in the group $\text{GL}_D(\mathbb{Z})$ of unimodular matrices of size $D$, i.e., $U$ is an integral matrix with determinant $\pm 1$. The map

\begin{equation}
U \rightarrow \mathbf{e}' = \begin{pmatrix} e'_1 \\ \vdots \\ e'_D \end{pmatrix} = U \mathbf{e}, \quad \text{where} \quad \mathbf{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_D \end{pmatrix},
\end{equation}

gives a bijection between the matrix group $\text{GL}_D(\mathbb{Z})$ and the bases $\mathbf{e}'$ of the cubic lattice $\mathbb{Z}^D$.

By Definition (1.8), a development $T$ that embeds into the torus $\mathbb{T}^D$ by $\mathbf{e} \mapsto \mathbb{T}^D$ possesses exchange vectors $v_0, v_1, \ldots, v_D$ of the form

\begin{equation}
v_k = n_k \alpha - a_{k1} e_1 - \cdots - a_{kD} e_D
\end{equation}

with integral coefficients $a_{kj}$, where $k = 0, 1, \ldots, D$ and $j = 1, \ldots, D$. Besides (3.2), in what follows we shall use another family of exchange vectors:

\begin{equation}
\mathbf{\hat{v}}_0 = \alpha, \quad \mathbf{\hat{v}}_1 = \alpha - e'_1, \ldots, \mathbf{\hat{v}}_D = \alpha - e'_D.
\end{equation}

Note that the vectors defined in (3.3) are determined by the choice of $\mathbf{e}'$ in (3.1). We rewrite equations (3.3) in a matrix form:

\begin{equation}
\mathbf{\hat{v}} = \mathbf{1}_{D+1} \alpha - \mathbf{\hat{e}}',
\end{equation}

where

\begin{equation}
\mathbf{\hat{v}} = \begin{pmatrix} \alpha \\ \alpha - e_1 \\ \vdots \\ \alpha - e_D \end{pmatrix}, \quad \mathbf{1}_{D+1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{\hat{e}}' = \begin{pmatrix} 0 \\ 0' \end{pmatrix}.
\end{equation}

In (3.4), $\mathbf{1}_{D+1} \alpha$ is a product of matrices, i.e., the column of vectors $\alpha$ with height $D + 1$. Similarly, equations (3.2) can be written as

\begin{equation}
\mathbf{v} = \mathbf{n} \alpha - A \mathbf{e}
\end{equation}

where

\begin{equation}
\mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_D \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_D \end{pmatrix}, \quad A = \begin{pmatrix} a_{01} & \cdots & a_{0D} \\ a_{11} & \cdots & a_{1D} \\ \vdots & \ddots & \vdots \\ a_{D1} & \cdots & a_{DD} \end{pmatrix}.
\end{equation}

In order to construct the orbits, we need a matrix

\begin{equation}
\mathbf{N} = \begin{pmatrix} n_{00} & n_{01} & \cdots & n_{0D} \\ n_{10} & n_{11} & \cdots & n_{1D} \\ \vdots & \vdots & \ddots & \vdots \\ n_{D0} & n_{D1} & \cdots & n_{DD} \end{pmatrix}
\end{equation}

satisfying

\begin{equation}
\mathbf{v} = \mathbf{N} \mathbf{\hat{v}}, \quad \mathbf{N} \mathbf{1}_{D+1} = \mathbf{n}
\end{equation}

and such that

\begin{equation}
\mathbf{N} \in \text{Mat}_{D+1,D+1}(\mathbb{Z}), \quad \mathbf{N} \geq 0.
\end{equation}
Here the inequality $N \geq 0$ means that the matrix $N$ has entries $n_{kk'} \geq 0$. By (3.1) and the first identity in (3.9), we have

$$v = N \hat{v} = N1_{D+1} + Ne'.$$

Combined with the second identity in (3.9), this implies that

(3.11) $$v = n\alpha - N\hat{e}'.$$

Comparing (3.11) with (3.6), we obtain

(3.12) $$Ae = Ne'.$$

We split the matrix $N$ into blocks:

$$N = (n_0 | N_1), \quad \text{where} \quad n_0 = \begin{pmatrix} n_{00} \\ n_{10} \\ \vdots \\ n_{D0} \end{pmatrix}.$$

From (3.12) and (3.6) it follows that

(3.13) $$Ae = N_1e'.$$

Using the relationship (3.1) between the bases $e$ and $e'$ and formula (3.13), we obtain

(3.14) $$Ae = N_1Ue.$$

It follows that $A = N_1U$. Since by assumption the matrix $U$ is nonsingular, we can rewrite this as follows:

(3.15) $$N_1 = AU^{-1}.$$

Let $E_D$ be the identity matrix of size $D$. Then

(3.16) $$N_1(-1_D | E_D) = \begin{pmatrix} -n_{01} - \cdots - n_{0D} \\ -n_{11} - \cdots - n_{1D} \\ \vdots \\ -n_{D1} - \cdots - n_{DD} \end{pmatrix} N_1,$$

where, as in (3.5), we denote by $1_D$ the unit column of size $D$. It follows that

(3.17) $$(n | 0) + N_1(-1_D | E_D) = \begin{pmatrix} n_0 - n_{01} - \cdots - n_{0D} \\ n_1 - n_{11} - \cdots - n_{1D} \\ \vdots \\ n_D - n_{D1} - \cdots - n_{DD} \end{pmatrix} N_1.$$

By the second condition in (3.9), we have

(3.18) $$n_{k0} + n_{k1} + \cdots + n_{kD} = n_k$$

for $k = 0, 1, \ldots, D$. Therefore, the matrix on the right-hand side of (3.17) equals $(n_0 | N_1) = N$, so that we have

(3.19) $$N = (n | 0) + N_1(-1_D | E_D).$$

**Proposition 3.1.** For a matrix $U$ in $GL_D(\mathbb{Z})$, consider the matrix $N = N_U$ defined by

(3.20) $$N = \hat{n} + AU^{-1}\hat{E},$$

where

$$\hat{n} = (n | 0) = \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_D \end{pmatrix}_{D+1,D+1}$$
and
\[ \hat{E} = (-1_D | E_D) = \begin{pmatrix} -1 & 1 & 0 \\ \vdots & \ddots & \vdots \\ -1 & 0 & 1 \end{pmatrix}_{D,D+1}. \]

Then the matrix \( N \) has integral entries \( n_{kk'} \in \mathbb{Z} \) for \( k, k' = 0, 1, \ldots, D \), and we have
\[ (3.21) \quad v = N\hat{v}, \quad N1_{D+1} = n, \]
where \( v \) and \( \hat{v} \) are the vector columns (3.7) and (3.5).

Thus, conditions (3.9) and (3.10) on a unimodular matrix \( U \in \text{GL}_D(\mathbb{Z}) \) are equivalent to a single condition
\[ (3.22) \quad N = N_U \geq 0 \]
on the coefficients \( n_{kk'} \geq 0 \) of the matrix \( N \) in (3.20).

Proof. Definition (3.20) shows that the matrix \( N \) is integral. The first equation in (3.21) follows from the relation \( e = U^{-1}e' \) and (3.2), while the second follows from (3.17).

Now, after reversing the argument, we see that in order to satisfy conditions (3.9) and (3.10) on a matrix \( U \), we need to require condition (3.22). \( \square \)

A basis \( e' = e'_U \) of the cubic lattice \( \mathbb{Z}^D \) that corresponds (see (3.1)) to a unimodular matrix \( U \) satisfying (3.22) will be called an admissible basis.

### 3.2. \( P \)-invariance.

Suppose that \( U_1 = PU \), where \( U \in \text{GL}_D(\mathbb{Z}) \) and \( P \) is a permutation matrix, which has one entry equal to 1 in every column, while the other entries are equal to 0. Using (3.20), we get
\[ (3.23) \quad N_{U_1} = \hat{n} + AU_1^{-1}\hat{E} = \hat{n} + AU^{-1}P^{-1}\hat{E}. \]

We use the commutation relation
\[ (3.24) \quad P^{-1}\hat{E} = \hat{E}\hat{P}^{-1}, \quad \text{where} \quad \hat{P} = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}. \]

Moreover, since
\[ \hat{n}\hat{P}^{-1} = (n | 0) \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix} = (n | 0), \]
we obtain
\[ (3.25) \quad \hat{n}\hat{P}^{-1} = \hat{n}. \]
Thus, (3.23)–(3.25) give us the identity
\[ \hat{n} + AU^{-1}P^{-1}\hat{E} = (\hat{n} + AU^{-1}\hat{E})\hat{P}^{-1}. \]

Together with (3.20), this implies the following proposition.

**Proposition 3.2.** 1. Suppose that matrices \( N_U \) and \( N_{U_1} \) are defined as in (3.20). Then
\[ (3.26) \quad N_{U_1} = N_U\hat{P}^{-1} \]
whenever \( U_1 = PU \) for some permutation matrix \( P \).

2. If a matrix \( U \in \text{GL}_D(\mathbb{Z}) \) satisfies condition (3.22), then so does the matrix \( U_1 = PU \in \text{GL}_D(\mathbb{Z}) \).
§4. Colored sequences

4.1. Coloring of iterations. Suppose that \( U \in \text{GL}_D(\mathbb{Z}) \) is a matrix satisfying condition (3.22), and that \( N = N_U = (n_{kk'})_{D+1, D+1} \) is the corresponding matrix as in (3.19). From (3.22) it follows that

\[
\begin{align*}
n_{kk'} & \geq 0, \quad n_{k0} + n_{k1} + \cdots + n_{kD} = n_k
\end{align*}
\]

for \( k, k' = 0, 1, \ldots, D \).

We denote by

\[
\begin{align*}
N_k &= \{1, 2, \ldots, n_k\}
\end{align*}
\]

the numbers of iterations in the orbits of tiles and introduce the partitions

\[
N_k = N_{k0} \sqcup N_{k1} \sqcup \cdots \sqcup N_{kD},
\]

where the set \( N_{kk'} \) contains \( \sharp N_{kk'} = n_{kk'} \) elements, and

\[
\begin{align*}
N_{k0} &= \{1, \ldots, n_{k0}\}, \quad N_{k1} = \{n_{k0} + 1, \ldots, n_{k0} + n_{k1}\}, \ldots, \\
N_{kD} &= \{n_{k0} + \cdots + n_{k,D-1} + 1, \ldots, n_k\}.
\end{align*}
\]

Let \( S_k \) be the symmetric group on the set \( N_k \). Choosing an arbitrary permutation \( s_k \in S_k \), we define a coloring of \( N_k \): an element \( i \in N_k \) will have color \( c(i) = k' \), where \( k' = 0, 1, \ldots, D \), such that

\[
\begin{align*}
c(i) = c_{s_k}(i) = k' \quad \text{if} \quad s_k(i) \in N_{kk'}.
\end{align*}
\]

The colored set \( N_k \) will be denoted by

\[
\begin{align*}
N_{k}^{\text{col}} &= N_{k}^{\text{col}}(s_k).
\end{align*}
\]

4.2. Invariance of the coloring. We define subgroups \( S_{kk'} \subset S_k \) in \( S_k \) for \( k' = 0, 1, \ldots, D \), by postulating that

1) any permutation \( s_{kk'} \in S_{kk'} \) gives rise to a bijection \( s_{kk'}: N_{kk'} \rightarrow N_{kk'} \) of the subset \( N_{kk'} \subset N_k \);

2) the restriction of \( s_{kk'} \) to the complement \( N_k \setminus N_{kk'} \) is the identity map, \( s_{kk'}(i) = i \) for all \( i \in N_k \setminus N_{kk'} \).

Consider the direct product

\[
\begin{align*}
S_{k*} &= S_{k0} \times S_{k1} \times \cdots \times S_{kD} \subset S_k,
\end{align*}
\]

which is a subgroup in \( S_k \). Definition (4.3) implies that for any permutations \( s_{k*} \in S_{k*} \) and \( s_k \in S_k \) we have the coloring invariance:

\[
\begin{align*}
\forall i \in N_k, \quad c_{s_{k*} \circ s_k}(i) = c_{s_k}(i)
\end{align*}
\]

for all \( i \in N_k \), and therefore,

\[
\begin{align*}
N_{k}^{\text{col}}(s_{k*} \cdot s_k) = N_{k}^{\text{col}}(s_k).
\end{align*}
\]

Thus, considering a coloring \( N_{k}^{\text{col}}(s_k) \) of the sets \( N_k \), we may assume that

\[
\begin{align*}
s_k \in S_{k*} \setminus S_k,
\end{align*}
\]

where \( S_{k*} \setminus S_k \) is the quotient group of \( S_k \) by the subgroup \( S_{k*} \) defined in (4.5).
5. Orbits and Toric Developments

5.1. Orbits. Suppose that \( T = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_D \subset \mathbb{R}^D \) is the exchanged development of the torus \( T_D^L \), as defined by (1.2). Moreover, suppose that a matrix \( U \in \text{GL}_D(\mathbb{Z}) \) satisfies condition (3.22), and \( s_k \in S_k \) is an arbitrary permutation for some \( k = 0, 1, \ldots, D \).

We define a map \( S \) by induction, setting
\[
S^j(T_k) = T_k \quad \text{for} \quad j = 0,
\]
\[
S^j(T_k) = S^{j-1}(T_k) + \hat{v}_{c(j)} \quad \text{for} \quad 1 \leq j \leq n_k,
\]
where \( c(j) = c_{s_k}(j) \) is the color introduced in (4.3).

The exchange condition (1.3) for the development \( T \) and condition (3.22) on the matrix \( U \) imply that
\[
S^{n_k}(T_k) = T_k + v_k \subset T.
\]

Consider the separate orbits
\[
\text{Orb}(T_k) = \{ S^j(T_k); \; j = 0, 1, \ldots, n_k - 1 \}
\]
for \( k = 0, 1, \ldots, D \), and their collection
\[
\text{Orb}(T) = \{ \text{Orb}(T_k); \; k = 0, 1, \ldots, D \},
\]
assuming that we are given a permutation \( s = (s_0, s_1, \ldots, s_D) \) belonging to the group
\[
S_{0,1,\ldots,D} = S_0 \times S_1 \times \cdots \times S_D.
\]

Denote by
\[
\hat{T} = \hat{T}_U \subset \mathbb{R}^D
\]
the union of all the sets in the orbit \( \text{Orb}(T) \); thus, by definition, the inclusion in (5.5) is true.

Consider the quotient map
\[
\hat{T} \overset{\iota}{\rightarrow} T^D; \; x \mapsto x = x \mod \mathbb{Z}^D
\]
of the set (5.5) onto the torus \( T^D = \mathbb{R}^D / \mathbb{Z}^D \).

From (1.8) it follows that the development \( T \) is embedded \( T \therefore T^D \) into the torus \( T^D \).

Proposition 5.1. Under the assumptions of Theorem 1.1, suppose that a unimodular matrix \( U \in \text{GL}_D(\mathbb{Z}) \) satisfies condition (3.22) of Proposition 3.1 and that the set \( \hat{T} = \hat{T}_U \) is as defined in (5.3). Then the following statements hold true.

1. If \( (j, k) \neq (j', k') \), then
\[
S^j(T_k) \cap S^{j'}(T_{k'}) = \emptyset.
\]

2. The map \( \hat{T} \overset{\iota}{\rightarrow} T^D \) defined in (5.6) is bijective.

3. a) The map
\[
S: \hat{T} \rightarrow \mathbb{R}^D,
\]
is well defined by (5.1), and moreover, b) the set \( \hat{T} \) is closed,

under the map (5.1).
4. The diagram

\[
\begin{align*}
\hat{T} & \xrightarrow{\iota} T^D \\
\downarrow S & \quad \downarrow S \\
\hat{T} & \xrightarrow{\iota} T^D
\end{align*}
\]

is commutative, where the map \( S \) (see (5.10)) is bijective, and \( S = S_\alpha \) is the translation \( (1.7) \) of the torus \( T^D \) by the vector \( \alpha \).

**Proof.** The first two parts follow from Theorem 1.1. The consistency of the definition (5.1) of the map (5.8) follows from (5.7), while the closedness (5.9) of the set \( \hat{T} \) under the map \( S \) follows from (5.2).

By (3.3), the translation vectors \( \hat{v}_{k'} \) in the definition (5.1) of the map \( S \) satisfy the congruences

\[
\hat{v}_{k'} \equiv \alpha \mod Z^D
\]

for all \( k' = 0, 1, \ldots, D \). The definition (5.6) of the quotient map \( \iota \) and the congruences (5.11) imply the commutativity of the diagram (5.10), which in its turn implies that the map \( S \) in (5.10) is a bijection. \( \square \)

5.2. Symmetrized toric development. We define a partition

\[
(5.12) \quad \hat{T} = \hat{T}_0 \sqcup \hat{T}_1 \sqcup \cdots \sqcup \hat{T}_D
\]

into sets \( \hat{T}_{k'} \) that are unions of the sets in

\[
\text{Orb}(T)_{k'} = \{ \text{Orb}(T_k); \ k = 0, 1, \ldots, D \} \subset \text{Orb}(T),
\]

where we set

\[
(5.13) \quad \text{Orb}(T_k)_{k'} = \{ S^j(T_k); \ c_{s_k}(j) = k', \ j = 0, 1, \ldots, n_k - 1 \} \subset \text{Orb}(T_k).
\]

It should be noted that in the definition (5.13) the index \( k' \) denotes the color of the translated set \( S^j(T_k) \). Then the map \( S \) acts on the set \( \hat{T} \subset \mathbb{R}^D \) as an exchange transformation

\[
(5.14) \quad S(\hat{T}_{k'}) = \hat{T}_{k'} + \hat{v}_{k'}
\]

for \( k' = 0, 1, \ldots, D \), where the exchanging vectors \( \hat{v}_{k'} \) are defined as in (3.3).

**Theorem 5.1.** Suppose that a matrix \( U \in \text{GL}_D(\mathbb{Z}) \) satisfies condition (3.22), and \( s = (s_0, s_1, \ldots, s_D) \) is an arbitrary permutation in the group \( S_{0,1,\ldots,D} \). Then the set \( \hat{T} \subset \mathbb{R}^D \) defined in (5.5) is an exchange transformation of the torus \( T^D \), i.e., there is a commutative diagram (5.10), where the map \( S \) is the exchange transformation (5.1).

**Proof.** This follows from Proposition 5.1 and the exchange transformation formula (5.1). \( \square \)

\section*{6. Distribution of Points on the Torus}

6.1. Distribution functions. Using diagram (5.10) and Theorem 5.1 we define a partition

\[
(6.1) \quad T = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_D
\]

of the torus \( T^D \) into sets

\[
(6.2) \quad T_k = \iota(\hat{T}_k) \subset T^D,
\]

where the \( \hat{T}_k \) are the sets in the partition (5.12).
In addition to (1.7), we introduce another translation
\[(6.3) \quad S_\beta(x) \equiv x + \beta \mod \mathbb{Z}^D\]
of the torus $\mathbb{T}^D$ by the vector
\[\beta = \frac{1}{h}(\alpha + b e'),\]
where $h$ is an arbitrary natural number, and
\[(6.4) \quad b e' = b_1 e'_1 + \cdots + b_D e'_D\]
is the product of the integral matrix $b = (b_1 \ldots b_d)$ and the column $e'$ defined in (3.1).
By the definition (6.4), the vector $b e'$ lies in the cubic lattice $\mathbb{Z}^D$.

We denote by
\[(6.5) \quad r_k(i, x_0) = \sharp\{ j; S_\beta^j(x_0) \in T_k, 0 \leq j < i \}\]
the distribution function $(k = 0, 1, \ldots, D)$, i.e., the number of the first $i$ points
\[x_0 = S_\beta^0(x_0), \quad x_1 = S_\beta^1(x_0), \ldots, \quad x_j = S_\beta^j(x_0), \ldots,\]
in the orbit $\text{Orb}(x_0, S_\beta)$ that lie in $T_k \subset \mathbb{T}^D$.

6.2. Deviation functions. Since by condition (3.1) the vectors $e'_1, \ldots, e'_D$ form a basis of the lattice $\mathbb{Z}^D$ and are thus linearly independent over $\mathbb{R}$, there exists a unique dual basis $e''_1, \ldots, e''_D$ defined by the equations
\[e''_k \cdot e'_m = \delta_{k,m},\]
where by $x \cdot y$ we denote the inner product $x \cdot y = x_1 y_1 + \cdots + x_D y_D$ of two vectors $x = (x_1, \ldots, x_D)$, $y = (y_1, \ldots, y_D)$ in $\mathbb{R}^D$.
We define the deviation functions
\[(6.6) \quad \delta_k(i, x_0) = r_k(i, x_0) - i a_k,\]
where
\[(6.7) \quad a_k = e''_k \cdot \alpha\]
for $k = 1, \ldots, D$, and
\[(6.8) \quad a_0 = 1 - a_0 - \cdots - a_D.\]
From (6.7) it follows that $a_1, \ldots, a_D$ are the coordinates of the vector $\alpha$ with respect to the basis $e'_1, \ldots, e'_D$.

6.3. Deviations for symmetrized toric developments. For an arbitrary set $X \subset \mathbb{R}^D$, we define the boundary values
\[m_l(X) = \inf_{x \in X} l \cdot x, \quad M_l(X) = \sup_{x \in X} l \cdot x,\]
where $l$ is a vector in $\mathbb{R}^D$.

**Theorem 6.1.** Under the assumptions of Theorem 1.1, suppose that a unimodular matrix $U \in \text{GL}_D(\mathbb{Z})$ satisfies condition (3.22), and that $\hat{T} = \hat{T}_U \subset \mathbb{R}^D$ is the development of the torus $\mathbb{T}^D = \mathbb{R}^D / \mathbb{Z}^D$ as in Theorem 5.1. Then the following statements hold true.

1. For any $k = 0, 1, \ldots, D$ we have
\[(6.9) \quad |\delta_k(i, x_0)| \leq c_k h\]
for an arbitrary initial point $x_0 \in \mathbb{T}^D$ and all $i = 0, 1, 2, \ldots$. In inequalities (6.9), the constants
\[(6.10) \quad c_k = c_{\hat{T},k} = M_{e''_k}(\hat{T}) - m_{e''_k}(\hat{T})\]
do not depend on the parameter $h$ of the translation vector $\beta$ occurring in (6.3) and on the iteration step $i$. The constants in (6.10) depend only on the size of the development $\hat{T}$ of the torus $T^D$.

2. If, moreover, we assume that in the partition (1.2) of the development $T$ every tile $T_k$ is measurable, then the deviations $\delta_k(i, x_0)$ defined by (6.6) are of the form

\[
\delta_k(i, x_0) = r_k(i, x_0) - i \text{vol} T_k
\]

for $k = 0, 1, \ldots, D$. Here $\text{vol} T_k$ denotes the volume of the set $T_k$ on the torus $T^D$, which is equal to the volume $\text{vol} \hat{T}_k$ of the corresponding set $\hat{T}_k \subset \hat{T}$ in (6.2).

Proof. This theorem is a particular case of the general Theorem 2 in [22] applied to the exchanged development $\hat{T}$ of the torus $T^D$. \qed

REFERENCES


Received 27/APR/2015

Translated by A. LUZGAREV