Алгебра и анализ
Том 29 (2017), № 2

St. Petersburg Math. J.
Vol. 29 (2018), No. 2, Pages 227–248
http://dx.doi.org/10.1090/spmj/1491
Article electronically published on March 12, 2018

WAVE MODEL OF THE STURM–LIOUVILLE OPERATOR ON THE HALF-LINE

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Dedicated to the memory of V. S. Buslaev

Abstract. The notion of the *wave spectrum* of a semibounded symmetric operator was introduced by one of the authors in 2013. The wave spectrum is a topological space determined by the operator in a canonical way. The definition involves a dynamical system associated with the operator: the wave spectrum is constructed from its reachable sets. In the paper, a description is given for the wave spectrum of the operator $L_0 = -\frac{d^2}{dx^2} + q$ that acts in the space $L_2(0,\infty)$ and has defect indices $(1,1)$. A functional (*wave*) model is constructed for the operator L_0^* in which the elements of the original $L_2(0,\infty)$ are realized as functions on the wave spectrum. This model turns out to be identical to the original L_0^*. The latter is fundamental in solving inverse problems: the wave model is determined by their data, which allows reconstruction of the original.

§0. Introduction

The notion of the wave spectrum of a symmetric semibounded operator was introduced in [14]. The wave spectrum is a topological space determined by the operator in a canonical way. The definition involves a dynamical system associated with the operator: states of the system serve as material for constructing this space. It is constituted of the atoms of the Hilbert lattice of subspaces determined by the reachable sets of the system and is endowed with an adequate topology.

The wave spectrum is an invariant of the operator: the wave spectra of unitarily-equivalent operators are canonically homeomorphic. At the same time, in important applications the wave spectrum of an operator acting in the space of functions turns out to be homeomorphic to the support of the functions that comprise the space. For example, the wave spectrum of the minimal Laplacian on a Riemannian manifold with boundary is essentially identical (isomorphic) to the manifold itself. This fact is used for solving inverse problems. In the problem of reconstruction of a manifold from its boundary data (for instance, the reaction operator), a unitary copy of the Laplacian is extracted from data, and then one can find its wave spectrum. The latter, by construction, is isometric to the manifold and thus gives the solution of the problem (cf. [14, 2]). These arguments constitute a form of the boundary control method (the BC-method). This approach to inverse problems is based on their deep relationship with the control theory [11, 12].

The notion of the wave spectrum arose as a result of generalizing the "experimental material" gathered from solving particular inverse problems with the BC-method. At

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2010 *Mathematics Subject Classification.* Primary 34B24.

Key words and phrases. Functional model of a symmetric operator, Green's system, wave spectrum, inverse problem.

The first author was supported by RFBR (grant no. 14-01-00535) and by Volkswagen Foundation-2016.

The second author was supported by RFBR (grants nos. 16-01-00443 and 16-01-00635).

some point it became clear that the procedure of solving is equivalent to constructing a certain functional model of a symmetric operator. In this model the elements of the original Hilbert space are realized as functions on the wave spectrum. An outline of this "wave" model was given in [14][1]; its usefulness and efficiency may be viewed as established facts. At the same time, in our opinion, the wave model is also interesting from the theoretical point of view. Its systematic study is our long-term goal.

We consider a particular example: a positive definite Sturm–Liouville operator $L_0 = -\frac{d^2}{dx^2}+q$ in $L_2(0,\infty)$ which has defect indices $(1,1)$. We construct the wave model of the operator L_0^*. As we proceed, we describe the elements of the general construction and in parallel clarify how they look like in our case. At some point, realizing the elements of the original $L_2(0,\infty)$ as complex-valued functions on the wave spectrum, we use the specifics of the Sturm–Liouville operator. In the general case the realization is more complex: the corresponding functions map to linear spaces of rather abstract nature [14]. The above-mentioned specifics make it possible to investigate the model completely. The wave spectrum turns out to be identical to the half-line $[0,\infty)$, and the model operator is related to the original L_0^* by a simple gauge transform. As a consequence, the potential q is easily recovered, which determines the original operator.

We dedicate this work to the memory of Vladimir Savel′evich Buslaev, a wonderful person, an excellent mathematician, one of our Teachers.

§1. Dynamical system with boundary control

1.1. The operator L_0. Let us describe the class of operators for which our definitions make sense. Let $\mathcal{H}$ be a (separable) Hilbert space and L_0 an operator in $\mathcal{H}$. We suppose that:

(1) L_0 is closed and densely defined: $\overline{\mathrm{Dom} L_0}=\mathcal{H}$;
(2) L_0 is positive definite: for some $\varkappa>0$ and every $y\in\mathrm{Dom}\,L_0$ we have $(L_0y,y)\geq\varkappa\|y\|^2$;
(3) L_0 has nonzero defect indices $n_\pm=\dim\mathrm{Ker}\,L_0^*\leq\infty$.

From the third assumption it follows that L_0 is unbounded. Let L denote the Friedrichs extension of $L_0\,:\,L_0\subset L\subset L_0^*$, $L^*=L$, and $(Ly,y)\geq\varkappa\|y\|^2$ for every $y\in\mathrm{Dom}\,L$ (cf. [4]). The inverse operator L^{-1} is bounded and is defined on the entire space $\mathcal{H}$.

• Throughout, H^s stands for Sobolev classes; $\mathbb{R}_+:=(0,\infty)$, $\bar{\mathbb{R}}_+:=[0,\infty)$. "Smooth" always means "C^∞-smooth".

In the case of the Sturm–Liouville operator, we have $\mathcal{H}=L_2(\mathbb{R}_+)$. The operator itself is $L_0\colon\mathcal{H}\to\mathcal{H}$,

$$\begin{aligned}&\mathrm{Dom}\,L_0=\left\{y\in\mathcal{H}\cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+)\,|\,y(0)=y'(0)=0;\quad -y''+qy\in\mathcal{H}\right\},\\ &L_0y:=-y''+qy,\end{aligned}\tag{1.1}$$

where $q=q(x)$ is a real-valued function (the *potential*) such that

$$\begin{aligned}&(1)\, q\in C^\infty(\bar{\mathbb{R}}_+),\\ &(2)\ \text{the limit point case occurs,}\\ &(3)\ \text{the operator } L_0 \text{ is positive definite.}\end{aligned}\tag{1.2}$$

In this case, the problem

$$-\phi''+q\phi=0, x>0\,;\qquad \phi(0)=1,\qquad \phi\in L_2(\mathbb{R}_+)\tag{1.3}$$

[1]The idea of this construction can be seen in the early work [1]. At the heuristic level, the wave spectrum was introduced in [12] for solving inverse problems. A formal definition (although with a different name) appeared later in [13].

has a unique solution $\phi(x)$, which is a smooth function. It is known that, first,

$$\begin{aligned} \operatorname{Dom} L_0^* &= \left\{ y \in \mathcal{H} \cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+) \,|\, -y'' + qy \in \mathcal{H} \right\}, \\ L_0^* y &:= -y'' + qy, \\ \operatorname{Ker} L_0^* &= \{ c\phi \,|\, c \in \mathbb{C} \}, \end{aligned} \tag{1.4}$$

and the defect indices of L_0 are $n_\pm = \dim \operatorname{Ker} L_0^* = 1$[2]; and second, the Friedrichs extension of the operator L_0 is $L : \mathcal{H} \to \mathcal{H}$,

$$\begin{aligned} \operatorname{Dom} L &= \left\{ y \in \mathcal{H} \cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+) \,|\, y(0) = 0; -y'' + qy \in \mathcal{H} \right\}, \\ Ly &:= -y'' + qy. \end{aligned} \tag{1.5}$$

We mention that the smoothness of q simplifies considerations; however, all the main results can be extended to the case where $q \in C_{\mathrm{loc}}\big(\bar{\mathbb{R}}_+\big)$ at least.

1.2. Green's system. The following definitions are close to those used in the classical work of A. N. Kochubeĭ [8] (see also [18, 16]).

Let $\mathcal{H}$ and $\mathcal{B}$ be Hilbert spaces, and let $A \colon \mathcal{H} \to \mathcal{H}$ and $\Gamma_i \colon \mathcal{H} \to \mathcal{B}$ $(i = 1, 2)$ be operators satisfying

$$\overline{\operatorname{Dom} A} = \mathcal{H}, \quad \operatorname{Dom} \Gamma_i \supset \operatorname{Dom} A, \quad \overline{\operatorname{Ran} \Gamma_1 + \operatorname{Ran} \Gamma_2} = \mathcal{B}.$$

A collection $\mathfrak{G} = \{\mathcal{H}, \mathcal{B}; A, \Gamma_1, \Gamma_2\}$ is called a *Green system* if its elements are related by the *Green formula*

$$(Au, v)_{\mathcal{H}} - (u, Av)_{\mathcal{H}} = (\Gamma_1 u, \Gamma_2 v)_{\mathcal{B}} - (\Gamma_2 u, \Gamma_1 v)_{\mathcal{B}} \tag{1.6}$$

for every $u, v \in \operatorname{Dom} A$. The space $\mathcal{H}$ is said to be *inner*, $\mathcal{B}$ is the *space of boundary values*, A is the *basic operator*, and the $\Gamma_{1,2}$ are the *boundary operators*.

1.3. The system $\mathfrak{G}_{L_0}$. With the operator L_0 satisfying conditions 1–3 of Subsection 1.1, a Green system can be associated in a canonical way. Let

$$\mathcal{K} := \operatorname{Ker} L_0^*.$$

Denote by P_K the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$, and let $\mathbb{O}$ and $\mathbb{I}$ denote the zero and identity operators. We also define the following operators:

$$\Gamma_1 := L^{-1} L_0^* - \mathbb{I}, \quad \Gamma_2 := P_K L_0^*. \tag{1.7}$$

As it was shown in [2, Lemma 1], the set $\mathfrak{G}_{L_0} := \{\mathcal{H}, \mathcal{K};\, L_0^*, \Gamma_1, \Gamma_2\}$ forms a Green system. In the same paper, the action of the boundary operators Γ_i was described in terms of *Vishik's decomposition*, which has the following form:

$$\operatorname{Dom} L_0^* = \operatorname{Dom} L_0 \dotplus L^{-1}\mathcal{K} \dotplus \mathcal{K} \tag{1.8}$$

(the sums are direct). If one applies this decomposition to an arbitrary $y \in \operatorname{Dom} L_0^*$,

$$y = y_0 + L^{-1} g_y + h_y, \quad y_0 \in \operatorname{Dom} L_0, \quad g_y, h_y \in \mathcal{K}, \tag{1.9}$$

then the boundary operators (1.7) act by the rule

$$\Gamma_1 y = -h_y, \quad \Gamma_2 y = g_y \tag{1.10}$$

(see [2, Subsection 2.3]).

• For the Sturm–Liouville operator (1.1) we have $\mathcal{K} = \{c\phi \,|\, c \in \mathbb{C}\}$. Let $\eta := L^{-1}\phi$ and observe that $\eta(0) = 0$ owing to (1.5), while $\eta'(0) \neq 0$. Indeed, assuming that $\eta(0) = \eta'(0) = 0$, we have $\eta \in \operatorname{Dom} L_0$ and $L_0 \eta = L_0 L^{-1} \phi = \phi$; since

$$0 = (L_0^* \phi, \eta) = (\phi, L_0 \eta) = (\phi, \phi) \neq 0,$$

we get a contradiction.

[2]This is true, for instance, if $q(x) > -cx^2$ with some $c > 0$: see [9, Chapter VII.26, Theorem 6].

It is easy to check that in our case the representations (1.9) and (1.10) take the form

$$\begin{aligned}
&y=\left\{y-y(0)\phi-\left[\frac{y'(0)-y(0)\phi'(0)}{\eta'(0)}\right]\eta\right\}+\left[\frac{y'(0)-y(0)\phi'(0)}{\eta'(0)}\right]\eta+y(0)\phi\,;\\
&\Gamma_1 y=-\,y(0)\phi,\qquad \Gamma_2 y=\left[\frac{y'(0)-y(0)\phi'(0)}{\eta'(0)}\right]\phi
\end{aligned}\tag{1.11}$$

(recall that $\phi(0)=1$).

Thus, the operator (1.1) canonically gives rise to the Green system with the inner space $L_2(\mathbb{R}_+)$, the basic operator (1.4), the boundary space $\{c\phi\,|\,c\in\mathbb{C}\}$, and the boundary operators (1.11).

1.4. The system α_{L_0}. In its turn, the system $\mathfrak{G}_{L_0}$ that corresponds to the operator L_0 determines a dynamical system

$$u_{tt}+L_0^*u=0,\qquad t>0,\tag{1.12}$$

$$u\big|_{t=0}=u_t\big|_{t=0}=0,\tag{1.13}$$

$$\Gamma_1 u=h,\qquad t\geq 0,\tag{1.14}$$

where $h=h(t)$ is the *boundary control* (a $\mathcal{K}$-valued function of time) and $u=u^h(t)$ is the solution (an $\mathcal{H}$-valued function of time). In the control theory, $u^h(\,\cdot\,)$ is called the trajectory, and $u^h(t)$ is the state of the system at the moment t. Aiming at applications, we call u^h the *wave*. System (1.12)–(1.14) is determined by the operator L_0, and we denote it by α_{L_0}.

Recall that L is the Friedrichs extension of the operator L_0. Let $L^{\frac12}$ denote the positive square root of L. Assume that the control h is smooth and vanishes near $t=0$. We denote by

$$\mathcal{M}:=\{h\in C^\infty([0,\infty);\mathcal{K})\mid \operatorname{supp}h\subset(0,\infty)\}\tag{1.15}$$

the linear span of such controls. As was shown in [2], for $h\in\mathcal{M}$, problem (1.12)–(1.14) has a unique classical solution $u^h\in C^\infty\big([0,\infty);\mathcal{H}\big)$. It vanishes near $t=0$ and admits the following representation:

$$u^h(t)=-h(t)+\int_0^t L^{-\frac12}\sin\left[(t-s)L^{\frac12}\right]h_{tt}(s)\,ds,\qquad t\geq 0.\tag{1.16}$$

For controls in the class $\big\{h\,|h,h_{tt}\in L_2^{\rm loc}\big([0,\infty);\mathcal{K}\big),h(0)=h_t(0)=0\big\}$, the (generalized) solution is defined as the right-hand side of (1.16). To distinguish generalized solutions from the classical ones, we call the latter *smooth waves*. In what follows they will play the role of a certain structure in $\mathcal{H}$.

• In the case of the Sturm–Liouville operator we have $\mathcal{K}=\{c\phi\,|\,c\in\mathbb{C}\}$, and the condition (1.14) takes the form $\Gamma_1 u=h(t)=f(t)\phi$, $t\geq 0$, with some complex-valued function f. Hence, system (1.12)–(1.14) is equivalent to the following initial boundary value problem:

$$u_{tt}-u_{xx}+qu=0,\qquad x>0,t>0,\tag{1.17}$$

$$u\big|_{t=0}=u_t\big|_{t=0}=0,\qquad x\geq 0,\tag{1.18}$$

$$u\big|_{x=0}=f,\qquad t\geq 0.\tag{1.19}$$

As an analog of (1.15), we introduce the linear space

$$\dot{\mathcal{M}}:=\big\{f\in C^\infty[0,\infty)\mid \operatorname{supp}f\subset(0,\infty)\big\}\tag{1.20}$$

of smooth controls that vanish near $t=0$. For $f\in\mathcal{M}$, problem (1.17)–(1.19) has a unique classical solution $u=u^f(x,t)$, which is smooth in both variables. For this solution, we have

$$u^f(x,t) = f(t-x) + \int_x^t w(x,s)\, f(t-s)\, ds, \qquad x \geq 0, \;\; t \geq 0, \tag{1.21}$$

where both summands on the right are assumed to vanish for $x > t$. The function w is defined for $0 \leq x \leq t$ and is smooth; it is related in a simple way to the classical Riemann function of the equation (1.17). This representation is employed to define the (generalized) solution corresponding to controls $f \in L_2^{\mathrm{loc}}[0,\infty)$: such a solution is *defined* as the right-hand side of (1.21).

The solution $u^f(\,\cdot\,,t)$ viewed as an $L_2(\mathbb{R}_+)$-valued function of time is the trajectory of the system α that corresponds to the operator (1.1). This case is specific in that the map $f \mapsto u^f$ is continuous: it is easily seen that $u^f \in C_{\mathrm{loc}}\big([0,\infty); L_2(\mathbb{R}_+)\big)$.

1.5. Controllability. We return to the system α_{L_0} in the general case. Fix $t = T \geq 0$; the set of states

$$\mathcal{U}^T_{L_0} := \{u^h(T) \mid h \in \mathcal{M}\} \tag{1.22}$$

is said to be *reachable* (at the time T). It is easy to check that $\mathcal{U}^T_{L_0}$ grows with T. Also, we define

$$\mathcal{U}_{L_0} := \bigcup_{T>0} \mathcal{U}^T_{L_0}, \quad \mathcal{D}_{L_0} := \mathcal{H} \ominus \bar{\mathcal{U}}_{L_0}, \tag{1.23}$$

the *total* reachable set and the *defect subspace*. The linear space of smooth waves is invariant under L_0^*. Indeed, for $u = u^h(T) \in \mathcal{U}^T_{L_0}$ we have

$$\begin{aligned} L_0^* u^h(T) &\overset{(1.12)}{=} -u^h_{tt}(T) \overset{(1.16)}{=} -u^{h_{tt}}(T) \in \mathcal{U}^T_{L_0}, \\ u^h(T) &\overset{(1.13)}{=} J^2[u^h_{tt}](T) \overset{(1.16)}{=} u^{J^2 h}_{tt}(T) \overset{(1.12)}{=} -L_0^* u^{J^2 h}(T) \in L_0^* \mathcal{U}^T_{L_0}, \end{aligned}$$

where $J := \int_0^t$ is integration in time. Consequently, $L_0^* \mathcal{U}_{L_0} = \mathcal{U}_{L_0}$.

Let us remark in advance that the functional model of the operator L_0^* which we are constructing is in fact a model of its *wave part* $L_0^*|_{\mathcal{U}_{L_0}}$. Related to this is the following question left unanswered: let $\bar{\mathcal{U}}_{L_0} = \mathcal{H}$, i.e., suppose that the part $L_0^*|_{\mathcal{U}_{L_0}}$ is densely defined; is it then true that its closure coincides with L_0^*? In all the examples we know the answer is in the positive.

A system α_{L_0} is said to be *controllable* if

$$\bar{\mathcal{U}}_{L_0} = \mathcal{H} \qquad (\mathcal{D}_{L_0} = \{0\}). \tag{1.24}$$

We formulate the criterion of controllability that was established in [2].

Recall the definitions. We say that an operator A induces a selfadjoint operator in a (nontrivial) subspace $\mathcal{L} \subset \mathcal{H}$ if $\overline{\mathcal{L} \cap \operatorname{Dom} A} = \mathcal{L}$, $A\big[\mathcal{L} \cap \operatorname{Dom} A\big] \subset \mathcal{L}$, and the operator $A|_{\mathcal{L} \cap \operatorname{Dom} A}$ is selfadjoint in $\mathcal{L}$. The operator A is said to be *completely nonselfadjoint* if A does not induce a selfadjoint operator in any subspace in $\mathcal{H}$. In [2, Theorem 1], the system α_{L_0} is controllable if and only if L_0 is a completely nonselfadjoint operator.

• In the case of the Sturm–Liouville operator (1.1), we show that the system α is controllable. Let $C^\infty_{\mathrm{fin}}(\bar{\mathbb{R}}_+)$ be the set of smooth functions with bounded support. Denote

$$C^\infty_T(\bar{\mathbb{R}}_+) := \{y \in C^\infty(\bar{\mathbb{R}}_+) \mid \operatorname{supp} y \subset [0,T)\}.$$

Obviously, $C^\infty_{\mathrm{fin}}(\bar{\mathbb{R}}_+) = \bigcup_{T>0} C^\infty_T(\bar{\mathbb{R}}_+)$.

Lemma 1. *Suppose that the operator L_0 has the form* (1.1) *and the potential q satisfies* (1.2). *Then the following relations hold*:

$$\mathcal{U}^T = C^\infty_T(\bar{\mathbb{R}}_+), \quad T > 0\,; \qquad \mathcal{U} = C^\infty_{\mathrm{fin}}(\bar{\mathbb{R}}_+), \tag{1.25}$$

where $\mathcal{U}^T$ *and* $\mathcal{U}$ *are as defined in* (1.22) *and* (1.23).

Proof. Fix $T > 0$ and pick $f \in \dot{\mathcal{M}}$. By (1.21), we have

$$u^f(x,T) = f(T-x) + \int_x^T w(x,s)\, f(T-s)\, ds, \qquad x \geq 0.$$

This shows that the wave $u^f(\,\cdot\,,T)$ is a smooth function vanishing near $x = T$. Hence, the left-hand side of the first identity in (1.25) is a subset of the right-hand side.

Let y belong to the right-hand side, and let $f = f(t)\big|_{0\leq t\leq T}$ be found from the Volterra integral equation of the second kind

$$f(T-x) + \int_x^T w(x,s)\, f(T-s)\, ds = y(x), \qquad 0 \leq x \leq T.$$

It is easily that f is a smooth function vanishing near $t = 0$. We extend f to the interval (T,∞) arbitrarily but preserving its smoothness. Then, by construction, $f \in \mathcal{M}$ and $y = u^f(\,\cdot\,,T) \in \mathcal{U}^T$. Therefore, the right-hand side of the first identity in (1.25) is a subset of the left-hand side. Thus, this identity is established.

The second identity follows from the first. $\square$

As a consequence, we have controllability: $\bar{\mathcal{U}} = \overline{C^\infty_{\rm fin}(\bar{\mathbb{R}}_+)} = L_2(\mathbb{R}_+)$. From controllability it follows that the operator (1.1) is completely nonselfadjoint. This fact can also be proved directly, without using dynamics. Moreover, using (1.25) it is not difficult to show that the closure of the wave part $L_0^*\big|_{\mathcal{U}}$ of the operator (1.4) coincides with L_0^* itself.

§2. The wave spectrum

2.1. Wave isotony. Let $\mathcal{P}$ and $\mathcal{Q}$ be partially ordered sets. A map $i\colon \mathcal{P} \to \mathcal{Q}$ is said to be *isotone* (order-preserving) if $p \preceq p'$ implies $i(p) \preceq i(p')$, see [3].

By an *isotony* we mean a family of isotone maps $\{i^t\}_{t\geq 0}$ such that $p \preceq p'$ and $t \leq t'$ imply $i^t(p) \preceq i^{t'}(p')$. In another formulation, an isotony is an isotone map of the set $\mathcal{P} \times [0,\infty)$ (with the natural order on it) into $\mathcal{Q}$.

A lattice is a partially ordered set in which every two elements x, y have the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ (see [3]). We will deal with lattices endowed with additional structures: complements, topology, etc.

Let $\mathfrak{L}(\mathcal{H})$ be the lattice of (closed) subspaces of $\mathcal{H}$ with the partial order $\subseteq$. Its is easy to check that $\mathcal{A} \vee \mathcal{B} = \overline{\{a + b \,|\, a \in \mathcal{A},\, b \in \mathcal{B}\}}$ é $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B}$. The lattice $\mathfrak{L}(\mathcal{H})$ is also a lattice with the least element $\{0\}$ and the greatest element $\mathcal{H}$, and with complements, $\mathcal{A}^\perp = \mathcal{H} \ominus \mathcal{A}$ (because $\mathcal{A}^\perp \vee \mathcal{A} = \mathcal{H}$, $\mathcal{A}^\perp \wedge \mathcal{A} = \{0\}$). By $P_{\mathcal{A}}$ we denote the (orthogonal) projection onto $\mathcal{H}$ on $\mathcal{A}$. The topology on $\mathfrak{L}(\mathcal{H})$ is defined by the strong operator convergence of projections: $\mathcal{A}_j \to \mathcal{A}$ if $P_{\mathcal{A}_j} \overset{s}{\to} P_{\mathcal{A}}$ as $j \to \infty$[3].

By a *lattice isotony* of $\mathfrak{L}(\mathcal{H})$ we call an isotony $I = \{I^t\}_{t\geq 0}\colon \mathfrak{L}(\mathcal{H}) \times [0,\infty) \to \mathfrak{L}(\mathcal{H})$ with the following additional properties: $I^0 = \mathrm{id}$ and $I^t\{0\} = \{0\}$. In what follows we deal only with such I.

Every operator L_0 as defined in Subsection 1.1 gives rise to a *wave isotony* I_{L_0} defined as follows. Consider the dynamical system

$$v_{tt} + Lv = g, \qquad t > 0, \tag{2.1}$$

$$v\big|_{t=0} = v_t\big|_{t=0} = 0, \tag{2.2}$$

[3]The strong operator topology is not first countable, and thus it cannot be described in terms of convergence of sequences. However, its restriction to the set of orthogonal projections (as well as to any subset of $\mathcal{B}(\mathcal{H})$ bounded in the operator norm) is first countable and even metrizable [17].

where g is an $\mathcal{H}$-valued function of time. If $g \in C^\infty\big([0,\infty);\mathcal{H}\big)$ vanishes near $t=0$, then this problem has a unique classical solution $v = v^g(t)$ that admits the Duhamel representation:

$$v^g(t) = \int_0^t L^{-\frac{1}{2}} \sin\big[(t-s)L^{\frac{1}{2}}\big] g(s)\, ds, \qquad t \geq 0 \tag{2.3}$$

(see [4]). For $g \in L_2^{\mathrm{loc}}\big([0,\infty);\mathcal{H}\big)$ the (generalized) solution is defined as the right-hand side of (2.3).

Fix a subspace $\mathcal{G} \in \mathfrak{L}(\mathcal{H})$ and consider $\mathcal{G}$-valued controls. The corresponding reachable sets of the system (2.1)–(2.2) are

$$\mathcal{V}^t_{\mathcal{G}} := \big\{v^g(t) \,|\, g \in L_2^{\mathrm{loc}}\big([0,\infty);\mathcal{G}\big)\big\}\,. \tag{2.4}$$

It is clear that $\mathcal{V}^t_{\mathcal{G}}$ grows with $\mathcal{G}$ and t. We introduce a family of maps $I_{L_0} = \{I^t_{L_0}\}_{t\geq 0}$:

$$I^0_{L_0} := \mathrm{id}; \qquad I^t_{L_0}\mathcal{G} := \overline{\mathcal{V}^t_{\mathcal{G}}}, \quad t > 0. \tag{2.5}$$

Proposition 1. *The family I_{L_0} is an isotony of the lattice $\mathfrak{L}(\mathcal{H})$.*

The proof can be found in [14]. Note that I_{L_0} is determined not by L_0, but rather by its Friedrichs extension L. It is clear that the wave isotony can be defined consistently for every selfadjoint operator semibounded from below. In applications, problem (2.1)–(2.2) describes propagation of waves excited by the sources g, so that the initial subspace $\mathcal{G}$ is extended by the waves v^g.

• We discuss the properties of the wave isotony for the Sturm–Liouville operator (1.1). For a subset E of the half-line, denote by E^r its r-neighborhood:

$$E^r := \{x \in \bar{\mathbb{R}}_+ \mid \mathrm{dist}(x,E) := \inf_{e\in E}|x-e| < r\}, \qquad r > 0.$$

Let $\Delta_{a,b}$ be one of the intervals (a,b), $[a,b)$, $(a,b]$, $[a,b]$ $(0 \leq a < b \leq \infty)$. Then, obviously,

$$\Delta^r_{a,b} = \begin{cases} (a-r,\, b+r), & a \geq r, \\ [0,\, b+r), & a < r. \end{cases} \tag{2.6}$$

If $x_0 \in \bar{\mathbb{R}}_+$ is a point, then

$$\{x_0\}^r = \begin{cases} (x_0-r,\, x_0+r), & x_0 \geq r, \\ [0,\, x_0+r), & x_0 < r. \end{cases} \tag{2.7}$$

For a measurable set $E \subset \mathbb{R}_+$, we denote $L_2(E) := \{y \in L_2(\mathbb{R}_+),\, |y|_{CE} = 0\}$, where $CE := \mathbb{R}_+ \setminus E$.

Lemma 2. *Under the conditions of Lemma* 1, *for every $0 \leq a < b \leq \infty$ and $T > 0$ we have*

$$I^T L_2(\Delta_{a,b}) = L_2(\Delta^T_{a,b}), \tag{2.8}$$

where I^T is defined by (2.5).

Proof. **1.** In our case, system (2.1)–(2.2) is equivalent to the initial boundary value problem

$$v_{tt} - v_{xx} + qv = g, \qquad x \in \mathbb{R}_+, t > 0, \tag{2.9}$$

$$v\big|_{t=0} = v_t\big|_{t=0} = 0, \qquad x \in \bar{\mathbb{R}}_+, \tag{2.10}$$

$$v\big|_{x=0} = 0, \qquad t \geq 0, \tag{2.11}$$

with the right-hand side $g = g(x,t)$ such that $g(\,\cdot\,,t) \in \mathcal{G}$ for every $t \geq 0$. Condition (2.11) follows from $v(\,\cdot\,,t) \in \mathrm{Dom}\, L$, in accordance with to (1.5). Problem (2.9)–(2.11) is well posed for every $g \in L_2^{\rm loc}\big(\mathbb{R}_+ \times [0,\infty)\big)$; for $g \in C_0^\infty\big(\mathbb{R}_+ \times (0,\infty)\big)$ its solution $v = v^g(x,t)$ is a classical one. Moreover, owing to the finiteness of the influence domain for the (hyperbolic) equation (2.9), if $\operatorname{supp} g(\,\cdot\,,t) \subset \overline{\Delta_{a,b}}$ for every $t > 0$, then $\operatorname{supp} v^g(\,\cdot\,,T) \subset \overline{\Delta^T_{a,b}}$ for every $T > 0$.

We fix $T > 0$ and choose $0 \le a < b \le \infty$. Consider $\mathcal{G} = L_2(\Delta_{a,b})$ and take $g \in L_2^{\rm loc}(\Delta_{a,b} \times [0,\infty))$. From (2.3) it follows that the map $L_2[0,T] \ni g\big|_{[0,T]} \mapsto v^g(\,\cdot\,,T) \in L_2(\mathbb{R}_+)$ is continuous. Also, we have $\operatorname{supp} v^g(\,\cdot\,,T) \subset \overline{\Delta^T_{a,b}}$. Therefore, $\mathcal{V}^T_{\mathcal{G}} \subset L_2(\Delta^T_{a,b})$. We shall show that $\mathcal{V}^T_{\mathcal{G}}$ is dense in $L_2(\Delta^T_{a,b})$.

2. Consider the auxiliary problem

$$w_{tt} - w_{xx} + qw = 0, \qquad x \in \mathbb{R}_+, 0 < t < T, \tag{2.12}$$

$$w\big|_{t=T} = 0, w_t\big|_{t=T} = y, \qquad x \in \bar{\mathbb{R}}_+, \tag{2.13}$$

$$w\big|_{x=0} = 0, \qquad 0 \le t \le T. \tag{2.14}$$

It is well posed for every $y \in L_2^{\rm loc}\big(\mathbb{R}_+\big)$, and for $y \in C_0^\infty(\mathbb{R}_+)$ its solution $w = w^y(x,t)$ is classical. The finiteness of the domain of influence shows that $\operatorname{supp} w^y(\,\cdot\,,t) \subset \overline{\Delta^{T-t}_{a,b}}$ for every $t > 0$ whenever $\operatorname{supp} y \subset \overline{\Delta_{a,b}}$.

We want to establish a relationship between the solutions of problems (2.9)–(2.11) and (2.12)–(2.14). Suppose $g \in C_0^\infty\big(\mathbb{R}_+ \times (0,\infty)\big)$ and $y \in C_0^\infty(\mathbb{R}_+)$, so that the solutions of both problems are classical (smooth). By the finiteness of the domain of influence, for every $t > 0$ the functions $v^g(\,\cdot\,,t)$ and $w^y(\,\cdot\,,t)$ have compact support in $\bar{\mathbb{R}}_+$. This fact justifies the following calculation.

Integrating by parts we have:

$$\begin{aligned}
\int_{\mathbb{R}_+ \times [0,T]} g w^y \, dx\, dt &\overset{(2.9)}{=} \int_{\mathbb{R}_+ \times [0,T]} [v^g_{tt} - v^g_{xx} + qv^g] w^y \, dx\, dt \\
&= \int_{\mathbb{R}_+} [v^g_t w^y - v^g w^y_t]\Big|_{t=0}^{t=T} dx - \int_0^T [v^g_x w^y - v^g w^y_x]\Big|_{x=0}^{x=\infty} dt \\
&\quad - \int_{\mathbb{R}_+ \times [0,T]} v^g [w^y_{tt} - w^y_{xx} + qw^y] \, dx\, dt \overset{(2.10)\text{–}(2.14)}{=} - \int_{\mathbb{R}_+} v^g(\,\cdot\,,T)\, y \, dx,
\end{aligned}$$

and

$$\int_{\mathbb{R}_+ \times [0,T]} g w^y \, dx\, dt = - \int_{\mathbb{R}_+} v^g(\,\cdot\,,T)\, y\, dx. \tag{2.15}$$

Since $C_0^\infty(\mathbb{R}_+)$ is dense in $L_2(\mathbb{R}_+)$, the last identity is valid for every $y \in L_2(\mathbb{R}_+)$.

Let $g \in C_0^\infty\big(\mathbb{R}_+ \times (0,\infty)\big)$ and $y \in L_2(\mathbb{R}_+)$; suppose additionally that $\operatorname{supp} g(\,\cdot\,,t) \subset \Delta_{a,b}$ for every $t > 0$. It follows that $\operatorname{supp} v^g(\,\cdot\,,T) \subset \overline{\Delta^T_{a,b}}$, and (2.15) takes the form

$$\int_{\Delta_{a,b} \times [0,T]} g w^y \, dx\, dt = - \big(v^g(\,\cdot\,,T), y\big)_{L^2(\Delta^T_{a,b})}. \tag{2.16}$$

3. Returning to the question about the density of $\mathcal{V}^T_{\mathcal{G}}$ in $L_2(\Delta^T_{a,b})$, we take $y \in L_2(\Delta^T_{a,b}) \ominus \overline{\mathcal{V}^T_{\mathcal{G}}}$ and show that $y = 0$.

By the choice of y, the right-hand side of (2.16) is 0. Since g is arbitrary, we conclude that

$$w^y = 0 \quad \text{in } \Delta_{a,b} \times [0,T]. \tag{2.17}$$

We extend the solution w^y to the times $T \le t \le 2T$ by oddness:

$$w^y(\,\cdot\,,t) := \begin{cases} w^y(\,\cdot\,,t), & 0 \le t < T, \\ w^y(\,\cdot\,,2T-t), & T \le t \le 2T. \end{cases}$$

This extension solves the following problem:

$$w_{tt} - w_{xx} + qw = 0, \qquad x \in \mathbb{R}_+, \ 0 < t < 2T, \tag{2.18}$$

$$w\big|_{t=T} = 0, w_t\big|_{t=T} = y, \qquad x \in \bar{\mathbb{R}}_+, \tag{2.19}$$

$$w\big|_{x=0} = 0, \qquad 0 \le t \le 2T. \tag{2.20}$$

We only need to check that the extended function w^y satisfies (2.18). This is easy, because the odd extension does not lead to jumps of w^y and w^y_t at $t = T$. By (2.17), we have

$$w^y = 0 \quad \text{in} \quad \Delta_{a,b} \times [0, 2T]. \tag{2.21}$$

The solution of equation (2.18) with the property (2.21) can be extended by zero from $\Delta_{a,b} \times [0, 2T]$ to the wider domain

$$\Omega^{2T}_{a,b} := \{(x,t) \,|\, \max\{0, |t-T| + (a-T)\} < x < b + T - |t-T|\}\,,$$

which is bounded by the corresponding characteristic lines of the equation (2.18). Indeed, fix a point (x_0, t_0) for which $b < x_0 < b+T-|t_0-T|$. This point belongs to $\Omega^{2T}_{a,b}$, but not to $\Delta_{a,b} \times [0, 2T]$. Take a small $\delta > 0$ such that the characteristic triangle (the influence cone) $\{(x,t) \,|b - \delta < x < x_0 - |t-t_0|\}$ is contained in $\Omega^{2T}_{a,b}$. By the finiteness of the domain of influence, the value $w^y(x_0, t_0)$ is determined by the Cauchy data w^y, w^y_x on the (vertical) base of the cone, which is a subset of the line $x = b - \delta$. This base is contained inside $\Delta_{a,b} \times [0, 2T]$, and, by (2.21), this data is zero. Hence, $w^y(x_0, t_0) = 0$. The points $(x_0, t_0) \in \Omega^{2T}_{a,b}$ with $x_0 < a$ (if any) can be treated similarly.

Thus, $w^y = w^y_t = 0$ everywhere in $\Omega^{2T}_{a,b}$. In particular, by (2.19), for $t = T$ we have $y = w^y_t(x,T) = 0$ for every $x \in \Delta^T_{a,b}$. Therefore, from $y \in L_2(\Delta^T_{a,b}) \ominus \overline{\mathcal{V}^T_{\mathcal{G}}}$ it follows that $y = 0$. Thus the relation $\overline{\mathcal{V}^T_{\mathcal{G}}} = L_2(\Delta^T_{a,b})$ is established, i.e., (2.8) is proved. ☐

2.2. Lattices, atoms, the wave spectrum. *A lattice* in $\mathfrak{L}(\mathcal{H})$ is a subset invariant under all the operations in $\mathfrak{L}(\mathcal{H})$ that were defined at the beginning of Subsection 2.1. Every lattice necessarily contains $\{0\}$ and $\mathcal{H}$.

Let $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ be a family of subspaces. By $\mathfrak{L}_{\mathfrak{M}}$ we denote the minimal lattice in $\mathfrak{L}(\mathcal{H})$ that contains $\mathfrak{M}$. It consists of all subspaces of the form $\bigvee_{1\le k\le n} \bigcap_{1\le l\le m} \mathcal{A}_{kl}$, where for each subspace $\mathcal{A}_{kl}$ either $\mathcal{A}_{kl} \in \mathfrak{M}$ or $\mathcal{A}^{\perp}_{kl} \in \mathfrak{M}$ (see [4]).

Let I be an isotony of the lattice $\mathfrak{L}(\mathcal{H})$. The family $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ is *invariant* under I if $I\mathfrak{M} := \{I^t\mathcal{M} \,|\, t \ge 0, \mathcal{M} \in \mathfrak{M}\} = \mathfrak{M}$. For every $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ there exists a minimal lattice $\mathfrak{L}^I_{\mathfrak{M}}$ that contains $\mathfrak{M}$ and is invariant under I. It is easy to check that this minimal lattice has the following constructive description. Define an operation σ on subsets of the lattice $\mathfrak{L}(\mathcal{H})$ by the rule $\sigma(\mathfrak{M}) := I\mathfrak{L}_{\mathfrak{M}}$. Then $\mathfrak{L}^I_{\mathfrak{M}} = \bigcup_{j\ge 1} \sigma^j(\mathfrak{M})$.

Let $\mathcal{F}\big([0,\infty); \mathfrak{L}(\mathcal{H})\big)$ be the set of $\mathfrak{L}(\mathcal{H})$-valued functions of t. It is a lattice with partial order, operations, and convergence defined pointwise:

$$\{f \le g\} \iff \{f(t) \subseteq g(t), t \ge 0\}, \quad (f \vee g)(t) := f(t) \vee g(t),$$
$$(f \wedge g)(t) := f(t) \cap g(t), \quad (f^{\perp})(t) := (f(t))^{\perp}, \quad (\lim f_j)(t) := \lim(f_j(t)).$$

The least and the greatest elements of this lattice are the functions $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ identically equal to $\{0\}$ and $\mathcal{H}$, respectively. If $\mathfrak{L} \subset \mathfrak{L}(\mathcal{H})$ is a lattice, then the set $\mathcal{F}([0,\infty); \mathfrak{L})$

consisting of $\mathfrak{L}$-valued functions is also a lattice. If $\mathfrak{L}$ is invariant under the isotony (of the lattice) I, then the set of motone increasing functions

$$\mathcal{F}_I\big([0,\infty);\mathfrak{L}\big) := \{f(t) = I^t\mathcal{L} \mid \mathcal{L} \in \mathfrak{L}\}$$

is contained in $\mathcal{F}([0,\infty);\mathfrak{L})$. In what follows an important role will be played by its completion $\overline{\mathcal{F}_I([0,\infty);\mathfrak{L})} \subset \mathcal{F}([0,\infty);\mathfrak{L}(\mathcal{H}))$, which is the set obtained by adding to $\mathcal{F}_I([0,\infty);\mathfrak{L})$ the limits of all converging sequences in $\mathcal{F}_I([0,\infty);\mathfrak{L})$.

In the most general setting, let $\mathcal{P}$ be a partially ordered set with the least element 0. The element $\omega \in \mathcal{P}$ is called an *atom* if $\omega \neq 0$ and from $0 \neq \omega' \preceq \omega$ it follows that $\omega' = \omega$ (see [3]). By $\operatorname{At}\mathcal{P}$ we denote the set of all atoms of $\mathcal{P}$.

Consider the system α_{L_0}. Recall that its reachable sets are defined by (1.22); let $\mathfrak{U}_{L_0} := \{\overline{\mathcal{U}^T_{L_0}}\}_{T\geq 0} \subset \mathfrak{L}(\mathcal{H})$ be the family of reachable subspaces (the closures of reachable sets). The family $\mathfrak{U}_{L_0}$ and the wave isotony I_{L_0} are determined by the operator L_0. As a consequence, this operator determines the (minimal) lattice $\mathfrak{L}_{L_0} := \mathfrak{L}^{I_{L_0}}_{\mathfrak{U}_{L_0}}$ that contains all reachable subspaces and is invariant under I_{L_0}. The lattice and the isotony determine the set of functions $\mathcal{F}_{I_{L_0}}([0,\infty);\mathfrak{L}_{L_0})$. Thus, there is a canonical correspondence between the operator L_0 and the set of atoms

$$\Omega_{L_0} := \operatorname{At}\overline{\mathcal{F}_{I_{L_0}}\big([0,\infty);\mathfrak{L}_{L_0}\big)}.$$

This set is called the *wave spectrum* of the operator L_0 and is the main object of our interest in this paper.

Certain additional assumptions about the operator L_0 ensure that $\Omega_{L_0} \neq \varnothing$, see [14]. There exist operators whose wave spectrum consists of a single point. In the general case the question as to whether Ω_{L_0} is nonempty remains open.

• Let us turn to the case of the Sturm–Liouville operator L_0 given by (1.1) and look at the objects defined above. In accordance with (1.25), the reachable subspaces of the corresponding system α are $\mathfrak{U} = \{L_2(\Delta_{0,T})\}_{T\geq 0}$. The action of the wave isotony I on subspaces $L_2(\Delta_{a,b})$ is described by Lemma 2.

We say that a set $E \subset \bar{\mathbb{R}}_+$ is *elementary* if $E = \bigcup_{j=1}^{n(E)} \Delta_{a_j,b_j}$, where $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_{n(E)} < b_{n(E)} \leq \infty$. Let $\mathcal{E}$ be the family of all elementary sets. Obviously, the metric extension $E \mapsto E^T = \{x \in \bar{\mathbb{R}}_+ \mid \operatorname{dist}(x,E) < T\}$ (see (2.6) and (2.7)) maps elementary sets to elementary sets. We say that the subspaces $L_2(E)$ with $E \in \mathcal{E}$ are *elementary.* The family of such subspaces forms the lattice $\mathfrak{L}_{\mathcal{E}} \subset \mathfrak{L}(\mathcal{H})$.

Lemma 3. *Under the conditions of Lemma* 1, *we have* $I^T L_2(E) = L_2(E^T)$ *for every* $E \in \mathcal{E}$.

Proof. The set E can be written as $E = \bigcup_{j=1}^{n(E)} \Delta_{a_j,b_j}$. Owing to the isotony property of I_{L_0}, from Lemma 2 it follows that $L_2(\Delta^T_{a_j,b_j}) = I^T L_2(\Delta_{a_j,b_j}) \subseteq I^T L_2(E)$ for every $j = 1,2,\dots,n(E)$, whence $L_2(E^T) \subseteq I^T L_2(E)$. Arguing as in the first part of the proof of Lemma 2, we conclude that $\mathcal{V}^T_{L_2(E)} \subseteq L_2(E^T)$, and, hence, $I^T(L_2(E)) \subseteq L_2(E^T)$. $\square$

It is important that $\mathcal{E}$ only contains unions of intervals of positive length (nondegenerate): for a degenerate interval (like $E = \{x\}$, $x \in \mathbb{R}_+$) the equality $I^T L_2(E) = L_2(E^T)$ fails obviously.

The construction of the minimal lattice $\mathfrak{L}_{\mathfrak{M}}$ for $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ as given above implies that $\mathfrak{L}_{\mathfrak{U}} = \mathfrak{L}_{\mathcal{E}}$. From Lemma 3 it follows directly that the lattice $\mathfrak{L}_{\mathcal{E}}$ is invariant under the wave isotony I, and, therefore, $\mathfrak{L} = \mathfrak{L}^I_{\mathfrak{U}} = \mathfrak{L}_{\mathcal{E}}$.

Below, m stands throughout for Lebesgue's measure on $\bar{\mathbb{R}}_+$, and

$$A \triangle B := [A \setminus B] \cup [B \setminus A]$$

is the symmetric difference of the sets A and B. Let $\mathrm{Leb}(\mathbb{R}_+)$ be the σ-algebra of Lebesgue measurable subsets of the half-line $\bar{\mathbb{R}}_+$.

Lemma 4. *Let* $\{E_n\}_{n=1}^\infty$ *be a sequence of sets belonging to* $\mathrm{Leb}(\mathbb{R}_+)$, $E \in \mathrm{Leb}(\mathbb{R}_+)$. *Then* $L_2(E_n) \xrightarrow{\mathfrak{L}(\mathcal{H})} L_2(E)$ *as* $n \to \infty$ *if and only if for every* $L > 0$ *we have*

$$m((E_n \triangle E) \cap (0, L)) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. If $L_2(E_n) \xrightarrow{\mathfrak{L}(\mathcal{H})} L_2(E)$ as $n \to \infty$, then the projections onto these subspaces converge in the strong sense: $P_{L_2(E_n)} \xrightarrow{s} P_{L_2(E)}$. Then for every $L > 0$ we have $P_{L_2(E_n)}\chi_{(0,L)} \xrightarrow{L_2(\mathbb{R}_+)} P_{L_2(E)}\chi_{(0,L)}$, where $\chi_{(0,L)}$ is the characteristic function of the interval $(0, L)$. This means that $\int_0^L |\chi_{E_n}(x) - \chi_E(x)|^2\, dx = m((E_n \triangle E) \cap (0, L)) \to 0$ as $n \to \infty$.

Now, suppose that for every $L > 0$ we have $m((E_n \triangle E) \cap (0, L)) \to 0$ as $n \to \infty$. Then $P_{L_2(E_n)}\chi_{(0,L)} \xrightarrow{L_2(\mathbb{R}_+)} P_{L_2(E)}\chi_{(0,L)}$. The linear span of the set $\{\chi_{(0,L)} | L > 0\}$ is dense in $L_2(\mathbb{R}_+)$. By the Banach–Steinhaus theorem we have $P_{L_2(E_n)} \xrightarrow{s} P_{L_2(E)}$ as $n \to \infty$, and this by definition means that $L_2(E_n) \xrightarrow{\mathfrak{L}(\mathcal{H})} L_2(E)$ as $n \to \infty$. □

In accordance with the lemma, we define convergence in $\mathrm{Leb}(\mathbb{R}_+)$ as follows: a sequence $\{E_n\}_{n=1}^\infty \subset \mathrm{Leb}(\mathbb{R}_+)$ *converges* to $E \in \mathrm{Leb}(\mathbb{R}_+)$ if for every $L > 0$ we have $m((E_n \triangle E) \cap (0, L)) \to 0$ as $n \to \infty$. Denote $\mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)} := \{L_2(E) | E \in \mathrm{Leb}(\mathbb{R}_+)\}$.

Lemma 5. *Under the conditions of Lemma* 1 *we have* $\bar{\mathfrak{L}} = \mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)}$, *where closure is taken in the sense of convergence in* $\mathfrak{L}(\mathcal{H})$.

Proof. First, we show that $\bar{\mathfrak{L}} \subseteq \mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)}$. Let $\{L_2(E_n)\}_{n=1}^\infty$ be a sequence in $\mathfrak{L}$ convergent in the sense of the topology on $\mathfrak{L}(\mathcal{H})$. We need to show that its limit $A \in \mathfrak{L}(\mathcal{H})$ belongs to $\mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)}$. The existence of the limit implies that for every $L > 0$ the sequence of functions $P_{L_2(E_n)}\chi_{(0,L)} = \chi_{E_n \cap (0,L)}$ converges in $L_2(\mathbb{R}_+)$ and, therefore, is fundamental. We have

$$\begin{aligned}\|\chi_{E_n\cap(0,L)} - \chi_{E_m\cap(0,L)}\|_{L_2(\mathbb{R}_+)} &= \int_0^L |\chi_{E_n}(x) - \chi_{E_m}(x)|^2\, dx \\ &= m((E_n \cap (0, L)) \triangle (E_m \cap (0, L))).\end{aligned} \tag{2.22}$$

For every $L > 0$ the function $\rho_L(F, G) = m(F \triangle G)$, $F, G \in \mathrm{Leb}(0, L)$, is a pseudometric on $\mathrm{Leb}(0, L)$. The equivalence relation $F \sim F'$ whenever $\rho_L(F, F') = 0$ determines the set of equivalence classes $\mathrm{Leb}^\sim(0, L)$, which is a complete metric space, see [7]. From (2.22) it follows that the sequence $\{E_n \cap (0, L)\}_{n=1}^\infty$ is fundamental in $\mathrm{Leb}(0, L)$. Thus, the sequence of equivalence classes $\{(E_n \cap (0, L))^\sim\}_{n=1}^\infty$ converges to some equivalence class $E^\sim(L)$. For $L_2 > L_1$ the intersection of every representative of the equivalence class $E^\sim(L_2)$ with the interval $(0, L_1)$ belongs to the equivalence class $E^\sim(L_1)$. Hence, there exists a set $E \in \mathrm{Leb}(\mathbb{R}_+)$ such that for every $L > 0$ one has $E \cap (0, L) \in E^\sim(L)$. This exactly means that $E_n \to E$ as $n \to \infty$ in the sense of the above definition, which by Lemma 4 means that $L_2(E_n) \xrightarrow{\mathfrak{L}(H)} L_2(E)$ as $n \to \infty$. Therefore, $A = L_2(E) \in \mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)}$.

Now we show that $\mathfrak{L}$ is dense in $\mathfrak{L}_{\mathrm{Leb}(\mathbb{R}_+)}$. This is equivalent to the density of $\mathcal{E}$ in $\mathrm{Leb}(\mathbb{R}_+)$ in the sense of the above definition. It suffices to show that for every $L > 0$ the set $\{E \in \mathcal{E} | E \subseteq (0, L)\}$ is dense in $\mathrm{Leb}(0, L)$ with respect to the pseudometric ρ_L. Since every measurable subset of $(0, L)$ can be approximated in ρ_L by open subsets of $(0, L)$, the open subsets are dense in $\mathrm{Leb}(0, L)$. Every bounded open set on the real line is an at most countable union of nonintersecting open intervals such that the sequence of their lengths is summable. Thus, every open subset of $(0, L)$ can be approximated in the pseudometric ρ_L by a finite union of intervals, so that this allows us to approximate every measurable subset of $(0, L)$. Consequently, $\mathcal{E}$ is dense in $\mathrm{Leb}(\mathbb{R}_+)$. $\square$

For $x \geq 0$ denote

$$\omega_x(t) := L_2(\{x\}^t); \tag{2.23}$$

then $\omega_x(t)$ is a monotone function of t with values in $\mathfrak{L}(\mathcal{H})$, i.e., it is an element of $\mathcal{F}\big([0,\infty); \mathfrak{L}\big) \subset \mathcal{F}\big([0,\infty); \mathfrak{L}(\mathcal{H})\big)$.

Lemma 6. *Under the conditions of Lemma* 1*, we have* $\omega_x \in \Omega$ *for every* $x \in \bar{\mathbb{R}}_+$*.*

Proof. First, we show that $\omega_x \in \overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$. If $x = 0$, then for every $t \geq 0$ we have

$$\{0\}^t = [0,t) = \lim_{n\to\infty} \left(0, t + \frac{1}{n}\right) = \lim_{n\to\infty} \left(0, \frac{1}{n}\right)^t,$$

where both limits are in the sense of convergence in $\mathrm{Leb}(\mathbb{R}_+)$. If $x > 0$, then, by (2.7), for every $t \geq 0$,

$$\begin{aligned}\{x\}^t = \lim_{n\to\infty} &\left(\max\left\{0, x\left(1 - \frac{1}{n}\right) - t\right\}, x\left(1 + \frac{1}{n}\right) + t\right)\\ &= \lim_{n\to\infty} \left(x\left(1 - \frac{1}{n}\right), x\left(1 + \frac{1}{n}\right)\right)^t.\end{aligned}$$

Hence, by Lemma 4, the corresponding subspaces converge as $n \to \infty$ in $\mathfrak{L}(\mathcal{H})$ for every $t \geq 0$, which means that $\omega_x \in \overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$.

Now we show that ω_x is an atom of the lattice $\overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$. Suppose there exists a nonzero element $\omega \in \overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$ such that $\omega \leq \omega_x$. For every $t \geq 0$ we can write $\omega(t) = L_2(E(t))$ with some measurable set $E(t) \subseteq \{x\}^t$. Since $\omega \in \overline{\mathcal{F}_I\big([0,\infty); \mathfrak{L}\big)}$, there exists a sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ such that for every $t \geq 0$ we have $I^t(L_2(E_n)) \xrightarrow{\mathfrak{L}(\mathcal{H})} L_2(E(t))$ or, by Lemma 4, $E_n^t \xrightarrow{\mathrm{Leb}(\mathbb{R}_+)} E(t)$ as $n \to \infty$. For $\delta \in (0,1)$, denote $F_n(\delta) := E_n \cap (x(1-\delta), x(1+\delta))$. For every $t \geq 0$ we have the inclusion $E(t) \subseteq \{x\}^t$. Thus, for every $L > 0$ we have $m\big(((F_n(\delta))^t \triangle E(t)) \cap (0,L)\big) \leq m\big((E_n^t \triangle E(t)) \cap (0,L)\big) \to 0$, which means that $(F_n(\delta))^t \xrightarrow{\mathrm{Leb}(\mathbb{R}_+)} E(t)$ as $n \to \infty$.

For every set $E \in \mathcal{E}$ the derivative $\frac{d(m(E^t))}{dt}$ is equal to the number of positive edges of the nonintersecting open intervals comprising the set E^t. For the set $F_n(\delta)$ this number is at least two for $t < x(1-\delta)$ and at least one for $t \geq x(1-\delta)$, so that $m((F_n(\delta))^t) \geq m(F_n(\delta)) + \min\{2t, x(1-\delta) + t\}$. Passing to the limit, for arbitrarily small $\delta > 0$ we get $m(E(t)) \geq \min\{2t, x(1-\delta) + t\}$. This means that $m(E(t)) \geq \min\{2t, x + t\} = m(\{x\}^t)$ and, since $E(t) \subseteq \{x\}^t$, we have $m(E(t) \triangle \{x\}^t) = 0$ for every $t \geq 0$. Therefore, every nonzero element $\omega \in \overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$ such that $\omega \leq \omega_x$ coincides with ω_x. Thus, ω_x is an atom of the lattice $\overline{\mathcal{F}_I([0,\infty); \mathfrak{L})}$. $\square$

The following result characterizes the wave spectrum of the operator (1.1).

Theorem 1. *Let* L_0 *be the operator given by* (1.1) *with* q *satisfying conditions* (1.2)*. Then the set* Ω *is in one-to-one correspondence with the half-line* $\bar{\mathbb{R}}_+$ *:*

$$\Omega = \{\omega_x \,|\, x \in \bar{\mathbb{R}}_+\},$$

where the elements ω_x *are as defined in* (2.23)*.*

Proof. By Lemma 6, $\{\omega_x \,|\, x \geq 0\} \subseteq \Omega$. To prove the reverse inclusion, take an atom $\omega \in \Omega$. For every $t \geq 0$ the subspace $\omega(t)$ has the form $\omega(t) = L_2(E(t))$, where $E(t)$ is some measurable set. Hence, there exists a sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ such that $E_n^t \xrightarrow{\mathrm{Leb}(\mathbb{R}_+)} E(t)$ as $n \to \infty$.

For every $L > 0$ we have $E_n^t \cap (0, L+t) \xrightarrow{\mathrm{Leb}(\mathbb{R}_+)} E(t) \cap (0, L+t)$ as $n \to \infty$. There exist $t_0 \geq 0$ and $L_0 > 0$ such that $m\big(E(t_0) \cap (0, L_0 + t_0)\big) > 0$. Hence, there exists N_0 such that $m\big(E_n^{t_0} \cap (0, L_0 + t_0)\big) > 0$ for every $n \geq N_0$. Since

$$E_n^{t_0} \cap (0, L_0 + t_0) = \big(E_n \cap (0, L_0 + 2t_0)\big)^{t_0} \cap (0, L_0 + t_0),$$

for every $n \geq N_0$ we have $m((E_n \cap (0, L_0 + 2t_0))^{t_0}) > 0$. We denote $L_1 := L_0 + 2t_0$ and $F_n := E_n \cap (0, L_1)$. Since $E_n \in \mathcal{E}$, we see that F_n contains no degenerate intervals; next, $F_n \neq \varnothing$ because $m(F_n^{t_0}) > 0$. Therefore, $F_n \in \mathcal{E}$ and $m(F_n) > 0$.

For every $L > 0$ and $n, m \in \mathbb{N}$ we have

$$\begin{aligned} m\big((F_n^t \triangle F_m^t) \cap (0, L)\big) &= m\big(\big(((E_n \cap (0, L_1))^t) \triangle (E_m \cap (0, L_1))^t\big) \cap (0, L)\big) \\ &\leq m\big((E_n^t \triangle E_m^t) \cap (0, L)\big). \end{aligned}$$

Since the sequence $\{E_n^t \cap (0, L)\}_{n=1}^{\infty}$ is fundamental in the pseudometric ρ_L for every $L > 0$, the sequence $\{F_n^t\}_{n=1}^{\infty}$ is also fundamental in ρ_L. Thus, it has a limit, which we denote by $F(t)$ (the set $F(t)$ is defined not uniquely, but up to a set of measure zero). Since for every $n \geq N_0$ we have $F(t) \setminus E(t) \subseteq (F(t) \setminus F_n^t) \cup (E_n^t \setminus E(t))$, whence

$$m\big(F(t) \setminus E(t)\big) \leq m\big((F(t) \setminus F_n^t)\big) + m\big((E_n^t \setminus E(t))\big),$$

we obtain $m(F(t) \setminus E(t)) = 0$. Since ω is an atom, we see that $\omega(t) = L_2(F(t))$.

For every $n \geq N_0$ we have $F_n \in \mathcal{E}$, $F_n \subseteq (0, L_1)$, $m(F_n) > 0$. For $t > L_1$, from Lemma 3 it follows that $F_n^t = (0, \sup F_n + t)$. The existence of the limit of F_n^t as $n \to \infty$ in $\mathrm{Leb}(\mathbb{R}_+)$ means that for every $\varepsilon > 0$ there exists $N_1(\varepsilon)$ such that for every $n, m \geq N_1(\varepsilon)$ we have

$$\begin{aligned} m(F_n^t \triangle F_m^t) &= m([\min\{\sup F_n, \sup F_m\} + t, \max\{\sup F_n, \sup F_m\} + t)) \\ &= |\sup F_n - \sup F_m| < \varepsilon. \end{aligned}$$

Therefore, $\{\sup F_n\}_{n=1}^{\infty}$ is a fundamental sequence of positive numbers. We denote its limit by L_2. For every $\varepsilon > 0$ there exists $N_2(\varepsilon)$ such that for every $n \geq N_2(\varepsilon)$ we have $\sup F_n \in (L_2 - \varepsilon, L_2 + \varepsilon)$. Then for $t > \varepsilon$ the following inclusion is valid:

$$\big(\max\{0, L_2 + \varepsilon - t\}, L_2 - \varepsilon + t\big) \subseteq F_n^t.$$

Since $F_n^t \xrightarrow{\mathrm{Leb}(\mathbb{R}_+)} F(t)$, we have

$$\begin{aligned} & m\big((\max\{0, L_2 + \varepsilon - t\}, L_2 - \varepsilon + t) \setminus F(t)\big) \\ &\quad \leq m\big((\max\{0, L_2 + \varepsilon - t\}, L_2 - \varepsilon + t) \setminus F_n^t\big) + m(F_n^t \setminus F(t)) \\ &\quad = m(F_n^t \setminus F(t)) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

and, consequently, $m\big((\max\{0, L_2 + \varepsilon - t\}, L_2 - \varepsilon + t) \setminus F(t)\big) = 0$. Since this is true for every $\varepsilon \in (0, t)$, we get

$$m\big((\max\{0, L_2 - t\}, L_2 + t) \setminus F(t)\big) = m\big(\{L_2\}^t \setminus F(t)\big) = 0$$

for every $t \geq 0$. Therefore, $\omega = \omega_{L_2}$ because ω is an atom. $\square$

2.3. The space Ω_{L_0}. We return to the case of a general L_0. The wave spectrum, if it is not empty, can be endowed naturally with certain structures.

Topology. By definition, the atoms are $\mathfrak{L}(\mathcal{H})$-valued functions of time. Fix an atom $\omega \in \Omega_{L_0}$: $\omega = \omega(t)$, $t \geq 0$. The set

$$B_r[\omega] := \{\omega' \in \Omega_{L_0} \,|\, \exists\, t > 0 \,:\, \{0\} \neq \omega'(t) \subseteq \omega(r)\} \qquad (r > 0) \tag{2.24}$$

is called a *ball*; ω and r are its center and radius.

Proposition 2. *The system of balls* $\{B_r[\omega] \,|\omega \in \Omega_{L_0}, r > 0\}$ *is a base of some topology on* Ω_{L_0}.

The proof was given in [2]; it checks the characteristic properties of a base. Thus, the wave spectrum becomes a topological space.

There exist other natural topologies on Ω_{L_0}. Relationships among them are yet to be revealed, cf. [14]. The ball topology now seems to us the most relevant. However, its general properties (the Hausdorff property, metrizability, etc.) have not been explored.

Metric. Under additional assumptions about atoms one can introduce a metric on Ω_{L_0}. Each atom $\omega \in \Omega_{L_0}$: $\omega = \omega(t)$, $t \geq 0$, gives rise to a positive operator in $\mathcal{H}$,

$$\tau_\omega := \int_{[0,\infty)} t\, dP_{\omega(t)}, \tag{2.25}$$

where $P_{\omega(t)}$ is the projection onto the subspace $\omega(t) \subseteq \mathcal{H}$. We call it an *eikonal*; this term is motivated by applications, see [14, 2]. We introduce the distance

$$\tau \colon \Omega_{L_0} \times \Omega_{L_0} \to [0, \infty), \quad \tau(\omega, \omega') := \|\tau_\omega - \tau_{\omega'}\|. \tag{2.26}$$

Below we shall see that this definition can be consistent even in the case of an unbounded τ_ω. However, in the general case we cannot exclude the pathologic situation where $\tau = \infty$. How the ball topology is related to the topology that corresponds to the metric (2.26), is also an open question.

The boundary. We return to the system α_{L_0} and the family $\{\overline{\mathcal{U}^T_{L_0}}\}_{T \geq 0} \subset \mathfrak{L}(\mathcal{H})$ of its reachable subspaces. The set of atoms

$$\partial\Omega_{L_0} := \left\{\omega \in \Omega_{L_0} \,|\, \omega(t) \subseteq \overline{\mathcal{U}^t_{L_0}},\, \forall\, t > 0\right\} \tag{2.27}$$

is called the *boundary* of the wave spectrum. Whether $\partial\Omega_{L_0}$ is always nonempty, is an open question.

• Let L_0 be the Sturm–Liouville operator (1.1). Let

$$\beta \colon \bar{\mathbb{R}}_+ \ni x \,\mapsto\, \omega_x \in \Omega \tag{2.28}$$

be the canonical bijection established by Theorem 1. Below, $\operatorname{dist}(x, x') = |x - x'|$ is the standard distance in $\bar{\mathbb{R}}_+$.

Lemma 7. *Let* $\omega \in \Omega$ *be an atom. The eikonal corresponding to it is the unbounded selfadjoint operator* τ_ω *with the domain*

$$\operatorname{Dom} \tau_\omega = \left\{ y \in L_2(\mathbb{R}_+) \,\Big|\, \int_0^\infty (1 + x)^2 |y(x)|^2 \, dx < \infty \right\}.$$

Its action is multiplication by the distance:

$$(\tau_\omega y)(x) = \operatorname{dist}(x, x_\omega)\, y(x), \quad x \in \bar{\mathbb{R}}_+, \tag{2.29}$$

where $x_\omega = \beta^{-1}(\omega)$.

Proof. Pick a function $y \in L_2(\mathbb{R}_+)$ with compact support.

By Theorem 1 we have $\omega(t) = L_2(\{x_\omega\}^t), t \geq 0$. Hence, the operator $P_{\omega(t)}$, which projects $L_2(\mathbb{R}_+)$ onto $L_2(\{x_\omega\}^t)$, acts by cutting functions to the neighborhood $\{x_\omega\}^t$:

$$(2.30)\qquad \left(P_{\omega(t)}y\right)(x) = \begin{cases} y(x) & \text{if} \quad \operatorname{dist}(x, x_\omega) < t \\ 0 & \text{if} \quad \operatorname{dist}(x, x_\omega) > t. \end{cases}$$

Let $T > 0$ be such that $\operatorname{supp} y \subset \{x_\omega\}^T$. Take a partition $\Xi := \{t_i\}_{i=0}^N : t_0 < t_1 < \cdots < t_N$ of the interval $\overline{\{x_\omega\}^T}$ and points $\widetilde{t}_i \in [t_{i-1}, t_i]$. The quantity $r_\Xi := \max_{1\leq i\leq N}(t_i - t_{i-1})$ is the rank of the partition. By the definition of the integral, in (2.25) we have

$$\tau_\omega y = \lim_{r_\Xi \to 0} \sum_{i=1}^N \widetilde{t}_i\, \Delta_i P_{\omega(t)} y,$$

where $\Delta_i P_{\omega(t)} := P_{\omega(t_i)} - P_{\omega(t_{i-1})}$ and convergence is in the $L_2(\mathbb{R}_+)$ norm. Since $\Delta_i P_{\omega(t)} \Delta_j P_{\omega(t)} = \mathbb{O}$ for $i \neq j$, the summands are pairwise orthogonal. By (2.30), they are equal to

$$\left(\widetilde{t}_i\, \Delta_i P_{\omega(t)} y\right)(x) = \begin{cases} \widetilde{t}_i y(x) & \text{if} \quad \operatorname{dist}(x, x_\omega) < t_i - t_{i-1} \\ 0 & \text{if} \quad \operatorname{dist}(x, x_\omega) > t_i - t_{i-1} \end{cases}.$$

In the first line we have $\widetilde{t}_i = \operatorname{dist}(x, x_\omega) + O(r_\Xi)$ uniformly in $x \in \operatorname{supp} y$ and $i = 1, \dots N$. Using the formula $y = \sum_{i=1}^N \Delta_i P_{\omega(t)} y$ and the orthogonality of summands, we get

$$\left\| \operatorname{dist}(\,\cdot\,, x_\omega)\, y - \sum_{i=1}^N \widetilde{t}_i\, \Delta_i P_{\omega(t)} y \right\|^2 = \left\| \sum_{i=1}^N \left[\operatorname{dist}(\,\cdot\,, x_\omega) - \widetilde{t}_i\right]\, \Delta_i P_{\omega(t)} y \right\|^2$$
$$= \sum_{i=1}^N \left[\operatorname{dist}(\,\cdot\,, x_\omega) - \widetilde{t}_i\right]^2 \left\|\Delta_i P_{\omega(t)} y\right\|^2 = O(r_\Xi^2) \sum_{i=1}^N \left\|\Delta_i P_{\omega(t)} y\right\|^2 = O(r_\Xi^2)\|y\|^2.$$

Passing to the limit as $r_\Xi \to 0$, we arrive at (2.29).

Closure extends τ_ω from functions with finite support to the natural domain $\operatorname{Dom}\tau_\omega = \left\{y \in L_2(\mathbb{R}_+) \mid \int_0^\infty [1 + \operatorname{dist}(x, x_\omega)]^2 |y(x)|^2\, dx < \infty\right\}$. The conditions $y \in \operatorname{Dom}\tau_\omega$ and $\int_0^\infty (1+x)^2 |y(x)|^2\, dx < \infty$ are obviously equivalent. □

Corollary 1. *The function* (2.26) *determines a metric on* Ω; *moreover,*

$$(2.31)\qquad \tau(\omega, \omega') = \operatorname{dist}(x_\omega, x_{\omega'}), \qquad \omega, \omega' \in \Omega.$$

Indeed,

$$\tau(\omega, \omega') = \|\tau_\omega - \tau_{\omega'}\| \overset{(2.29)}{=} \sup_{x \in \bar{\mathbb{R}}_+} |\operatorname{dist}(x, x_\omega) - \operatorname{dist}(x, x_{\omega'})| = \operatorname{dist}(x_\omega, x_{\omega'}).$$

From (2.31) we conclude that the bijection β is an isometry from $\bar{\mathbb{R}}_+$ (with the metric dist) to Ω (with the metric τ). The following facts can be seen from this.

Proposition 3. *The balls* (2.24) *are identical with the balls corresponding to the* τ*-metric:* $B_r[\omega] = \{\omega' \in \Omega \,|\, \tau(\omega, \omega') < r\}$, *so that the ball topology on* Ω *coincides with the metric topology. There exists a unique measure* ν *on* Ω *such that*

$$(2.32)\qquad \nu\left(B_r[\omega]\right) = m(\{x_\omega\}^r) = r + \min\{r, x_\omega\}.$$

The boundary $\partial\Omega$ *of the wave spectrum consists of the single atom* $\omega_0 = \beta(0)$. *The function* $\tau\colon \Omega \to [0, \infty)$, $\tau(\omega) := \tau(\omega, \omega_0) = x_\omega$ *is a global coordinate on* Ω.

We omit the simple check of these facts. We only note that the fact that $\partial\Omega = \{\omega_0\}$ follows from the definition of the boundary (2.27) and the relations

$$\overline{\mathcal{U}^T} \overset{(1.25)}{=} \overline{C_T^\infty(\bar{\mathbb{R}}_+)} = L_2(\Delta_{0,T}) = L_2(\{0\}^T) = \omega_0(T) \qquad (T \geq 0),$$

the last of which was established in Theorem 1.

We write $\bar{\mathbb{R}}_+[\,\cdot\,]$ to specify the variable that we consider. The coordinatization

$$\Omega \ni \omega \;\mapsto\; \tau(\omega) \in [0, \infty) =: \bar{\mathbb{R}}_+[\tau] \tag{2.33}$$

makes the wave spectrum an isometric copy of the original half-line $\bar{\mathbb{R}}_+[x]$. Summarizing, we see that the wave spectrum of the Sturm–Liouville operator on the half-line with the defect indices $(1, 1)$ is in fact identical to the half-line itself.

§3. The wave model

3.1. The spaces $\widetilde{\mathcal{H}}$ and $\mathcal{H}^{\mathrm{w}}$. Let L_0 be an operator in $\mathcal{H}$ with a nonzero wave spectrum. The wave model is devised to realize elements $y \in \mathcal{H}$ as functions $\widetilde{y}(\,\cdot\,)$ on Ω_{L_0} with values in "natural" auxiliary spaces. A universal way to map $y \mapsto \widetilde{y}(\,\cdot\,)$ was proposed in [14] and is described below.

Germs. Fix $\omega \in \Omega_{L_0} : \omega = \{\omega(t)\}_{t\geq 0}$. Recall that $P_{\omega(t)}$ is the projection onto $\omega(t)$ in $\mathcal{H}$. We say that elements $y, y' \in \mathcal{H}$ coincide on ω (and write $y \overset{\omega}{=} y'$) if there exists $\varepsilon = \varepsilon(\omega, y) > 0$ such that $P_{\omega(t)}y = P_{\omega(t)}y'$ for $0 \leq t < \varepsilon$. Obviously, coincidence on ω is an equivalence relation. The corresponding equivalence class is called the *germ* of the element y on the atom ω and is denoted by $\widetilde{y}(\omega)$. The set of germs $\mathcal{G}_\omega := \{\widetilde{y}(\omega)\,|\,y \in \mathcal{H}\}$ forms the *stalk* above ω which obviously has the structure of a linear space.

We call the space of "functions" $\widetilde{\mathcal{H}} := \{\widetilde{y}(\,\cdot\,)\,|\,y \in \mathcal{H}\}$ with algebraic operations defined pointwise the *model* space and its elements $\widetilde{y}$ are *models* of $y \in \mathcal{H}$. Transition to the model is realized by the operator $W\colon \mathcal{H} \to \widetilde{\mathcal{H}}$, $Wy := \widetilde{y}(\,\cdot\,)$. It is linear and, in known applications, injective. The noninjectivity of W would mean the existence of a nonzero $y \in \mathcal{H}$ and a function $\varepsilon = \varepsilon(\omega)$ such that $y \perp \vee_{\omega\in\Omega_{L_0}} \vee_{0\leq t<\varepsilon(\omega)} \omega(t)$, which could be interpreted as the absence of *completeness* of the system of atoms. In the applications that we know completeness occurs, but whether the same is true in the general case is an open question.

The transition operator W has additional properties if the space $\widetilde{\mathcal{H}}$ is equipped with a Hilbert structure. One of the ways to define such a structure is the following. Let $\operatorname{Ker} W = \{0\}$. Take by definition $(\widetilde{y}, \widetilde{w})_{\widetilde{\mathcal{H}}} := (y, w)_{\mathcal{H}}$; then W is unitary. If $\operatorname{Ker} W \neq \{0\}$, then by restricting W to $\mathcal{H} \ominus \operatorname{Ker} W$ we obtain a partial isometry. This trick is used in the model theory (see, e.g., [10]); it is universal, but not very meaningful. Should some canonical Hilbert structure be found in stalks $\mathcal{G}_\omega$, then one could hope for realization of $\mathcal{H}$ as $\widetilde{\mathcal{H}} = \oplus \int_{\Omega_{L_0}} \mathcal{G}_\omega \, d\mu(\omega)$ (with an adequate measure μ) such that $W\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ would be unitary. At present such a structure cannot be seen, but chances appear under additional assumptions about atoms. The studied examples motivate the following heuristic construction.

Values on atoms. Recall that in system (1.12)–(1.14) for every control $h \in \mathcal{M}$ there exists a corresponding classical solution $u^h \in C^\infty_{\mathrm{loc}}([0, \infty); \mathcal{H})$ called a smooth wave (see 1.4). Such waves constitute reachable sets $\mathcal{U}^T_{L_0}$ and $\mathcal{U}_{L_0}$ (see (1.22)).

Suppose that the operator L_0 is completely nonselfadjoint, so that the controllability (1.24) occurs. Additionally assume that:

(A) there exists a subset Ω^e of Ω_{L_0} such that the system of atoms constituting Ω^e is complete, they all are continuous at zero, and $\omega(0) = \lim_{t\to+0} \omega(t) = \{0\}$;

(B) there exists an element $e \in \mathcal{H}$ such that the limits $\lim_{t\to+0} \frac{\|P_{\omega(t)}u\|}{\|P_{\omega(t)}e\|}$ exist and are finite for every $u \in \mathcal{U}_{L_0}$, $\omega \in \Omega^e$. We call such e a *gauge* element.

In the general case, the existence of gauge elements is not proved; however, in examples they can be found, and it is even possible to choose $e \in \operatorname{Ker} L_0^*$. The linear space $\mathcal{U}_{L_0}$ can be called the *smooth structure* determined by L_0, owing to the role of condition **(B)** that will be seen below.

Fix $\omega \in \Omega^e$; let $\widetilde{\mathcal{U}}_{L_0,\omega} := \{\widetilde{u}(\omega)\,|u \in \mathcal{U}_{L_0}\} \subset \mathcal{G}_\omega$ be the linear space of germs of smooth waves. On it, we define the following sesquilinear form:

$$\langle \widetilde{u}(\omega), \widetilde{u}'(\omega)\rangle := \lim_{t\to+0} \frac{(P_{\omega(t)}u, u')}{(P_{\omega(t)}e, e)}, \qquad u, u' \in \mathcal{U}_{L_0}, \omega \in \Omega^e. \tag{3.1}$$

Consider its (linear) subset $\widetilde{\mathcal{U}}^0_{L_0,\omega} := \{\widetilde{u}(\omega)\,|u \in \mathcal{U}_{L_0}, \langle \widetilde{u}(\omega), \widetilde{u}(\omega)\rangle = 0\}$ and the factor space $\mathcal{U}_{L_0,\omega} := \widetilde{\mathcal{U}}_{L_0,\omega}/\widetilde{\mathcal{U}}^0_{L_0,\omega}$; let $[u](\omega)$ denote the equivalence class of the element $\widetilde{u}(\omega)$. We call $[u](\omega)$ the *value* of the wave $u \in \mathcal{U}_{L_0}$ on the atom ω. The form (3.1) induces a natural pre-Hilbert structure on $\mathcal{U}_{L_0,\omega}$. Taking completion with respect to the corresponding norm, we obtain the Hilbert *space of values*. We keep the notation $\mathcal{U}_{L_0,\omega}$ for it. It should be noted that every wave $u \in \mathcal{U}_{L_0}$ can be represented as $u = u^h(T)$, and so evolution of waves in system (1.12)–(1.14) is reflected in evolution of values $[u^h](\omega, T)$ on the atoms $\omega \in \Omega^e$.

The wave representation. In addition to **(A)** and **(B)**, we make another assumption: **(C)** there exists a measure μ on Ω_{L_0} such that $\mu(\Omega_{L_0} \setminus \Omega^e) = 0$ and

$$(u, u')_{\mathcal{H}} = \int_{\Omega_{L_0}} \big\langle [u](\omega), [u'](\omega)\big\rangle \, d\mu(\omega), \qquad u, u' \in \mathcal{U}_{L_0}. \tag{3.2}$$

In all examples we know such measures can be found. It is not known whether conditions **(A)** and **(B)** guarantee the existence of μ in the general situation.

We call the space $\mathcal{H}^{\mathrm{w}} := \oplus \int_{\Omega_{L_0}} \mathcal{U}_{L_0,\omega}\, d\mu(\omega)$ the *wave representation* of the original $\mathcal{H}$. From the definitions it is clear that the operator $U\colon \mathcal{H} \to \mathcal{H}^{\mathrm{w}}$,

$$\mathcal{H} \supset \mathcal{U}_{L_0} \ni u \overset{U}{\mapsto} [u](\,\cdot\,) \in \mathcal{H}^{\mathrm{w}},$$

which realizes this representation, is isometric and extends up to a unitary operator from $\mathcal{U}_{L_0}$ to the entire $\mathcal{H}$. U acts by applying W and factorizing the germs.

The passage from germs $\widetilde{u}$ to values $[u](\,\cdot\,)$ is aimed at the following. In all known examples, describing elements of $\mathcal{H}$ by sections of the bundle $\bigcup_{\omega\in\Omega_{L_0}} \{\omega, \mathcal{G}_\omega\}$ is redundant. Passing to values removes this redundancy, owing to factorization. We show this in the example of the Sturm–Liouville operator.

• Let L_0 be the operator (1.1). In this case $\mathcal{H} = L_2(\mathbb{R}_+)$.

Germs. Pick an atom $\omega \in \Omega$. Let $y, y' \in \mathcal{H}$ be two functions. By Theorem 1, $y \overset{\omega}{=} y'$ means that y and y' coincide in some neighborhood of the point $x_\omega \in \bar{\mathbb{R}}_+$. Thus, the germ $\widetilde{y}(\omega)$ can be identified canonically with the usual germ of the function $y(\,\cdot\,)$ at the point x_ω, and the model space $\widetilde{\mathcal{H}}$ with the stalk of square-integrable functions above x_ω. Then, the stalks $\mathcal{G}_\omega$ are spaces of infinite dimension.

From the same Theorem 1 it easily follows that the system of atoms composing the wave spectrum Ω is complete. Thus, the operator $W\colon y \mapsto \widetilde{y}$ is injective. At the same time, since $\dim \mathcal{G}_\omega = \infty$, modeling scalar functions y by the elements of the germ $\widetilde{y}$ is obviously redundant. This motivates the passage from germs to values.

Values. In our case condition **(A)** is satisfied.

We check condition **(B)**. The set of smooth waves is $\mathcal{U} = C^\infty_{\rm fin}(\bar{\mathbb{R}}_+)$, see (1.25). Recall that the function ϕ is the solution of problem (1.3). Pick a nonzero element $e \in \operatorname{Ker} L_0^*$. By (1.4), we have $e = c\phi$ with some constant $c \neq 0$. In accordance with the general theory of ordinary differential equations, the function e is smooth and may only have simple zeros accumulating only to ∞. Denote $N^e := \{x \in \mathbb{R}_+ \,|\, e(x) = 0\}$.

Let $\omega = \{\omega(t)\}_{t\geq 0} : \omega(t) = L_2(\{x_\omega\}^t)$ be an atom such that $x_\omega \notin N^e$. For the smooth waves $u, u' \in \mathcal{U}$ we have

$$\tag{3.3} \frac{(P_{\omega(t)}u, u')}{(P_{\omega(t)}e, e)} = \frac{\int_{\{x_\omega\}^t} u(x)\overline{u'(x)}\,dx}{\int_{\{x_\omega\}^t} e(x)\overline{e(x)}\,dx} \underset{t\to+0}{\longrightarrow} \left(\frac{u(x_\omega)}{e(x_\omega)}\right)\overline{\left(\frac{u'(x_\omega)}{e(x_\omega)}\right)}.$$

Thus, the function e fits for the role of the gauge element. Taking $\Omega^e = \Omega_{L_0} \setminus \{\omega \,|\, x_\omega \in N^e\}$, we conclude that condition **(B)** is satisfied.

By (3.3), for the germs $\widetilde{u}(\omega), \widetilde{u}'(\omega) \in \widetilde{\mathcal{U}}_\omega \subset \mathcal{G}_\omega$ we have

$$\tag{3.4} \langle \widetilde{u}(\omega), \widetilde{u}'(\omega)\rangle = \left(\frac{u(x_\omega)}{e(x_\omega)}\right)\overline{\left(\frac{u'(x_\omega)}{e(x_\omega)}\right)}.$$

Clearly, the condition $\widetilde{u}(\omega) \in \widetilde{\mathcal{U}}^0_\omega$, which by definition means that $\langle \widetilde{u}(\omega), \widetilde{u}(\omega)\rangle = 0$, is equivalent to $u(x_\omega) = 0$. It also follows that the correspondence

$$\tag{3.5} \mathcal{U}_\omega = \widetilde{\mathcal{U}}_\omega / \widetilde{\mathcal{U}}^0_\omega \ni [u](\omega) \mapsto \lim_{t\to+0} \frac{(P_{\omega(t)}u, e)}{(P_{\omega(t)}e, e)} = \frac{u(x_\omega)}{e(x_\omega)} \in \mathbb{C}$$

is an isometry. Thus, for $\omega \in \Omega^e$ the space of values $\mathcal{U}_\omega$ is *one-dimensional.* The same correspondence gives the canonical coordinatization of $\mathcal{U}_\omega$. Other coordinatizations are also possible and have the form $[u](\omega) \mapsto e^{i\theta(\omega)}\frac{u(x_\omega)}{e(x_\omega)}$ with real-valued functions $\theta(\,\cdot\,)$.

The wave representation. Recall that the measure ν is defined in Proposition 3. For the smooth waves $u, u' \in \mathcal{U}$ we have

$$\begin{aligned}(u, u') &= \int_{\mathbb{R}_+} u(x)\overline{u'(x)}\,dx = \int_{\mathbb{R}_+} \left(\frac{u(x_\omega)}{e(x_\omega)}\right)\overline{\left(\frac{u'(x_\omega)}{e(x_\omega)}\right)}|e(x)|^2\,dx \\ &\overset{(3.4),(3.5)}{=} \int_\Omega \langle [u](\omega), [u'](\omega)\rangle\,d\mu(\omega),\end{aligned}$$

where $\mu(\omega) := |e(x_\omega)|^2 d\nu(\omega)$. Thus, condition **(C)** is satisfied, and the correspondence

$$\tag{3.6} \mathcal{H} = L_2(\bar{\mathbb{R}}_+) \ni u \overset{U}{\mapsto} [u](\,\cdot\,) \in L_{2,\,\mu}(\Omega) =: \mathcal{H}^{\rm w}$$

is an isometry. It is defined on smooth waves and extends from $\mathcal{U}$ up to a unitary operator $U\colon \mathcal{H} \to \mathcal{H}^{\rm w}$, which gives the wave representation of the elements of the original $\mathcal{H}$.

The coordinate representation. Coordinatizations of the spectrum (2.33) and of the value spaces (3.5) determine an isometry

$$\tag{3.7} \mathcal{H}^{\rm w} \ni [u](\,\cdot\,) \overset{V}{\mapsto} u[\,\cdot\,] \in L_{2,\,\rho}(\bar{\mathbb{R}}_+[\tau]) =: \mathcal{H}^{\rm c},$$

where $[u](\tau) := \frac{u(\tau)}{e(\tau)}$ and $d\rho := |e(\tau)|^2 d\tau$. This gives the wave representation of elements of the space $\mathcal{H}^{\rm w}$.

The composition $Y := VU$, $Y\colon \mathcal{H} \to \mathcal{H}^{\rm c}$ extends from smooth waves up to a unitary operator that takes functions in the original $L_2\big(\bar{\mathbb{R}}_+[x]\big)$ to functions in $L_{2,\,\rho}(\bar{\mathbb{R}}_+[\tau])$ by the rule

$$\tag{3.8} (Yy)\,(\tau) = y[\tau] := \frac{y(\tau)}{e(\tau)}, \qquad \tau \geq 0.$$

The obvious similarity of the original space to the space of coordinate representation and the simplicity of the correspondence $y \mapsto y[\,\cdot\,]$ are important facts used for solving inverse problems.

3.2. The operator $\widetilde{L}_0^*$. Now we return to the general case considered at the beginning of Subsection 3.1. Assume that the controllability $\overline{\mathcal{U}}_{L_0} = \mathcal{H}$ occurs and the operator of transition to the model W is injective. In this situation the set of pairs $\big\{\{Wu, WL_0^*u\} \,|\, u \in \mathcal{U}_{L_0}\big\}$ is a graph of an operator acting in the model space $\widetilde{\mathcal{H}}$. We call this operator the *wave model* of the operator L_0^* and denote it by $\widetilde{L}_0^+$. Note that it would be more consistent to talk about the model not of L_0^* itself, but rather of its wave part $L_0^*|_{\mathcal{U}_{L_0}}$ (see the remark in Subsection 1.4). We ignore this inaccuracy in order not to overload terminology.

Since every smooth wave is $u = u^h(T)$ and

$$L_0^* u^h(T) \overset{(1.12)}{=} -u^h_{tt}(T) \overset{(1.16)}{=} -u^{h_{tt}}(T),$$

the wave model can be defined as the operator with the graph

$$\operatorname{graph} \widetilde{L}_0^+ = \big\{\{Wu^h(T), -Wu^{h_{tt}}(T)\} \,|\, h \in \mathcal{M}, T \geq 0\big\}\,.$$

It is hard to expect the model to have rich properties in such generality. Its *locality* can be conjectured: if $\widetilde{y} \in \operatorname{Dom} \widetilde{L}_0^+$ and $\widetilde{y}|_A = 0$ on an open set $A \subset \Omega_{L_0}$, then $\widetilde{L}_0^+ \widetilde{y}|_A = 0$. In the known examples locality does occur.

If the wave representation (3.6) is defined, then the corresponding version of the wave model appears, $L_{0\,\mathrm{w}}^+ := UL_0^*|_{\mathcal{U}_{L_0}} U^*$, which acts in the space $\mathcal{H}^{\mathrm{w}}$. It is defined by its graph

$$\operatorname{graph} L_{0\,\mathrm{w}}^+ = \big\{\{Uu^h(T), -Uu^{h_{tt}}(T)\} \,|\, h \in \mathcal{M}, T \geq 0\big\}\,.$$

• In the case of the Sturm–Liouville operator, the coordinate realization of the wave model $L_{0\,\mathrm{c}}^+ := YL_0^*|_{\mathcal{U}_{L_0}} Y^*$ is defined consistently. It acts in the space $\mathcal{H}^{\mathrm{c}} = L_{2,\,\rho}(\overline{\mathbb{R}}_+[\tau])$, is defined by its graph

$$\operatorname{graph} L_{0\,\mathrm{c}}^+ = \big\{\{Yu^h(T), -Yu^{h_{tt}}(T)\} \,|\, h \in \mathcal{M},\, T \geq 0\big\}$$

and, by (1.4) and (3.8), is the differential operator

$$\begin{aligned}
\left(L_{0\,\mathrm{c}}^+ y[\,\cdot\,]\right)[\tau] &= \left\{\frac{1}{e(\tau)} \left(-\frac{d^2}{d\tau^2} + q(\tau)\right) e(\tau)\right\} y[\tau] \\
&= -y''[\tau] + p(\tau)\, y'[\tau] + Q(\tau) y[\tau], \quad \tau \geq 0,\ \tau \in \Omega^e.
\end{aligned} \tag{3.9}$$

with the coefficients

$$p(\tau) := -2\,\frac{e'(\tau)}{e(\tau)}, \quad Q(\tau) := q(\tau) - \frac{e''(\tau)}{e(\tau)}. \tag{3.10}$$

This operator is not closed, but its closure $L_{0\,\mathrm{c}}^* = \overline{L_{0\,\mathrm{c}}^+}$ is unitarily equivalent to the operator L_0^* by the remark at the end of Subsection 1.5. The construction of the canonical Green system as in Subsection 1.3 results in the following (recall that $(Yy)(\tau) = \frac{y(\tau)}{e(\tau)}$, $(Y^*y)(\tau) = y(\tau)e(\tau)$):

$$\begin{aligned}
K_c &= YK = \{\mathrm{const}\}, \\
\Gamma_{1\,\mathrm{c}} &= Y\Gamma_1 Y^* \colon y \mapsto -y(0), \\
\Gamma_{2\,\mathrm{c}} &= Y\Gamma_2 Y^* \colon y \mapsto \frac{y'(0)}{\eta'(0)}.
\end{aligned} \tag{3.11}$$

3.3. The inverse problem. The functional model $\widetilde{L}_0^+$ gives a unified approach to a rather wide class of boundary inverse problems. Putting off generalizations, we demonstrate the idea of the approach with our example.

• **Auxiliary model.** Consider the boundary value problem

$$-\psi'' + q\psi = \lambda\psi, \qquad x > 0, \tag{3.12}$$

$$\psi(0) = 0, \quad \psi'(0) = 1. \tag{3.13}$$

Its solution $\psi = \psi(x, \lambda)$ is a function that is smooth in x and entire in $\lambda \in \mathbb{C}$. In particular, for $q = 0$ we have $\psi = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}$.

Our operator L_0 has defect indices $(1, 1)$. Therefore, there exists a unique *spectral function* σ such that the formulas

$$\breve{y}(\lambda) = \int_0^\infty y(x)\,\psi(x,\lambda)\,dx, \qquad y(x) = \int_{-\infty}^\infty \breve{y}(\lambda)\,\psi(x,\lambda)\,d\sigma(\lambda)$$

establish an isometry of the spaces $L_2(\mathbb{R}_+[x])$ and $L_{2,\,\sigma}(\mathbb{R}[\lambda])$ (see [9, Chapter VIII]). In other words, the operator $\Phi\colon y \mapsto \breve{y}$ is unitary.

Let us find the Φ-representation of waves corresponding to problem (1.17)–(1.19) with a smooth control $f \in \dot{\mathcal{M}}$. For $\breve{u}^f(\,\cdot\,, t) = \Phi u^f(\,\cdot\,, t)$, using the finiteness of the support of $u^f(\,\cdot\,, t)$, we have:

$$\begin{aligned}
\breve{u}^f_{tt}(\lambda,t) &= \int_0^\infty u^f_{tt}(x,t)\psi(x,\lambda)\,dx \overset{(1.17)}{=} \int_0^\infty \left[u^f_{xx}(x,t) - q(x)u^f(x,t)\right]\psi(x,\lambda)\,dx \\
&= -u^f_x(0,t)\psi(0,\lambda) + u^f(0,t)\psi'(0,\lambda) + \int_0^\infty u^f(x,t)\left[\psi''(x,\lambda) - q(x)\psi(x,\lambda)\right]dx \\
&\overset{(1.17),(1.19),(3.12),(3.13)}{=} f(t) - \lambda\int_0^\infty u^f(x,t)\psi(x,\lambda)\,dx = f(t) - \lambda\breve{u}^f(\lambda,t).
\end{aligned}$$

Integrating and using (1.18), we see that

$$\breve{u}^f(\lambda,t) = \int_0^t \lambda^{-\frac12}\,\sin[\lambda^{\frac12}\,(t-s)]f(s)\,ds, \qquad t \ge 0. \tag{3.14}$$

For the inverse problem the operator $\breve{L}_0^+ := \Phi\,L_0^*|_{\mathcal{U}}\,\Phi^*$ plays the role of an *auxiliary model* of the original L_0^*. It acts in $L_{2,\,\sigma}(\mathbb{R}[\lambda])$ by the rule

$$\begin{aligned}
\breve{L}_0^+\breve{u}^f(\lambda,t) &= \left[L_0^* u^f(\,\cdot\,,t)\right]\breve{}\,(\lambda) = -[u^f_{tt}(\,\cdot\,,t)]\breve{}\,(\lambda) = -[u^{f_{tt}}(\,\cdot\,,t)]\breve{}\,(\lambda) \\
&= -\breve{u}^{f_{tt}}(\lambda,t) \overset{(3.14)}{=} -\int_0^t \lambda^{-\frac12}\,\sin[\lambda^{\frac12}\,(t-s)]f''(s)\,ds
\end{aligned} \tag{3.15}$$

and is defined by its graph

$$\operatorname{graph}\breve{L}_0^+ = \left\{\{\breve{u}^f(\lambda,T) - \breve{u}^{f_{tt}}(\lambda,T)\} \mid f \in \dot{\mathcal{M}}, T \ge 0\right\}. \tag{3.16}$$

Recovering the potential. The classical inverse spectral problem for the Sturm–Liouville operator on the half-line is to determine the potential $q\big|_{x\in\bar{\mathbb{R}}_+}$ from the given spectral function $\sigma\big|_{\lambda\in\mathbb{R}}$ (see [9, Chapter VIII]). To solve this problem, we can use the following approach.

Step 1. Using (3.14) and (3.15), we find the operator $\breve{L}_0^+$ in the space $L_{2,\,\sigma}(\mathbb{R}[\lambda])$ from its graph (3.16).

Step 2. We construct the wave model of the operator $\breve{L}_0^+$ and consider its coordinate realization. Owing to the invariance of the construction of the wave model, this leads to the operator $L^*_{0\,\mathrm{c}}$ acting in $L_{2,\,\rho}(\mathbb{R}_+[\tau])$.

Step 3. The representation (3.9) allows us to find the coefficients p and Q. From (3.10) we get $e(\tau) = C \exp\left\{-\int_0^\tau \frac{p(s)}{2}\,ds\right\}$. Finally, we recover $q(\tau) = Q(\tau) + \frac{e''(\tau)}{e(\tau)}$.

Since the construction of the wave model is not simple, this approach is, of course, too involved compared to the classical procedure that involves the Gelfand–Levitan equation, see [9]. In the present paper we only want to demonstrate, on a relatively simple example of the Sturm–Liouville operator, the construction in all details and to show how it solves inverse problems.

At the same time, the wave model has some advantages. It can be used for solving problems with *any* data, whenever they determine the operator L_0 (or, equivalently, L_0^*) up to unitary equivalence. The spectral data, scattering data, Weyl function, characteristic function (see [10, 16, 18]) can be viewed as such. The universality of the model makes it unnecessary to convert the data of one type into another. Moreover, the wave model is efficient for recovering objects of greater complexity, namely, Riemmanian manifolds [14].

In the future we plan to study the construction of the wave model itself, as well as its possible applications. It would be of interest to construct, on the basis of the wave model, a functional model of an abstract Green system $\mathfrak{G}_{L_0}$, see Subsection 1.3. This interest is motivated by the existence of the *boundary* of the wave spectrum (2.27).

There are relationships, which deserve to be studied, between the wave model and operator C^*-algebras, see [15]. The source of these relationships is the correspondence $\omega \leftrightarrow \tau_\omega$ between atoms and eikonals, see (2.25).

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Received 20/OCT/2016

Translated by THE AUTHORS