Алгебра и анализ
Том 29 (2017), № 2

St. Petersburg Math. J.
Vol. 29 (2018), No. 2, Pages 249–266
http://dx.doi.org/10.1090/spmj/1492
Article electronically published on March 12, 2018

ON GLOBAL ATTRACTORS AND RADIATION DAMPING FOR NONRELATIVISTIC PARTICLE COUPLED TO SCALAR FIELD

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To the memory of Vladimir Buslaev

Abstract. The Hamiltonian system of a scalar wave field and a single nonrelativistic particle coupled in a translation invariant manner is considered. The particle is also subject to a confining external potential. The stationary solutions of the system are Coulomb type wave fields centered at those particle positions for which the external force vanishes. It is proved that the solutions of finite energy converge, in suitable local energy seminorms, to the set $\mathcal{S}$ of all stationary states in the long time limit $t \to \pm\infty$. Next it is shown that the rate of relaxation to a stable stationary state is determined by the spatial decay of initial data. The convergence is followed by the radiation of the dispersion wave that is a solution of the free wave equation.

Similar relaxation has been proved previously for the case of a relativistic particle when the speed of the particle is less than the wave speed. Now these results are extended to a nonrelativistic particle with velocity, including that greater than the wave speed. However, the research is restricted to the plane particle trajectories in $\mathbb{R}^3$. Extension to the general case remains an open problem.

§1. Introduction

We consider the Hamiltonian system of a real scalar field $\varphi(x)$ on $\mathbb{R}^3$ and an extended nonrelativistic particle with the center position $q \in \mathbb{R}^3$ and with the charge density $\rho(x-q)$. The field is governed by the wave equation with a source. The particle is subject to the wave field and also to an external potential V, which is confining in the sense of (1.9). The interaction between the particle and the scalar field is local, translation invariant, and linear in the field. We study the long-time behavior of the coupled system. Our main results are the asymptotics

$$\dot{q}(t) \to 0, \quad \ddot{q}(t) \to 0, \quad t \to \pm\infty, \tag{1.1}$$

and the convergence of the field to the corresponding Coulombic potential. Moreover, we establish the rate of that convergence in the case where $q_\pm$ is a nondegenerate local minimum of the potential V.

Let $\pi(x)$ be the canonically conjugate field to $\varphi(x)$, and let p be the momentum of the particle. Then the Hamiltonian (energy functional) reads

$$\mathcal{H}(\varphi, q, \pi, p) \equiv \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int (|\pi(x)|^2 + |\nabla\varphi(x)|^2)\,dx + \int \varphi(x)\rho(x-q)\,dx. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35Q60, 78A40, 78M35.

Key words and phrases. Hamiltonian system, nonrelativistic particle, wave equation with a source, extended electron.

The research was carried out at the IITP RAS and was supported by the Russian Foundation for Sciences (project no. 14-50-00150).

Taking formally variational derivatives in (1.2), the coupled dynamics becomes

$$\left.\begin{aligned} \dot\varphi(x,t) &= \pi(x,t), \quad \dot\pi(x,t) = \Delta\varphi(x,t) - \rho(x-q(t)), \\ \dot q(t) &= p(t), \quad \dot p(t) = -\nabla V(q(t)) + \int \varphi(x,t)\nabla\rho(x-q(t))\,dx. \end{aligned}\right| \tag{1.3}$$

For smooth $\varphi(x)$ vanishing at infinity, the Hamiltonian can be rewritten as

$$\begin{aligned} \mathcal{H}(\varphi,q,\pi,p) \equiv{}& \frac{1}{2}p^2 + V(q) \\ &+ \frac{1}{2}\int (|\pi(x)|^2 + |\nabla[\varphi(x) - \Delta^{-1}\rho(x-q)]|^2)\,dx + \frac{1}{2}\langle\rho,\Delta^{-1}\rho\rangle, \end{aligned} \tag{1.4}$$

where

$$\frac{1}{2}\langle\rho,\Delta^{-1}\rho\rangle = -\frac{1}{8\pi}\iint \frac{\rho(x)\rho(y)}{|x-y|}\,dx\,dy \le 0. \tag{1.5}$$

Thus, the energy (1.4) is bounded from below if $|\langle\rho,\Delta^{-1}\rho\rangle| < \infty$, which provides *a priori* estimates for solutions of (1.3), and hence guarantees the existence of global solutions. Otherwise, the dynamics is not well defined. For example, $\langle\rho,\Delta^{-1}\rho\rangle = -\infty$ for the point particle with $\rho(x) = \delta(x)$:

$$\langle\delta,\Delta^{-1}\delta\rangle = -(2\pi)^{-3}\int \frac{1}{k^2}\,dk = -\infty. \tag{1.6}$$

For the first time, this "ultraviolet divergence" was a point particle in classical electrodynamics, where $-\langle\rho,\Delta^{-1}\rho\rangle$ is proportional to the energy of the particle in its own electrostatic field. Not only the self-energy (1.6) is infinite, but also the particle acquires an infinite effective mass. These infinities inspired the introduction of the "extended electron" by Abraham [1]. Our system (1.3) is a scalar analog of the Abraham electrodynamics with the extended electron, see [23, 28].

The stationary solutions for (1.3) are easily determined. Denote

$$s_q(x) = -\int \frac{\rho(y-q)}{4\pi|y-x|}\,dy, \quad x,q\in\mathbb{R}^3. \tag{1.7}$$

Let $Z = \{q\in\mathbb{R}^3 : \nabla V(q) = 0\}$ be the set of critical points for V. Then the set of all stationary states is given by

$$\mathcal{S} = \big\{(\varphi,\pi,q,p) = (s_q,0,q,0) := S_q \,|\, q\in Z\big\}. \tag{1.8}$$

One natural goal is to investigate the domain of attraction for $\mathcal{S}$ and in particular to prove that each finite energy solution of (1.3) converges to some stationary states $S_{q_\pm} = (s_{q_\pm},0,q_\pm,0)\in\mathcal{S}$ in the long time limit $t\to\pm\infty$.

To state our main results, we need some assumptions on V and ρ. We assume that

$$V\in C^2(\mathbb{R}^3), \quad \lim_{|q|\to\infty} V(q) = \infty, \tag{1.9}$$

$$\rho\in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = 0 \ \text{ for } \ |x|\ge R_\rho, \quad \rho(x) = \rho_r(|x|). \tag{1.10}$$

Moreover, we suppose that the following Wiener condition is fulfilled:

$$\widehat\rho(k) \ne 0 \ \text{ for } \ k\in\mathbb{R}^3. \tag{1.11}$$

It is an analog of the Fermi Golden Rule: the coupling term $\rho(x-q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum for the linear part of the equation (cf. [26, 27]). As we shall see, the Wiener condition (1.11) is very essential for our asymptotic analysis.

For technical reasons, we restrict ourselves to the case where the particle moves in the plane, i.e., we suppose that $q(t) = (q^1(t), q^2(t), q^3(t)) \in R^3$ is such that

$$q^3(t) = 0, \quad t \in \mathbb{R}. \tag{1.12}$$

For example, this condition is fulfilled if the initial fields $\varphi_0(x) = \varphi(x, 0)$ and $\pi_0(x) = \pi(x, 0)$ are symmetric in x^3, and

$$q^3(0) = p^3(0) = 0 \ \text{ and } \ \partial_{x^3} V(x^1, x^2, 0) = 0 \ \text{ for } \ (x^1, x^2) \in \mathbb{R}^2. \tag{1.13}$$

In the first part of the paper we prove that the set $\mathcal{S}$ is an attracting set for each trajectory $Y(t) = (\varphi(t), \pi(t), q(t), p(t))$. Namely, we consider the initial data $Y(0) = (\varphi_0, \pi_0, q_0, p_0)$ with

$$\varphi_0 \in C^2(\mathbb{R}^3), \quad \pi_0 \in C^1(\mathbb{R}^3), \tag{1.14}$$

such that

$$\begin{aligned} |\nabla\varphi_0(x)| + |\pi_0(x)| + |x|\big(|\nabla\nabla\varphi_0(x)| + |\nabla\pi_0(x)|\big) = \mathcal{O}(|x|^{-\sigma}),& \\ |x| \to \infty, \ \text{where } \sigma > 3/2,& \end{aligned} \tag{1.15}$$

which guarantees the finiteness of the energy (1.2). Next, we prove the relaxation (1.1). Further, we prove the long-time attraction

$$Y(t) \to \mathcal{S}, \quad t \to \pm\infty, \tag{1.16}$$

where the fields converge in the local energy seminorms. If, moreover, the set $\mathcal{S}$ is discrete, then (1.16) implies

$$Y(t) \to S_{q_\pm}, \quad t \to \pm\infty, \tag{1.17}$$

where the stationary states $S_{q_\pm} \in \mathcal{S}$ depend on the solution $Y(t)$ in question.

In the second part of the paper we specify the rate of convergence in (1.17) to a stationary state S_{q_+} in the case where $q_* \in Z$ is a nondegenerate minimum of the potential, i.e.,

$$d^2V(q_+) > 0. \tag{1.18}$$

where $d^2V(q_+)$ is the Hessian. We suppose that the initial fields belong to the weighted space $\mathring{H}^1_\alpha \oplus L^2_\alpha$ with some $\alpha > 1$ (see Definition 2.1). Then for any $\varepsilon > 0$ we have

$$\begin{aligned} &\dot q(t) = \mathcal{O}(|t|^{-\alpha+\varepsilon}), \quad q(t) = q_+ + \mathcal{O}(|t|^{-\alpha+\varepsilon}), \\ &\big\|(\varphi(t), \pi(t)) - (s_{q_+}, 0)\big\|_{\mathring{H}^1_{-\alpha}\oplus L^2_{-\alpha}} = \mathcal{O}(t^{-\alpha+\varepsilon}), \quad t \to \infty. \end{aligned} \tag{1.19}$$

Moreover, in this case the scattering asymptotics hold,

$$(\varphi(x,t), \pi(x,t)) = (s_{q_+}, 0) + W(t)\Phi_+ + r(x,t). \tag{1.20}$$

Here $W(t)$ is the dynamical group of the free wave equation, $\Phi \in \mathring{H}^1 \oplus L^2$ is the corresponding asymptotic state, and

$$\|r(t)\|_{\mathring{H}^1\oplus L^2} = \mathcal{O}(|t|^{-\alpha+1+\varepsilon}), \quad t \to \infty. \tag{1.21}$$

The investigation is inspired by the fundamental problems of the field theory and quantum mechanics. Namely, the relaxation of the acceleration (1.1) is known as *radiation damping* in classical electrodynamics since Lorentz and Abraham [1], however for the first time it was proved in [23, 22] for the case of the relativistic particle with $\dot q = p/\sqrt{p^2+1}$. Second, the asymptotics (1.17) provide a dynamical model of Bohr's transitions to quantum stationary states, see the details in [17, 18].

Our extension to the nonrelativistic particle is not straightforward and is important in connection with the Cherenkov radiation. The main difficulty is due to the singular nature of the radiation for $|\dot q(t)| \geq 1$.

Traditionally, the classical Larmor and Liénard formulas [6, (14.22)] and [6, (14.24)] are accepted for the power of radiation of a point particle. These formulas contain the factor $(1-\beta\cdot\omega)^{-3}$ (cf. our formula (5.4)) where $\beta = v/c$ and ω is the direction of the radiation. Here $v = \dot{q}(\tau)$ is the particle velocity at the "retarded time" τ and c is the propagation speed of the wave field describing the dispersive medium. These formulas are deduced from the Liénard–Wiechert expressions for the retarded potentials neglecting the initial fields. Moreover, these formulas neglect the back field-reaction though it should be the key reason for the relaxation. The main problem is that this back field-reaction is infinite for the point particles. In (1.3) we have set $c = 1$. In general, c is less than the speed of light in vacuum, so the particle velocities $\dot{q}(t) > 1$ are possible. Then the factor $(1-\beta\cdot\omega)^{-3}$ in the Larmor formula becomes infinite for some directions ω.

A rigorous meaning to these calculations for the relativistic particle has been suggested first in [23, 22] for the Abraham model of the "extended electron" under the Wiener condition (1.11). A survey can be found in [28].

For the nonrelativistic Abraham type model (1.3) with the "extended electron" the radiation remains finite due to smoothing by the coupling function ρ. Nevertheless, the case where $|\dot{q}(t)| > 1$ rises many open questions.

Our main novelties in the present paper are the following.

I. Global attraction of finite energy solutions to stationary states for the case of nonrelativistic particle.

II. The asymptotics (1.19)–(1.21) in the weighted Sobolev norms for the case of the nonrelativistic particle.

Let us comment on previous results in these directions. The global attractions (1.16) and (1.17) were proved in [22, 23] for the system of type (1.3) with the relativistic particle and for the similar Maxwell–Lorentz system. In [20], the global attraction to solitons was proved for system (1.3) without external potential under the Wiener condition (1.11). In [11] this result was extended to similar Maxwell–Lorentz system. In [7, 8, 9, 10], the global attraction to solitons was proved for system (1.3) and similar systems with the Klein–Gordon and Maxwell equations with small ρ. In [12, 13, 14, 15, 16], the global attraction to solitary waves was proved for the Klein–Gordon and Dirac equations coupled to $U(1)$-invariant nonlinear oscillators.

The asymptotics of type (1.19)–(1.21) were established in [22] for the case of relativistic particle in local energy seminorms for initial fields with compact support. In [21] we have proved the asymptotic stability of the stationary states for system (1.3) in the weighted Sobolev norms.

In a series of papers, Egli, Fröhlich, Gang, Sigal, and Soffer have established convergence to a soliton for the system of type (1.3) with the Schrödinger equation instead of the wave equation. The main result of [5] is the long time convergence to a soliton with a subsonic speed for initial solitons with supersonic speeds. The convergence is associated with the Cherenkov radiation, see [5] and the references therein.

The asymptotics of type (1.20) were proved by Soffer and Weinstein for nonlinear Schrödinger equations with a potential [29, 30], and for translation invariant nonlinear Schrödinger equations by Buslaev, Perelman, and Sulem [2, 3, 4].

Now let us comment on our methods. For the proof of (1.1) we estimate the energy dissipation by decomposing φ into a near and a far field. Energy is radiated in the far field. Since the Hamiltonian is bounded from below, such radiation cannot go on forever and a certain "energy radiation functional" must be bounded. This radiation functional can be written as a convolution. By a Wiener Tauberian theorem, using (1.11), we conclude (1.1) for $\ddot{q}$. Therefore, (1.1) also holds true for $\dot{q}$, because $|q(t)|$ is bounded by some $\bar{q}_0 < \infty$ due to (1.9). Finally, we deduce (1.16) and (1.17) from (1.1) and integral representations for the fields. This strategy is close to [22, 23, 28]; however, the singularity of the radiation at $|\dot{q}(t)| \geq 1$ requires suitable modifications in application of the Wiener Tauberian theorem. We suggest the modification for the plane particle trajectories (1.12). Extension to the general case remains an open problem.

We prove the asymptotics (1.19)–(1.21) by developing the methods of [22] and controlling the nonlinear part of (1.3) by the dispersion decay for the linearized equation, which we established in [21]. Let us emphasize however, that the asymptotics (1.19)–(1.21) are quite different from the asymptotic stability proved in [21].

The plan of our paper is as follows. In §2 we introduce appropriate function spaces and formulate our main results. In §3 we refine known results on the long range asymptotics of the Liénard–Wiechert potentials. In §4 we calculate the energy radiation integral. We use this formula in §5 to prove the velocity relaxation. In §6 we prove the attraction to stationary states. In §7 we consider linearization at a stationary state. In §8 we prove a version of the strong Huygens principle for nonlinear system (1.3). In §§9,10 we deduce the asymptotics (1.19)–(1.21).

§2. Existence of dynamics and main results

We consider the Cauchy problem for the Hamiltonian system (1.3), which can be written as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y_0. \tag{2.1}$$

Here $Y(t) = (\varphi(t), \pi(t), q(t), p(t))$, $Y_0 = (\varphi_0, \pi_0, q_0, p_0)$, and all derivatives are understood in the sense of distributions.

Now we introduce a suitable phase space. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{L^2}$, and let H^1 denote the Sobolev space $H^1 = \{\psi \in L^2 : |\nabla\psi| \in L^2\}$ with the norm $\|\psi\|_{H^1} = \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}$. For $\alpha \in \mathbb{R}$, we denote by L^2_α the weighted Sobolev spaces L^2_α with the norms $\|\psi\|_{L^2_\alpha} := \|(1+|x|)^\alpha\psi\|_{L^2}$.

Denote by $\mathring{H}^1$ the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\nabla\varphi(x)\|_{L^2}$. Equivalently, by Sobolev's embedding theorem,

$$\mathring{H}^1 = \big\{\varphi(x) \in L^6(\mathbb{R}^3) \,:\, |\nabla\varphi(x)| \in L^2\big\}.$$

Denote by $\mathring{H}^1_\alpha$ the completion of the real space $C_0^\infty(\mathbb{R}^3)$ with the norm

$$\|(1+|x|)^\alpha\nabla\varphi(x)\|_{L^2}.$$

For any $R > 0$, let $\|\varphi\|_{L^2(B_R)}$ denote the norm in $L^2(B_R)$, where $B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}$. Then the seminorms $\|\varphi\|_{H^1(B_R)} = \|\nabla\varphi\|_{L^2(B_R)} + \|\varphi\|_{L^2(B_R)}$ are continuous on $\mathring{H}^1$.

Definition 2.1.

i) The phase space $\mathcal{E}$ is the real Hilbert space $\mathring{H}^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, \pi, q, p)$ with the finite norm

$$\|Y\|_{\mathcal{E}} = \|\nabla\psi\|_{L^2} + \|\pi\|_{L^2} + |q| + |p|.$$

ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|Y\|_R = \|\varphi\|_{H^1(B_R)} + \|\pi\|_{L^2(B_R)} + |q| + |p|, \quad R > 0. \tag{2.2}$$

iii) $\mathcal{E}_\alpha$ with $\alpha \in \mathbb{R}$ is the Hilbert space $\mathring{H}^1_\alpha \oplus L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm

$$\|Y\|_\alpha = \|\,Y\|_{\mathcal{E}_\alpha} = \|\nabla\psi\|_{L^2_\alpha} + \|\pi\|_{L^2_\alpha} + |q| + |p|. \tag{2.3}$$

iv) $\mathcal{F}_\alpha$ is the space $\mathring{H}^1_\alpha \oplus L^2_\alpha$ of fields $F = (\psi, \pi)$ with the finite norm

$$\|\,F\|_\alpha = \|\,F\|_{\mathcal{F}_\alpha} = \|\nabla\psi\|_{L^2_\alpha} + \|\pi\|_{L^2_\alpha}. \tag{2.4}$$

Note that we use the same notation for the norms in the space $\mathcal{F}_\alpha$ as in the space $\mathcal{E}_\alpha$ defined in (2.3). We hope it will create no misunderstanding because it will be always clear from the context if we deal with fields only, and therefore with the space $\mathcal{F}_\alpha$, or with fields-particles, and therefore with elements of the space $\mathcal{E}_\alpha$.

Note that both spaces $\mathcal{E}_F$ and $\mathcal{E}$ are metrisable, $\mathring{H}^1$ is not contained in L^2 and, for instance, $\|S_q\|_{L^2} = \infty$. On the other hand, $S_q \in \mathcal{E}$. Therefore, $\mathcal{E}$ is the space of finite energy states. The Hamiltonian functional (1.4) is continuous on the space $\mathcal{E}$ and is bounded from below. In the point charge limit the lower bound tends to $-\infty$ by (1.6).

Lemma 2.2 (see [22, Lemma 2.1]). *Let conditions* (1.9) *and* (1.10) *be satisfied. Then*:

(i) *For every* $Y_0 \in \mathcal{E}$ *the Cauchy problem* (2.1) *has a unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$.

(ii) *For every* $t \in \mathbb{R}$ *the map* $U(t)\colon Y_0 \mapsto Y(t)$ *is continuous both on* $\mathcal{E}$ *and on* $\mathcal{E}_F$.

(iii) *The energy is conserved, i.e.*,

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0) \quad \text{for} \quad t \in \mathbb{R}. \tag{2.5}$$

(iv) *The following a priori estimates hold true*:

$$\|Y(t)\|_{\mathcal{E}} \le C(Y_0), \quad t \in \mathbb{R}. \tag{2.6}$$

(v) *The time derivatives* $q^{(k)}(t)$, $k = 0, 1, 2, 3$, *are uniformly bounded, i.e., there are constants* $\bar{q}_k > 0$ *depending only on the initial data and such that*

$$|q^{(k)}(t)| \le \bar{q}_k \quad \text{for} \quad t \in \mathbb{R}. \tag{2.7}$$

Our first main result is the following theorem.

Theorem 2.3. *Under conditions* (1.9)–(1.12) *and* (1.14)–(1.15), *the following is true for the corresponding solution* $Y(t) \in \mathcal{E}$ *of the Cauchy problem* (2.1).

i) *The attraction holds*

$$Y(t) \xrightarrow{\mathcal{E}}_F \mathcal{S}, \quad t \to \pm\infty. \tag{2.8}$$

ii) *If, moreover, the set* $\mathcal{S}$ *is discrete, then* (2.8) *implies a similar convergence*

$$Y(t) \xrightarrow{\mathcal{E}}_F S_{\pm}, \quad t \to \pm\infty. \tag{2.9}$$

Our second main result refines the asymptotics (2.8)–(2.9) for initial fields belonging to the Sobolev weighted spaces.

Theorem 2.4. *Under conditions* (1.9)–(1.11), *let* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *be a solution of the Cauchy problem* (2.1) *with* $Y_0 \in \mathcal{E}_\alpha$, *where* $\alpha > 1$. *Suppose that*

$$Y(t) \xrightarrow{\mathcal{E}}_F S_{q_+}, \quad t \to \infty, \tag{2.10}$$

where the limit point $q_+ \in Z$ *satisfies* (1.18). *Then*:

i) *For every* $\varepsilon > 0$,

$$\|Y(t) - S_{q_+}\|_{-\alpha} = \mathcal{O}(t^{-\alpha+\varepsilon}), \quad t \to \infty. \tag{2.11}$$

ii) *For every* $\varepsilon > 0$, *the scattering asymptotics is valid,*

$$(\varphi(x,t), \pi(x,t)) = (s_{q_+}, 0) + W(t)\Phi_+ + r(x,t), \tag{2.12}$$

where $\Phi_+ \in \mathring{H}^1 \oplus L^2$, *and*

$$\|r(t)\|_{\mathring{H}^1 \oplus L^2} = \mathcal{O}(|t|^{-\alpha+1+\varepsilon}), \quad t \to \infty. \tag{2.13}$$

§3. Liénard–Wiechert asymptotics

The solution of the nonhomogeneous wave equation from system (1.3) is the sum of two terms. The first is the retarded Liénard–Wiechert potential (3.1), which is the solution of the nonhomogeneous wave equation with zero initial data. The second term is the solution of the homogeneous equation with the initial data of the total field. This term is given by the Kirchhoff formula (3.13).

The second term does not affect the long-time asymptotics of the solution due to the strong Huygens principle. Thus, exactly the retarded Liénard–Wiechert potential is responsible for the long-time asymptotics.

In this section we refine the results of [22, 23] on the long time and long range asymptotics of the Liénard–Wiechert potentials

$$\varphi_r(x,t) = -\frac{1}{4\pi}\int dy\,\frac{\theta(t-|x-y|)}{|x-y|}\rho(y-q(t-|x-y|)),\quad \pi_r(x,t)=\dot\varphi_r(x,t). \tag{3.1}$$

These asymptotics will play the key role in the subsequent calculation of the energy radiation, which is used in the proof of the relaxation (1.1). Furthermore, we estimate the energy radiation corresponding to the Kirchhoff integral (3.13).

First, we find asymptotics of the retarded potentials in the wave zone $|x|\sim t\to\infty$. These asymptotics and their proofs are similar to those in [23, Lemma 3.2].

Lemma 3.1. *Under conditions* (1.9) *and* (1.10), *there exists $T_r>0$ such that the following asymptotics hold uniformly in $t\in[T_r,T]$: for every fixed $T>T_r$,*

$$\pi_r(x,|x|+t) = \overline\pi(\omega(x),t)|x|^{-1}+\mathcal O(|x|^{-2}), \tag{3.2}$$

$$\nabla\varphi_r(x,|x|+t) = -\omega(x)\overline\pi(\omega(x),t)|x|^{-1}+\mathcal O(|x|^{-2}) \tag{3.3}$$

as $|x|\to\infty$ with a function $\overline\pi(\omega,t)$. Here $\omega(x)=x/|x|$.

Proof. The integrand of (3.1) vanishes for $|y|>T_r:=\bar q_0+R_\rho$. Then for $t-|x|>T_r$ we have

$$|x-y|\le|x|+|y|\le t-T_r+T_r\le t,$$

and hence (3.1) implies that

$$\pi_r(x,t) = -\int dy\,\frac{1}{4\pi|x-y|}\nabla\rho(y-q(\tau))\cdot\dot q(\tau), \tag{3.4}$$

where $\tau=t-|x-y|$. Similarly, for $t-|x|>T_r$ we obtain

$$\begin{aligned}\nabla\varphi_r(x,t) &= \int dy\,\frac{1}{4\pi|x-y|}n\nabla\rho(y-q(\tau))\cdot\dot q(\tau)+\mathcal O(|x|^{-2})\\ &= -\omega(x)\pi_r(x,t)+\mathcal O(|x|^{-2}),\end{aligned} \tag{3.5}$$

because $n=\frac{x-y}{|x-y|}=\omega(x)+\mathcal O(|x|^{-1})$ for bounded y. Now we substitute $|x|+t$ for t in (3.4), (3.5) to get the asymptotics (3.2), (3.3) for $t>T_r$. Then τ becomes

$$\tau=|x|+t-|x-y|=t+\omega(x)\cdot y+\mathcal O(|x|^{-1})=\overline\tau+\mathcal O(|x|^{-1}),\quad \overline\tau=t+\omega\cdot y, \tag{3.6}$$

because

$$|x|-|x-y|=|x|-\sqrt{|x|^2-2x\cdot y+|y|^2}\sim|x|\left(\frac{x\cdot y}{|x|^2}+\frac{|y|^2}{2|x|^2}\right)=\omega(x)\cdot y+\mathcal O(|x|^{-1}).$$

Hence, (3.4) implies (3.2) with

$$\overline\pi(\omega,t) = -\frac{1}{4\pi}\int dy\,f\nabla\rho(y-q(\overline\tau))\cdot\dot q(\overline\tau). \tag{3.7}$$

Then (3.5) gives (3.3) immediately. □

Note that the asymptotics (3.2)–(3.3) are valid without condition (1.12). However, this condition allows us to represent (3.7) in a more efficient way for ω close to $(0, 0, \pm 1)$, see the next lemma. Namely, denote

$$\Theta = \begin{cases} 0 & \text{if } \bar{q}_1 < 1, \\ \varepsilon + \sqrt{1 - (\bar{q}_1)^{-2}} & \text{if } \bar{q}_1 \geq 1, \end{cases} \tag{3.8}$$

with an arbitrarily small $0 < \varepsilon < 1 - \sqrt{1 - (\bar{q}_1)^{-2}}$. Then for $\omega = (\omega^1, \omega^2, \omega^3)$ with $|\omega^3| \geq \Theta$ we obtain

$$|\omega \cdot \dot{q}| = |\dot{q}||\cos(\omega, \dot{q})| \leq \bar{q}_1 \sqrt{1 - (\omega^3)^2} \leq \bar{q}_1 \sqrt{1 - \Theta^2} < 1. \tag{3.9}$$

Lemma 3.2. *Suppose that conditions* (1.9), (1.10), *and* (1.12) *are satisfied. Then for any ω with $|\omega^3| \geq \Theta$ we have*

$$\bar{\pi}(\omega, t) = \frac{1}{4\pi} \int dy\, \rho(y - q(\bar{\tau})) \frac{\omega \cdot \ddot{q}(\tau)}{(1 - \omega \cdot \dot{q}(\tau))^2}. \tag{3.10}$$

Proof. We observe that

$$\nabla_y \rho(y - q(\bar{\tau})) \cdot \dot{q}(\bar{\tau}) = \nabla \rho(y - q(\bar{\tau})) \cdot \dot{q}(\bar{\tau})(1 - \omega \cdot \dot{q}(\bar{\tau})).$$

Then (3.9) implies

$$\begin{aligned} \int dy\, \nabla \rho(y - q(\bar{\tau})) \cdot \dot{q}(\bar{\tau}) &= \int dy\, \nabla_y \rho(y - q(\bar{\tau})) \cdot \dot{q}(\bar{\tau}) \frac{1}{1 - \omega \cdot \dot{q}(\bar{\tau})} \\ &= - \int dy\, \rho(y - q(\bar{\tau})) \sum_{j=1}^{2} \frac{\partial}{\partial y^j} \frac{\dot{q}^j(\bar{\tau})}{1 - \omega \cdot \dot{q}(\bar{\tau})}. \end{aligned} \tag{3.11}$$

Differentiating, we get

$$\sum_{j=1}^{2} \frac{\partial}{\partial y^j} \frac{\dot{q}^j}{1 - \omega \cdot \dot{q}} = \frac{\omega \cdot \ddot{q}}{(1 - \omega \cdot \dot{q})^2}. \tag{3.12}$$

Then, obviously, (3.7) agrees with (3.10). □

Denote $(\varphi_K(t), \pi_K(t)) := W(t)[(\varphi_0, \pi_0)]$, where $\varphi_K(x, t)$ is the Kirchhoff integral

$$\varphi_K(x, t) = \frac{1}{4\pi t} \int_{S_t(x)} d^2 y\, \pi_0(y) + \frac{\partial}{\partial t} \Big(\frac{1}{4\pi t} \int_{S_t(x)} d^2 y\, \varphi_0(y) \Big), \tag{3.13}$$

and $\pi_K(x, t) = \dot{\varphi}_K(x, t)$. Here $S_t(x)$ denotes the sphere $\{y : |y - x| = t\}$ and $d^2 y$ is the corresponding surface area element. Below we shall use the following lemma.

Lemma 3.3. *Suppose* (φ_0, π_0) *satisfies* (1.14) *and* (1.15). *Then there exists $I_0 < \infty$ such that for every $R > 0$ and every $T > T_0 \geq 0$ we have*

$$\int_{R+T_0}^{R+T} dt \int_{S_R} d^2 x \left(|\pi_K(x, t)|^2 + |\nabla \varphi_K(x, t)|^2 \right) \leq I_0. \tag{3.14}$$

Here and in what follows $S_R = S_R(0)$.

Proof. Formula (3.13) implies

$$\varphi_K(x, t) = \frac{t}{4\pi} \int_{S_1} d^2 z\, \pi_0(x + tz) + \frac{1}{4\pi} \int_{S_1} d^2 z\, \varphi_0(x + tz) + \frac{t}{4\pi} \int_{S_1} d^2 z\, \nabla \varphi_0(x + tz) \cdot z.$$

Therefore,

$$\nabla \varphi_K(x, t) = \frac{t}{4\pi} \int_{S_1} d^2 z\, \nabla \pi_0(x + tz) + \frac{1}{4\pi} \int_{S_1} d^2 z\, \nabla \varphi_0(x + tz) + \frac{t}{4\pi} \int_{S_1} d^2 z\, \nabla_x (\nabla \varphi_0(x + tz) \cdot z).$$

Here all derivatives are understood in the classical sense. A similar representation is true for $\pi_K(x,t)$. Hence, taking the assumption (1.15) into account, we obtain

$$|\pi_K(x,t)| + |\nabla\varphi_K(x,t)| \le C\sum_{s=0}^{1} t^s \int_{S_1} d^2z\,|x+tz|^{-\sigma-s}, \quad \sigma > 3/2. \tag{3.15}$$

Next, for $\sigma \ne 2$ we have

$$\int_{S_1} d^2z\,|x+tz|^{-\sigma-s} = \frac{2\pi}{(\sigma+s-2)|x|t}\big((t-|x|)^{2-\sigma-s} - (t+|x|)^{2-\sigma-s}\big), \quad s=0,1.$$

Therefore,

$$\begin{aligned}
&\int_{R+T_0}^{R+T} dt \int_{S_R} d^2x\,\big(|\pi_K(x,t)|^2 + |\nabla\varphi_K(x,t)|^2\big)\\
&\quad\le C\int_{R+T_0}^{R+T}\left[\frac{(t+R)^{4-2\sigma}+(t-R)^{4-2\sigma}}{t^2} + (t-R)^{2-2\sigma}\right]dt\\
&\quad\le C_1\int_{R+T_0}^{R+T} dt\left[\Big(1+\frac{R}{t}\Big)^2 + \Big(1-\frac{R}{t}\Big)^2 + 1\right](t-R)^{2-2\sigma} < \infty. \qquad \square
\end{aligned}$$

§4. Scattering of energy to infinity

In this section we establish a lower bound on the total energy radiated to infinity in terms of a "radiation integral". Since the energy is bounded *a priori*, this integral must be finite, which is then our main input for proving Theorem 2.3.

Proposition 4.1. *Under conditions* (1.9), (1.10), (1.14), (1.15), *let*

$$Y(t) = \big(\varphi(t), \pi(t), q(t), p(t)\big) \in C(\mathbb{R}, \mathcal{E})$$

be the solution of (1.2) *with initial data* $Y(0) = (\varphi_0, \pi_0, q_0, p_0)$. *Then*

$$\int_0^\infty dt \int_{S_1} d^2\,\omega|\overline{\pi}(\omega,t)|^2 < \infty. \tag{4.1}$$

Proof. Step i). The energy $\mathcal{H}_R(t)$ at time $t\in\mathbb{R}$ in the ball B_R with radius $R > \bar{q}_0 + R_\rho$ is defined by

(4.2)
$$\mathcal{H}_R(t) = \frac{1}{2}\int_{B_R} dx\,\big(|\pi(x,t)|^2 + |\nabla\varphi(x,t)|^2\big) + \frac{1}{2}p^2(t) + V(q(t)) + \int_{\mathbb{R}^3} dx\,\varphi(x,t)\rho(x-q(t)).$$

We fix $R>0$ and consider the total radiated energy $\mathcal{H}_R(R+T_0) - \mathcal{H}_R(R+T)$ from the ball B_R during the time interval $[R+T_0, R+T]$, where $T > T_0 \ge 0$. This quantity is bounded *a priori*, because $\mathcal{H}_R(R+T_0)$ and $\mathcal{H}_R(R+T)$ are bounded by (2.6). Hence,

$$\mathcal{H}_R(R+T_0) - \mathcal{H}_R(R+T) \le I < \infty, \tag{4.3}$$

where I does not depend on T_0, T, and R.

Step ii). Note that the function $\varphi(x,t) = \varphi_r(x,t) + \varphi_K(x,t)$ is C^1-differentiable in the region $t > |x|$ by (1.15), (3.1), and (3.13). Hence, differentiating (4.2) with respect to t and integrating by parts, we get

$$\frac{d}{dt}\mathcal{H}_R(t) = \int_{S_R} d^2x\,\omega(x)\cdot\pi(x,t)\nabla\varphi(x,t), \quad t > R. \tag{4.4}$$

Now (4.4) and (4.3) imply

$$-\int_{R+T_0}^{R+T} dt \int_{S_R} d^2x\,\omega(x)\cdot\pi(x,t)\nabla\varphi(x,t) \le I. \tag{4.5}$$

Step iii). We check that (4.5) leads to (4.1) in the limits as $R \to \infty$ and then $T \to \infty$. Indeed, substituting

$$\pi = \pi_r + \pi_K, \quad \varphi = \varphi_r + \varphi_K \tag{4.6}$$

in (4.5), we obtain

$$-\int_{R+T_0}^{R+T} dt \int_{S_R} d^2x\, \omega(x) \cdot (\pi_r \nabla\varphi_r + \pi_K \nabla\varphi_r + \pi_r \nabla\varphi_K + \pi_K \nabla\varphi_K) \le I. \tag{4.7}$$

Then Lemmas 3.1 and 3.3 show that, for every fixed $T > T_0 := T_r$,

$$\int_{T_r}^{T} dt \int_{S_1} d^2\omega\, |\overline{\pi}(\omega, t)|^2 \le I_1 + T\mathcal{O}(R^{-1}), \tag{4.8}$$

where $I_1 < \infty$ does not depend on T and R. This follows by the Cauchy–Schwarz inequality. Taking the limit as $R \to \infty$ and then $T \to \infty$, we obtain (4.1). □

§5. Relaxation of the particle acceleration and velocity

In this section we deduce the relaxation $\dot q(t) \to 0$, $\ddot q(t) \to 0$ as $t \to \infty$ using Proposition 4.1. First, the function

$$\overline{\pi}(\omega, t) = \frac{1}{4\pi} \int dy\, \rho(y - q(t + \omega \cdot y)) \frac{\omega \cdot \ddot q(t + \omega \cdot y)}{(1 - \omega \cdot \dot q(t + \omega \cdot y))^2} \tag{5.1}$$

is globally Lipschitz continuous in ω and t for $|\omega^3| \ge \Theta$ due to (3.9) and the bounds (2.7) with $k = 2, 3$. Hence, Proposition 4.1 implies that

$$\lim_{t\to\infty} \overline{\pi}(\omega, t) = 0 \tag{5.2}$$

uniformly in $\omega \in \Omega(\Theta) := \{\omega \in S_1 : |\omega^3| \ge \Theta\}$. Denote $r(t) = \omega \cdot q(t) \in \mathbb{R}$, $s = \omega \cdot y$, and $\rho_a(q^3) = \int dq^1\, dq^2\, \rho(q^1, q^2, q^3)$, and decompose the y-integration in (5.1) along and transversal to ω. Then

$$\begin{aligned} \overline{\pi}(\omega, t) &= \int ds\, \rho_a(s - r(t+s)) \frac{\ddot r(t+s)}{(1 - \dot r(t+s))^2} \\ &= \int d\tau\, \rho_a(t - (\tau - r(\tau))) \frac{\ddot r(\tau)}{(1 - \dot r(\tau))^2} \\ &= \int d\theta\, \rho_a(t - \theta) g_\omega(\theta) = \rho_a * g_\omega(t), \quad \omega \in \Omega(\Theta). \end{aligned} \tag{5.3}$$

Here we have substituted $\theta = \theta(\tau) = \tau - r(\tau)$, which is a nondegenerate diffeomorphism because $\dot r \le \overline{r} < 1$ by (3.9), and we set

$$g_\omega(\theta) = \frac{\ddot r(\tau(\theta))}{(1 - \dot r(\tau(\theta)))^3}, \quad \omega \in \Omega(\Theta). \tag{5.4}$$

Now we extend $q(t)$ smoothly to zero for $t < 0$. Then $\widetilde{\rho} * g_\omega(t)$ is defined for all t and agrees with $\overline{\pi}(\omega, t)$ for sufficiently large t. Hence, (5.2) reads as a convolution limit,

$$\lim_{t\to\infty} \rho_a * g_\omega(t) = 0, \quad \omega \in \Omega(\Theta). \tag{5.5}$$

Now note that inequalities (2.7) with $k = 2, 3$ show that $g_\omega'(\theta)$ is bounded. Hence, by Pitt's extension of Wiener's Tauberian theorem, cf. [25, Theorem 9.7(b)], (5.5) and (1.11) imply that

$$\lim_{\theta\to\infty} g_\omega(\theta) = 0, \quad \omega \in \Omega(\Theta). \tag{5.6}$$

Lemma 5.1. *Under conditions* (1.9)–(1.12) *and* (1.14)–(1.15)*, let* $Y(t) \in \mathcal{E}$ *be the corresponding solution of the Cauchy problem* (2.1). *Then*

$$\lim_{t\to\infty} \ddot{q}(t) = 0. \tag{5.7}$$

Proof. We have (5.6) for any $\omega \in S_1$ with $|\omega^3| \geq \Theta$ (see (3.8)). Moreover, $\theta(t) \to \infty$ as $t \to \infty$. Hence, $\ddot{r}(t) = \omega \cdot \ddot{q}(t) \to 0$ as $t \to \infty$ for any $\omega \in \Omega(\Theta)$. □

Remarks 5.2.

(i) For a point charge $\rho(x) = \delta(x)$ we have $\rho_a(s) = \delta(s)$. Hence, (5.5) implies (5.6) directly, without application of the Wiener Tauberian Theorem.

(ii) Condition (1.11) is necessary for the implication (5.6)⇒(5.7). Indeed, if (1.11) is violated, then $\widehat{\rho}_a(\xi) = 0$ for some $\xi \in \mathbb{R}$, and with the choice $g(\theta) = \exp(i\xi\theta)$ we have $\rho_a * g(t) = 0$, whereas g does not decay to zero.

Corollary 5.3. *Under the assumptions of Lemma* 5.1*, we have*

$$\lim_{t\to\infty} \dot{q}(t) = 0. \tag{5.8}$$

Proof. (5.7) implies (5.8) because $|q(t)| \leq \bar{q}_0$ by (2.7) with $k = 0$. □

§6. Transitions to stationary states

Here we prove our main Theorem 2.3. First, we show that the set

$$\mathcal{A} = \{S_q \,:\, q = (q^1, q^2, 0) \in \mathbb{R}^3,\ |q| \leq \bar{q}_0\} \tag{6.1}$$

is an attracting subset. It is compact in $\mathcal{E}_F$ because $\mathcal{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$.

Lemma 6.1. *Under the assumptions of Theorem* 2.3*, we have*

$$Y(t) \xrightarrow{\mathcal{E}}_F \mathcal{A}, \quad t \to \pm\infty. \tag{6.2}$$

Proof. It suffices to verify that, for every $R > 0$,

$$\|Y(t) - S_{q(t)}\|_R = \|\varphi(t) - s_{q(t)}\|_{H^1(B_R)} + \|\pi(t)\|_{L^2(B_R)} + |p(t)| \to 0 \quad \text{as } t \to \infty. \tag{6.3}$$

We estimate each term separately.

i) The convergence (5.8) shows that $|p(t)| \to 0$ as $t \to \infty$.

ii) The integral representation (3.4) implies that for $|x| < R$ and $t > R + T_r$, $T_r = \bar{q}_0 + R_\rho$, we have

$$|\pi_r(x,t)| \leq C \max_{\tau\in[t-R-T_r,t]} |\dot{q}(\tau)| \int_{|y|<T_r} dy\, \frac{1}{|x-y|} |\nabla\rho(y - q(t - |x-y|))|.$$

Here the integral is bounded uniformly in $t > R + T_r$ for $x \in B_R$, and therefore (5.8) implies that $\|\pi_r(t)\|_{L^2(B_R)} \to 0$ as $t \to \infty$. Hence, $\|\pi(t)\|_{L^2(B_R)} \to 0$ by (4.6) and (3.15).

iii) For $t > R + T_r$ and $|x| < R$ the integral representation (3.1) implies that

$$\varphi_r(x,t) - s_{q(t)}(x) = -\int_{|y|<T_r} dy\, \frac{1}{4\pi|x-y|} \big(\rho(y - q(t - |x-y|)) - \rho(y - q(t))\big).$$

The difference $q(t-|x-y|) - q(t)$ may be written as an integral depending only on $\dot{q}(\tau)$ for $\tau \in [t - R - T_r, t]$, which tends to zero as $t \to \infty$ uniformly in $x \in B_R$ due to (5.8). Hence, $\|\varphi_r(t) - s_{q(t)}\|_{L^2(B_R)} \to 0$ as $t \to \infty$. Then $\|\varphi(t) - s_{q(t)}\|_{L^2(B_R)} \to 0$ by (4.6) and (3.15). This proves the claim, because $\|\nabla(\varphi(t) - s_{q(t)})\|_{L^2(B_R)}$ can be estimated in a similar way. □

Now we prove the convergences (2.8).

Lemma 6.2. *Under the assumptions of Theorem* 2.3, *we have*

$$Y(t) \xrightarrow{\mathcal{E}}_F \mathcal{S}, \quad t \to \pm\infty. \tag{6.4}$$

Proof. Lemma 6.1 implies that the orbit $O(Y) := \{Y(t) : t \in \mathbb{R}\}$ is precompact in $\mathcal{E}_F$ because $\mathcal{A}$ is a compact set in $\mathcal{E}_F$. Let Ω be the set of all omega-limit points of the orbit in $\mathcal{E}_F$, namely, $\bar{Y} \in \Omega$ means by definition that

$$Y(t_k) \xrightarrow{\mathcal{E}}_F \bar{Y}, \quad t_k \to \infty. \tag{6.5}$$

It suffices to prove that $\Omega \subset \mathcal{S}$, i.e., that any omega-limit point $\bar{Y}$ coincides with S_{q_+} for some $q_+ \in Z$.

First, Lemma 6.1 implies $\bar{Y} \in \mathcal{A}$. Next, Ω is invariant with respect to the dynamical group $U(t)$ with $t \in \mathbb{R}$ by the continuity of $U(t)$ in $\mathcal{E}_F$. Hence, there exists a C^2-curve $t \mapsto Q(t) \in \mathbb{R}^3$ such that $U(t)\bar{Y} = S_{Q(t)}$, in accordance with Definition (6.1). However, for $S_{Q(t)}$ to be a solution of (1.3) we must have $\dot{Q}(t) \equiv 0$, whence $Q(t) \equiv q_+ \in Z$. Therefore, $\bar{Y} = S_{q_+} \in \mathcal{S}$. □

Finally, we formalize the implication (2.8) ⇒ (2.9) by the following definition. Let $\mathcal{T}$ be a subset of a metrizable space $\mathcal{F}$.

Definition 6.3. $\mathcal{T}$ is a trapping set in $\mathcal{F}$ if for every continuous curve $Y(t) \in C(\mathbb{R}, \mathcal{F})$ with a precompact orbit $O(Y)$ the convergence $Y(t) \xrightarrow{\mathcal{F}} \mathcal{T}$ as $t \to \infty$ implies the convergence $Y(t) \xrightarrow{\mathcal{F}} T$ as $t \to \infty$ to some point $T \in \mathcal{T}$.

For example, every discrete set in $\mathbb{R}^3$ is a trapping set in $\mathbb{R}^3$.

Lemma 6.4. *Under the assumptions of Theorem* 2.3, *let Z be a trapping set in* $\mathbb{R}^3$. *Then there exist stationary states $S^\pm \in \mathcal{S}$ depending on Y_0 such that* (2.9) *holds true.*

Proof. The set Z is the image of the set $\mathcal{S}$ under the map $I\colon (\varphi, \pi, q, p) \mapsto q$. This map is continuous $\mathcal{E}_F \to \mathbb{R}^3$, and it is injective on $\mathcal{S}$. Therefore, $\mathcal{S}$ is a trapping set in $\mathcal{E}_F$, because Z is a trapping set in $\mathbb{R}^3$. Hence (2.8) implies (2.9). □

§7. Linearization at a stationary state

In the rest of the paper we prove Theorem 2.4. If the particle is close to a stable minimum of V, we expect the nonlinear evolution to be dominated by the linearized dynamics. In this case the rate of the convergence (2.9) corresponds to the decay rate of initial fields.

For notational simplicity, we assume isotropy in the following sense:

$$\partial_i \partial_j V(q_+) = \nu_0^2 \delta_{ij}, \quad i, j = 1, 2, 3, \quad \nu_0 > 0. \tag{7.1}$$

Without loss of generality we take $q_+ = 0$. Let $S_0 = (s_0, 0, 0, 0)$ be the stationary state of (1.3) corresponding to $q_+ = 0$. To linearize (1.3) at S_0, we set $\varphi(x,t) = s_0(x) + \psi(x,t)$. Then (1.3) becomes

$$\left.\begin{aligned} &\dot{\psi}(x,t) = \pi(x,t), \quad \dot{\pi}(x,t) = \Delta\psi(x,t) + \rho(x) - \rho(x - q(t)), \\ &\dot{q}(t) = p(t), \quad \dot{p}(t) = -\nabla V(q(t)) + \int d^3x\, \psi(x,t)\, \nabla\rho(x - q(t)) \\ &\qquad\qquad + \int d^3x\, s_0(x)[\nabla\rho(x - q(t)) - \nabla\rho(x)]. \end{aligned}\right| \tag{7.2}$$

We denote $X(t) = Y(t) - S_0 = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$ and rewrite the nonlinear system (7.2) in the form

$$\dot{X}(t) = AX(t) + B(X(t)). \tag{7.3}$$

Here the linear operator A reads

$$A\colon (\psi,\pi,q,p)\mapsto \Big(\pi, \Delta\psi+\nabla\rho\cdot q, p, -(\nu_0^2+\nu_1^2)q+\int d^3x\,\psi(x)\nabla\rho(x)\Big),$$

with

$$\nu_1^2\delta_{ij}=\frac{1}{3}\,\|\rho\|_{L^2}^2\delta_{ij}=-\int d^3x\,\partial_i s_0(x)\partial_j\rho(x). \tag{7.4}$$

The factor 1/3 is due to the spherical symmetry of $\rho(x)$ (see (1.10)). The nonlinear part is given by

$$\begin{aligned}B(X)=\Big(0,\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q,\ 0,&\\ -\nabla V(q)+\nu_0^2q+\int d^3x\,\psi(x)[\nabla\rho(x-q)-\nabla\rho(x)]&\\ +\int d^3x\,\nabla s_0(x)[\rho(x)-\rho(x-q)-\nabla\rho(x)\cdot q]\Big).&\end{aligned} \tag{7.5}$$

Consider the Cauchy problem for the linear equation

$$\dot Z(t)=AZ(t),\quad Z=(\Psi,\Pi,Q,P),\quad t\in\mathbb{R}, \tag{7.6}$$

with the initial condition

$$Z|_{t=0}=Z_0=(\Psi_0,\Pi_0,Q_0,P_0). \tag{7.7}$$

System (7.6) is a formal Hamiltonian system with the quadratic Hamiltonian

$$\mathcal{H}_0(Z)=\frac{1}{2}\Big(P^2+(\nu_0^2+\nu_1^2)Q^2+\int d^3x\,(|\Pi(x)|^2+|\nabla\Psi(x)|^2-2\Psi(x)\nabla\rho(x)\cdot Q)\Big), \tag{7.8}$$

which is the formal Taylor expansion of $\mathcal{H}(Y_0+Z)$ up to the second order at $Z=0$.

Lemma 7.1. *Suppose that condition* (1.10) *is fulfilled and* $Z_0\in\mathcal{E}$. *Then*:
(i) *The Cauchy problem* (7.6), (7.7) *has a unique solution* $Z(t)\in C(\mathbb{R},\mathcal{E})$.
(ii) *For every* t, *the map* $U_0(t)\colon Z_0\mapsto Z(t)$ *is continuous both on* $\mathcal{E}$ *and* $\mathcal{E}_F$.
(iii) *The energy* $\mathcal{H}_0$ *is conserved, i.e.,*

$$\mathcal{H}_0(Z(t))=\mathcal{H}_0(Z_0),\quad t\in\mathbb{R}. \tag{7.9}$$

iv) *The estimate*

$$\|Z(t)\|_{\mathcal{E}}\le C,\quad t\in\mathbb{R}, \tag{7.10}$$

is true with C *depending only on the norm* $\|Z_0\|_{\mathcal{E}}$ *of the initial state.*

The key role in the proof is played by the positivity of the Hamiltonian (7.8):

$$2\mathcal{H}_0(Z)=P^2+\nu_0^2Q^2+\int d^3x\,(|\Pi(x)|^2+|\nabla\Psi(x)+\rho(x)Q)|^2\ge 0.$$

Thus, (7.10) follows from (7.9) because $\nu_0>0$. The positivity of $\mathcal{H}_0$ is also obvious from (1.4).

In [21], we proved the following long-time decay of the linearized dynamics in the weighted Sobolev norms.

Proposition 7.2. *Under conditions* (1.10)–(1.11), *let* $Z_0\in\mathcal{E}_\alpha$ *with some* $\alpha>1$. *Then*

$$\|U_0(t)Z_0\|_{-\alpha}\le C(\rho,\alpha)(1+|t|)^{-\alpha}\|Z_0\|_\alpha,\quad t\in\mathbb{R}. \tag{7.11}$$

Similar decay also holds for the dynamical group $W(t)$ of 3D free wave equation.

Proposition 7.3 (cf. [24, Proposition 2.1] and [19])**.** *Let* $(\varphi_0, \pi_0) \in \mathcal{F}_\alpha$ *with some* $\alpha > 1$. *Then*

$$\|W(t)(\varphi_0, \pi_0)\|_{-\alpha} \le C(\alpha)(1+|t|)^{-\alpha}\|(\varphi_0, \pi_0)\|_\alpha, \quad t \in \mathbb{R}. \tag{7.12}$$

We will use these two decays in the next section.

§8. A nonlinear Huygens principle

The following lemma is a version of the strong Huygens principle for the nonlinear system (1.3). Let M_* be a fixed number, $M_* > 3R_\rho + 1$.

Lemma 8.1. *Under the assumptions of Theorem* 2.4, *let* $\delta > 0$ *be an arbitrary fixed number. Then for sufficiently large* $t_* > 0$ *system* (1.3) *admits a solution*

$$Y_*(t) = (\varphi_*(x,t), \pi_*(x,t), q_*(t), p_*(t)) \in C([t_*, \infty), \mathcal{E})$$

such that:

(i) $Y_*(t)$ *coincides with* $Y(t)$ *in some future cone,*

$$\begin{aligned} \varphi_*(x,t) &= \varphi(x,t) \quad \textit{for} \quad |x| < t - t_*, \\ q_*(t) &= q(t) \quad \textit{for} \quad t > t_*; \end{aligned} \tag{8.1}$$

(ii) $Y_*(t_*)$ *admits a decomposition* $Y_*(t_*) = S_0 + K_0 + Z_0$, *where* $Z_0 = (\Psi_0, \Pi_0, Q_0, P_0)$ *satisfies*

$$\Psi_0(x) = \Pi_0(x) = 0 \quad \textit{for} \quad |x| \ge M_*, \tag{8.2}$$

$$\|Z_0\|_\alpha \le \delta, \tag{8.3}$$

and K_0 *satisfies*

$$\|U_0(\tau)K_0\|_{-\alpha} \le C(1 + t_* + \tau)^{-\alpha}, \quad \tau > 0, \tag{8.4}$$

where $C = C(\alpha)$ *does not depend on* δ.

Proof. The convergence (2.10) with $q_+ = 0$ implies that for every $\epsilon > 0$ there exist t_ϵ such that

$$|q(t)| + |\dot q(t)| < \epsilon \quad \text{for} \quad t > t_\epsilon. \tag{8.5}$$

We may assume that $t_\epsilon > 1/\epsilon$. Denote

$$t_{0,\epsilon} = t_\epsilon + R_\rho, \quad t_{1,\epsilon} = t_{0,\epsilon} + 1, \quad t_{2,\epsilon} = t_{1,\epsilon} + \epsilon + R_\rho, \quad t_{3,\epsilon} = t_{2,\epsilon} + \epsilon + R_\rho. \tag{8.6}$$

Then there exist a function $q_\epsilon(\cdot) \in C^1(\mathbb{R})$ such that

$$q_\epsilon(t) = \begin{cases} q(t) & \text{if} \ \ t > t_{1,\epsilon}, \\ 0 & \text{if} \ \ t < t_{0,\epsilon}, \end{cases} \quad \text{and} \quad |q_\epsilon(t)| + |\dot q_\epsilon(t)| < \epsilon \quad \text{for all} \quad t \in \mathbb{R}, \tag{8.7}$$

by suitable interpolation. Now we define a modification $\varphi_\varepsilon(x,t)$ of the solution $\varphi(x,t) = \varphi_r(x,t) + \varphi_K(x,t)$:

$$\varphi_\epsilon(x,t) = \varphi_{r,\varepsilon}(x,t) + \varphi_K(x,t) \quad \text{for} \quad x \in \mathbb{R}^3 \quad \text{and} \quad t > 0, \tag{8.8}$$

where

$$\varphi_{r,\epsilon}(x,t) = -\int d^3y \, \frac{1}{4\pi|x-y|}\rho(y - q_\epsilon(t - |x-y|)). \tag{8.9}$$

For $|x| < t - t_{2,\epsilon}$ and $|y| \le R_\rho + \epsilon$, we have

$$t - |x-y| > t - (|x| + |y|) > t - (t - t_{2,\epsilon} + R_\rho + \epsilon) = t_{1,\epsilon}.$$

Then (8.9), (3.1), and (8.7) imply

$$\varphi_{r,\epsilon}(x,t) = \varphi_r(x,t) \quad \text{for} \quad |x| < t - t_{2,\epsilon}. \tag{8.10}$$

Next, for $|x| > t - t_\epsilon$ and $|y| \le R_\rho$ we obtain

$$t - |x-y| < t - (|x| - |y|) < t - (t - t_\epsilon - R_\rho) = t_{0,\epsilon}.$$

Then $q_\epsilon(t - |x-y|) = 0$ by (8.7), and hence

$$\varphi_{r,\epsilon}(x,t) = s_0(x) \quad \text{for} \quad |x| > t - t_\epsilon. \tag{8.11}$$

Moreover, $\varphi_{r,\epsilon}(\cdot,\cdot) \in C^1(\mathbb{R}^4)$, and (8.7) implies

$$\sup_{\substack{x\in\mathbb{R}^3 \\ t\in\mathbb{R}}} (|\dot\varphi_{r,\epsilon}(x,t)| + |\nabla\varphi_{r,\epsilon}(x,t) - \nabla s_0(x)| + |\varphi_{r,\epsilon}(x,t) - s_0(x)|) = \mathcal{O}(\epsilon). \tag{8.12}$$

Now we define $t_* := t_{3,\epsilon}$, and

$$\begin{aligned} Y_*(t) &= (\varphi_\epsilon(t), \dot\varphi_\epsilon(t), q(t), p(t)), \\ K_0 &= (\varphi_K(t_*), \dot\varphi_K(t_*), 0, 0), \\ Z_0 &= (\varphi_{r,\epsilon}(t_*) - s_0, \dot\varphi_{r,\epsilon}(t_*), q(t_*), p(t_*)). \end{aligned} \tag{8.13}$$

It is easy to check that t_* and $Y_*(t)$, K_0, and Z_0 satisfy all requirements of Lemma 8.1, provided $\epsilon > 0$ is sufficiently small.

First, $Y_*(x,t)$ is a solution to (1.3) for $t > t_*$. Indeed, for $|x| < \epsilon + R_\rho$ one has $t - |x-y| > t_{3,\epsilon} - 2\epsilon - 2R_\rho = t_{1,\epsilon}$. Since (8.6) implies that $q_\epsilon(t-|x-y|) = q(t-|x-y|)$, we have $\varphi_\epsilon(x,t) = \varphi(x,t)$ then. Hence, $Y_*(t) = Y(t)$ in the region $|x| < \epsilon + R_\rho$.

On the other hand, (8.5) and (8.7) imply

$$\rho(x - q_\epsilon(t)) = \rho(x - q(t)) = 0 \quad \text{for} \quad |x| > \epsilon + R_\rho \quad \text{and} \quad t > t_\epsilon.$$

Hence, $\varphi_{r,\epsilon}(x,t)$ satisfies the equation

$$\ddot\varphi(x,t) = \Delta\varphi(x,t) \quad \text{for} \quad |x| > \epsilon + R_\rho \quad \text{and} \quad t > t_\epsilon. \tag{8.14}$$

Therefore, $Y_*(t)$ satisfies (1.3) in the region $|x| > \epsilon + R_\rho$. Now (8.1) follows from (8.7) and (8.10), (8.2) for $M_* = 3R_\rho + 2\epsilon + 1$ follows from (8.11), and (8.3) follows from (8.2) and (8.12).

It remains to prove (8.4). We deduce this estimate from the decay (7.12) for the linearized dynamics $U(t)$ and the decay (7.11) for $W(t)$. Denote

$$U(\tau)K_0 = (\Psi(x,\tau), \Pi(x,\tau), Q(\tau), P(\tau)).$$

From [21, formulas (4.18), (4.19), (4.25)] it follows that

$$\begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} = \mathcal{L} * \begin{pmatrix} 0 \\ f_k \end{pmatrix} (\tau)$$

where

$$f_k(\tau) = \langle W(\tau)[\phi_k(t_*), \dot\phi_k(t_*)], \nabla\rho\rangle = \langle W(\tau + t_*)[\phi_0, \pi_0], \nabla\rho\rangle,$$

and

$$(\Psi(\tau), \Pi(\tau)) = W(\tau + t_*)[\phi_0, \pi_0] + \int_0^\tau W(\tau - s)[0, Q(s)\cdot\nabla\rho]\, ds.$$

Moreover, in accordance with [21, formula (4.20)], for $\mathcal{L}(t)$ we have the decay

$$\mathcal{L}(t) = \mathcal{O}(|t|)^{-N}, \quad t \to \infty, \quad \text{for all} \quad N > 0.$$

Then (8.4) follows. □

§9. The rate of convergence

Here we prove Theorem 2.4 i). Due to (8.1), it suffices to prove that for any $\varepsilon > 0$ we have

$$\|Y_*(t) - S_0\|_{-\alpha} = \mathcal{O}(t^{-\alpha+\varepsilon}), \quad t \to \infty. \tag{9.1}$$

Denote $X(\tau) = Y_*(t_* + \tau) - S_0$. Then $X(0) = K_0 + Z_0$, and the integrated version of (7.3) reads

$$X(\tau) = U_0(\tau)K_0 + U_0(\tau)Z_0 + \int_0^\tau ds\, U_0(\tau - s)B(X(s)), \quad \tau > 0. \tag{9.2}$$

Next, (7.11), (7.5), (8.2), and (8.4) imply

$$\begin{aligned}\|X(\tau)\|_{-\alpha} \le C\Big((t_* + \tau + 1)^{-\alpha} + (1+\tau)^{-\alpha}\|Z_0\|_\alpha \\ + \int_0^\tau ds\,(1+\tau-s)^{-\alpha}\|X(s)\|^2_{-\alpha}\Big), \quad \tau > 0.\end{aligned} \tag{9.3}$$

We fix an arbitrary $\varepsilon \in (0, 1/2)$ and introduce the majorant

$$m(t) = \sup_{0\le s\le t} (1+s)^{\alpha-\varepsilon}\|X(s)\|_{-\alpha}. \tag{9.4}$$

Let μ be any fixed positive number, and let T_μ be the exit time:

$$T_\mu = \sup\{t > 0 \,:\, m(t) \le \mu\}. \tag{9.5}$$

Multiplying the two sides of (9.3) by $(1+\tau)^{\alpha-\varepsilon}$, and taking the supremum over $\tau \in [0, T_\mu]$, we get

$$m(\tau) \le C\Big(\frac{(1+\tau)^{\alpha-\varepsilon}}{(1+t_*+\tau)^\alpha} + \delta + \int_0^\tau ds \frac{(1+\tau)^{\alpha-\varepsilon}}{(1+\tau-s)^\alpha}\frac{m^2(s)}{(1+s)^{2\alpha-2\varepsilon}}\Big), \quad \tau \le T_\mu. \tag{9.6}$$

Note that, for every $\varepsilon > 0$,

$$\sup_{\tau>0} \frac{(1+\tau)^{\alpha-\varepsilon}}{(1+t_*+\tau)^\alpha} \to 0, \quad t_* \to \infty. \tag{9.7}$$

Consequently, since $m(t)$ is a monotone increasing function, for sufficiently large t_* we get

$$m(\tau) \le C(\delta + Cm^2(\tau)), \quad \tau \le T_\mu. \tag{9.8}$$

This inequality implies that $m(\tau)$ is bounded for $\tau \le T_\mu$, and moreover,

$$m(\tau) \le C_1\delta, \quad \tau \le T, \tag{9.9}$$

if δ is sufficiently small. The constant C_1 in (9.9) does not depend on T. By Lemma 8.1, we can choose t_* so large that $\delta < \mu/(2C_1)$. Then (9.9) implies that $T = \infty$ and (9.9) holds true for all $\tau > 0$ if t_* is sufficiently large. □

§10. Scattering asymptotics

Here we prove Theorem 2.4 ii). We only prove the asymptotics (2.12)–(2.13) for $t \to +\infty$ because system (1.3) is time reversible. Denote

$$\Phi(x,t) = (\Phi_1(x,t), \Phi_2(x,t)) = (\varphi(x,t), \pi(x,t)) - (s_{q_+}, 0).$$

Then the asymptotics (2.12)–(2.13) are equivalent to

$$\Phi(t) = W(t)\Phi_+ + r(t), \quad \|r(t)\|_{\mathring{H}^1\oplus L^2} = \mathcal{O}(t^{-\alpha+1+\varepsilon}), \quad t \to +\infty.$$

This is equivalent to

$$W(-t)\Phi(t) = \Phi_+ + r_1(t), \quad \|r_1(t)\|_{\mathring{H}^1\oplus L^2} = \mathcal{O}(t^{-\alpha+1+\varepsilon}), \quad t \to +\infty, \tag{10.1}$$

due to the unitarity of $W(t)$ on $\mathring{H}^1 \oplus L^2$. The first two equations of (1.3) imply

$$\dot{\Phi}_1(x,t) = \Phi_2(x,t), \quad \dot{\Phi}_2(x,t) = \Delta\Phi_1(x,t) + \rho(x-q_+) - \rho(x-q(t)).$$

Then

$$\Phi(t) = W(t)\Phi(0) - \int_0^t W(t-s)[(0, \rho(x-q_+) - \rho(x-q(s)))]\, ds. \tag{10.2}$$

Therefore,

$$W(-t)\Phi(t) = \Phi(0) - \int_0^t W(-s)R(s)\, ds, \quad R(s) = (0, \rho(x-q_+) - \rho(x-q(s))), \tag{10.3}$$

where the integral converges in $\mathring{H}^1 \oplus L^2$ with the rate $\mathcal{O}(t^{-\alpha+1+\varepsilon})$. Indeed,

$$\|W(-s)R(s)\|_{\mathring{H}^1\oplus L^2} = \mathcal{O}(s^{-\alpha+\varepsilon}), \quad 0 < \varepsilon < \alpha - 1,$$

by the unitarity of $W(-s)$ and the decay rate $\|R(s)\|_{\mathring{H}^1\oplus L^2} = \mathcal{O}(s^{-\alpha+\varepsilon})$, which follows from conditions (1.10) on ρ and the asymptotics (2.11). Setting

$$\Phi_+ = \Phi(0) - \int_0^\infty W(-s)R(s)\, ds, \quad r_1(t) = \int_t^\infty W(-s)R(s)\, ds,$$

we obtain (10.1).

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Received 21/NOV/2016

Originally published in English