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FUNCTIONAL DIFFERENCE EQUATIONS IN THE PROBLEM ON THE FORCED OSCILLATIONS OF A FLUID IN AN INFINITE POOL WITH CONICAL BOTTOM

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Dedicated to the memory of V. S. Buslaev

Abstract. The model problem under study concerns the stationary forced oscillations of a fluid of small amplitude under the action of the field of gravity in an infinite pool with sources located on its conical bottom with infiltration. A classical solution of that problem is studied in the linear approximation. By the use of the Mellin transform and expansion in spherical functions, the problem is reduced to a set of systems of functional difference equations with meromorphic coefficients that are combinations of associated Legendre functions and their derivatives. Then, the problem on systems of difference equations reduces to singular integral equations. For this, in particular, solutions of some auxiliary first order functional equations with meromorphic coefficients are computed. It is shown that the system of integral equations in question is Fredholm with index zero. Within some assumptions, the classical solution of the problem exists and is unique. Some estimates of the classical solution in the vicinity of the conic point and at infinity are obtained.

§1. Introduction. Nonstationary problem

The object of study is a linearized model of the forced oscillations of a fluid in the gravitation field, see [1]. The movement of the fluid is assumed to be irrotational and having small amplitude of oscillations. The fluid is assumed to be incompressible, nonviscous, and such that the forces of the surface tension can be neglected. The fluid fills in the domain W (Figure 1) between the free surface F and the conical bottom B. The vertex of the cone belongs to F. The symmetry axis OZ is directed vertically downwards along the vector of the gravity force.

The fluid movement is described by the velocity potential $U(X,Y,Z,t)$, where t is time and X,Y,Z are the Cartesian coordinates (as in Figure 1); the velocity $\vec{V}$ of movement is related to the potential by the formula

$$\vec{V}(X,Y,Z,t) = \nabla U(X,Y,Z,t).$$

The velocity potential satisfies the Laplace equation in W, see [1]:

$$\Delta U(X,Y,Z,t) = 0,$$

where $\Delta = \partial_X^2 + \partial_Y^2 + \partial_Z^2, \quad t > 0$, and the dynamic boundary condition on the free surface F ($Z = 0$, $X \neq 0$, $Y \neq 0$):

$$U_{tt} - g\,U_Z = 0, \quad t > 0,$$

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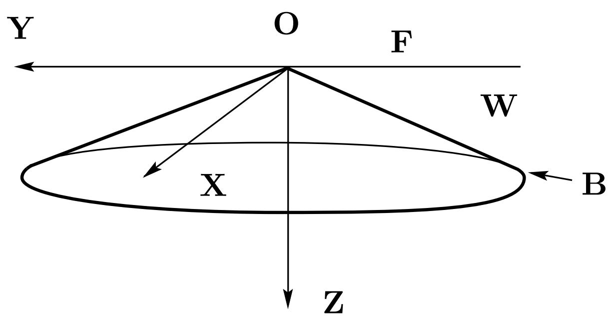

FIGURE 1. Oscillations of fluid in a pool with conic bottom

where g is the gravitational acceleration[1].

The boundary condition on the conic surface of the bottom B takes the form

$$U_n + \eta^{-1} U_t = \mathcal{F}, \quad t > 0,$$

where U_n is the derivative of the potential with respect to the normal to B directed into W and $\mathcal{F}$ is the distribution of sources on the bottom (the acting force). Next, the potential and its t-derivative are assumed to satisfy $U(X, Y, 0, 0) = 0$ and $U_t(X, Y, 0, 0) = 0$, i.e., are assumed to be zero on F at the initial moment $t = 0$ (the free surface is immovable and occupies the position of equilibrium at the initial moment). The boundary condition on the bottom requires a comment. The summand $\eta^{-1} U_t$ is responsible for the process of infiltration of the fluid across the surface of the bottom, the parameter η specifies the 'velocity' of infiltration. In practice, the quantity η^{-1} turns out to be small and the infiltration effect is negligible. In the model of rigid bottom, this term is absent, and, formally, $\eta^{-1} = 0$. Note that the density of energy is assumed to be locally integrable, also in the vicinity of the conic point.

The acting force $\mathcal{F} = \mathcal{F}(s, t)$ is equal to zero for $t < 0$, $s \in B$. Provided the source acts for $t > 0$ as

$$\mathcal{F}(s, t) = \operatorname{Re}\left\{\exp(-\mathrm{i}\Omega t)\, f(s)\right\},$$

where Ω is the constant circular frequency, it is natural to expect that, for sufficiently large t, the potential of the steady oscillations can be described by the function

$$U(X, Y, Z, t) = \operatorname{Re}\left\{\exp(-\mathrm{i}\Omega t)\, u(X, Y, Z)\right\},$$

where $u(X, Y, Z)$ is a complex-valued harmonic function describing the stationary oscillations of the fluid. Formulation and investigation of the problem for the potential $u(X, Y, Z)$, and the study of the corresponding systems of functional equations are the principal subjects in the present paper.

It should be noted that in the nonstationary formulation the problem under study has some relationship with the description of the initial stage of development of a tsunami wave, see [2], generated by sources $\mathcal{F} = \mathcal{F}(s, t)$ on the bottom, when it runs over a shallow of a conical shape in the ocean. Thus, in linear approximation, the deviation $H = H(X, Y, t)$ of the free surface from the equilibrium position $Z = 0$ is related to the potential by the formula

$$H(X, Y, t) = -g^{-1} U_t(X, Y, 0, t).$$

This quantity describes the shape of the gravitational oscillations of the surface of the fluid arising under the influence of the sources on the bottom B. Although such a model

[1]An inhomogeneous boundary condition on F could also be considered by the incorporation of an inhomogeneity term $\mathcal{H}$ into the right-hand side of the boundary condition.

seems to be oversimplified for the description of a tsunami wave, from the mathematical point of view it leads to substantial constructions. In particular, in the stationary forced oscillations problem, we meet the necessity of the study of systems of functional-difference equations with meromorphic coefficients.

A new circumstance in this work is that the meromorphic coefficients of our functional difference equations are given by rational combinations of special functions, the associated Legendre functions and their derivatives. This leads to some technical complications, which nevertheless can be overcome. We remark that, lately, functional difference equations have been arising in various problems of theoretical and mathematical physics, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], so that various new analytic and asymptotic methods are needed for their study.

The problem treated in the next section is closely related to the study of solutions of elliptic problems in a domain with piecewise smooth boundary, see [14]. However, contrary to the monograph [14], where elliptic problems in a cone with "ideal" boundary Dirichlet or Neumann conditions were considered, in this paper we deal with boundary conditions of the third kind on the boundary of the domain W, depending on the scalar parameters called surface impedances in physics. The study of the asymptotics of the problem's solution in the vicinity of the conic point and at infinity reduces to the investigation of a system of functional difference equations with meromorphic coefficients. It should be noted that precisely the a system of functional difference equations that arises in the context of the problem on forced oscillations of a fluid in a cone-shaped pool is the principal object of study in this paper. Apparently, depending on the character of the questions under study, elliptic problems in the cone-shaped domain with impedance type boundary conditions also admit investigation with the help of methods described in the monograph [14].

In §2, the problem of stationary forced oscillations of a fluid in the domain W is formulated and the uniqueness of a classical rapidly vanishing solution is proved. In §3, the Mellin transformation is exploited in order to separate the radial variable, and the problem for the Mellin transform is formulated in a layer of the unit sphere, with boundary conditions on the circles of the layer that are nonlocal with respect to the separation parameter. The Mellin transform in question is assumed to be a meromorphic function with respect to the separation parameter, and it is sought in the form of an expansion in spherical functions with undetermined coefficients. Substitution of that expansion into the boundary conditions on the boundaries of the layer leads to a system of functional difference equations for the unknown coefficients, which are meromorphic functions of the separation parameter. In its turn, the problem for the system of functional difference equations reduces further in §4, where, in particular, some auxiliary first order functional difference equations are solved, and zeros of some combinations of the associated Legendre functions and their derivatives are studied in connection with these equations. In §5, the desired solutions of the reduced functional difference equations are represented in terms of their values on the imaginary axis, and a system of singular integral equations on the imaginary axis is deduced for the unknown solutions. This system of singular integral equations is studied in §6. Its Fredholm property is proved and the index is shown to be zero. Under some conditions on the distribution of sources, the singular integral equations in question admit a unique solution in a required class, so that the classical solution of the problem also exists, which is demonstrated in the §7. For this, in particular, the behavior of the classical solution for the problem on the stationary forced oscillations is studied in the vicinity of the conic point and at infinity.

§2. The problem on stationary forced oscillations of a fluid and the uniqueness of a rapidly decaying solution

We introduce the spherical coordinates

$$X = r\cos\varphi\sin\theta, \quad Y = r\sin\varphi\sin\theta, \quad Z = r\cos\theta$$

in the domain W occupied by the fluid, $\omega = (\theta, \varphi)$,

$$W = \left\{(r, \omega) \,:\, r > 0, \ -\pi < \varphi \le \pi, \ \theta_1 < \theta < \frac{\pi}{2}\right\},$$

$\theta = \theta_1$ is the equation of the surface B of the bottom, $0 < \theta_1 < \frac{\pi}{2}$.

The potential $u(r, \theta, \varphi)$ of the stationary oscillations we are looking for satisfies the Laplace equation

$$\tag{1} \begin{gathered} \Delta u(r, \theta, \varphi) = 0, \\ \Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega, \quad \Delta_\omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \end{gathered}$$

in W, the Steklov boundary condition on the free surface F ($\theta = \frac{\pi}{2}$),

$$\tag{2} \left(\frac{1}{r}\frac{\partial}{\partial\theta} u(r, \theta, \varphi) - K\, u(r, \theta, \varphi)\right)\bigg|_{\theta = \frac{\pi}{2}} = 0,$$

where the parameter $K = \frac{\Omega^2}{g}$ is positive, and the following boundary condition on the surface of the bottom:

$$\tag{3} \left(\frac{1}{r}\frac{\partial}{\partial\theta} u(r, \theta, \varphi) - \mathrm{i}\varkappa\, u(r, \theta, \varphi)\right)\bigg|_{\theta = \theta_1} = f(r, \varphi),$$

$\varkappa = \Omega/\eta > 0$. Note that $\frac{1}{r}\frac{\partial u}{\partial\theta}\big|_{\theta = \frac{\pi}{2}} = -\frac{\partial u}{\partial Z}\big|_F$. We want to find a classical solution of (1)–(3) that is 2π-periodic with respect to φ,

$$\tag{4} u(r, \theta, \varphi) = u(r, \theta, \varphi + 2\pi),$$

and satisfies the Meixner condition at the vertex O of the conical domain W. These conditions can be written in the form

$$\tag{5} |u(r, \theta, \varphi)| \le \mathrm{const}\, r^\delta, \qquad r|\nabla u(r, \theta, \varphi)| \le \mathrm{const}\, r^\delta,$$

uniformly with respect to the angular variables for some $\delta > -\frac{1}{2}$. The function $f(r, \varphi)$ describes the sources on the bottom and is 2π-periodic in φ. The fact that this function has compact support with respect to r and φ is natural from the point of view of physical applications. Keeping this in mind, below we formulate, nevertheless, not so restrictive conditions on the set of sources f.

At infinity ($r \to \infty$), the conditions

$$\tag{6} |u(r, \theta, \varphi)| \le C\, r^{-q}, \qquad |r\nabla u(r, \theta, \varphi)| \le C\, r^{-q}, \quad q > 0,$$

are satisfied uniformly in $\omega = (\theta, \varphi) \in \Sigma$, $\Sigma = S^2 \cap W$, S^2 is the unit sphere with the center at O.[2]

We say that a classical solution is *rapidly decaying* if in (6) we have

$$\tag{7} q > 1.$$

It turns out that, provided a rapidly decaying classical solution of problem (1)–(6) exists, it is unique.

Theorem 2.1. *If a classical solution $u = u(r, \theta, \varphi)$ of the homogeneous problem* (1)–(6) *($f = 0$, $K > 0$, $\varkappa \ne 0$) is rapidly decaying in the sense of* (7)*, then $u \equiv 0$.*

[2] We call the domain Σ the 'layer on the unit sphere'.

Indeed, we employ Green's formula with integration over the domain $W_{\rho,R} = W \cap (b_R \setminus b_\rho)$, where $b_a = \{r < a\}$ is the ball of radius a with center at the origin, $R > \rho > 0$. Using the equation, we have

$$0 = \int_{W_{\rho,R}} \Delta u\, \overline{u}\, dX\, dY\, dZ = -\int_{W_{\rho,R}} |\nabla u|^2\, dX\, dY\, dZ + \int_{\partial W_{\rho,R}} \frac{\partial u}{\partial n} \overline{u}\, dS,$$

where n is the external normal to $\partial W_{\rho,R}$. The homogeneous boundary conditions on F and B yields

$$-\int_{W_{\rho,R}} |\nabla u|^2\, dX\, dY\, dZ + K \int_{F_{\rho,R}} |u|^2\, dS - \mathrm{i}\varkappa \int_{B_{\rho,R}} |u|^2\, dS \\ + \int_{\Sigma_\rho} \frac{\partial u}{\partial n} \overline{u}\, dS + \int_{\Sigma_R} \frac{\partial u}{\partial n} \overline{u}\, dS = 0,$$

where $F_{\rho,R}$ and $B_{\rho,R}$ are the parts of the surfaces F and B (respectively) that are in the interior of the sphere Σ_R and in the exterior of the sphere Σ_ρ. Letting $\rho \to 0$ and $R \to \infty$ and using the Meixner conditions (5) and conditions (6) and (7) at infinity in the last two integrals respectively, we arrive at the relation

$$\int_W |\nabla u|^2\, dX\, dY\, dZ - K \int_F |u|^2\, dS + \mathrm{i}\varkappa \int_B |u|^2\, dS = 0.$$

Separating the real and imaginary parts in this identity, we see that

$$\int_W |\nabla u|^2\, dX\, dY\, dZ = K \int_F |u|^2\, dS$$

and

$$\varkappa \int_B |u|^2\, dS = 0.$$

The latter identity implies

$$u|_B = 0,$$

and from the homogeneous boundary condition on the bottom we have

$$\left.\frac{\partial u}{\partial n}\right|_B = 0.$$

Should the surface B be internal and located in W, the uniqueness theorem for the Cauchy problem (see [15, pp. 165–166]) for an elliptic equation would imply that $u \equiv 0$ in our problem. We extend u by zero to the interior of the cone bounded by B. The function u becomes harmonic in the half-space $Z > 0$ (see [16]). Now the fact that $u \equiv 0$ follows from the result of [15] mentioned above.[3]

Remark 2.2. The restrictions imposed on the impedances $K > 0$, $\varkappa \neq 0$ are essential. If $\varkappa = 0$, then, as was shown in [12], the corresponding homogeneous problem (1)–(6) has nontrivial kernel of decaying solutions for any $K > 0$. Therefore, the impedance $\varkappa$ can be viewed as a parameter regularizing the problem.

§3. Separation of the radial variable in the layer Σ on the unit sphere

We seek the solution $u(r,\omega)$ in W in the form of the Mellin integral

$$u(r,\omega) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} u_\nu(\omega)\, r^{\nu-\frac{1}{2}}\, \mathrm{d}\nu, \tag{8}$$

[3]The author is grateful to A. A. Fedotov for his remark, which enabled us to simplify the proof.

where c is a constant. The inverse transform has the form

$$u_\nu(\omega) = \int_0^\infty u(r,\omega)\, r^{-\nu-\frac{1}{2}}\, \mathrm{d}r. \tag{9}$$

For any $\omega \in \Sigma$, the Mellin transform $u_\nu(\omega)$ is regular (holomorphic) with respect to ν in the strip $\Pi(1/2-b, 1/2+a)$, where we have used the notation $\Pi(x_1,x_2) = \{\nu \in \mathrm{C} : x_1 < \mathrm{Re}(\nu) < x_2\}$. The constants a and b specify the width of the regularity strip, and $1/2 - b < c < 1/2 + a$, where $a \geq \delta$, $b \geq q$ (see (5) and (6)). In accordance with the properties of the Mellin transformation, see [17], we obtain the estimates

$$|u(r,\theta,\varphi)| \leq C\, r^a$$

as $r \to 0$ and

$$|u(r,\theta,\varphi)| \leq C\, r^{-b}$$

as $r \to \infty$, with a constant C independent of $\omega = (\theta, \varphi)$. We assume that $u_\nu(\omega)$ is twice continuously differentiable with respect to $(\theta,\varphi) \in \Sigma$ (see Figure 2) for all regular values of ν and meromorphic with respect to ν for all $\omega \in \Sigma$. Moreover, we assume that

$$u_\nu(\omega) \to 0, \quad |\nu| \to \infty,$$

in the strip $\Pi(1/2-b, 1/2+a)$ uniformly with respect to $\omega \in \Sigma$, so that the Mellin transform and the inverse formula exist.

Substituting (8) in equation (1), we get

$$\begin{aligned}
\Delta u &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right) \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} u_\nu(\omega)\, r^{\nu-\frac{1}{2}}\, \mathrm{d}\nu \\
&= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left(\Delta_\omega u_\nu(\omega) + \left[\nu^2 - \frac{1}{4}\right] u_\nu(\omega)\right) r^{\nu-\frac{5}{2}}\, \mathrm{d}\nu = 0.
\end{aligned}$$

Obviously, whenever $u_\nu(\omega)$ satisfies the equation

$$\left(\Delta_\omega + \nu^2 - \frac{1}{4}\right) u_\nu(\omega) = 0, \tag{10}$$

$\omega \in \Sigma$, the original $u(r,\theta,\varphi)$ is a solution of the Laplace equation.

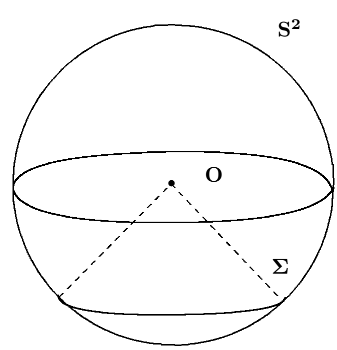

FIGURE 2. A spherical layer

We turn to the boundary conditions. Substituting the Mellin transform to the condition on the free surface, we obtain

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\frac{\partial u_\nu(\omega)}{\partial\theta}\,r^{\nu-\frac{3}{2}} - K u_\nu(\omega)\,r^{\nu-\frac{1}{2}}\right)\mathrm{d}\nu\Big|_{\theta=\frac{\pi}{2}}$$
$$= \left\{\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty-1}^{+\mathrm{i}\infty-1}\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\,r^{\nu-\frac{1}{2}}\,\mathrm{d}\nu - \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} K u_\nu(\omega)\,r^{\nu-\frac{1}{2}}\,\mathrm{d}\nu\right\}\Big|_{\theta=\frac{\pi}{2}} = 0,$$

where we have changed the integration variable $\nu-1\mapsto\nu$ in the first summand. We deform the straightlinear integration contour $(-\mathrm{i}\infty-1,+\mathrm{i}\infty-1)$ into the imaginary axis, assuming that no poles of $\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\Big|_{\theta=\frac{\pi}{2}}$ are intersected in this case. This assumption, i.e., the regularity of $\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\big|_{\theta=\frac{\pi}{2}}$ in the strip $\Pi(-\varepsilon-1,\varepsilon)$ for some small $\varepsilon>0$, is viewed as an additional constraint for the class of transforms $u_\nu(\omega)$ described above.

Therefore, the boundary condition (2)

$$\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left(\frac{\partial u_{\nu+1}(\omega)}{\partial\theta} - K u_\nu(\omega)\right)\Big|_{\theta=\frac{\pi}{2}} r^{\nu-\frac{1}{2}}\,\mathrm{d}\nu = 0,$$

is fulfilled provided

$$\left(\frac{\partial u_{\nu+1}(\omega)}{\partial\theta} - K u_\nu(\omega)\right)\Big|_{\theta=\frac{\pi}{2}} = 0. \tag{11}$$

Remark 3.1. The boundary condition (11) is nonlocal with respect to the parameter ν of separation of variables, which reflects the fact that no separation of the radial and angular variables is possible in the mixed boundary condition in the traditional meaning of this term.

Proceeding similarly with the boundary condition on the bottom, we assume the regularity of $\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\big|_{\theta=\theta_1}$ in the strip $\Pi(-\varepsilon-1,\varepsilon)$ for some small $\varepsilon>0$. We have

$$\left(\frac{\partial u_{\nu+1}(\omega)}{\partial\theta} - \mathrm{i}\varkappa\, u_\nu(\omega)\right)\Big|_{\theta=\theta_1} = \psi_\nu(\varphi). \tag{12}$$

In (12), the right-hand side $\psi_\nu(\varphi)$ is the Mellin transform of the function of source $f(r,\varphi)$, which we simply call 'source' as well.

We need to obtain a more detailed description of the class of *rapidly decaying* sources f, i.e., such that as $r\to\infty$ we have $|f(r,\varphi)|\le O(r^{-1-\varepsilon})$ uniformly in $\varphi\in(0,2\pi]$ for some small $\varepsilon>0$. Precisely such sources are considered in this paper. Let a source be represented by the integral

$$f(r,\varphi) = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\psi_\nu(\varphi)\,r^{\nu-\frac{1}{2}}\,\mathrm{d}\nu, \tag{13}$$

then the inverse transform is

$$\psi_\nu(\varphi) = \int_0^\infty f(r,\varphi)\,r^{-\nu-\frac{1}{2}}\,\mathrm{d}r. \tag{14}$$

For any $\varphi\in(0,2\pi]$, the Mellin transform $\psi_\nu(\varphi)$ of a *rapidly decaying* source is regular (holomorphic) with respect to ν in the strip $\Pi(-1/2-\varepsilon,1+\varepsilon)$, $\varepsilon>0$, satisfies the estimate[4]

$$|\psi_\nu(\varphi)|\le C\,|\nu|^{-3/2-\delta_1},\quad \delta_1>0, \tag{15}$$

[4]This restriction is related to the smoothness of f and is of technical nature: it could be relaxed should it be necessary for some applications.

in that strip uniformly in φ, and extends up to a meromorphic function on the entire complex plane. The transform is a 2π-periodic function of φ and is represented by a trigonometric polynomial

$$\psi_\nu(\varphi) = \sum_{n=-N}^{N} \mathrm{e}^{-in\varphi}\Psi_n(\nu), \quad \Psi_n(\nu) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{in\varphi}\,\psi_\nu(\varphi)\,\mathrm{d}\varphi,$$

where N is an arbitrary natural number. In the last identity, instead of the complete Fourier series, an arbitrary finite segment of it is considered, which leads to some technical simplifications,[5] but is not a serious constraint from the viewpoint of applications. As a consequence of this, the source f is also assumed to be a trigonometric polynomial in φ obeying the estimates

$$|f(r,\varphi)| \le C\, r^{1/2+\varepsilon}$$

as $r \to 0$ and

$$|f(r,\varphi)| \le C\, r^{-1-\varepsilon}$$

as $r \to \infty$ uniformly in φ. Certainly, the class of sources can be enlarged substantially, but, however, this is not our goal in this paper.

Lemma 3.2. *Suppose that the Mellin transform $u_\nu(\omega)$ satisfies the conditions mentioned above (including the regularity of $\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\big|_{\theta=\frac{\pi}{2}}$ and $\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\big|_{\theta=\theta_1}$ in the strip $\Pi(-\varepsilon-1,\varepsilon)$ for some small $\varepsilon>0$) and that the function $\psi_\nu(\varphi)$ is specified by a rapidly decaying source f. Moreover, suppose that $u_\nu(\omega)$ is a classical solution of problem* (10)–(12)*. Then the potential $u(r,\omega)$ in* (8) *is a classical solution of problem* (1)–(6).

It is well known that a linear combination of spherical functions

$$\mathrm{e}^{-in\varphi}\, P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta), \quad \mathrm{e}^{-in\varphi}\, P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta), \quad n = 0, \pm1, \pm2, \dots$$

solves equation (10), where the $P^{-|n|}_{\nu-\frac{1}{2}}(\,\cdot\,)$ are the associated Legendre functions, see [18]. A solution of problem (10)–(12) in the spherical layer Σ is naturally sought in the form

$$u_\nu(\omega) = \sum_{n=-N}^{N} \mathrm{e}^{-in\varphi}\left(A_n(\nu)\frac{P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)} + B_n(\nu)\frac{P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)}\right), \tag{16}$$

where

$$\mathrm{d}_\theta P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) := \frac{\mathrm{d}}{\mathrm{d}\theta}P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) = -\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac{1}{2}}(x)\Big|_{x=\cos\theta}.$$

The unknown coefficients $A_n(\nu)$ and $B_n(\nu)$ are to be found from the boundary conditions.

[5]In this case there is no need to take care of the uniform convergence of the Fourier series and the corresponding uniform estimates.

§4. Problems for systems of functional difference equations for $A_n(\nu)$ and $B_n(\nu)$ and solution of auxiliary equations

Direct substitution of expression (16) into the boundary conditions (11), (12) leads to the following functional difference equations $(n = 0, \pm 1, \dots, N)$[6]

$$
\begin{aligned}
&A_n(\nu+1)\frac{\mathrm{d}_\theta P^{-|n|}_{\nu+\frac12}(\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu+\frac12}(\cos\theta_1)} + B_n(\nu+1)\frac{\mathrm{d}_\theta P^{-|n|}_{\nu+\frac12}(-\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu+\frac12}(-\cos\theta_1)}\\
&\quad - K\left(A_n(\nu)\frac{P^{-|n|}_{\nu-\frac12}(\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1)} + B_n(\nu)\frac{P^{-|n|}_{\nu-\frac12}(-\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}\right) = 0,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&A_n(\nu+1) + B_n(\nu+1)\\
&\quad - \mathrm{i}\varkappa\left(A_n(\nu)\frac{P^{-|n|}_{\nu-\frac12}(\cos\theta_1)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1)} + B_n(\nu)\frac{P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}\right) = \Psi_n(\nu).
\end{aligned}
\tag{18}
$$

It turns out that, instead of equations (17) and (18), it is convenient to study equivalent equations for the unknown functions $U_n(\nu)$ and $V_n(\nu)$ related to $A_n(\nu)$ and $B_n(\nu)$ by the formulas

$$
U_n(\nu) = A_n(\nu)\,\frac{\mathrm{d}_\theta P^{-|n|}_{\nu-\frac12}(\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1)} + B_n(\nu)\,\frac{\mathrm{d}_\theta P^{-|n|}_{\nu-\frac12}(-\cos\theta)\big|_{\theta=\frac{\pi}{2}}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)},
\tag{19}
$$

$$
V_n(\nu) = A_n(\nu) + B_n(\nu).
\tag{20}
$$

The reason for introducing $U_n(\nu)$ and $V_n(\nu)$ will be clarified below. Note that, using [18], we have

$$
\begin{aligned}
P^{-|n|}_{\nu-\frac12}(0) &= \frac{2^{-|n|}\sqrt{\pi}}{\Gamma\big(\frac{\nu+|n|-\frac12}{2}+1\big)\Gamma\big(\frac{-\nu+|n|+\frac32}{2}\big)},\\
\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-\frac12}(x)\Big|_{x=0} &= \frac{2^{1-|n|}(-\sqrt{\pi})}{\Gamma\big(\frac{\nu+|n|+\frac12}{2}\big)\Gamma\big(\frac{-\nu+|n|+\frac12}{2}\big)}.
\end{aligned}
$$

The formulas for $A_n(\nu)$ and $B_n(\nu)$ in terms of $U_n(\nu)$ and $V_n(\nu)$ follow from (19), (20) and take the form

$$
A_n(\nu) = \left(U_n(\nu) - V_n(\nu)\,\frac{\mathrm{d}_x P^{-|n|}_{\nu-\frac12}(x)\big|_{x=0}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}\right)[D_n(\nu)]^{-1},
\tag{21}
$$

$$
B_n(\nu) = \left(-U_n(\nu) - V_n(\nu)\,\frac{\mathrm{d}_x P^{-|n|}_{\nu-\frac12}(x)\big|_{x=0}}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1)}\right)[D_n(\nu)]^{-1},
\tag{22}
$$

where

$$
[D_n(\nu)]^{-1} = \frac{-1}{\mathrm{d}_x P^{-|n|}_{\nu-\frac12}(x)\big|_{x=0}}\,\frac{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1)\,\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)}{\big[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac12}(-\cos\theta_1)\big]}.
$$

[6]It should be noted that, instead of a simple linear algebraic system of equations that arises traditionally in the framework of the method of variables' separation for the ideal boundary conditions, we obtain a system of functional difference equations for the calculation of $A_n(\,\cdot\,), B_n(\,\cdot\,)$ in view of the nonlocality with respect to ν of the boundary conditions (11), (12).

Note that $D_0(\nu)$ has neither zeros nor poles at the points $\nu = \pm 1/2$ and in the strip $\Pi(-1/2, 1/2)$, which can be shown with the help of formula 8.842(1) in [18], although $\mathrm{d}_{\theta_1} P_{\nu-\frac{1}{2}}(\pm\cos\theta_1)$ vanish for $\nu = \pm 1/2$, see also (6.10), (6.11) in [12].

In terms of the unknowns $U_n(\nu)$ and $V_n(\nu)$, equations (17), (18) take the form

$$U_n(\nu+1) - K\left(T_{11}(\nu)U_n(\nu) + T_{12}(\nu)V_n(\nu)\right) = 0, \tag{23}$$

$$V_n(\nu+1) - \mathrm{i}\varkappa(T_{21}(\nu)V_n(\nu) + T_{22}(\nu)V_n(\nu)) = \Psi_n(\nu), \tag{24}$$

where

$$T_{11}(\nu) = \frac{P^{-|n|}_{\nu-\frac{1}{2}}(0)}{\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}} \frac{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]},$$

$$T_{12}(\nu) = \frac{2P^{-|n|}_{\nu-\frac{1}{2}}(0)}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]},$$

$$T_{21}(\nu) = \frac{1}{\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}} \frac{\mathcal{W}_n(\nu)}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]},$$

$$T_{22}(\nu) = \frac{\left[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]},$$

$$\mathcal{W}_n(\nu) = P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\,\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1)\,\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1).$$

Let the meromorphic solutions $U_n(\nu)$ and $V_n(\nu)$ of equations (23), (24) be holomorphic in the strip $\Pi(-\varepsilon, 1+\varepsilon)$ for some small $\varepsilon > 0$. Then $U_n(\nu+1)$ and $V_n(\nu+1)$ are holomorphic with respect to ν in the strip $\Pi(-1-\varepsilon, \varepsilon)$ whence we see that the functions

$$\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\Big|_{\theta=\frac{\pi}{2}} = \sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}n\varphi} U_n(\nu+1),$$

$$\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}\Big|_{\theta=\theta_1} = \sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}n\varphi} V_n(\nu+1)$$

are regular in the same strip, which was assumed in Lemma 3.2. This explains the convenience of introducing $U_n(\nu)$ and $V_n(\nu)$.

We turn to estimating $U_n(\nu)$ and $V_n(\nu)$ at $\pm\mathrm{i}\infty$ in the strip $\Pi(-\varepsilon, 1+\varepsilon)$. We employ the fact that the classical solution $u(r,\omega)$ in (16) is continuos in F and B, which implies the absolute integrability of $u_\nu(\omega)|_{\theta=\pi/2}$ and $u_\nu(\omega)|_{\theta=\theta_1}$ for $\nu \in (-\mathrm{i}\infty, \mathrm{i}\infty)$, which is also uniform in φ. The asymptotics of the associated Legendre functions (see [18]) looks like this:

$$P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi\sin\theta}}\,\frac{\Gamma(\nu+|n|+1/2)}{\Gamma(\nu+1)}\,\cos(\nu\theta - \pi|n|/2 - \pi/4)\left[1 + O(1/|\nu|)\right],$$

$0 < \theta < \pi$, $\nu \to \pm\mathrm{i}\infty$, n is fixed arbitrary. It is easy to verify that the following estimates suffice for the integrability of $u_\nu(\omega)|_{\theta=\pi/2}$ and $u_\nu(\omega)|_{\theta=\theta_1}$ uniform in φ:

$$|A_n(\nu)| \le C\,|\nu|^{-\delta_1}|\exp(\pm\mathrm{i}\nu[\pi/2-\theta_1])|,$$

$$|B_n(\nu)| \le C\,|\nu|^{-\delta_1}, \quad \delta_1 > 0,$$

as $\nu \to \pm\mathrm{i}\infty$ in the strip $\Pi(-\varepsilon, 1+\varepsilon)$, and we also assume (see (15)) that

$$|\Psi_n(\nu)| \le C\,|\nu|^{-3/2-\delta_1}.$$

These estimates mean that

$$|U_n(\nu)| \le C\,|\nu|^{-\delta_1}, \quad |V_n(\nu)| \le C\,|\nu|^{-\delta_1}, \quad \delta_1 > 0, \tag{25}$$

as $\nu \to \pm \mathrm{i}\infty$ in the strip $\Pi(-\varepsilon, 1+\varepsilon)$.

Remark 4.1. These estimates make it possible to describe classes of functions in which the singular integral equations deduced from the problem for functional equations are to be studied.

Under the assumptions made above, the following is true.

Theorem 4.2. *Suppose that the meromorphic functions $U_n(\nu)$ and $V_n(\nu)$ satisfy the conditions mentioned above and solve the system of functional equations* (23), (24) *for $n = 0, \pm 1 \ldots, \pm N$. The Mellin transform $u_\nu(\omega)$ in* (16) *solves problem* (10)–(12).

We want to reshape system (23), (24) to integral equations. For this, as a preliminary, we simplify system (23), (24) extracting the 'principal' terms in the equations. Observe that

$$T_{11}(\nu) = O(1/\nu), \quad T_{12}(\nu) = O(\exp(\pm \mathrm{i}\nu[\pi/2 - \theta_1])/\nu),$$
$$T_{22}(\nu) = O(1/\nu), \quad T_{21}(\nu) = O(\exp(\pm \mathrm{i}\nu[\pi/2 - \theta_1])/\nu)$$

as $\nu \to \pm\mathrm{i}\infty$ along the imaginary axis. In the matrix $T = \{T_{ik}\}$, the diagonal entries are asymptotically leading. Taking this into account, we construct special solutions of the auxiliary equations

$$w_n(\nu+1) - K\,T_{11}(\nu)\,w_n(\nu) = 0, \tag{26}$$
$$v_n(\nu+1) - \mathrm{i}\varkappa\,T_{22}(\nu)\,v_n(\nu) = 0 \tag{27}$$

in order to reduce (23), (24) to the form enabling us to deduce singular integral equations with the Fredholm property. Equation (26) was studied carefully in [12], and the results are as follows.

Lemma 4.3. *If a meromorphic solution $w_n(\nu)$ of equation* (23) *is holomorphic in the strip $\Pi(-\varepsilon, 1+\varepsilon)$ and has there the asymptotics $w_n(\nu) = O(\sqrt{|\nu|})$ as $\nu \to \pm\mathrm{i}\infty$, then in this strip $w_n(\nu)$ admits the representation* $(n = 0, \pm 1 \ldots, \pm N)$

$$w_n(\nu) = \exp(\nu \log K)\, \mathrm{d}_x P^{-|n|}_{\nu - \frac{1}{2}}(x)\big|_{x=0} \Gamma(-\nu + |n| + 1/2)\, s_n(\nu - 1/2),$$

where the meromorphic function $s_n(\nu)$ is a holomorphic solution of the equation

$$\begin{gathered} s_n\left(\nu + \frac{1}{2}\right) = \frac{h_n^-(\nu)}{h_n^+(\nu)}\, s_n\left(\nu - \frac{1}{2}\right), \\ h_n^{\pm}(\nu) = \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1) \pm \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1), \end{gathered} \tag{28}$$

bounded in the strip $\Pi(-C_n - 1/2, C_n + 1/2)$, ($C_n > 1$) and having no zeros in that strip. In the strip $\Pi(-C_n - 1/2, C_n + 1/2)$ the function $s_n(\nu)$ in (28) *admits the integral representation*

$$s_n(\nu) = \exp\left\{ \frac{\pi}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Lambda_n(\xi)\,\mathrm{d}\xi}{\cos^2(\pi[\xi - \nu])} \right\},$$
$$\Lambda_n(\xi) = \int_{-\mathrm{i}\infty}^{\xi} \log\left\{ \frac{h_n^-(\tau)}{h_n^+(\tau)} \right\} \mathrm{d}\tau,$$

where the cuts specifying a regular branch of the logarithm pass from the real zeros $\nu_m^{\pm}(|n|)$ *and* $-\nu_m^{\pm}(|n|)$ $(\nu_m^{\pm}(|n|) > 0)$ *of the functions* $h_n^{\pm}(\nu)$ *at* $+\infty$ *and* $-\infty$ *respectively,*

$$\log\left\{\frac{h_n^-(\nu)}{h_n^+(\nu)}\right\} = O(\exp[-\mathrm{i}\nu(\pi - 2\theta_1)]), \quad \nu \to -\mathrm{i}\infty.$$

Turning to equation (27), we represent it in the form[7]

$$v_n(\nu+1) - \varkappa\,\frac{1}{\nu+a}\left(\mathrm{i}(\nu+a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right) v_n(\nu) = 0, \tag{29}$$

where $p_n^+(\nu) = P_{\nu-\frac12}^{-|n|}(\cos\theta_1) + P_{\nu-\frac12}^{-|n|}(-\cos\theta_1)$, $a > 0$. The reason for representing equation (29) in such a form is of technical nature and is related to the fact that, unlike $\mathrm{i}\nu\,\frac{p_n^+(\nu)}{h_n^+(\nu)}$, the factor $\left(\mathrm{i}(\nu+a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right)$ is holomorphic in the vicinity of the imaginary axis and is not zero including the infinity. We solve the equation in a desired class and then let $a \to +0$. The substitution

$$v_n(\nu) = \exp(\nu\log\varkappa)\,\frac{1}{\Gamma(\nu+a)}\,\sigma_n(\nu, a) \tag{30}$$

reduces (29) to the equation

$$\sigma_n(\nu+1, a) - \left(\mathrm{i}(\nu+a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right)\sigma_n(\nu, a) = 0, \tag{31}$$

where the coefficient has the asymptotics

$$\left(\mathrm{i}(\nu+a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right) = \pm 1 + O(1/\nu), \quad \nu \to \pm\mathrm{i}\infty.$$

Note that the parameter a is chosen so that this coefficient have no zeros on the imaginary axis.

For the construction of the desired solution of equation (31), we must study the zeros of the functions $p_n^+(\nu)$ and $h_n^+(\nu)$. The zeros $\mu_m^+(|n|)$ of the equation $h_n^+(\nu) = 0$ were studied in [12], they are simple and located on the real axis, being symmetric with respect to the origin because $h_n^+(\nu)$ is even, and admit the estimate $|\mu_m^+(|n|)| > \sqrt{\frac14 + |n|^2}$, $|n| > 0$. Observe, however, that $|\mu_m^+(0)| \geq 1/2$.

The behavior of zeros of the entire function $p_n^+(\nu)$ can be studied much as the behavior of $\mu_m^+(|n|)$ was studied in [12]. Indeed, consider the regular Sturm–Liouville problems ($|n| = 0, 1, 2, \ldots$, $x = \cos\theta$)

$$-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}Y^+(x) + \frac{|n|^2}{1-x^2}\,Y^+(x) = \lambda_\nu^+ Y^+(x), \quad x \in (0, x_1),\ \ x_1 = \cos\theta_1,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}Y^+(x)\Big|_{x=0} = 0, \qquad Y^+(x)\Big|_{x=\cos\theta_1} = 0,$$

where $Y^+(x) = P_{\nu-\frac12}^{-|n|}(-x) + P_{\nu-\frac12}^{-|n|}(x)$ and $\lambda_\nu^+(|n|) = \nu_+^2(|n|) - \frac14$ is the spectral parameter depending on $|n|$. We denote by $\mathcal{L}^+$ the differential operator in the first summand of the equation. Standard calculations lead to the expression

$$\begin{aligned}\int_0^{x_1} \mathcal{L}^+ Y^+(x)\overline{Y^+(x)}\,\mathrm{d}x &= \int_0^{x_1} (1-x^2)\left|\frac{\mathrm{d}Y^+(x)}{\mathrm{d}x}\right|^2 \mathrm{d}x \\ &= \int_0^{x_1} \left[\left(\nu_+^2 - \frac14\right)|Y^+(x)|^2 - \frac{|n|^2}{1-x^2}\,\left|Y^+(x)\right|^2\right]\mathrm{d}x.\end{aligned}$$

[7]Notice that this way to construct a solution with the desired properties has an alternative.

The spectrum of the problem in question is discrete and simple, and the $\nu_m^+(|n|)$ corresponding to nontrivial $Y^+(\cdot)$ are real and satisfy the inequality

$$(\nu_m^+(|n|))^2 \geq \frac{1}{4} + |n|^2 + |n|^2 \int_0^{x_1} \frac{x^2\,|Y^+(x)|^2}{1-x^2}\,\mathrm{d}x \left[\int_0^{x_1} |Y^+(x)|^2\,\mathrm{d}x\right]^{-1}.$$

Also, they are roots of the equation

$$\left(P_{\nu-\frac{1}{2}}^{-|n|}(-x) + P_{\nu-\frac{1}{2}}^{-|n|}(x)\right)\Big|_{x=\cos\theta_1} = 0$$

and admit the estimates

$$|\nu_m^+(|n|)| > \sqrt{\frac{1}{4} + |n|^2}, \quad |n| = 0, 1, 2, \dots.$$

Note that for $n = 0$ the relation $\left(P_{\nu-\frac{1}{2}}(-\cos\theta_1) + P_{\nu-\frac{1}{2}}(\cos\theta_1)\right)\big|_{\nu=1/2} = 2$ leads to the strict inequalities $|\nu_m^+(0)| > \frac{1}{2}$ instead of $|\nu_m^+(0)| \geq \frac{1}{2}$.

All zeros of the entire functions $\nu+a$, $p_n^+(\nu)$, and $h_n^+(\nu)$ except for $h_0^+(\nu)$ with $\nu = \pm 1/2$ lie outside the strip $\Pi(-\varepsilon, \varepsilon)$ ($\varepsilon > 0$ is small) including the imaginary axis. We make cuts from these zeros to $\pm\infty$ so that they do not intersect this strip. A branch

$$l_n(\nu) = \log\left(\mathrm{i}(\nu + a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right)$$

will be fixed by the condition $\log(\dots) \to 0$ as $\nu \to \mathrm{i}\infty$. Note that $\log(\dots) \to -\mathrm{i}\pi$ as $\nu \to -\mathrm{i}\infty$.

We seek a solution of equation (31) in the form

$$\sigma_n(\nu, a) = \exp\{\tau_n(\nu - 1/2, a)\}, \tag{32}$$

where the $\tau_n(\nu, a)$ satisfy the equation

$$\tau_n(\nu + 1/2, a) - \tau_n(\nu - 1/2, a) = \log\left(\mathrm{i}(\nu + a)\,\frac{p_n^+(\nu)}{h_n^+(\nu)}\right) := l_n(\nu)$$

and, as a consequence,

$$\tau_n'(\nu + 1/2, a) - \tau_n'(\nu - 1/2, a) = \frac{\mathrm{d}l_n(\nu)}{\mathrm{d}\nu}.$$

The right-hand side of the last equation is holomorphic in the strip $\Pi(-1/2, 1/2)$ and is estimated as $O(1/\nu^2)$ as $\nu \to \mathrm{i}\infty$. Its solution has the form (see, e.g., [12])

$$\tau_n'(\nu) = \frac{(-1)}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}l_n(\xi)}{\mathrm{d}\xi} \tan\left(\pi[\xi - \nu]\right)\mathrm{d}\xi.$$

Integrating this with respect to ν, we obtain (see (32))

$$\sigma_n(\nu, a) = \exp\{\tau_n(\nu - 1/2, a)\} = \exp\left\{\frac{\mathrm{i}}{2\pi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}l_n(\xi)}{\mathrm{d}\xi} \log\left[\frac{\sin\left(\pi[\nu - \xi]\right)}{\cos(\pi\xi)}\right]\mathrm{d}\xi\right\}. \tag{33}$$

Now let $a \to +0$; then $v_n(\nu)$ in (30) is holomorphic and has no zeros for $\nu \in \Pi(-\varepsilon, 1+\varepsilon)$.[8] (Moreover, the pole of $\frac{\mathrm{d}l_n(\xi)}{\mathrm{d}\xi}$ in the integrand in (33) at the point $\xi = 0$ is on the imaginary axis, but the integration contour in (33) goes on the right of it along an arc of small radius.) Meromorphic continuation of $v_n(\nu)$ to the entire complex plane is performed with the use of the functional equation (27). Observe that $\sigma_n(\nu, +0) = \exp(\pm\frac{\mathrm{i}\pi\nu}{2}[1 + O(1/\nu)])$, $\nu \to \pm\mathrm{i}\infty$, and it has a simple pole at $\nu = 0$, so that $v_n(\nu)$ is holomorphic and has no zeros in the vicinity of $\nu = 0$.

[8]The point $\nu = 0$ is a removable singularity of $v_n(\nu) = \exp(\nu \log \varkappa)\frac{1}{\Gamma(\nu)}\sigma_n(\nu, +0)$ because it is a simple zero of $1/\Gamma(\nu)$ and a simple pole of $\sigma_n(\nu, +0)$.

The above argument shows that the following is true.

Lemma 4.4. *If a meromorphic solution $v_n(\nu)$ of equation* (24) *is holomorphic in the strip $\Pi(-\varepsilon, 1+\varepsilon)$ and has the asymptotics $v_n(\nu) = O(\sqrt{|\nu|})$ as $\nu \to \pm i\infty$, then in that strip $v_n(\nu)$ admits the representation* ($n = 0, \pm 1 \dots, \pm N$, $a = +0$),

$$v_n(\nu) = \exp(\nu \log \varkappa) \frac{1}{\Gamma(\nu)} \sigma_n(\nu, +0),$$

(*see* (30)), *where $\sigma_n(\nu, +0)$ is a meromorphic function* (*see* (32)) *holomorphic in the strip $\Pi(-\varepsilon, 1+\varepsilon)$ except for $\nu = 0$, representable as in* (33), *where $a = +0$, and having no zeros in that strip.*

§5. Reduction of the functional difference equations to a system of singular integral equations

We move the summands containing the factors T_{12}, T_{21} exponentially decaying at $\pm i\infty$ to the right-hand side of (23) and (24). Using the auxiliary functions $w_n(\nu), v_n(\nu)$ from Lemmas 4.3 and 4.4, we introduce new unknowns $\alpha_n(\nu), \beta_n(\nu)$ by the formulas

$$U_n(\nu) = \alpha_n(\nu) w_n(\nu) \cos^2 \pi\nu, \quad V_n(\nu) = \beta_n(\nu) v_n(\nu) \cos^2 \pi\nu,$$

and substitute them in (23),(24), obtaining

$$\alpha_n(\nu+1) - \alpha(\nu) = K T_{12}(\nu) v_n(\nu) \beta(\nu) / w_n(\nu+1), \tag{34}$$

$$\beta_n(\nu+1) - \beta_n(\nu) = [i\varkappa T_{21}(\nu)\, w_n(\nu) \alpha(\nu) + \Psi_n(\nu)/\cos^2 \pi\nu] / v_n(\nu+1). \tag{35}$$

Estimates (25) show that, as $\nu \to \pm i\infty$ in the strip $\Pi(-\varepsilon, 1+\varepsilon)$, the unknowns $\alpha_n(\nu)$, $\beta_n(\nu)$ satisfy

$$|\alpha_n(\nu)| \le C \frac{|\nu|^{-1/2-\delta_1}}{|\cos^2 \pi\nu|}, \quad |\beta_n(\nu)| \le C \frac{|\nu|^{-1/2-\delta_1}}{|\cos^2 \pi\nu|}, \quad \delta_1 > 0. \tag{36}$$

Remark 5.1. In particular, estimates (36) mean that $\alpha_n(\nu)$ and $\beta_n(\nu)$ belong to $L_2(iR)$, where $iR = (-i\infty, i\infty)$, because they are holomorphic in the vicinity of the imaginary axis. The right-hand sides of equations (34), (35) decay on the imaginary axis as $\nu \to \pm i\infty$ not slower than $O(|\nu|^{-1-\delta_1})$, $\delta_1 > 0$; actually, they decay exponentially.

We shall use a simple statement (see, e.g., [3, 12]) about the solution of the difference equation

$$\tau(\nu+1) - \tau(\nu) = h(\nu)$$

with a function $h(\nu)$ holomorphic in the vicinity of the imaginary axis (specifically, in $\Pi(-\varepsilon, +\varepsilon)$) and admitting the estimate $|h(\nu)| \le O(|\nu|^{-1-\delta_1})$ there as $\nu \to \pm i\infty$. This equation has a bounded solution

$$\tau(\nu) = \frac{1}{2i} \int_{-i\infty}^{i\infty} \cot(\pi[\xi - \nu])\, h(\xi)\, d\xi$$

holomorphic in the strip $\nu \in \Pi(-\varepsilon, 1+\varepsilon)$. This statement can be verified with the help of the residues theorem. If ν tends to the imaginary axis, $\nu \to iR$, approaching it from the right side ($\nu + 0$), then the action of the integral operator on the right-hand side is understood as

$$\frac{1}{2i} \int_{-i\infty}^{i\infty} \cot(\pi[\xi - (\nu+0)])\, h(\xi)\, d\xi = -\frac{h(\nu)}{2} + \frac{1}{2i} \text{v.p.} \int_{-i\infty}^{i\infty} \cot(\pi[\xi - \nu])\, h(\xi)\, d\xi,$$

where the last summand is an integral in the sense of the Cauchy principal value.

Applying this statement to (34), (35), we obtain an integral representation

$$\alpha(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (\cot(\pi[\xi - (\nu + 0)]) - \tan\pi\nu) \frac{KT_{12}(\xi)\, v_n(\xi) \beta_n(\xi)}{w_n(\xi + 1)}\, \mathrm{d}\xi, \tag{37}$$

$$\beta_n(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (\cot(\pi[\xi - (\nu + 0)]) - \tan\pi\nu) \left[\frac{\mathrm{i}\varkappa T_{21}(\xi)\, w_n(\xi)\, \alpha_n(\xi)}{v_n(\xi + 1)} + \frac{\Psi_n(\xi)}{\cos^2(\pi\xi) v_n(\xi + 1)}\right] \mathrm{d}\xi, \tag{38}$$

where $\nu \in \Pi(-\varepsilon, 1+\varepsilon)$, and the summand Const $\tan\pi\nu$ is a solution of the homogeneous equation $\tau(\nu+1) - \tau(\nu) = 0$ and is added in (37), (38) in order that the unknowns $\alpha_n(\nu), \beta_n(\nu)$ decay along the imaginary axis at infinity.

The elementary identity

$$\cot(\pi[\xi - \nu]) - \tan\pi\nu = \frac{\cos(\pi\xi)}{\sin(\pi[\xi - \nu]) \cos(\pi\nu)}$$

allows us to deduce the following integral representations from (37), (38):

$$U_n(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\xi}{\sin(\pi[\xi - \nu - 0])} \frac{T_{12}(\xi) \cos(\pi\nu) w_n(\nu)}{T_{11}(\xi) \cos(\pi\xi) w_n(\xi)} V_n(\xi), \tag{39}$$

$$V_n(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\xi}{\sin(\pi[\xi - \nu - 0])} \left[\frac{T_{21}(\xi) \cos(\pi\nu) v_n(\nu)}{T_{22}(\xi) \cos(\pi\xi) v_n(\xi)} U_n(\xi) + \frac{\cos(\pi\nu) v_n(\nu)\, \Psi_n(\xi)}{\mathrm{i}\varkappa \cos(\pi\xi) v_n(\xi) T_{22}(\xi)}\right]. \tag{40}$$

From (39) and (40) it is obvious that if $U_n(\nu), V_n(\nu)$ are given on the imaginary axis by the right-hand sides and are holomorphic near that axis, then these functions on the left-hand side are holomorphically continued to the entire strip $\nu \in \Pi(-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$. However, if ν belongs to the imaginary axis, we get singular integral equations for computing $U_n(\nu), V_n(\nu)$. Since $U_n(\nu)$ and $V_n(\nu)$ are holomorphic for $\nu \in \Pi(-\varepsilon, 1+\varepsilon)$ (see (39), (40)), they can be analytically continued to the strip $\nu \in \Pi(-1-\varepsilon, 0)$. Analytic continuation can be also constructed with the help of the functional equations for $U_n(\nu)$, $V_n(\nu)$.

Instead of $U_n(\nu)$ and $V_n(\nu)$, in (39), (40) it is convenient to introduce $a_n(\nu)$, $b_n(\nu)$ by the formulas

$$U_n(\nu) = a_n(\nu) \cos^2 \pi\nu, V_n(\nu) = b_n(\nu) \cos^2 \pi\nu.$$

These new functions satisfy a system of integral equations in $L_2(\mathrm{i}R) \times L_2(\mathrm{i}R)$:

$$a_n(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{1}{\sin(\pi[\xi - \nu - 0])} \frac{T_{12}(\xi) \cos(\pi\xi) w_n(\nu)}{T_{11}(\xi) \cos(\pi\nu) w_n(\xi)} b_n(\xi)\, \mathrm{d}\xi, \tag{41}$$

$$b_n(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{1}{\sin(\pi[\xi - \nu - 0])} \left[\frac{T_{21}(\xi) \cos(\pi\xi) v_n(\nu)}{T_{22}(\xi) \cos(\pi\nu) v_n(\xi)} a_n(\xi)\right] \mathrm{d}\xi + \Phi(\nu), \tag{42}$$

where the free term is defined as

$$\Phi(\nu) = \frac{1}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{1}{\sin(\pi[\xi - \nu - 0])} \left[\frac{v_n(\nu)\, \Psi_n(\xi)/(\mathrm{i}\varkappa)}{\cos(\pi\xi) \cos(\pi\nu) v_n(\xi) T_{22}(\xi)}\right] \mathrm{d}\xi,$$

and

$$\frac{T_{12}(\nu)}{T_{11}(\nu)} = \frac{2\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]},$$
$$\frac{T_{21}(\nu)}{T_{22}(\nu)} = \frac{1}{\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}} \frac{\mathcal{W}_n(\nu)}{\left[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}.$$

Equations (41), (42) can be written in a matrix form standard for singular integral equations with the Cauchy kernel:

$$H_n(\nu) + \frac{1}{\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathcal{K}_n(\nu,\xi)}{[\xi-\nu-0]} H_n(\xi)\,\mathrm{d}\xi = F_n(\nu), \tag{43}$$

where $H_n(\nu) = (a_n(\nu), b_n(\nu))^T$, $F_n(\nu) = (0, \Phi_n(\nu))^T$ and

$$\mathcal{K}_n(\nu,\xi) = -\frac{1}{2}\,\frac{\pi[\xi-\nu]}{\sin(\pi[\xi-\nu])} \begin{pmatrix} 0 & \frac{T_{12}(\xi)\cos(\pi\xi)w_n(\nu)}{T_{11}(\xi)\cos(\pi\nu)w_n(\xi)} \\ \frac{T_{21}(\xi)\cos(\pi\xi)v_n(\nu)}{T_{22}(\xi)\cos(\pi\nu)v_n(\xi)} & 0 \end{pmatrix}.$$

Extracting the characteristic and the regular part of the singular operator in (43), we obtain

$$(I - \mathcal{K}_n(\nu,\nu))H_n(\nu) + \frac{\text{v.p.}}{\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathcal{K}_n(\nu,\nu)H_n(\xi)}{[\xi-\nu]}\,\mathrm{d}\xi + \mathcal{T}_n H_n(\nu) = F_n(\nu), \tag{44}$$

where $\mathcal{T}_n$ is a regular integral operator,

$$\mathcal{T}_n H_n(\nu) = \frac{1}{\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathcal{K}_n(\nu,\xi) - \mathcal{K}_n(\nu,\nu)}{\xi-\nu} H_n(\xi)\,\mathrm{d}\xi.$$

We shall study the singular integral equation (44) in the space $L_2(\mathrm{i}R) \times L_2(\mathrm{i}R)$.

§6. Fredholm property and the index of the singular integral operator

Theorem 6.1.

1) *The index of the characteristic part of the singular integral operator in* (44) *is finite and is equal to zero.*
2) *The operator $\mathcal{T}_n$ is a Hilbert–Schmidt operator in $L_2(\mathrm{i}R) \times L_2(\mathrm{i}R)$.*
3) *The singular integral operator in* (44) *is Fredholm and has zero index.*

Obviously, the third statement is a direct consequence of first two, so that we turn to their proof.

The symbol of the singular integral operator in question is the matrix-valued function

$$I - \mathcal{K}_n(\nu,\nu) + t\mathcal{K}_n(\nu,\nu),$$

$t = \pm 1$, $\nu \in \mathrm{i}R$. The characteristic operator in (44) is Fredholm if and only if $\det(I - \mathcal{K}_n(\nu,\nu) + \mathcal{K}_n(\nu,\nu))$ and $\det(I - 2\mathcal{K}_n(\nu,\nu))$ and continuos, see [19]. The first relation is obvious, we verify the second. A direct calculation leads to the expression

$$\det(I - 2\mathcal{K}_n(\nu,\nu)) = \frac{\left[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}{\left[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]} \frac{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}{\left[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\right]}.$$

When ν runs along the imaginary axis, the factors in the numerator and denominator are nonzero because, as it has been verified, the zeros of these factors are real and lie outside the strip $\Pi(-1/2, 1/2)$.

The index of the characteristic singular integral operator in (44) is equal to

$$\operatorname{ind}(\det\{I - 2\mathcal{K}_n(\nu,\nu)\}) = 0.$$

Indeed, the asymptotics of the associated Legendre functions as $\nu \to \pm \mathrm{i}\infty$, $\nu \in \mathrm{i}R$, shows that $\det\{1 - 2\mathcal{K}_n(\nu,\nu)\}$ tends to 1 and remains real[9] and positive for $\nu \in \mathrm{i}R$, because the factors in the determinant do not change their signs.

Turning to the compactness of the operator $\mathcal{T}_n$, we verify the Hilbert–Schmidt estimate for its kernel

$$\frac{1}{\mathrm{i}\pi}\frac{\mathcal{K}_n(\nu,\xi) - \mathcal{K}_n(\nu,\nu)}{\xi - \nu}$$
$$= \frac{\mathrm{i}}{2\pi[\xi-\nu]}\begin{pmatrix} 0 & \frac{\pi[\xi-\nu]}{\sin(\pi[\xi-\nu])}\frac{T_{12}(\xi)\cos(\pi\xi)w_n(\nu)}{T_{11}(\xi)\cos(\pi\nu)w_n(\xi)} - \frac{T_{12}(\nu)}{T_{11}(\nu)} \\ \frac{\pi[\xi-\nu]}{\sin(\pi[\xi-\nu])}\frac{T_{21}(\xi)\cos(\pi\xi)v_n(\nu)}{T_{22}(\xi)\cos(\pi\nu)v_n(\xi)} - \frac{T_{21}(\nu)}{T_{22}(\nu)} & 0 \end{pmatrix}$$

Obviously, the proof of the desired result reduces to Hilbert–Schmidt estimates for the entries of the matrix kernel of the operator

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{1}{|\xi-\eta|^2}\left|\frac{\pi[\xi-\nu]}{\sin(\pi[\xi-\nu])}\,\frac{\Sigma_{ik}(\xi)\cos(\pi\xi)w_n(\nu)}{\cos(\pi\nu)w_n(\xi)} - \Sigma_{ik}(\nu)\right|^2 |\mathrm{d}\xi|\,|\mathrm{d}\nu| < \infty,$$

where $i,k = 1,2$ $(i \neq k)$,

$$\Sigma_{ik}(\nu) = \frac{T_{ik}(\nu)}{T_{ii}(\nu)} = O(\exp(\pm\mathrm{i}\nu\chi)), \quad \nu \to \pm\mathrm{i}\infty,$$

$\chi = \pi/2 - \theta_1$. It is easily seen that, since the integrand is continuous, when we integrate over a compact set, the integral is finite. Therefore, it suffices to check that

$$\iint_{p^2+q^2\geq R^2}\frac{1}{|p-q|^2}\left|\frac{\pi[p-q]}{\sinh(\pi[p-q])}\,\frac{\sigma_{ik}(p)\cosh(\pi p)W_n(q)}{\cosh(\pi q)W_n(p)} - \sigma_{ik}(q)\right|^2 \mathrm{d}p\,\mathrm{d}q < \infty,$$

where R is sufficiently large, $\xi = \mathrm{i}p$, $\nu = \mathrm{i}q$, $W_n(q) = w_n(\nu)$, $\sigma_{ik}(q) = \Sigma_{ik}(\nu)$. When we integrate over the subdomain $\{p^2+q^2 \geq R^2\}$ in the second and fourth quadrant, i.e., for $p > 0$, $q < 0$ or $p < 0$, $q > 0$, the double integral is easily shown to be bounded. This follows from the fact that $p \neq q$ and the functions $\sigma_{ik}(q)$ decay exponentially at infinity. Therefore, it remains to study the convergence of the integral in the closed first and third quadrants. These two cases are treated similarly, so that we restrict ourselves to the integral over $D_R = \{p^2+q^2 \geq R^2,\, p > 0,\, q > 0\}$,

$$J_R := \iint_{D_R}\frac{|\sigma_{ik}(q)|^2}{|p-q|^2}\left|\frac{\pi[p-q]}{\sinh(\pi[p-q])}\,\frac{Z_{ik}(p)}{Z_{ik}(q)} - 1\right|^2 \mathrm{d}p\,\mathrm{d}q,$$

where $\sigma_{ik}(q) = O(\exp(\mp q\chi))$, $q \to \pm\infty$,

$$Z_{ik}(p) := \sigma_{ik}(p)\cosh(\pi p)/W_n(p) = O(\exp(\pm p[\pi-\chi])/\sqrt{|p|}), \quad p \to \pm\infty.$$

Introducing the polar coordinates $p = \rho\cos\psi$, $q = \rho\sin\psi$, $\psi \in [0,\pi/2]$, we split the segment of integration over ψ into two parts: $(\psi_1(\epsilon), \psi_2(\epsilon))$, where $\psi_1(\epsilon) = \pi/4 - \epsilon$, $\psi_2(\epsilon) = \pi/4 + \epsilon$, $\epsilon > 0$ is small, and its complement P_ϵ to $[0,\pi/2]$. In J_R we pass to the iterated integral, splitting it into two summands with integration over $(\psi_1(\epsilon), \psi_2(\epsilon))$ and over P_ϵ. In the summand with integration over P_ϵ we have $\cos\psi \neq \sin\psi$ $(p \neq q)$, the iterated integral converges, and this summand is bounded by a constant C_1. The

[9] The functions $P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta)$ are real for $\nu \in \mathrm{i}R$.

second summand with integration over $(\psi_1(\epsilon),\psi_2(\epsilon))$ can be transformed so as to admit the estimates

$$J_R \le C_1 + C\int_{\psi_1(\epsilon)}^{\psi_2(\epsilon)}\int_{\rho\ge R}\frac{\exp(-2\chi\rho\sin\psi)}{\rho[\cos\psi-\sin\psi]^2}\left|\frac{\pi\rho[\cos\psi-\sin\psi]}{\sin(\pi\rho[\cos\psi-\sin\psi])}\frac{Z_{ik}(\rho\cos\psi)}{Z_{ik}(\rho\sin\psi)}-1\right|^2 \mathrm{d}\rho\,\mathrm{d}\psi.$$

We estimate the iterated integral on the right-hand side of this inequality, denoting it by J_R^ϵ. Changing the order of integration in J_R^ϵ, we introduce the new variable ϕ of integration by the formula $\phi=\psi-\pi/4$, $(\cos\psi-\sin\psi=-\sqrt{2}\sin\phi)$:

$$J_R^\epsilon = \int_{\rho\ge R}\rho\exp(-\chi\rho\sqrt{2})\int_{-\epsilon}^{\epsilon}\left|\frac{\pi\exp(-\chi\rho\Phi_2(\phi))}{\sinh(\pi\rho[-\sqrt{2}\sin\phi])}\frac{Z_{ik}\big(\frac{\rho}{\sqrt{2}}+\rho\Phi_1(\phi)\big)}{Z_{ik}\big(\frac{\rho}{\sqrt{2}}+\rho\Phi_2(\phi)\big)}-\frac{1}{\rho[-\sqrt{2}\sin\phi]}\right|^2 \mathrm{d}\phi\,\mathrm{d}\rho,$$

where the notation is introduced in accordance with the relations $\cos\psi = 1/\sqrt{2}+\Phi_1(\phi)$, $\sin\psi = 1/\sqrt{2}+\Phi_2(\phi)$, $\Phi_1(\phi)=(-\sin\phi+\cos\phi-1)/\sqrt{2}$, $\Phi_2(\phi)=(\sin\phi+\cos\phi-1)/\sqrt{2}$. In the inner integral we change the integration variable by the formula $\tau=\rho[-\sqrt{2}\sin\phi]$. We have

$$J_R^\epsilon \le C\int_{\rho\ge R}\mathrm{d}\rho\,\exp(-\sqrt{2}\chi\rho)\,I(\rho,\epsilon),$$

where

$$I(\rho,\epsilon)=\int_{-\sqrt{2}\rho\sin\epsilon}^{\sqrt{2}\rho\sin\epsilon}\left|\frac{\pi\,\exp(-\chi[\tau/2+\tau\Psi(\tau/\rho)])}{\sinh(\pi\tau)}\frac{Z_{ik}(\rho/\sqrt{2}+\tau/2+\tau\Psi(\tau/\rho))}{Z_{ik}(\rho/\sqrt{2}-\tau/2+\tau\Psi(\tau/\rho))}-\frac{1}{\tau}\right|^2\mathrm{d}\tau,$$

where the function $\Psi(x)$, analytic for small x, is defined by the relation

$$\rho\Phi_1(\phi)\big|_{\tau=\rho[-\sqrt{2}\sin\phi]} = \tau/2+\tau\Psi(\tau/\rho),$$

or

$$\rho\Phi_2(\phi)\big|_{\tau=\rho[-\sqrt{2}\sin\phi]} = -\tau/2+\tau\Psi(\tau/\rho),$$

$\Psi(x)=O(x)$ as $x\to 0$. The integrand in $I(\rho,\epsilon)$ is continuous in τ for all $\rho\ge R$ and is bounded with respect to ρ $(\rho\ge R)$ for all τ on the segment of integration, so that $I(\rho,\epsilon)$ admits the estimate

$$I(\rho,\epsilon)\le \text{const}\,\rho.$$

The resulting inequality allows us to assert that

$$J_R^\epsilon \le C_2\int_{\rho\ge R}\rho\exp(-\sqrt{2}\chi\rho)\,\mathrm{d}\rho \le \text{Const},$$

which completes the proof of Theorem 6.1.

Remark 6.2. In accordance with the general theory [19], a two-sided regularizer of (44) can be constructed in explicit terms.

§7. Estimates for the solution of the problem in the vicinity of the conic point and at infinity. Existence of the classical solution

Assume that the integral equation (44) is solvable and hence, there exist meromorphic solutions $A_n(\nu), B_n(\nu)$ of the functional difference equations in the required class of functions. Then the solution of the problem on forced oscillations of a fluid is constructed by formulas (8) and (16). In order to estimate the solution in the vicinity of the conic point and at infinity, we use the representation (8), (16), where the coefficients $A_n(\nu), B_n(\nu)$ are solutions of functional difference equations. Calculation of the asymptotics as $r\to 0$ and $r\to\infty$ is based on the arguments traditional for the Mellin integral representations. For $\omega\in\Sigma$, the integral (8) converges exponentially if $r>0$. The integration contour i$\mathbb{R}$ can be deformed into a parallel straight line $(-\mathrm{i}\infty+\gamma,\mathrm{i}\infty+\gamma)$ to the right of the imaginary axis, $\gamma > 0$, or to the left, $\gamma < 0$. Moreover, the poles of the Mellin transform $u_\nu(\omega)$ are intersected at points ν_p,

$$
\begin{aligned}
u(r,\omega) &= \sum_p \operatorname{sgn}(-\gamma)\ \operatorname{res}_{\nu_p}\{u_\nu(\omega)\}\, r^{\nu_p-1/2} + u_\gamma(r,\omega),\\
u_\gamma(r,\omega) &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} u_\nu(\omega)\, r^{\nu-\frac{1}{2}}\, \mathrm{d}\nu,
\end{aligned}
\tag{45}
$$

Summation in (45) is over all poles located in the strip between the imaginary axis and the vertical line $\mathrm{iR} + \gamma$, and

$$|u_\gamma(r,\omega)| \le C\, r^{\gamma-1/2}$$

uniformly with respect to $\omega \in \Sigma$.

Taking these simple arguments into account, it is easy to evaluate the behavior of the solution as $r \to 0$. The pole nearest to the imaginary axis in the domain $\mathrm{Re}(\nu) > 0$ is located at the point $\nu = \nu_+ > 1/2$, see (16), corresponds to the summand with $n = 0$, see (21), (22), and is a root of a transcendental equation. Observe that $\nu = 1/2$ is not a pole in (16), because in (21), (22) (for $n = 0$) we have $U_0(\nu)\big|_{\nu=1/2} = 0$, $V_0(\nu)\big|_{\nu=1/2} = 0$, see (39), (40). Since $U_n(\nu)$ and $V_n(\nu)$ are holomorphic in $\Pi(-\varepsilon, 1+\varepsilon)$, we can use (21), (22), from (45) to find

$$u(r,\omega) = -\operatorname{res}_{\nu=\nu_+}\{u_\nu(\omega)\}\, r^{\nu_+-1/2} + O(r^{\nu_*-1/2}), \quad r \to 0. \tag{46}$$

Note that ν_* is the pole of $u_\nu(\omega)$ nearest to ν_+ to the right of ν_+.

For calculation of asymptotics as $r \to \infty$, we need to study the poles of $u_\nu(\omega)$ located to the left of the imaginary axis. Let $-\nu_q$ be the negative pole of the Mellin transform[10] nearest to the imaginary axis. Deforming the integration contour as above, we arrive at the asymptotics

$$u(r,\omega) = \operatorname{res}_{\nu=-\nu_q}\{u_\nu(\omega)\}\, r^{-\nu_q-1/2} + O(r^{-\nu_{**}-1/2}), \quad r \to \infty, \tag{47}$$

where $-\nu_{**}$ is the next (by the order) negative pole.

We want to check that the classical solution of problem (1)–(6) obtained in this way not only satisfies the Meixner conditions (see (46)) but also decays rapidly in the sense of the definition in §2 (see (7)). For this, it is necessary to verify that $\nu_q > 1/2$ in the asymptotics (47). In order to prove the inequality $\nu_q > 1/2$, it suffices to consider $n = 0$ in (16) and to check that the pole[11] at the point $\nu = -1/2$ for the summands with $n = 0$ is compensated for by the zero at $\nu = -1/2$ of the functions $A_0(\nu), B_0(\nu)$. In view of the explicit linear relations (21), (22) linking $A_0(\nu), B_0(\nu)$ and $U_0(\nu), V_0(\nu)$, it suffices to show that $U_0(\nu), V_0(\nu)$ vanish at this point. This would mean that the Mellin transform $u_\nu(\omega)$ is holomorphic with respect to ν in the strip $\Pi(-1/2-\varepsilon, 0)$ for some $\varepsilon > 0$, thus proving the inequality $\nu_q > 1/2$.

For this, we rewrite the functional difference equations (23), (24) in the form

$$\begin{Bmatrix} U_n(\nu) \\ V_n(\nu) \end{Bmatrix} = \begin{pmatrix} T_{11}(\nu) & T_{12}(\nu) \\ T_{21}(\nu) & T_{22}(\nu) \end{pmatrix}^{-1} \begin{Bmatrix} K^{-1}\, U_n(\nu+1) \\ (\mathrm{i}\varkappa)^{-1}\,(V_n(\nu+1) - \Psi_n(\nu)) \end{Bmatrix}. \tag{48}$$

[10]This pole is simple.

[11]There is a zero of the denominators $\mathrm{d}_{\theta_1} P_{\nu-\frac{1}{2}}(\pm\cos\theta_1)$ at this point, while the other summands are regular in the strip $\Pi(-1/2-\varepsilon, 0)$.

The entries of the inverse matrix $\{T^{-1}\}_{ik}$ on the right-hand side of the equations have the form

$$\{T^{-1}\}_{11} = \frac{\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}}{P^{-|n|}_{\nu-\frac{1}{2}}(0)} \frac{\big[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) + P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]}{\big[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]},$$

$$\{T^{-1}\}_{12} = \frac{(-2)\,\mathrm{d}_x P^{-|n|}_{\nu-\frac{1}{2}}(x)\big|_{x=0}}{\big[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]},$$

$$\{T^{-1}\}_{21} = \frac{(-1)}{P^{-|n|}_{\nu-\frac{1}{2}}(0)} \frac{\mathcal{W}_n(\nu)}{\big[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]},$$

$$\{T^{-1}\}_{22} = \frac{\big[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]}{\big[P^{-|n|}_{\nu-\frac{1}{2}}(\cos\theta_1) - P^{-|n|}_{\nu-\frac{1}{2}}(-\cos\theta_1)\big]}$$

and are holomorphic in the vicinity of $-1/2$ (a removable singularity for $n = 0$) in the strip $\Pi(-1/2 - \varepsilon, 0)$. If the argument ν on the left-hand side of (48) is in the strip $\Pi(-1 - \varepsilon, 0)$, then the argument of $U_n(\,\cdot\,), V_n(\,\cdot\,)$ on the right-hand side of (48) varies in the strip $\Pi(-\varepsilon, 1)$. Recall that it suffices to consider the summand with $n = 0$ in (16), because the other summands are regular in this strip. If $\nu = -1/2$ and $n = 0$ in (48), then $U_0(\nu+1)\big|_{\nu=-1/2} = 0$ and $V_0(\nu+1)\big|_{\nu=-1/2} = 0$ on the right-hand side, but, generally speaking, $\Psi_0(-1/2) \neq 0$. We impose an additional constraint on the class of sources by assuming that

$$\Psi_0(-1/2) = 0, \quad \text{or, equivalently} \quad \int_0^\infty \int_0^{2\pi} f(r,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}r = 0. \tag{49}$$

Now in a standard way we can prove that under certain conditions the problem on forced oscillations of a fluid admits a classical rapidly decaying solution.

Theorem 7.1. *Let $f(r,\varphi)$ belong to the class of rapidly vanishing sources satisfying conditions* (49)*, and let $K > 0$, $\varkappa \neq 0$; then there exists a unique rapidly decaying classical solution of the problem* (1)–(6) *having the asymptotics* (46) *as $r \to 0$ and* (47) *as $r \to \infty$.*

Indeed, the uniqueness of a rapidly decaying solution was proved in Theorem 2.1. Since the integral equation (44) is Fredholm with zero index, unique solvability in $L_2(\mathrm{i}R) \times L_2(\mathrm{i}R)$ follows from the fact that the homogeneous equation has only a trivial solution. The latter circumstance is implied by the proved uniqueness of a rapidly decaying classical solution of the problem: if the integral equation (44) has a nontrivial solution constructed by the explicit formulas (8) and (16), then the classical solution will be a rapidly decaying solution of the homogeneous problem, which contradicts Theorem 2.1. Therefore, the equation has a unique solution, which allows us to construct the solution $A_n(\nu), B_n(\nu)$ of the problem for functional equations in the required class and to recover the Mellin transform (16) and the rapidly decaying classical solution (8) of problem (1)–(6).

Some comments are in order. As has already been noted, the parameter $\varkappa$ plays the role of a parameter regularizing the problem in the following sense. If $\varkappa = 0$, then, as was shown in [12], the problem has a nontrivial kernel, so that it cannot be well posed in the sense of Hadamard. If $\varkappa \neq 0$, the problem of finding the classical solution of (1)–(6) admits reduction to a Fredholm singular integral equation with zero index. Thus, in this sense the problem is Fredholm with zero index. If the sources also decay rapidly and condition (49) is fulfilled (a source is "orthogonal" to constants), then the rapidly decaying solution for such sources exists and is unique.

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