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THE MAXWELL SYSTEM IN WAVEGUIDES WITH SEVERAL CYLINDRICAL OUTLETS TO INFINITY AND NONHOMOGENEOUS ANISOTROPIC FILLING

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To the memory of V. S. Buslaev

Abstract. A waveguide occupies a domain G in $\mathbb{R}^3$ with several cylindrical outlets to infinity; the boundary ∂G is assumed to be smooth. The dielectric ε and magnetic μ permittivities are matrix-valued functions smooth and positive definite in $\overline{G}$. At every cylindrical outlet, the matrices ε and μ tend, at infinity, to limit matrices independent of the axial variable. The limit matrices can be arbitrary smooth and positive definite matrix-valued functions of the transverse coordinates in the corresponding cylinder. In such a waveguide, the stationary Maxwell system with perfectly conducting boundary conditions and a real spectral parameter is considered. In the presence of charges and currents, the corresponding boundary value problem with radiation conditions turns out to be well posed. A unitary scattering matrix is also defined. The Maxwell system is extended to an elliptic system. The results for the Maxwell system are derived from those obtained for the elliptic problem.

§1. Introduction

Let $G \subset \mathbb{R}^3$ be a domain with smooth boundary ∂G coinciding outside a large ball with the union of finitely many nonintersecting semicylinders $\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\}$, $r = 1, \dots, \mathcal{T}$. The cross-sections Ω^r of the cylindrical outlets are bounded domains in $\mathbb{R}^2$ with smooth boundary.

In the domain G, we consider the stationary Maxwell system

$$(1.1)\qquad \begin{aligned} i\varepsilon(x)^{-1}\operatorname{curl} u^2(x) - ku^1(x) = f(x), \quad -i\operatorname{div}(\mu(x)u^2(x)) = 0,\\ -i\mu(x)^{-1}\operatorname{curl} u^1(x) - ku^2(x) = 0, \quad i\operatorname{div}(\varepsilon(x)u^1(x)) = h(x), \quad x \in G, \end{aligned}$$

with the boundary conditions

$$(1.2)\qquad u^1_{\tau_2}(x) = 0, \quad u^1_{\tau_1}(x) = 0, \quad (\mu u^2)_\nu(x) = 0, \quad x \in \partial G.$$

Here u^1 and u^2 are vector-valued functions on $\bar{G}$ that stand for the electric and magnetic vectors, respectively. The spectral parameter $k \in \mathbb{R}$ is assumed to be fixed and, as a rule, will not be shown. The tangent vectors $\tau_1(x)$, $\tau_2(x)$, and the outer normal $\nu(x)$ to ∂G make up a right triple of orthogonal unit vectors; u_e denotes $\langle u, e\rangle$ for the vector $e = \tau_1, \tau_2, \nu$, while $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the inner product in $\mathbb{C}^3$. The dielectric ε and magnetic μ permittivities are matrix-valued functions, smooth and positive definite on $\bar{G}$. At every cylindrical outlet Π_+^r, $r = 1, \dots, \mathcal{T}$, as $t^r \to +\infty$, the matrices $\varepsilon(y^r, t^r)$ and $\mu(y^r, t^r)$ uniformly tend to the limit matrices $\varepsilon^r(y^r)$ and $\mu^r(y^r)$ at exponential rate; these ε^r and μ^r are matrix-valued functions smooth and positive definite on $\bar{\Omega}^r$. More

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precisely, we assume that, for a certain $\delta > 0$, the entries of the mentioned matrices satisfy

$$|\varepsilon_{jl}(y^r,t^r) - \varepsilon^r_{jl}(y^r)| + |\nabla(\varepsilon_{jl}(y^r,t^r) - \varepsilon^r_{jl}(y^r))| = O(\exp(-\delta t^r)),$$
$$|\mu_{jl}(y^r,t^r) - \mu^r_{jl}(y^r)| + |\nabla(\mu_{jl}(y^r,t^r) - \mu^r_{jl}(y^r))| = O(\exp(-\delta t^r)), \quad t^r \to +\infty,$$

uniformly with respect to $y^r \in \bar{\Omega}^r$; here $j,l = 1,2,3$, while (y^r,t^r) are local coordinates at the cylindrical outlet $G \cap \Pi^r_+ = \{(y^r,t^r) : y^r \in \Omega^r, t^r > 0\}$. Problem (1.1), (1.2) is overdetermined; the compatibility condition

$$\operatorname{div}(\varepsilon(x)f(x)) - ik\,h(x) = 0, \quad x \in G, \tag{1.3}$$

is necessary for the solvability of this problem.

Let us briefly describe the results of this paper. In every cylinder $\Pi^r = \Omega^r \times \mathbb{R}$, we consider a model problem of the form (1.1), (1.2) with the limit matrices ε^r, μ^r (depending on $y^r \in \Omega^r$ only). A spectral parameter $k \in \mathbb{R}$ is fixed. The homogeneous model problem can have at most finitely many, if any, linearly independent solutions of the form

$$\exp(i\lambda t^r) \sum_{q=0}^{\varkappa-1} \frac{(it^r)^q}{q!} \varphi^{(\varkappa-1-q)}(y^r), \tag{1.4}$$

where λ is real and $\varkappa \geq 1$. (This fact has been known for the waveguides described by elliptic equations, see, e.g., [1]. It can be obtained for the Maxwell system by extending the system to an elliptic one.) Solutions of the form (1.4) with $\varkappa > 1$ can exist only for isolated k; such k are called thresholds. (For elliptic systems, this was stated in [2] and can be obtained for the Maxwell system from the elliptic extension mentioned above.) We denote by W^r the linear space spanned by the solutions of the form (1.4) for all possible real λ. The elements of W^r are called waves. In W^r there exists a basis consisting of waves subject to certain orthogonality and normalization conditions. Such a basis contains equally many "incoming" (from $+\infty$) and "outgoing" (to $+\infty$) waves.

Let $\eta \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that $0 \leq \eta(t) \leq 1$ for $t \in \mathbb{R}$, $\eta(t) = 0$ for $t < 0$, and $\eta(t) = 1$ for $t > 1$. Set $\eta_T(t) = \eta(t - T)$, where T is a sufficient large number. For an arbitrary wave $w \in W^r$, we introduce the function

$$G \cap \Pi^r_+ \ni (y^r,t^r) \mapsto \eta_T(t^r)w(y^r,t^r),$$

and extend it by zero to the domain G. All functions obtained in this way are called waves in G. We say that the waves in G obtained from the basis incoming (outgoing) waves in the spaces $W^1,\dots,W^{\mathcal{T}}$ are incoming (outgoing) and denote them by $u^+_1,\dots,u^+_{\mathfrak{m}}$ $(u^-_1,\dots,u^-_{\mathfrak{m}})$.

A nonzero function Y is called a continuous spectrum eigenfunction (CSE) of problem (1.1), (1.2) in the domain G if Y satisfies the corresponding homogeneous problem and the conditions $Y \notin L_2(G)$ and $|Y(x)| \leq \mathrm{Const}(1 + |x|)^N$ with $N < \infty$. If for a given spectral parameter k such a solution exists, then k, by definition, belongs to the continuous spectrum. (Note, that for an elliptic problem in a waveguide, the existence of such a solution is equivalent to the fact that the range of the problem operator is nonclosed in $L_2(G)$, see [1]. Therefore, for the elliptic systems, the two definitions of continuous spectrum (in terms of polynomially bounded solutions and operator range) coincide. For the Maxwell operator, generally speaking, these two definitions do not coincide. For the problem under consideration, the continuous spectrum definition in terms of polynomially bounded solution is more natural.)

Let $E_c(k)$ denote the space spanned by the continuous spectrum eigenfunctions. If for some k there exists a solution Z of the homogeneous problem (1.1), (1.2), $Z \in L_2(G)$, then these k and Z are called an eigenvalue and an eigenfunction of problem (1.1), (1.2).

We denote by $E_p(k)$ the space of eigenfunctions; the eigenspace is finite-dimensional. If k is a point of the continuous spectrum and an eigenvalue of problem (1.1), (1.2), then $E_p(k) \subset E_c(k)$.

Theorem 1.1. 1. *Let k be a point of the continuous spectrum of problem* (1.1), (1.2), *while k is not an eigenvalue. Then in the space $E_c(k)$ there exists a basis $Y_1^+, \dots, Y_{\mathfrak{m}}^+$ with the asymptotics*

$$Y_j^+(x) = u_j^+ + \sum_{l=1}^{\mathfrak{m}} s_{jl} u_l^- + O(e^{-\alpha|x|}), \quad j = 1, \dots, \mathfrak{m}, \tag{1.5}$$

as $|x| \to \infty$, where α is a sufficiently small positive number. The matrix s with the entries s_{jl} is unitary.
2. *Let k be a point of the continuous spectrum and an eigenvalue of problem* (1.1), (1.2). *Then in the quotient space $E_c(k)/E_p(k)$ there exists a basis with representatives $Y_1^+, \dots, Y_{\mathfrak{m}}^+$ subject to* (1.5). *The matrix s with the entries s_{jl} is independent of the choice of representatives and is unitary.*

Definition 1.2. The matrix s mentioned in Theorem 1.1 is called the scattering matrix of problem (1.1), (1.2).

We now describe a well-posed problem of the form (1.1), (1.2) with intrinsic radiation conditions. Denote by $C_c^\infty(\bar{G})$ the set of smooth functions with compact support and by $H_\alpha^l(G)$, $l \geq 0$ the closure of $C_c^\infty(\bar{G})$ in the norm

$$\|u; H_\alpha^l(G)\| := \|\rho_\alpha u; H^l(G)\| = \left(\sum_{|\sigma|=0}^{l} \int_G |D^\sigma(\rho_\alpha u)|^2 \, dx \right)^{1/2}.$$

Here $\rho_\alpha \in C^\infty(\bar{G})$ is a positive function that coincides on $G \cap \Pi_+^r$ with the mapping $(y^r, t^r) \mapsto \exp(\alpha t^r)$, $r = 1, \dots, \mathcal{T}$, and α is the same number as in (1.5).

Theorem 1.3. *Let $Z_1, \dots, Z_d$ be a basis in $E_p(k)$. Assume that $f \in H_\alpha^l(G; \mathbb{C}^3)$ and $h \in H_\alpha^l(G; \mathbb{C})$ satisfy the compatibility condition* (1.3) *and the conditions $(F, Z_j)_G = 0$, $j = 1, \dots, d$, where $F := (\varepsilon f, 0)$ is a vector-valued function with 6 components. Then there exists a solution $U = (u^1, u^2)$ of problem* (1.1), (1.2) *subject to the radiation conditions*

$$V := U - \sum_{j=1}^{\mathfrak{m}} c_j u_j^- \in H_\alpha^{l+1}(G; \mathbb{C}^6). \tag{1.6}$$

Here $c_j = i(F, Y_j^-)_G$, where $Y_j^- := \sum_{l=1}^{\mathfrak{m}} (s^{-1})_{jl} Y_l^+$, while the Y_l^+ are elements of the space $E_c(k)$ and satisfy (1.5).

The solution U is determined up to an arbitrary element of the space $E_p(k)$, and

$$\begin{aligned} &\left\| V; H_\alpha^{l+1}(G; \mathbb{C}^6) \right\| + \sum_{j=1}^{\mathfrak{m}} |c_j| \\ &\qquad \leq \text{const} \left(\|f; H_\alpha^l(G; \mathbb{C}^3)\| + \|h; H_\alpha^l(G; \mathbb{C})\| + \|\rho_\alpha V; L_2(G; \mathbb{C}^6)\| \right). \end{aligned} \tag{1.7}$$

A solution U_0 subject to the additional conditions $(\pi U_0, Z_j)_G = 0$, $j = 1, \dots, d$, is unique (*here $\pi = \operatorname{diag}(\varepsilon, \mu)$ is a (6×6) matrix-valued function*); *for the function V in* (1.6) *with U changed for U_0, estimate* (1.7) *is valid with the right-hand side replaced by* $\text{const}(\|f; H_\alpha^l(G; \mathbb{C}^3)\| + \|h; H_\alpha^l(G; \mathbb{C})\|)$.

In the paper, the (overdetermined) Maxwell system is extended to an elliptic system. Then we analyze the elliptic system and clarify its specific properties coming from the Maxwell system. The information on the problem for the Maxwell system is derived from that obtained for the elliptic one. The elliptic extension is fairly traditional, see, e.g., [3, 4, 5, 6]; certain links of the extended Maxwell system with some other operators in mathematical physics were discussed in [7]. In [8], the elliptic extension was used for studying the Maxwell system in empty waveguides with several cylindrical outlets to infinity, the unitary scattering matrix was introduced, and the radiation principle was established for the Maxwell system.

Among numerous mathematical publications devoted to the Maxwell system in waveguides, we mention two lines of investigation. One line is related to the Wiener–Hopf technique and the mode matching method; in the cases under consideration the Maxwell system reduces to the Helmholtz equations and the waveguide in question consists of model domains. Some surveys of the methods were given in [9, 11, 10]. In the other line of work, the cylindrical waveguides are considered with filling independent of the axial coordinate of the cylinder; some local perturbations (in a bounded domain) are admitted in the waveguide form and filling. The case where the electrical and magnetic permittivities ε and μ are constant outside a bounded part of a waveguide was studied quite thoroughly, see the monographs [12] and [13]. In [14, 15], and [16], the Maxwell system was considered in cylindrical waveguides, while outside a bounded part of a waveguide, the permittivities ε and μ are block diagonal matrices depending on the transverse coordinates only; the block of size 1×1 corresponds to the axial direction and the block of size 2×2 is related to the transverse directions. We mention also the paper [17], where the waveguide is a cylinder with circular cross-section and ε and μ are numbers depending on the radial coordinate. We use neither results nor methods of the papers mentioned in this paragraph.

We briefly describe the sections of this paper. The elliptic extension of the Maxwell system is introduced in §2. For such an extension we formulate some known results in the theory of selfadjoint boundary value problems for elliptic systems in domains with several cylindrical outlets to infinity, see [1]. In particular, we introduce the space of waves and the scattering matrix, and present a well-posed problem with radiation conditions. The principal results of the paper are proved in §3. In Subsections 3.1–3.3, we establish the existence of a special basis in the space of continuous spectrum eigenfunctions for the extended (elliptic) Maxwell system. The scattering matrix in such a basis is block diagonal, and one of its blocks plays the role of the scattering matrix for the original Maxwell system; this proves Theorem 1.1. In Subsection 3.4, the radiation principle for the Maxwell system is derived from that for the elliptic system; in doing so, we establish the radiation principle for a problem of the form

$$\begin{aligned} i\varepsilon(x)^{-1}\operatorname{curl} u^2(x) - ku^1(x) &= f^1(x), \quad -i\operatorname{div}(\mu(x)u^2(x)) = h^1(x), \\ -i\mu(x)^{-1}\operatorname{curl} u^1(x) - ku^2(x) &= f^2(x), \quad i\operatorname{div}(\varepsilon(x)u^1(x)) = h^2(x), \quad x \in G, \end{aligned} \tag{1.8}$$

with boundary conditions

$$-u^1_{\tau_2}(x) = g^1(x), \quad u^1_{\tau_1}(x) = g^2(x), \quad (\mu u^2)_\nu(x) = g^3(x), \quad x \in \partial G, \tag{1.9}$$

and with right-hand side subject to the compatibility condition

$$\begin{aligned} \operatorname{div}(\varepsilon(x)f^1(x)) - ik\,h^2(x) = 0, &\quad x \in G, \\ \operatorname{div}(\mu(x)f^2(x)) + ik\,h^1(x) = 0, &\quad x \in G, \\ i\,(\mu f^2)_\nu(x) + \operatorname{div}_2 g(x) + ik\,g^3(x) = 0, &\quad x \in \partial G, \end{aligned} \tag{1.10}$$

where $g(x) = (g^1(x), g^2(x))$ and div_2 denotes the divergence on the surface ∂G. Theorem 1.2 , corresponding to the "physical" statement (1.1), (1.2), is a special case of this result for $f^2 = 0$, $h^1 = 0$, $g^1 = 0$, $g^2 = 0$, and $g^3 = 0$.

Throughout the paper, it is assumed that the spectral parameter belongs to the continuous spectrum of problem (1.1), (1.2), while in §3 we suppose, moreover, that $k \neq 0$. For the empty waveguide, in [8] it was shown that $k = 0$ is an isolated point of multiplicity $N-1$ in the continuous spectrum of problem (1.1), (1.2), where $N \geq 2$ is the number of cylindrical outlets (for $N = 1$, a neighborhood of zero is free from the continuous spectrum). For the waveguides with inhomogeneous anisotropic filling considered in the present paper, the point $k = 0$ was not investigated.

§2. Elliptic Problem

We replace problem (1.8), (1.9) with its elliptic extension

$$
\begin{aligned}
i\varepsilon(x)^{-1}\operatorname{curl} u^2(x) + i\,\nabla a^2(x) - ku^1(x) &= f^1(x),\\
-i\operatorname{div}(\mu(x)\,u^2(x)) - ka^1(x) &= h^1(x),\\
-i\mu(x)^{-1}\operatorname{curl} u^1(x) - i\,\nabla a^1(x) - ku^2(x) &= f^2(x), \quad x \in G,\\
i\operatorname{div}(\varepsilon(x)\,u^1(x)) - ka^2(x) &= h^2(x),
\end{aligned}
\tag{2.1}
$$

with the following boundary conditions on ∂G:

$$
\begin{aligned}
-\langle u^1(x), \tau_2(x)\rangle &= g^1(x),\\
\langle u^1(x), \tau_1(x)\rangle &= g^2(x),\\
\langle \mu(x)u^2(x), \nu(x)\rangle &= g^3(x),\\
a^2(x) &= g^4(x).
\end{aligned}
\tag{2.2}
$$

We write this elliptic problem in the form

$$
\mathcal{A}(x, D_x)\mathcal{U}(x) = \mathcal{F}(x), \quad x \in G, \qquad \mathcal{B}(x)\mathcal{U}(x) = \mathcal{G}(x), \quad x \in \partial G,
\tag{2.3}
$$

where

$$
\mathcal{U} = (u^1, a^1, u^2, a^2), \quad \mathcal{F} = (f^1, h^1, f^2, h^2), \quad \mathcal{G} = (g^1, g^2, g^3, g^4),
$$

while the u^j and f^j are vector-valued functions with three components, the a^j and h^j are scalar functions in the domain G, $j = 1, 2$, and the g^l are scalar functions on ∂G, $l = 1, \dots, 4$. For any vector-valued functions $\mathcal{U} = (u^1, a^1, u^2, a^2)$ and $\mathcal{V} = (v^1, b^1, v^2, b^2)$ in $C_c^\infty(\bar{G}; \mathbb{C}^8)$ we have the Green formula

$$
(\mathcal{A}\mathcal{U}, \mathcal{V})_{G,\varpi} + (\mathcal{B}\mathcal{U}, \mathcal{Q}\mathcal{V})_{\partial G} = (\mathcal{U}, \mathcal{A}\mathcal{V})_{G,\varpi} + (\mathcal{Q}\mathcal{U}, \mathcal{B}\mathcal{V})_{\partial G}
\tag{2.4}
$$

with the following $\mathcal{Q}\mathcal{U} = -i(\langle u^2, \tau_1\rangle, \langle u^2, \tau_2\rangle, a^1, -\langle \varepsilon u^1, \nu\rangle)$. Here $(\,\cdot\,,\,\cdot\,)_{G,\varpi}$ is the inner product in $L_2(G; \mathbb{C}^8)$ with the weight $\varpi(x) = \operatorname{diag}(\varepsilon(x), 1, \mu(x), 1)$,

$$
(\mathcal{U}, \mathcal{V})_{G,\varpi} = (\varepsilon u^1, v^1)_G + (a^1, b^1)_G + (\mu u^2, v^2)_G + (a^2, b^2)_G,
$$

$(\,\cdot\,,\,\cdot\,)_G$ is the standard inner product in $L_2(G; \mathbb{C}^d)$, $d = 1, 3$, and $(\,\cdot\,,\,\cdot\,)_{\partial G}$ is the standard inner product in $L_2(\partial G; \mathbb{C}^4)$. (In what follows, in the notation for a space of vector-valued functions we do not indicate the number of components.) Problem (2.3) is selfadjoint with respect to the Green formula (2.4).

The asymptotics of solutions of problem (2.3) at infinity will be described in terms of "incoming and outgoing waves". These notions are necessary for the definition of the scattering matrix and the statement of "intrinsic radiation principle".

2.1. Incoming and outgoing wawes. In the cylinder $\Pi = \{(y,t) : y \in \Omega, t \in \mathbb{R}\}$, we consider a problem of the form (2.1), (2.2) with matrices $\varepsilon(y)$ and $\mu(y)$ independent of the axial coordinate:

$$\begin{aligned}
i\varepsilon(y)^{-1}\operatorname{curl} u^2(y,t) + i\,\nabla a^2(y,t) - ku^1(y,t) &= f^1(y,t),\\
-i\operatorname{div}(\mu(y)\,u^2(y,t)) - ka^1(y,t) &= h^1(y,t),\\
-i\mu(y)^{-1}\operatorname{curl} u^1(y,t) - i\,\nabla a^1(y,t) - ku^2(y,t) &= f^2(y,t), \quad (y,t)\in\Pi,\\
i\operatorname{div}(\varepsilon(y)\,u^1(y,t)) - ka^2(y,t) &= h^2(y,t),\\
-\langle u^1(y,t), \tau_2(y)\rangle &= g^1(y,t),\\
\langle u^1(y,t), \tau_1(y)\rangle &= g^2(y,t),\\
\langle \mu(y)u^2(y,t), \nu(y)\rangle &= g^3(y,t), \quad (y,t)\in\partial\Pi.\\
a^2(y,t) &= g^4(y,t),
\end{aligned}$$

We shall write this elliptic problem in the form

$$\begin{aligned}
A(y,D_y,D_t)\mathcal{U}(y,t) &= \mathcal{F}(y,t), \quad (y,t)\in\Pi,\\
B(y)\mathcal{U}(y,t) &= \mathcal{G}(y,t), \quad (y,t)\in\partial\Pi.
\end{aligned} \tag{2.5}$$

Problem (2.5) is selfadjoint with respect to the Green formula

$$(A\mathcal{U},\mathcal{V})_{\Pi,\varpi} + (B\mathcal{U},Q\mathcal{V})_{\partial\Pi} = (\mathcal{U},A\mathcal{V})_{\Pi,\varpi} + (Q\mathcal{U},B\mathcal{V})_{\partial\Pi} \tag{2.6}$$

with the same notation as in (2.4), G and $\mathcal{Q}$ being changed for Π and Q. We introduce the operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(y,D_y,\lambda)$,

$$\mathfrak{A}(y,D_y,\lambda)\varphi(y) = \exp(-i\lambda t)A(y,D_y,D_t)\left(\exp(i\lambda t)\varphi(y)\right), \quad y\in\Omega, \tag{2.7}$$

with the domain

$$\mathcal{D}(\mathfrak{A}) = \{\varphi \in C^1(\bar\Omega,\mathbb{C}^8) : B(y)\varphi(y) = 0,\, y\in\partial\Omega\}.$$

The spectrum of the pencil (2.7) consists of isolated eigenvalues $\lambda \in \mathbb{C}$ of finite full multiplicity. For some $c > 0$, the double cone $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < c|\operatorname{Re}\lambda|\}$ contains at most finitely many (if any) eigenvalues (see, e.g., [18]). If λ_0 is an eigenvalue of the pencil (2.7) and $\varphi^{(0)}$, $\varphi^{(1)}\dots\varphi^{(\varkappa-1)}$ is the Jordan chain corresponding to λ_0 with eigenvector $\varphi^{(0)}$ and generalized eigenvectors $\varphi^{(1)}\dots\varphi^{(\varkappa-1)}$, then the function

$$(y,t) \mapsto \exp(i\lambda_0 t)\sum_{q=0}^{\varkappa-1}\frac{(it)^q}{q!}\varphi^{(\varkappa-1-q)}(y) \tag{2.8}$$

satisfies the homogeneous problem (2.5). Let W stand for the linear space of solutions of the form (2.8) corresponding to all real eigenvalues of the pencil (2.7). The dimension of W is equal to the (finite) sum of the full multiplicities of the real eigenvalues. The elements of W are called waves.

Now we define incoming and outgoing waves. Let $\eta \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that $0 \le \eta(t) \le 1$ for $t \in \mathbb{R}$, $\eta(t) = 0$ for $t < 0$, and $\eta(t) = 1$ for $t > 1$; we set $\eta_T(t) = \eta(t-T)$. Moreover, we introduce $\Pi(S) = \{(y,t)\in\Pi : t < S\}$. For $T < S-1$ and functions $\mathcal{U}$ and $\mathcal{V}$ in W, we define the form

$$F_S(\eta_T\mathcal{U},\eta_T\mathcal{V}) = (A\eta_T\mathcal{U},\eta_T\mathcal{V})_{\Pi(S),\varpi} - (\eta_T\mathcal{U},A\eta_T\mathcal{V})_{\Pi(S),\varpi}. \tag{2.9}$$

By using the Green formula (2.6), it can be verified that $F_S(\eta_T\mathcal{U},\eta_T\mathcal{V})$ is independent of T and S (for $T < S-1$). In what follows, we drop the index S from the notation of the form (2.9). We have

$$iF(\eta_T\mathcal{V},\eta_T\mathcal{U}) = \overline{iF(\eta_T\mathcal{U},\eta_T\mathcal{V})}, \tag{2.10}$$

so that $iF(\eta_T\mathcal{U},\eta_T\mathcal{U})$ is a real number. A wave $\mathcal{U}\in W$ is said to be incoming from $+\infty$ (outgoing to $+\infty$) if the number $iF(\eta_T\mathcal{U},\eta_T\mathcal{U})$ is negative (positive). By definition, waves $\mathcal{U}$ and $\mathcal{V}$ are orthogonal if $iF(\eta_T\mathcal{U},\eta_T\mathcal{V})=0$.

Proposition 2.1 (Theorem 5.3.2 in [1]). 1. *The space W is of even dimension 2ς.*
2. *In W there exists a basis $\{\mathcal{U}_j^{\pm}\}_{j=1}^{\varsigma}$ subject to the relations*

$$iF(\eta_T\mathcal{U}_j^{\pm},\eta_T\mathcal{U}_l^{\mp})=0,\quad iF(\eta_T\mathcal{U}_j^{\pm},\eta_T\mathcal{U}_l^{\pm})=\mp\delta_{jl},\quad j,l=1,\dots,\varsigma,$$

where $\mathcal{U}_j^{+}$ and $\mathcal{U}_j^{-}$ are incoming and outgoing waves, respectively.
3. *Any orthogonal basis of W consists of incoming and outgoing waves and contains ς incoming waves and ς outgoing waves.*

We comment on Proposition 2.1. The form $iF(\,\cdot\,,\,\cdot\,)$ is Hermitian and nondegenerate. Therefore, any orthogonal basis consists of incoming and outgoing waves. Since the number of incoming (outgoing) waves is independent of the choice of such a basis, item 3 follows from item 2.

For each of the cylindrical outlets $\Pi_+^r, r=1,\dots,\mathcal{T}$, we consider the model problem (2.5) in the cylinder Π^r with "limit coefficients" $\varepsilon^r(y^r)$ and $\mu^r(y^r)$ and form the space W^r of waves in Π^r. Taking a large number T and $\mathcal{V}\in W^r$, we introduce the function

$$G\cap\Pi_+^r\ni(y^r,t^r)\mapsto\eta_T(t^r)\mathcal{V}(y^r,t^r)$$

on $G\cap\Pi_+^r$ and extend it by zero to the remaining part of the domain G. The resulting function is called a wave in the waveguide G. The linear span of the waves corresponding to all possible elements of the spaces W^r, $r=1,\dots,\mathcal{T}$, is called the space of waves in G and is denoted by $\mathbb{W}$.

Assume that $S>T+1$ and $\Pi_+^r(S)=\{(y^r,t^r)\in G\cap\Pi_+^r\,:\,t^r>S\}$. Also, let $G(S)=\overline{G\setminus\bigcup_{r=1}^{\mathcal{T}}\Pi_+^r(S)}$ be the domain G with truncated cylindrical outlets and $(\partial G)(S):=\partial(G(S))\cap\partial G$. For waves $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{W}$, we introduce the form

$$q_S(\mathcal{U},\mathcal{V})=(\mathcal{A}\mathcal{U},\mathcal{V})_{G(S),\varpi}+(\mathcal{B}\mathcal{U},\mathcal{Q}\mathcal{V})_{(\partial G)(S)}-(\mathcal{U},\mathcal{A}\mathcal{V})_{G(S),\varpi}-(\mathcal{Q}\mathcal{U},\mathcal{B}\mathcal{V})_{(\partial G)(S)}. \tag{2.11}$$

Unlike the form (2.9), the new form (2.11) contains some terms related to $(\partial G)(S)$, because the boundary operators $\mathcal{B}$ and $\mathcal{Q}$ depend on the axial variables t^r in Π_+^r and the waves do not satisfy the homogeneous boundary conditions. The form $q_S(\mathcal{U},\mathcal{V})$ is independent of the number T in the definition of the space $\mathbb{W}$, but it depends on S. However, the following claim is valid.

Proposition 2.2. *For any $\mathcal{U}$ and $\mathcal{V}$ in $\mathbb{W}$ there exists a finite limit*

$$q(\mathcal{U},\mathcal{V}):=\lim_{S\to\infty}q_S(\mathcal{U},\mathcal{V}). \tag{2.12}$$

If the supports of $\mathcal{U}$ and $\mathcal{V}$ are in $G\cap\Pi_+^r$ and $G\cap\Pi_+^s$, respectively, then

$$q(\mathcal{U},\mathcal{V})=\delta_{rs}F^r(\mathcal{U},\mathcal{V}),$$

where $F^r(\,\cdot\,,\,\cdot\,)$ is the form (2.9) corresponding to the cylinder Π^r.

Thus, the form (2.12) takes finite values on $\mathbb{W}$. A wave $\mathcal{U}\in\mathbb{W}$ is said to be incoming (outgoing) if $iq(\mathcal{U},\mathcal{U})$ is a negative (positive) number. We say that two waves $\mathcal{U}$ and $\mathcal{V}$ are orthogonal if $iq(\mathcal{U},\mathcal{V})=0$. Taking Propositions 2.1 and 2.2 into account, we arrive at the following statement.

Proposition 2.3.
1. *The dimension of the space* $\mathbb{W}$ *is even,* $\dim \mathbb{W} = 2\Upsilon := 2(\varsigma^1 + \dots + \varsigma^{\mathcal{T}})$, *where* $\varsigma^r = \dim W^r$.
2. *In the space* $\mathbb{W}$ *there exists a basis* $\mathcal{U}_1^+, \dots, \mathcal{U}_\Upsilon^+, \mathcal{U}_1^-, \dots, \mathcal{U}_\Upsilon^-$ *subject to the relations*

$$iq(\mathcal{U}_j^\pm, \mathcal{U}_l^\mp) = 0, \quad iq(\mathcal{U}_j^\pm, \mathcal{U}_l^\pm) = \mp\delta_{jl}, \quad j, l = 1, \dots, \Upsilon,$$

where $\mathcal{U}_j^+$ ($\mathcal{U}_j^-$) *is an incoming* (*outgoing*) *wave.*
3. *Any orthogonal basis of the space* $\mathbb{W}$ *consists of incoming and outgoing waves*; *in such a basis, the number of incoming waves is equal to the number of outgoing waves.*

2.2. Continuous spectrum eigenfunctions. Scattering matrix. Before a formal definition of the scattering matrix based on Proposition 2.4, we describe this matrix without any preliminaries. For the homogeneous problem (2.3), there exist solutions $\mathcal{Y}_j^+$, $j = 1, \dots, \Upsilon$, that are smooth in $\bar{G}$ and admit, for large $|x|$, the representations

$$\mathcal{Y}_j^+(x) = \mathcal{U}_j^+(x) + \sum_{l=1}^{\Upsilon} S_{jl}(k)\mathcal{U}_l^-(x) + O(\exp(-\alpha|x|)),$$

where $\mathcal{U}_1^+, \dots, \mathcal{U}_\Upsilon^+, \mathcal{U}_1^-, \dots, \mathcal{U}_\Upsilon^-$ is the basis from Proposition 2.3, α is a sufficiently small positive number, and k is the spectral parameter of problem (2.1), (2.2). The matrix $S(k) = (S_{jl})_{j,l=1}^{\Upsilon}$ is unitary and is called the scattering matrix.

The number k can be an eigenvalue of problem (2.3). Then this eigenvalue gives rise to finitely many linearly independent eigenfunctions. All such functions decay exponentially as $x \to \infty$. Generally, $\mathcal{Y}_j^+$ can be chosen up to adding an eigenfunction, but, however, the scattering matrix $S(k)$ is uniquely determined also in this case.

Now we pass to a more formal exposition. For $\beta \in \mathbb{R}$, introduce a positive function $\rho_\beta \in C^\infty(\bar{G})$ that coincides on $G \cap \Pi_+^r$ with the mapping $(y^r, t^r) \mapsto \exp(\beta t^r)$, $r = 1, \dots, \mathcal{T}$. By definition, the space $H_\beta^l(G)$, $l \geq 0$, is the closure of the linear space $C_c^\infty(\bar{G})$ in the norm

$$\|u; H_\beta^l(G)\| := \|\rho_\beta u; H^l(G)\| = \left(\sum_{|\sigma|=0}^{l} \int_G |D^\sigma(\rho_\beta u)|^2\, dx\right)^{1/2}. \tag{2.13}$$

Let $H_\beta^{l+1/2}(\partial G)$ denote the space of traces of the functions in $H_\beta^{l+1}(G)$ on ∂G.

The operator $\{\mathcal{A}, \mathcal{B}\}$ of problem (2.3) implements a continuous mapping

$$\mathcal{L}_\beta \colon H_\beta^{l+1}(G) \to H_\beta^l(G) \times H_\beta^{l+1/2}(\partial G) =: \mathcal{H}_\beta^l(G) \tag{2.14}$$

for any $\beta \in \mathbb{R}$ and $l = 0, 1, \dots$. If the line $\mathbb{R} + i\beta$ contains no eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^{\mathcal{T}}$, then the operator (2.14) is Fredholm (that is, its range is closed, while the kernel and cokernel are finite-dimensional). We choose $\alpha > 0$ so small that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq \alpha\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r$, $r = 1, \dots, \mathcal{T}$, except the real ones. Then $\dim\ker\mathcal{L}_{-\alpha} - \dim\ker\mathcal{L}_\alpha = \Upsilon$. A point k is called an eigenvalue of problem (2.3) if the space $\ker\mathcal{L}_\alpha$ is nontrivial; the elements of $\ker\mathcal{L}_\alpha$ are called eigenfunctions and $\dim\ker\mathcal{L}_\alpha$ is called the multiplicity of k. The eigenvalues are isolated and of finite multiplicity; the set of eigenvalues is called the point spectrum of problem (2.3). By definition, a point k belongs to the continuous spectrum of problem (2.3) if $\ker\mathcal{L}_{-\alpha} \neq \ker\mathcal{L}_\alpha$ for that k; the elements of the set $\ker\mathcal{L}_{-\alpha} \setminus \ker\mathcal{L}_\alpha$ are called the continuous spectrum eigenfunctions.

Proposition 2.4. *In the quotient space* $\ker \mathcal{L}_{-\alpha}/\ker \mathcal{L}_{\alpha}$, *there exists a basis with representatives* $\mathcal{Y}_1^+, \dots, \mathcal{Y}_\Upsilon^+$ *such that*

$$\mathcal{Y}_j^+ - \mathcal{U}_j^+ - \sum_{l=1}^{\Upsilon} S_{jl}(k)\mathcal{U}_l^- \in H_\alpha^1(G), \quad j = 1, \dots, \Upsilon. \tag{2.15}$$

The matrix $S(k)$ *with the elements* $S_{jl}(k)$ *is independent of the choice of the representatives mentioned above and is unitary; it is called the scattering matrix.*

2.3. The radiation principle. Let $\mathcal{U}_1^+, \dots, \mathcal{U}_\Upsilon^+, \mathcal{U}_1^-, \dots, \mathcal{U}_\Upsilon^-$ be the basis from Proposition 2.3, and let $\mathfrak{S}$ denote the linear span of the functions $\mathcal{U}_1^-, \dots, \mathcal{U}_\Upsilon^-$. On the space $\mathfrak{S} \dotplus H_\alpha^{l+1}(G)$, we consider the restriction $\mathbb{L}$ of the operator $\mathcal{L}_{-\alpha}$, which is a continuous mapping

$$\mathbb{L}\colon \mathfrak{S} \dotplus H_\alpha^{l+1}(G) \to H_\alpha^l(G) \times H_\alpha^{l+1/2}(\partial G).$$

Proposition 2.5. *Assume that* $\mathcal{Z}_1, \dots, \mathcal{Z}_\Theta$ *is a basis in the space* $\ker \mathcal{L}_\alpha$ *and* $\{\mathcal{F}, \mathcal{G}\} \in H_\alpha^l(G) \times H_\alpha^{l+1/2}(\partial G)$, *while*

$$(\mathcal{F}, \mathcal{Z}_j)_{G,\varpi} + (\mathcal{G}, \mathcal{Q}\mathcal{Z}_j)_{\partial G} = 0, \quad j = 1, \dots, \Theta,$$

where $\mathcal{Q}$ *is the operator in the Green formula* (2.4). *Then the following statements hold.*

1^0. *The equation* $\mathbb{L}\mathcal{U} = \{\mathcal{F}, \mathcal{G}\}$ *admits a solution* $\mathcal{U} \in \mathfrak{S} \dotplus H_\alpha^{l+1}(G)$ *determined up to an arbitrary element of* $\ker \mathcal{L}_\alpha$.

2^0. *We have*

$$\mathcal{V} := \mathcal{U} - c_1\mathcal{U}_1^- - \dots - c_\Upsilon \mathcal{U}_\Upsilon^- \in H_\alpha^{l+1}(G), \tag{2.16}$$

where

$$c_j = i(\mathcal{F}, \mathcal{Y}_j^-)_{G,\varpi} + i(\mathcal{G}, \mathcal{Q}\mathcal{Y}_j^-)_{\partial G}, \quad j = 1, \dots, \Upsilon, \tag{2.17}$$

the $\mathcal{Y}_j^-$ *are the elements of the space* $\ker \mathcal{L}_{-\alpha}$ *given by*

$$\mathcal{Y}_j^- = \sum_{l=1}^{\Upsilon} (S^{-1})_{jl} \mathcal{Y}_l^+, \tag{2.18}$$

and the $\mathcal{Y}_l^+$ *are the elements of* $\ker \mathcal{L}_{-\alpha}$ *subject to* (2.15).

3^0. *For the solution* $\mathcal{U}$ *we have*

$$\begin{aligned} &\|\mathcal{V}; H_\alpha^{l+1}(G)\| + |c_1| + \dots + |c_\Upsilon| \\ &\qquad \le \operatorname{const} \left(\|\mathcal{F}; H_\alpha^l(G)\| + \|\mathcal{G}; H_\alpha^{l+1/2}(\partial G)\| + \|\rho_\alpha \mathcal{V}; L_2(G)\| \right). \end{aligned} \tag{2.19}$$

A solution $\mathcal{U}_0$ *satisfying the additional conditions* $(\mathcal{U}_0, \mathcal{Z}_j)_{G,\varpi} = 0$, $j = 1, \dots, \Theta$, *is unique and satisfies estimate* (2.19) *with right-hand side replaced by*

$$\operatorname{const} \left(\|\mathcal{F}; H_\alpha^l(G)\| + \|\mathcal{G}; H_\alpha^{l+1/2}(\partial G)\| \right).$$

Relation (2.16) contains the representatives of outgoing waves only; such an inclusion is called the intrinsic radiation condition.

§3. Return to the Maxwell system

3.1. Model problem in a cylinder. In this section, we return to problem (2.5) in a cylinder Π with coefficients independent of the axial coordinate; we shall use the same notation as in Subsection 2.1.

Proposition 3.1. *Assume that $k \neq 0$ and a smooth function $\mathcal{U} = (u^1, a^1, u^2, a^2)$ satisfies the equations*

$$
\begin{aligned}
i\varepsilon^{-1}\operatorname{curl} u^2 + i\,\nabla a^2 - ku^1 = 0, \quad & -i\operatorname{div}(\mu\, u^2) - ka^1 = 0,\\
-i\mu^{-1}\operatorname{curl} u^1 - i\,\nabla a^1 - ku^2 = 0, \quad & \quad i\operatorname{div}(\varepsilon\, u^1) - ka^2 = 0
\end{aligned} \tag{3.1}
$$

in the cylinder Π and the boundary conditions on $\partial\Pi$

$$
\begin{aligned}
-\langle u^1, \tau_2\rangle = 0, \quad & \langle u^1, \tau_1\rangle = 0,\\
\langle \mu u^2, \nu\rangle = 0, \quad & a^2 = 0;
\end{aligned} \tag{3.2}
$$

it is not excluded that $a^1 \equiv 0$ and/or $a^2 \equiv 0$. Then the following statements hold.

1. *The functions a^1 and a^2 satisfy the homogeneous problems*

$$-\operatorname{div}\mu\nabla a^1 - k^2 a^1 = 0 \ \text{in}\ \Pi, \quad \langle \mu\nabla a^1, \nu\rangle = 0 \ \text{on}\ \partial\Pi, \tag{3.3}$$

$$-\operatorname{div}\varepsilon\nabla a^2 - k^2 a^2 = 0 \ \text{in}\ \Pi, \qquad a^2 = 0 \ \text{on}\ \partial\Pi. \tag{3.4}$$

2. *The representations $\mathcal{U} = \mathcal{U}_{\mathcal{N}} + \mathcal{U}_{\mathcal{D}} + \mathcal{U}_{\mathcal{M}}$ are valid, where*

$$
\begin{aligned}
\mathcal{U}_{\mathcal{N}} &= \big(0, a^1, -(i/k)\nabla a^1, 0\big),\\
\mathcal{U}_{\mathcal{D}} &= \big((i/k)\nabla a^2, 0, 0, a^2\big),\\
\mathcal{U}_{\mathcal{M}} &= \big((i/k)\varepsilon^{-1}\ \text{curl}\ u^2, 0, -(i/k)\mu^{-1}\ \text{curl}\ u^1, 0\big).
\end{aligned} \tag{3.5}
$$

The vector-valued functions $\mathcal{U}_{\mathcal{N}}, \mathcal{U}_{\mathcal{D}}$, and $\mathcal{U}_{\mathcal{M}}$ satisfy problem (3.1), (3.2), *and at least one of these functions is nonzero.*

Proof. We apply $\operatorname{div}\varepsilon$ to the first curl-equation (3.1) and add the second div-equation multiplied by $-ik$, obtaining

$$i\big(\operatorname{div}\varepsilon(x)\nabla + k^2\big)a^2(x) = 0, \quad x \in \Pi.$$

Similarly, from the second curl-equation and the first div-equation we obtain

$$-i\big(\operatorname{div}\mu(x)\nabla + k^2\big)a^1(x) = 0, \quad x \in \Pi.$$

The boundary condition (3.4) is contained in (3.2). To derive (3.3), we multiply the second curl-equation (3.1) by μ. We restrict the result to $\partial\Pi$ and then project it to the normal direction:

$$-i(\operatorname{curl} u^1)_\nu - i(\mu\nabla a^1)_\nu - k(\mu u^2)_\nu = 0 \ \text{ on }\ \partial\Pi. \tag{3.6}$$

To calculate $(\operatorname{curl} u^1)_\nu$, we introduce local orthogonal curvilinear coordinates in a small neighborhood of ∂G. At every point $x \in G$ of this neighborhood, we set

$$s_3 := -\operatorname{dist}(x, \partial G).$$

Let $x_0 \in \partial G$ be the projection of x to ∂G: $\operatorname{dist}(x, \partial G) = \operatorname{dist}(x, x_0)$. On ∂G, we introduce a smooth manifold structure: choose a covering of ∂G by neighborhoods and in each neighborhood introduce coordinates s_1, s_2: $x_0 = x_0(s_1, s_2)$. Assume that the coordinates (s_1, s_2) are orthogonal, that is, $\langle \partial x_0/\partial s_1, \partial x_0/\partial s_2\rangle = 0$. Then the coordinates (s_1, s_2, s_3) are also orthogonal. We define $H_j(s_1, s_2, s_3) := \langle \partial x/\partial s_j, \partial x/\partial s_j\rangle^{1/2}$ and $e_{s_j} := H_j^{-1}\partial x/\partial s_j$, $j = 1, 2, 3$. In (1.9), we set $\nu(s_1, s_2) := e_{s_3}(s_1, s_2, 0)$, $\tau_1(s_1, s_2) := e_{s_1}(s_1, s_2, 0)$, and $\tau_2(s_1, s_2) := e_{s_2}(s_1, s_2, 0)$. Now we have

$$(\operatorname{rot} u^1)_{s_3} = \big(H_1\, H_2\big)^{-1}\left(\frac{\partial}{\partial s_1}\big(H_2 u^1_{s_2}\big) - \frac{\partial}{\partial s_2}\big(H_1 u^1_{s_1}\big)\right). \tag{3.7}$$

In accordance with (3.2), $u^1_{s_2} = u^1_{s_1} = 0$, whence $\partial(H_2\, u^1_{s_2})/\partial s_1 = 0 = \partial(H_1\, u^1_{s_1})/\partial s_2$ on $\partial\Pi$. Therefore,

$$(\operatorname{rot} u^1)_\nu = 0 \ \text{ on }\ \partial\Pi. \tag{3.8}$$

Taking (3.6), (3.8), and the third identity in (3.2) into account, we obtain

$$(\mu\nabla a^1)_\nu = ik\,(\mu u^2)_\nu - (\operatorname{curl} u^1)_\nu = 0 \quad \text{on } \partial\Pi,$$

which completes the proof of (3.3) and (3.4). The expansion $\mathcal{U} = \mathcal{U}_\mathcal{N} + \mathcal{U}_\mathcal{D} + \mathcal{U}_\mathcal{M}$ immediately follows from the curl-equations (3.1). From (3.5), (3.3), and (3.4) it follows that the vector-valued functions $\mathcal{U}_\mathcal{N}$ and $\mathcal{U}_\mathcal{D}$ satisfy problem (3.1), (3.2). Therefore, $\mathcal{U}_\mathcal{M} = \mathcal{U} - \mathcal{U}_\mathcal{N} - \mathcal{U}_\mathcal{D}$ also satisfies this problem. □

Problems of the forms (3.3) and (3.4) are elliptic and selfadjoint with respect to the Green formulas in the cylinder Π and in the domain G. As in §2, for both problems one can introduce a space of waves, describe a well-posed problem with radiation conditions, and define a scattering matrix. The objects related to the Neumann problem (3.3) will be denoted by $\mathfrak{N}$, while those for the Dirichlet problem (3.4) by $\mathfrak{D}$. In particular, let $W_\mathfrak{N}$ and $W_\mathfrak{D}$ be the spaces of waves for (3.3) and (3.4), respectively, in the cylinder Π. The next result follows from Proposition 3.1.

Proposition 3.2. *Assume that $k \neq 0$. The space W admits decomposition into the direct sum of subspaces*

$$W = W_\mathcal{M} \dotplus W_\mathcal{N} \dotplus W_\mathcal{D}, \tag{3.9}$$

where

$$\begin{aligned} W_\mathcal{M} &:= \big\{(u^1, 0, u^2, 0) \in W\big\},\\ W_\mathcal{N} &:= \big\{(0, a^1, -(i/k)\nabla a^1, 0),\ a^1 \in W_\mathfrak{N}\big\},\\ W_\mathcal{D} &:= \big\{((i/k)\nabla a^2, 0, 0, a^2),\ a^2 \in W_\mathfrak{D}\big\}. \end{aligned} \tag{3.10}$$

The forms $F_\mathfrak{N}(\eta_T\,\cdot\,, \eta_T\,\cdot\,)$ and $F_\mathfrak{D}(\eta_T\,\cdot\,, \eta_T\,\cdot\,)$ take finite values on $W_\mathfrak{N}$ and $W_\mathfrak{D}$, respectively, where

$$\begin{aligned} F_\mathfrak{N}(u,v) &= \big(-(\operatorname{div}\mu\nabla + k^2)u, v\big)_{\Pi(S)} + \big(\langle\mu\nabla u, \nu\rangle, v\big)_{(\partial\Pi)(S)}\\ &\quad - \big(u, -(\operatorname{div}\mu\nabla + k^2)v\big)_{\Pi(S)} - \big(u, \langle\mu\nabla v, \nu\rangle\big)_{(\partial\Pi)(S)},\\ F_\mathfrak{D}(u,v) &= \big(-(\operatorname{div}\varepsilon\nabla + k^2)u, v\big)_{\Pi(S)} + \big(u, -\langle\varepsilon\nabla v, \nu\rangle\big)_{(\partial\Pi)(S)}\\ &\quad - \big(u, -(\operatorname{div}\varepsilon\nabla + k^2)v\big)_{\Pi(S)} - \big(-\langle\varepsilon\nabla u, \nu\rangle, v\big)_{(\partial\Pi)(S)}. \end{aligned}$$

As before, the values of the forms are independent of the choice of T and $S > T+1$. Also, we introduce the form

$$F(\mathcal{X}, \mathcal{Y}) = (A\mathcal{X}, \mathcal{Y})_{\Pi(S),\varpi} + (B\mathcal{X}, Q\mathcal{Y})_{(\partial\Pi)(S)} - (\mathcal{X}, A\mathcal{Y})_{\Pi(S),\varpi} - (Q\mathcal{X}, B\mathcal{Y})_{(\partial\Pi)(S)}, \tag{3.11}$$

which will be considered on functions of the form $\mathcal{X} = \eta_T\mathcal{U} + \mathcal{U}_0$, $\mathcal{Y} = \eta_T\mathcal{U} + c\mathcal{V}_0$, where $\mathcal{U}$, $\mathcal{V} \in W$ and $\mathcal{U}_0$, $\mathcal{V}_0 \in C_c^\infty(\overline{\Pi(T+1)})$ (smooth with compact support). Under these conditions, the form $F(\mathcal{X}, \mathcal{Y})$ is independent of T and S, and

$$F(\eta_T\mathcal{U} + \mathcal{U}_0, \eta_T\mathcal{V} + \mathcal{V}_0) = F(\eta_T\mathcal{U}, \eta_T\mathcal{V}). \tag{3.12}$$

Proposition 3.3.
1. *The spaces $W_\mathcal{M}$, $W_\mathcal{N}$, and $W_\mathcal{D}$ are orthogonal with respect to the form $F(\eta_T\,\cdot\,, \eta_T\,\cdot\,)$; in other words, for any $\mathcal{U}_\mathcal{M} \in W_\mathcal{M}$, $\mathcal{U}_\mathcal{N} \in W_\mathcal{N}$, and $\mathcal{U}_\mathcal{D} \in W_\mathcal{D}$ we have*

$$F(\eta_T\mathcal{U}_\mathcal{M}, \eta_T\mathcal{U}_\mathcal{N}) = 0, \quad F(\eta_T\mathcal{U}_\mathcal{N}, \eta_T\mathcal{U}_\mathcal{D}) = 0, \quad F(\eta_T\mathcal{U}_\mathcal{D}, \eta_T\mathcal{U}_\mathcal{M}) = 0.$$

2. *For any $\mathcal{U}_\mathcal{N} = \big(0, 1, -(i/k)\nabla, 0\big)u_\mathfrak{N}$ and $\mathcal{V}_\mathcal{N} = \big(0, 1, -(i/k)\nabla, 0\big)v_\mathfrak{N}$ in $W_\mathcal{N}$ we have*

$$F(\eta_T\mathcal{U}_\mathcal{N}, \eta_T\mathcal{V}_\mathcal{N}) = k^{-1} F_\mathfrak{N}(\eta_T u_\mathfrak{N}, \eta_T v_\mathfrak{N}). \tag{3.13}$$

3. *For any $\mathcal{U}_\mathcal{D} = \big((i/k)\nabla, 0, 0, 1\big)u_\mathfrak{D}$ and $\mathcal{V}_\mathcal{D} = \big((i/k)\nabla, 0, 0, 1\big)v_\mathfrak{D}$ in $W_\mathcal{D}$ we have*

$$F(\eta_T\mathcal{U}_\mathcal{D}, \eta_T\mathcal{V}_\mathcal{D}) = k^{-1} F_\mathfrak{D}(\eta_T u_\mathfrak{D}, \eta_T v_\mathfrak{D}). \tag{3.14}$$

Proof. Let $\mathcal{U}_{\mathcal{M}}$, $\mathcal{U}_{\mathcal{N}}$, and $\mathcal{U}_{\mathcal{D}}$ be arbitrary elements of $W_{\mathcal{M}}$, $W_{\mathcal{N}}$, and $W_{\mathcal{D}}$, respectively. By the definition (3.10) of the space $W_{\mathcal{N}}$,

$$\mathcal{U}_{\mathcal{N}} = (0, 1, -(i/k)\nabla, 0)u_{\mathfrak{N}}$$

with $u_{\mathfrak{N}}$ in the space $W_{\mathfrak{N}}$. Then

$$\eta_T\mathcal{U}_{\mathcal{N}} - \big(0, 1, -(i/k)\nabla, 0\big)(\eta_T u_{\mathfrak{N}}) = \big(0, 0, -(i/k)[\eta_T, \nabla], 0\big)u_{\mathfrak{N}} \in C_c^\infty(\bar{\Pi}) \tag{3.15}$$

with $[\eta_T, \nabla] = \eta_T\nabla - \nabla\,\eta_T$. From (3.12) and (3.15) it follows that when computing $F(\,\cdot\,,\,\cdot\,)$, we can change $\eta_T\mathcal{U}_{\mathcal{N}}$ for $\widetilde{\mathcal{U}}_{\mathcal{N}} := (0, 1, -(i/k)\nabla, 0)(\eta_T u_{\mathfrak{N}})$. Then

$$\begin{aligned}
A\widetilde{\mathcal{U}}_{\mathcal{N}} &= \big(0, -(1/k)\ \mathrm{div}\,\mu\nabla - k, 0, 0\big)(\eta_T u_{\mathfrak{N}}),\\
B\widetilde{\mathcal{U}}_{\mathcal{N}} &= \big(0, 0, -(i/k)\langle\mu\nabla\,\cdot\,, \nu\rangle, 0\big)(\eta_T u_{\mathfrak{N}}),\\
Q\widetilde{\mathcal{U}}_{\mathcal{N}} &= -i\big(-(i/k)\langle\nabla\,\cdot\,, \tau_1\rangle, -(i/k)\langle\nabla\,\cdot\,, \tau_2\rangle, 1, 0\big)(\eta_T u_{\mathfrak{N}}).
\end{aligned} \tag{3.16}$$

Moreover, we can change $\eta_T\mathcal{U}_{\mathcal{D}} = \eta_T((i/k)\nabla, 0, 0, 1)u_{\mathfrak{D}}$ for $\widetilde{\mathcal{U}}_{\mathcal{D}} := ((i/k)\nabla, 0, 0, 1)(\eta_T u_{\mathfrak{D}})$, obtaining

$$\begin{aligned}
A\widetilde{\mathcal{U}}_{\mathcal{D}} &= \big(0, 0, 0, -(1/k)\,\mathrm{div}\,\varepsilon\nabla - k\big)(\eta_T u_{\mathfrak{D}}),\\
B\widetilde{\mathcal{U}}_{\mathcal{D}} &= (0, 0, 0, 0),\\
Q\widetilde{\mathcal{U}}_{\mathcal{D}} &= -i\big(0, 0, 0, -(i/k)\langle\varepsilon\nabla\,\cdot\,, \nu\rangle\big)(\eta_T u_{\mathfrak{D}}).
\end{aligned}$$

Finally, we set $\widetilde{\mathcal{U}}_{\mathcal{M}} := \eta_T\mathcal{U}_{\mathcal{M}} = \eta_T(u^1, 0, u^2, 0)$. Then

$$\begin{aligned}
A\widetilde{\mathcal{U}}_{\mathcal{M}} &= (f^1, h^1, f^2, h^2),\\
B\widetilde{\mathcal{U}}_{\mathcal{M}} &= (0, 0, 0, 0),\\
Q\widetilde{\mathcal{U}}_{\mathcal{M}} &= -i\eta_T\big(\langle u^2, \tau_1\rangle, \langle u^2, \tau_2\rangle, 0, -\langle\varepsilon u^1, \nu\rangle\big),
\end{aligned}$$

while the functions f^1, f^2, h^1, h^2 have compact support and satisfy the compatibility conditions (1.10) (the proof of this fact is given in Proposition 3.15 below). First, we verify that $F(\widetilde{\mathcal{U}}_{\mathcal{N}}, \widetilde{\mathcal{U}}_{\mathcal{D}}) = 0$. For this, we calculate separately all terms in (2.9):

$$\begin{aligned}
&(B\widetilde{\mathcal{U}}_{\mathcal{N}}, Q\widetilde{\mathcal{U}}_{\mathcal{D}})_{(\partial\Pi)(S)} = 0, \qquad (A\widetilde{\mathcal{U}}_{\mathcal{N}}, \widetilde{\mathcal{U}}_{\mathcal{D}})_{\Pi(S),\varpi} = 0,\\
&(Q\widetilde{\mathcal{U}}_{\mathcal{N}}, B\widetilde{\mathcal{U}}_{\mathcal{D}})_{(\partial\Pi)(S)} = 0, \qquad (\widetilde{\mathcal{U}}_{\mathcal{N}}, A\widetilde{\mathcal{U}}_{\mathcal{D}})_{\Pi(S),\varpi} = 0.
\end{aligned}$$

Now we calculate $F(\widetilde{\mathcal{U}}_{\mathcal{M}}, \widetilde{\mathcal{U}}_{\mathcal{N}})$:

$$\begin{gathered}
(B\widetilde{\mathcal{U}}_{\mathcal{N}}, Q\widetilde{\mathcal{U}}_{\mathcal{M}})_{(\partial\Pi)(S)} = 0, \quad (A\widetilde{\mathcal{U}}_{\mathcal{N}}, \widetilde{\mathcal{U}}_{\mathcal{M}})_{\Pi(S),\varpi} = 0, \quad (Q\widetilde{\mathcal{U}}_{\mathcal{N}}, B\widetilde{\mathcal{U}}_{\mathcal{M}})_{(\partial\Pi)(S)} = 0,\\
(\widetilde{\mathcal{U}}_{\mathcal{N}}, A\widetilde{\mathcal{U}}_{\mathcal{M}})_{\Pi(S),\varpi} = (\eta_T u_{\mathfrak{N}}, h^1)_{\Pi(S)} + (-(i/k)\,\mu\,\nabla(\eta_T u_{\mathfrak{N}}), f^2)_{\Pi(S)}.
\end{gathered}$$

We show that the right-hand side $(\eta_T u_{\mathfrak{N}}, h^1)_{\Pi(S)} + (-(i/k)\,\mu\,\nabla(\eta_T u_{\mathfrak{N}}), f^2)_{\Pi(S)}$ of the forth identity vanishes. Since h^1 and f^2 have compact support and satisfy the compatibility conditions (1.10), we can integrate the second term by parts:

$$(\eta_T u_{\mathfrak{N}}, h^1)_{\Pi(S)} + \big(-(i/k)\,\mu\,\nabla(\eta_T u_{\mathfrak{N}}), f^2\big)_{\Pi(S)} = \big(\eta_T u_{\mathfrak{N}}, h^1 - (i/k)\ \mathrm{div}(\mu f^2)\big)_{\Pi(S)} = 0.$$

Thus, $F(\widetilde{\mathcal{U}}_{\mathcal{M}}, \widetilde{\mathcal{U}}_{\mathcal{N}}) = 0$. Similarly, $F(\widetilde{\mathcal{U}}_{\mathcal{D}}, \widetilde{\mathcal{U}}_{\mathcal{M}}) = 0$. It remains to verify items 2 and 3. From (3.16) it follows that

$$\begin{aligned}
(A\widetilde{\mathcal{U}}_{\mathcal{N}}, \widetilde{\mathcal{V}}_{\mathcal{N}})_{\Pi(S),\varpi} &= k^{-1}\big(-(\mathrm{div}\,\mu\nabla + k^2)(\eta_T u_{\mathfrak{N}}), \eta_T v_{\mathfrak{N}}\big)_{\Pi(S)},\\
(B\widetilde{\mathcal{U}}_{\mathcal{N}}, Q\widetilde{\mathcal{V}}_{\mathcal{N}})_{(\partial\Pi)(S)} &= k^{-1}\big(\langle\mu\nabla(\eta_T u_{\mathfrak{N}}), \nu\rangle, \eta_T v_{\mathfrak{N}}\big)_{(\partial\Pi)(S)},
\end{aligned}$$

where $\widetilde{\mathcal{V}}_{\mathcal{N}} = (0, 1, -(i/k)\nabla, 0)(\eta_T v_{\mathfrak{N}})$. Calculating the third and fourth terms in (3.11), we arrive at (3.13). In a similar way we prove (3.14). □

Proposition 3.4. *Assume that $k \neq 0$. Then the following statements hold.*

1. *In every space $W_{\mathcal{P}}$, $\mathcal{P} = \mathcal{M}, \mathcal{N}, \mathcal{D}$, there exists a basis $\{\mathcal{U}^{\pm}_{\mathcal{P},j}\}_{j=1}^{\varsigma_{\mathcal{P}}}$ such that*

$$iF(\eta_T \mathcal{U}^{\pm}_{\mathcal{P},j}, \eta_T \mathcal{U}^{\mp}_{\mathcal{P},l}) = 0, \quad iF(\eta_T \mathcal{U}^{\pm}_{\mathcal{P},j}, \eta_T \mathcal{U}^{\pm}_{\mathcal{P},l}) = \mp\delta_{jl}, \quad j, l = 1, \dots, \varsigma_{\mathcal{P}}. \tag{3.17}$$

2. *Any basis in $W_{\mathcal{P}}$ with $\mathcal{P} = \mathcal{M}, \mathcal{N}$, or $\mathcal{D}$ consisting of orthogonal waves contains $\varsigma_{\mathcal{P}}$ incoming waves and $\varsigma_{\mathcal{P}}$ outgoing waves.*

Proof. Recalling what was said after Proposition 2.1, we see that it suffices to prove item 1, which implies item 2. Without loss of generality we assume that $k > 0$. Proposition 2.1 is a result of the general elliptic theory, so that it is valid for problems (3.3) and (3.4). We use this proposition and choose a basis $\{u^{\pm}_{\mathfrak{N},j}\}_{j=1}^{\varsigma_{\mathfrak{N}}}$ in the space $W_{\mathfrak{N}}$ subject to the relations

$$iF_{\mathfrak{N}}(\eta_T u^{\pm}_{\mathfrak{N},j}, \eta_T u^{\mp}_{\mathfrak{N},l}) = 0, \quad iF_{\mathfrak{N}}(\eta_T u^{\pm}_{\mathfrak{N},j}, \eta_T u^{\pm}_{\mathfrak{N},l}) = \mp\delta_{jl}, \quad j, l = 1, \dots, \varsigma_{\mathfrak{N}}.$$

By (3.10), the dimension $2\varsigma_{\mathcal{N}}$ of $W_{\mathcal{N}}$ is equal to the dimension $2\varsigma_{\mathfrak{N}}$ of $W_{\mathfrak{N}}$, and the collection

$$\mathcal{U}^{\pm}_{\mathcal{N},j} = k^{1/2}(0, 1, -(i/k)\nabla, 0) u^{\pm}_{\mathfrak{N},j}, \quad j = 1, \dots, \varsigma_{\mathfrak{N}},$$

is a basis in $W_{\mathcal{N}}$. Using (3.13), we obtain

$$iF(\eta_T \mathcal{U}^{\pm}_{\mathcal{N},j}, \eta_T \mathcal{U}^{\mp}_{\mathcal{N},l}) = 0, \quad iF(\eta_T \mathcal{U}^{\pm}_{\mathcal{N},j}, \eta_T \mathcal{U}^{\pm}_{\mathcal{N},l}) = \mp\delta_{jl}, \quad j, l = 1, \dots, \varsigma_{\mathfrak{N}}.$$

Now we choose a basis $\{u^{\pm}_{\mathfrak{D},j}\}_{j=1}^{\varsigma_{\mathfrak{D}}}$ in the space $W_{\mathfrak{D}}$ satisfying

$$iF_{\mathfrak{D}}(\eta_T u^{\pm}_{\mathfrak{D},j}, \eta_T u^{\mp}_{\mathfrak{D},l}) = 0, \quad iF_{\mathfrak{D}}(\eta_T u^{\pm}_{\mathfrak{D},j}, \eta_T u^{\pm}_{\mathfrak{D},l}) = \mp\delta_{jl}, \quad j, l = 1, \dots, \varsigma_{\mathfrak{D}}.$$

In accordance with (3.10) and (3.14), the collection

$$\mathcal{U}^{\pm}_{\mathcal{D},j} = k^{1/2}((i/k)\nabla, 0, 0, 1) u^{\pm}_{\mathfrak{D},j}, \quad j = 1, \dots, \varsigma_{\mathfrak{D}},$$

is a basis in $W_{\mathcal{D}}$ subject to (3.17).

We turn to the space $W_{\mathcal{M}}$. The form iF is Hermitian and nondegenerate (see (2.10)). Its restriction $iF|W_{\mathcal{M}}$ to $W_{\mathcal{M}}$ keeps these properties. Therefore, in $W_{\mathcal{M}}$ we can find a basis in which the matrix of $iF|W_{\mathcal{M}}$ is diagonal and every diagonal element is real and nonzero. Such a basis consists of orthogonal incoming and outgoing waves (containing equally many both kinds of waves). We denote the number of incoming (outgoing) waves by $\varsigma_{\mathcal{M}}$ and obtain $\varsigma_{\mathcal{M}} = \varsigma - \varsigma_{\mathfrak{N}} - \varsigma_{\mathfrak{D}}$, whence $\dim W_{\mathcal{M}} = 2\varsigma_{\mathcal{M}}$. □

3.2. Orthogonal decompositions for the space of waves in G. In this section, for the space $\mathbb{W}$ of waves in G we introduce the decomposition

$$\mathbb{W} = \mathbb{W}_{\mathcal{M}} \dotplus \mathbb{W}_{\mathcal{N}} \dotplus \mathbb{W}_{\mathcal{D}} \tag{3.18}$$

(an analog of the decomposition (3.9)). We also discuss the properties of $\mathbb{W}$ similar to those of the space W of waves in a cylinder that were established in Subsection 3.1.

In each of the cylinders $\Pi^r = \Omega^r \times \mathbb{R}$, $r = 1, \dots, \mathcal{T}$, we consider problem (2.5) with coefficients ε^r and μ^r independent of the axial variable t^r. In accordance with (3.9), the space W^r of waves for this problem is decomposed into the sum

$$W^r = W^r_{\mathcal{M}} \dotplus W^r_{\mathcal{N}} \dotplus W^r_{\mathcal{D}}.$$

Let $\mathcal{P}$ stand for any of the indexes $\mathcal{M}, \mathcal{N}, \mathcal{D}$. For every $\mathcal{V}$ in $W^r_{\mathcal{P}}$, like in Subsection 2.2, we introduce

$$G \cap \Pi^r_+ \ni (y^r, t^r) \mapsto \eta_T(t^r)\mathcal{V}(y^r, t^r)$$

and continue it by zero to the rest of the domain G. The resulting function is called a wave in the waveguide G. We denote by $\mathbb{W}_{\mathcal{P}}$ the linear hull of the waves corresponding to elements of $W^r_{\mathcal{P}}$, $r = 1, \dots, \mathcal{T}$. Now, the definitions given and Proposition 3.2 lead to the following claim.

Proposition 3.5. *Let $k \neq 0$. Then the space $\mathbb{W}$ admits decomposition into the direct sum* (3.18).

We describe the spaces $\mathbb{W}_{\mathcal{M}}$, $\mathbb{W}_{\mathcal{N}}$, and $\mathbb{W}_{\mathcal{D}}$ in more detail. For this, we introduce the spaces of waves in G for the elliptic boundary value problems

$$-\operatorname{div}\mu\nabla a^1 - k^2a^1 = 0 \text{ in } G, \quad \langle \mu\nabla a^1, \nu\rangle = 0 \text{ on } \partial G, \tag{3.19}$$

$$-\operatorname{div}\varepsilon\nabla a^2 - k^2a^2 = 0 \text{ in } G, \quad a^2 = 0 \text{ on } \partial G \tag{3.20}$$

(see Subsection 2.2) and denote these spaces by $\mathbb{W}_{\mathfrak{N}}$ and $\mathbb{W}_{\mathfrak{D}}$, respectively.

Proposition 3.6. 1. *We have*

$$\mathbb{W}_{\mathcal{M}} = \{(u^1, 0, u^2, 0) \in \mathbb{W}\}.$$

2. *For every wave $\mathcal{U}_{\mathcal{N}} \in \mathbb{W}_{\mathcal{N}}$ there exists a unique wave $u_{\mathfrak{N}} \in \mathbb{W}_{\mathfrak{N}}$ such that*

$$\mathcal{U}_{\mathcal{N}} - (0, 1, -(i/k)\nabla, 0)u_{\mathfrak{N}} \in C_c^\infty(\bar{G}); \tag{3.21}$$

conversely, for every wave $u_{\mathfrak{N}} \in \mathbb{W}_{\mathfrak{N}}$ there exists a unique wave $\mathcal{U}_{\mathcal{N}} \in \mathbb{W}_{\mathcal{N}}$ satisfying (3.21).
3. *For every wave $\mathcal{U}_{\mathcal{D}} \in \mathbb{W}_{\mathcal{D}}$ there exists a unique wave $u_{\mathfrak{D}} \in \mathbb{W}_{\mathfrak{D}}$ such that*

$$\mathcal{U}_{\mathcal{D}} - ((i/k)\nabla, 0, 0, 1)u_{\mathfrak{D}} \in C_c^\infty(\bar{G}); \tag{3.22}$$

conversely, for every wave $u_{\mathfrak{D}} \in \mathbb{W}_{\mathfrak{D}}$ there exists a unique wave $\mathcal{U}_{\mathcal{D}} \in \mathbb{W}_{\mathcal{D}}$ satisfying (3.22).

Proof. 1. An arbitrary wave $\mathcal{U} \in \mathbb{W}$ can be represented in the form $\mathcal{U} = \mathcal{U}^1 + \cdots + \mathcal{U}^{\mathcal{T}}$, where $\mathcal{U}^r$ is a wave in $\mathbb{W}$ with support in $G \cap \Pi_+^r$, $r = 1, \dots, \mathcal{T}$. On $G \cap \Pi_+^r$, the wave $\mathcal{U}^r$ is given by the formula $\mathcal{U}^r(y^r, t^r) = \eta_T(t^r)\mathcal{V}^r(y^r, t^r)$ with $\mathcal{V}^r \in W^r$. The wave $\mathcal{U}$ belongs to $\mathbb{W}_{\mathcal{M}}$ if and only if $\mathcal{V}^r$ belongs to $W_{\mathcal{M}}^r$ for each $r = 1, \dots, \mathcal{T}$. On the other hand, the wave $\mathcal{U}$ is of the form $\mathcal{U} = (u^1, 0, u^2, 0)$ if so is every function $\mathcal{V}^r$, $r = 1, \dots, \mathcal{T}$. In view of (3.10), the two conditions are equivalent, which completes the proof of item 1.

We restrict ourselves to verifying item 2, because item 3 can be proved in much the same way. Let $\mathcal{U}_{\mathcal{N}}$ be an arbitrary element of the space $\mathbb{W}_{\mathcal{N}}$. We may assume that the wave $\mathcal{U}_{\mathcal{N}}$ is supported in $G \cap \Pi_+^r$ and is specified there by the formula $\mathcal{U}_{\mathcal{N}}(y^r, t^r) = \eta_T(t^r)\mathcal{V}_{\mathcal{N}}(y^r, t^r)$ with $\mathcal{V}_{\mathcal{N}} \in W_{\mathcal{N}}^r$. By Proposition 3.2,

$$\mathcal{V}_{\mathcal{N}} = (0, 1, -(i/k)\nabla, 0)v_{\mathfrak{N}}, \tag{3.23}$$

where $v_{\mathfrak{N}}$ is an element of the space $W_{\mathfrak{N}}^r$ of waves for problem (3.3) in the cylinder Π^r. On $G \cap \Pi_+^r$, we have

$$\eta_T\mathcal{V}_{\mathcal{N}} - \big(0, \eta_T v_{\mathfrak{N}}, -(i/k)\nabla(\eta_T v_{\mathfrak{N}}), 0\big) = \big(0, 0, -(i/k)[\eta_T, \nabla]v_{\mathfrak{N}}, 0\big) \in C_c^\infty(\overline{G \cap \Pi_+^r}). \tag{3.24}$$

Extending the function

$$u_{\mathfrak{N}}(y^r, t^r) = \eta_T(t^r)v_{\mathfrak{N}}(y^r, t^r), \quad (y^r, t^r) \in G \cap \Pi_+^r, \tag{3.25}$$

by zero to the remaining part of G, we get $u_{\mathfrak{N}} \in \mathbb{W}_{\mathfrak{N}}$. Formula (3.24) becomes

$$\mathcal{U}_{\mathcal{N}} - \big(0, u_{\mathfrak{N}}, -(i/k)\nabla u_{\mathfrak{N}}, 0\big) \in C_c^\infty(\bar{G}). \tag{3.26}$$

Conversely, let a wave $u_{\mathfrak{N}} \in \mathbb{W}_{\mathfrak{N}}$ be supported in one of the cylindrical ends $G \cap \Pi_+^r$ and let it be given there by formula (3.25) with $v_{\mathfrak{N}} \in W_{\mathfrak{N}}^r$. We define a function $\mathcal{V}_{\mathcal{N}} \in W_{\mathcal{N}}^r$ by (3.23) and introduce the wave $\mathcal{U}_{\mathcal{N}} \in \mathbb{W}_{\mathcal{N}}$ by $\mathcal{U}_{\mathcal{N}}(y^r, t^r) = \eta_T(t^r)\mathcal{V}_{\mathcal{N}}(y^r, t^r)$. The relation (3.26) is true, and the proof is complete. $\square$

For problems (3.19) and (3.20), we introduce forms similar to the form (2.12). For any $u, v \in \mathbb{W}_{\mathfrak{N}}$ there exists a finite limit

$$q_{\mathfrak{N}}(u,v) = \lim_{S\to\infty} \big\{ \big(-(\operatorname{div}\mu\nabla + k^2)u, v\big)_{G(S)} + (\langle \mu\nabla u, \nu\rangle, v)_{(\partial G)(S)} - \big(u, -(\operatorname{div}\mu\nabla + k^2)v\big)_{G(S)} - (u, \langle \mu\nabla v, \nu\rangle)_{(\partial G)(S)} \big\},$$

where $G(S)$ and $(\partial G)(S)$ are the same as in (2.11). For any $u, v \in \mathbb{W}_{\mathfrak{D}}$ there exists a finite limit

$$q_{\mathfrak{D}}(u,v) = \lim_{S\to\infty} \big\{ \big(-(\operatorname{div}\varepsilon\nabla + k^2)u, v\big)_{G(S)} + (u, -\langle \varepsilon\nabla v, \nu\rangle)_{(\partial G)(S)} - \big(u, -(\operatorname{div}\varepsilon\nabla + k^2)v\big)_{G(S)} - (-\langle \varepsilon\nabla u, \nu\rangle, v)_{(\partial G)(S)} \big\}.$$

Propositions 2.2 and 3.3 imply the following assertion.

Proposition 3.7.
1. *The spaces $\mathbb{W}_{\mathcal{M}}$, $\mathbb{W}_{\mathcal{N}}$, and $\mathbb{W}_{\mathcal{D}}$ are pairwise orthogonal with respect to the form $q(\,\cdot\,,\,\cdot\,)$, i.e., for any $\mathcal{U}_{\mathcal{M}} \in \mathbb{W}_{\mathcal{M}}$, $\mathcal{U}_{\mathcal{N}} \in \mathbb{W}_{\mathcal{N}}$, and $\mathcal{U}_{\mathcal{D}} \in \mathbb{W}_{\mathcal{D}}$ we have*

$$q(\mathcal{U}_{\mathcal{M}}, \mathcal{U}_{\mathcal{N}}) = 0, \quad q(\mathcal{U}_{\mathcal{N}}, \mathcal{U}_{\mathcal{D}}) = 0, \quad q(\mathcal{U}_{\mathcal{D}}, \mathcal{U}_{\mathcal{M}}) = 0.$$

2. *For any $\mathcal{U}_{\mathcal{N}}$ and $\mathcal{V}_{\mathcal{N}} \in \mathbb{W}_{\mathcal{N}}$ we have*

$$q(\mathcal{U}_{\mathcal{N}}, \mathcal{V}_{\mathcal{N}}) = k^{-1}\, q_{\mathfrak{N}}(u_{\mathfrak{N}}, v_{\mathfrak{N}}), \tag{3.27}$$

where $u_{\mathfrak{N}}$ and $v_{\mathfrak{N}} \in \mathbb{W}_{\mathfrak{N}}$ are the waves of problem (3.19) *related to $\mathcal{U}_{\mathcal{N}}$ and $\mathcal{V}_{\mathcal{N}}$ as in* (3.21).

3. *For any $\mathcal{U}_{\mathcal{D}}$ and $\mathcal{V}_{\mathcal{D}} \in \mathbb{W}_{\mathcal{D}}$, we have*

$$q(\mathcal{U}_{\mathcal{D}}, \mathcal{V}_{\mathcal{D}}) = k^{-1}\, q_{\mathfrak{D}}(u_{\mathfrak{D}}, v_{\mathfrak{D}}), \tag{3.28}$$

where $u_{\mathfrak{D}}$ and $v_{\mathfrak{D}} \in \mathbb{W}_{\mathfrak{D}}$ are the waves of problem (3.20) *related to $\mathcal{U}_{\mathcal{D}}$ and $\mathcal{V}_{\mathcal{D}}$ as in* (3.22).

The proof is left to the reader. Propositions 3.4 and 2.2 imply the following statement.

Proposition 3.8.
1. *For every $\mathcal{P} = \mathcal{M}, \mathcal{N}$, or $\mathcal{D}$, in the space $\mathbb{W}_{\mathcal{P}}$ there exists a basis $\{\mathcal{U}^{\pm}_{\mathcal{P},j}\}_{j=1}^{\Upsilon_{\mathcal{P}}}$ such that*

$$iq(\mathcal{U}^{\pm}_{\mathcal{P},j}, \mathcal{U}^{\mp}_{\mathcal{P},l}) = 0, \quad iq(\mathcal{U}^{\pm}_{\mathcal{P},j}, \mathcal{U}^{\pm}_{\mathcal{P},l}) = \mp\delta_{jl}, \quad j,l = 1,\dots,\Upsilon_{\mathcal{P}}. \tag{3.29}$$

2. *Any basis of the space $\mathbb{W}_{\mathcal{P}}$ consisting of orthogonal waves contains $\Upsilon_{\mathcal{P}}$ incoming and $\Upsilon_{\mathcal{P}}$ outgoing waves, $\mathcal{P} = \mathcal{M}, \mathcal{N}, \mathcal{D}$.*

Verification of this proposition is also left to the reader. Like in Proposition 3.4, the basis of the space $\mathbb{W}_{\mathcal{N}}$ ($\mathbb{W}_{\mathcal{D}}$) in Proposition 3.8 is related to a similar basis of the space $\mathbb{W}_{\mathfrak{N}}$ ($\mathbb{W}_{\mathfrak{D}}$). For definiteness, let $k > 0$. By Proposition 3.6, the dimensions $2\Upsilon_{\mathcal{N}}$ and $2\Upsilon_{\mathfrak{N}}$ of $\mathbb{W}_{\mathcal{N}}$ and $\mathbb{W}_{\mathfrak{N}}$ are the same, and the bases of these spaces are related to each other by (3.21). Thus, the bases $\{\mathcal{U}^{\pm}_{\mathcal{N},j}\}_{j=1}^{\Upsilon_{\mathcal{N}}}$ and $\{u^{\pm}_{\mathfrak{N},j}\}_{j=1}^{\Upsilon_{\mathfrak{N}}}$ in $\mathbb{W}_{\mathcal{N}}$ and $\mathbb{W}_{\mathfrak{N}}$ satisfy

$$\mathcal{U}^{\pm}_{\mathcal{N},j} - k^{1/2}(0, 1, -(i/k)\nabla, 0)u^{\pm}_{\mathfrak{N},j} \in C^{\infty}_c(\bar{G}), \quad j = 1,\dots,\Upsilon_{\mathcal{N}}. \tag{3.30}$$

From (3.29) and (3.27) it follows that

$$iq_{\mathfrak{N}}(u^{\pm}_{\mathfrak{N},j}, u^{\mp}_{\mathfrak{N},l}) = 0, \quad iq_{\mathfrak{N}}(u^{\pm}_{\mathfrak{N},j}, u^{\pm}_{\mathfrak{N},l}) = \mp\delta_{jl}, \quad j,l = 1,\dots,\Upsilon_{\mathfrak{N}}.$$

For the bases $\{\mathcal{U}^{\pm}_{\mathcal{D},j}\}_{j=1}^{\Upsilon_{\mathcal{D}}}$ and $\{u^{\pm}_{\mathfrak{D},j}\}_{j=1}^{\Upsilon_{\mathfrak{D}}}$ in $\mathbb{W}_{\mathcal{D}}$ and $\mathbb{W}_{\mathfrak{D}}$ we have (cf. (3.22))

$$\mathcal{U}^{\pm}_{\mathcal{D},j} - k^{1/2}((i/k)\nabla, 0, 0, 1)u^{\pm}_{\mathfrak{D},j} \in C^{\infty}_c(\bar{G}), \quad j = 1,\dots,\Upsilon_{\mathcal{D}}. \tag{3.31}$$

From (3.29) and (3.28) we see that

$$iq_{\mathfrak{D}}(u^{\pm}_{\mathfrak{D},j}, u^{\mp}_{\mathfrak{D},l}) = 0, \quad iq_{\mathfrak{D}}(u^{\pm}_{\mathfrak{D},j}, u^{\pm}_{\mathfrak{D},l}) = \mp\delta_{jl}, \quad j,l = 1,\dots,\Upsilon_{\mathfrak{D}}.$$

3.3. Scattering matrix for problem (1.8)**,** (1.9)**.** In this subsection, we make use of the information obtained for the elliptic problem (2.3) to define the point and continuous spectra of the original Maxwell system (1.8), (1.9) and to introduce a scattering matrix on the continuous spectrum. For this, in Proposition 3.10 we represent the kernel $\ker \mathcal{L}_\beta$ of the operator (2.14) as a direct sum of subspaces,

$$\ker \mathcal{L}_\beta = E_{\mathcal{M}}(\beta) \dotplus E_{\mathcal{N}}(\beta) \dotplus E_{\mathcal{D}}(\beta). \tag{3.32}$$

This decomposition is similar to that in (3.9) for the kernel of the model problem (2.5) in a cylinder. With the help of the decomposition (3.32), it is established that the scattering matrix for problem (2.3) is block-diagonal, and that one of its blocks plays the role of the scattering matrix for problem (1.8), (1.9).

Proposition 3.9. *Let $k \neq 0$, and let the line $\{\lambda \in \mathbb{C} : \operatorname{Im} \lambda = \beta\}$ be free from the eigenvalues of the pencils $\mathfrak{A}^r$, $r = 1, \dots, \mathcal{T}$. Assume that $\mathcal{U} = (u^1, a^1, u^2, a^2) \in H^1_\beta(G)$ satisfies the system*

$$\begin{aligned} i\,\varepsilon^{-1} \operatorname{curl} u^2 + i\,\nabla a^2 - ku^1 = 0, &\quad -i \operatorname{div}(\mu u^2) - ka^1 = 0, \\ -i\,\mu^{-1} \operatorname{curl} u^1 - i\,\nabla a^1 - ku^2 = 0, &\quad i \operatorname{div}(\varepsilon u^1) - ka^2 = 0 \end{aligned} \tag{3.33}$$

in G and the boundary conditions:

$$-\langle u^1, \tau_2 \rangle = 0, \quad \langle u^1, \tau_1 \rangle = 0, \quad \langle \mu\, u^2, \nu \rangle = 0, \quad a^2 = 0 \tag{3.34}$$

on ∂G. Then the following statements hold.
1. *The vector-valued function $\mathcal{U}$ belongs to $H^l_\beta(G)$ for $l = 1, 2, \dots$, and the functions a^1 and a^2 satisfy problems* (3.19) *and* (3.20), *respectively.*
2. *The representation $\mathcal{U} = \mathcal{U}_{\mathcal{M}} + \mathcal{U}_{\mathcal{N}} + \mathcal{U}_{\mathcal{D}}$ is valid with*

$$\begin{aligned} \mathcal{U}_{\mathcal{M}} &:= \big((i/k)\,\varepsilon^{-1} \operatorname{rot} u^2, 0, -(i/k)\,\mu^{-1} \operatorname{rot} u^1, 0\big), \\ \mathcal{U}_{\mathcal{N}} &:= \big(0, a^1, -(i/k)\nabla a^1, 0\big), \\ \mathcal{U}_{\mathcal{D}} &:= \big((i/k)\nabla a^2, 0, 0, a^2\big). \end{aligned} \tag{3.35}$$

The vector-valued functions $\mathcal{U}_{\mathcal{M}}$, $\mathcal{U}_{\mathcal{N}}$, and $\mathcal{U}_{\mathcal{D}}$ belong to $H^l_\beta(G)$ for $l = 1, 2, \dots$ and satisfy problem (3.33), (3.34); *at least one of these vector-valued functions does not vanish identically.*

Proof. Since the boundary value problem (3.33), (3.34) is elliptic and the line $\mathbb{R} + i\beta$ is free from the eigenvalues of the pencils $\mathfrak{A}^r$, $r = 1, \dots, \mathcal{T}$, the solution $\mathcal{U} \in H^1_\beta(G)$ of the problem belongs to $H^l_\beta(G)$ for every $l = 1, 2, \dots$. Now, the proof can repeat the proof of Proposition 3.1 word for word, with Π changed for G. $\square$

Proposition 3.9 allows us to establish a relationship between the kernel of problem (2.3) and the kernels of problems (3.19) and (3.20). For this, we introduce the operator $\mathcal{L}^{\mathfrak{N}}_\beta$ of the boundary value problem (3.19),

$$\mathcal{L}^{\mathfrak{N}}_\beta \colon H^2_\beta(G) \to H^0_\beta(G) \times H^{1/2}_\beta(\partial G), \tag{3.36}$$

where $H^l_\beta(G)$ is the space of (scalar) functions with norm (2.13), and $H^{l+1/2}_\beta(\partial G)$ stands for the space of traces on ∂G of the functions in $H^{l+1}_\beta(G)$, $l = 0, 1, \dots$. We also introduce the operator

$$\mathcal{L}^{\mathfrak{D}}_\beta \colon H^2_\beta(G) \to H^0_\beta(G) \times H^{3/2}_\beta(\partial G) \tag{3.37}$$

of the boundary value problem (3.20). We define the continuous and point spectra of problem (3.19) (of problem (3.20)) by replacing, in the definitions of Subsection 2.2, the operator $\mathcal{L}_\beta$ occurring in (2.14) with the operator $\mathcal{L}^{\mathfrak{N}}_\beta$ given by (3.36) (with the operator $\mathcal{L}^{\mathfrak{D}}_\beta$ given by (3.37)). Proposition 3.9 yields the following statement.

Proposition 3.10. *Let $k \neq 0$, and let the line $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \beta\}$ be free from the eigenvalues of the pencils $\mathfrak{A}^r$, $r = 1, \dots, \mathcal{T}$. Then the kernel $\ker \mathcal{L}_\beta$ of the operator (2.14) admits decomposition in the direct sum*

$$\ker \mathcal{L}_\beta = E_{\mathcal{M}}(\beta) \dotplus E_{\mathcal{N}}(\beta) \dotplus E_{\mathcal{D}}(\beta)$$

of the subspaces spanned, respectively, by functions $\mathcal{U}_{\mathcal{M}}$, $\mathcal{U}_{\mathcal{N}}$, and $\mathcal{U}_{\mathcal{D}} \in H^1_\beta(G)$ of the form (3.35). Moreover,

$$\begin{aligned} E_{\mathcal{M}}(\beta) &= \big\{(u^1, 0, u^2, 0) \in \ker \mathcal{L}_\beta\big\}, \\ E_{\mathcal{N}}(\beta) &= \big\{(0, a^1, -(i/k)\nabla a^1, 0),\ a^1 \in \ker \mathcal{L}^{\mathfrak{N}}_\beta\big\}, \\ E_{\mathcal{D}}(\beta) &= \big\{((i/k)\nabla a^2, 0, 0, a^2),\ a^2 \in \ker \mathcal{L}^{\mathfrak{D}}_\beta\big\}. \end{aligned} \tag{3.38}$$

Let α be the exponent mentioned in Subsection 2.2 (such that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq \alpha\}$ contains only real eigenvalues of the pencils $\mathfrak{A}^r$, $r = 1, \dots, \mathcal{T}$). Recall that the spectral parameter k belongs to the point (continuous) spectrum of problem (2.3) if and only if $\ker \mathcal{L}_\alpha \neq \{0\}$ ($\ker \mathcal{L}_{-\alpha} \neq \ker \mathcal{L}_\alpha$). The point spectrum of problem (2.3) consists of isolated eigenvalues of finite multiplicity. By Proposition 3.10, for such eigenvalues we have the decomposition

$$\ker \mathcal{L}_\alpha = E_{\mathcal{M}}(\alpha) \dotplus E_{\mathcal{N}}(\alpha) \dotplus E_{\mathcal{D}}(\alpha); \tag{3.39}$$

generally speaking, some of the spaces on the right may be trivial. The space $E_{\mathcal{M}}(\alpha)$ consists of solutions of the form $\mathcal{U} = (u^1, 0, u^2, 0)$ of the homogeneous problem (2.3), and the functions $U = (u^1, u^2)$ satisfy the homogeneous problem (1.8), (1.9). The correspondence $U \leftrightarrow \mathcal{U}$ allows us to identify the eigenspace of problem (1.8), (1.9) with the space $E_{\mathcal{M}}(\alpha)$. By definition, a point k is an eigenvalue of problem (1.8), (1.9) if the space $E_{\mathcal{M}}(\alpha)$ is nontrivial. The second (third) formula in (3.38) implements a one-to-one map from the space $E_{\mathcal{N}}(\alpha)$ ($E_{\mathcal{D}}(\alpha)$) to the eigenspace $\ker \mathcal{L}^{\mathfrak{N}}_\alpha$ ($\ker \mathcal{L}^{\mathfrak{D}}_\alpha$) of problem (3.19) (of problem (3.20)). Thus, a point k is an eigenvalue of problem (2.3) if and only if k is an eigenvalue of at least one of the three problems: problem (3.19), problem (3.20), and problem (1.8), (1.9).

Suppose that k belongs to the continuous spectrum of problem (2.3). Applying Proposition 3.10 with $\beta = -\alpha$, we see that

$$\ker \mathcal{L}_{-\alpha} = E_{\mathcal{M}}(-\alpha) \dotplus E_{\mathcal{N}}(-\alpha) \dotplus E_{\mathcal{D}}(-\alpha).$$

By (3.38), k is a point of the continuous spectrum for problem (1.8), (1.9) (problem (3.19) or problem (3.20)) if and only if the condition $E_{\mathcal{P}}(-\alpha) \neq E_{\mathcal{P}}(\alpha)$ is fulfilled with $\mathcal{P} = \mathcal{M}$ ($\mathcal{N}$ or $\mathcal{D}$). Hence, a point k belongs to the continuous spectrum of problem (2.3) if k belongs to the continuous spectrum of at least one of the three problems: problem (3.19), problem (3.20), and problem (1.8), (1.9).

To define a scattering matrix for problem (1.8), (1.9), we are going to prove that in the quotient space $E_{\mathcal{M}}(-\alpha)/E_{\mathcal{M}}(\alpha)$ there exists a special basis subject to relations of the form (2.15). In Subsection 3.2, it was proved that the space of waves $\mathbb{W}$ is the direct sum (3.18) of subspaces that are pairwise orthogonal with respect to the form $q(\,\cdot\,,\,\cdot\,)$. In each of the spaces $\mathbb{W}_{\mathcal{P}}$, $\mathcal{P} = \mathcal{M}, \mathcal{N}, \mathcal{D}$, we chose a basis $\{\mathcal{U}^{\pm}_{\mathcal{P},j}\}_{j=1}^{\Upsilon_{\mathcal{P}}}$ subject to the orthogonality and normalization conditions (3.29). In Proposition 3.11 below we shall show that every incoming wave $\mathcal{U}^{+}_{\mathcal{P},j}$ is scattered only to the outgoing waves $\mathcal{U}^{-}_{\mathcal{P},l}$, $l = 1, \dots, \Upsilon_{\mathcal{P}}$, from the same subspace $\mathbb{W}_{\mathcal{P}}$. In other words, the scattering matrix for problem (2.3) is block-diagonal with three blocks on the principal diagonal. Since this matrix is unitary, each of the blocks is unitary.

We define the scattering matrix for problem (2.3) in accordance with Proposition 2.4. The basis $\{\mathcal{U}^{\pm}_{\mathcal{P},j}\}$ in the space of waves $\mathbb{W}$ is now enumerated with two indices $\mathcal{P}$ and j, where $\mathcal{P} = \mathcal{M},\mathcal{N},\mathcal{D}$ and $j = 1,\dots,\Upsilon_{\mathcal{P}}$. We want to introduce a consistent enumeration of continuous spectrum eigenfunctions. We denote by $\mathcal{Y}^{+}_{\mathcal{P},j}$ the continuous spectrum eigenfunction with the asymptotics (2.15) that contains the wave $\mathcal{U}^{+}_{\mathcal{P},j}$. Let $S^{\mathcal{P}\mathcal{R}}_{jl}$ stand for the coefficient of the outgoing wave $\mathcal{U}^{-}_{\mathcal{R},l}$ in the asymptotics of $\mathcal{Y}^{+}_{\mathcal{P},j}$: $\mathcal{N},\mathcal{D}$, $j = 1,\dots,\Upsilon_{\mathcal{P}}$, $l = 1,\dots,\Upsilon_{\mathcal{R}}$:

$$\mathcal{Y}^{+}_{\mathcal{P},j} - \mathcal{U}^{+}_{\mathcal{P},j} - \sum_{\mathcal{R}=\mathcal{M},\mathcal{N},\mathcal{D}} \sum_{l=1}^{\Upsilon_{\mathcal{R}}} S^{\mathcal{P}\mathcal{R}}_{jl}\mathcal{U}^{-}_{\mathcal{R},l} \in H^1_\alpha(G), \quad j = 1,\dots,\Upsilon_{\mathcal{P}},\ \mathcal{P} = \mathcal{M},\mathcal{N},\mathcal{D}. \tag{3.40}$$

The matrix $S^{\mathcal{P}\mathcal{R}}$, $\mathcal{P},\mathcal{R} = \mathcal{M},\mathcal{N},\mathcal{D}$, is a block of the scattering matrix

$$S = \begin{pmatrix} S^{\mathcal{M}\mathcal{M}} & S^{\mathcal{M}\mathcal{N}} & S^{\mathcal{M}\mathcal{D}} \\ S^{\mathcal{N}\mathcal{M}} & S^{\mathcal{N}\mathcal{N}} & S^{\mathcal{N}\mathcal{D}} \\ S^{\mathcal{D}\mathcal{M}} & S^{\mathcal{D}\mathcal{N}} & S^{\mathcal{D}\mathcal{D}} \end{pmatrix}. \tag{3.41}$$

It should be noted that the index $\mathcal{P}$ does not guarantee that $\mathcal{Y}^{+}_{\mathcal{P},j} \in E_{\mathcal{P}}(-\alpha)$ and serves so far for the consistent enumeration of the continuous spectrum eigenfunctions $\mathcal{Y}^{+}_{\mathcal{P},j}$ and the waves $\mathcal{U}^{+}_{\mathcal{P},j}$. We are going to establish the fact that $\mathcal{Y}^{+}_{\mathcal{P},j} \in E_{\mathcal{P}}(-\alpha)$ in Propositions 3.12 and 3.13 later on.

Proposition 3.11.
Let $k > 0$. Then the following statement hold.
1. *The block $S^{\mathcal{N}} := S^{\mathcal{N}\mathcal{N}}$ of the matrix S in* (3.41) *coincides with the scattering matrix for the Neumann problem* (3.19).
2. *The block $S^{\mathcal{D}} := S^{\mathcal{D}\mathcal{D}}$ of the matrix S in* (3.41) *coincides with the scattering matrix for the Dirichlet problem* (3.20).
3. *The functions $\mathcal{Y}^{+}_{\mathcal{P},j}$, $\mathcal{P} = \mathcal{M},\mathcal{N},\mathcal{D}$, $j = 1,\dots,\Upsilon_{\mathcal{P}}$, in Proposition* 2.4 *are such that*

$$\mathcal{Y}^{+}_{\mathcal{P},j} - \mathcal{U}^{+}_{\mathcal{P},j} - \sum_{l=1}^{\Upsilon_{\mathcal{P}}} S^{\mathcal{P}}_{jl}\mathcal{U}^{-}_{\mathcal{P},l} \in H^1_\alpha(G), \quad j = 1,\dots,\Upsilon_{\mathcal{P}},\ \mathcal{P} = \mathcal{M},\mathcal{N},\mathcal{D}. \tag{3.42}$$

Thus, the matrix S is block-diagonal $S = \operatorname{diag}(S^{\mathcal{M}}, S^{\mathcal{N}}, S^{\mathcal{D}})$, and the matrices $S^{\mathcal{N}}$, $S^{\mathcal{D}}$, and $S^{\mathcal{M}} := S^{\mathcal{M}\mathcal{M}}$ are unitary.

Proof. Items 1 and 2 can be proved in much the same way and here we only prove item 1. We employ the relationship between problem (2.3) and the Neumann problem (3.19) established in Propositions 3.6 and 3.10. Given the basis $\{\mathcal{U}^{\pm}_{\mathcal{N},j}\}_{j=1}^{\Upsilon_{\mathcal{N}}}$ in $\mathbb{W}_{\mathcal{N}}$, we can choose a basis $\{u^{\pm}_{\mathfrak{N},j}\}_{j=1}^{\Upsilon_{\mathfrak{N}}}$ in $\mathbb{W}_{\mathfrak{N}}$ to satisfy relations (3.30). (Recall that $\Upsilon_{\mathcal{N}} = \Upsilon_{\mathfrak{N}}$, that is, the dimensions of the spaces $\mathbb{W}_{\mathcal{N}}$ and $\mathbb{W}_{\mathfrak{N}}$ coincide.) By Proposition 2.4, in the space $\ker \mathcal{L}^{\mathfrak{N}}_{-\alpha}/\ker \mathcal{L}^{\mathfrak{N}}_{\alpha}$ there exists a basis with representatives $y^{+}_{\mathfrak{N},1},\dots,y^{+}_{\mathfrak{N},\Upsilon_{\mathfrak{N}}} \in \ker \mathcal{L}^{\mathfrak{N}}_{-\alpha}$ subject to

$$y^{+}_{\mathfrak{N},j} - u^{+}_{\mathfrak{N},j} - \sum_{l=1}^{\Upsilon_{\mathfrak{N}}} S^{\mathfrak{N}}_{jl} u^{-}_{\mathfrak{N},l} \in H^2_\alpha(G), \quad j = 1,\dots,\Upsilon_{\mathfrak{N}},$$

where $S^{\mathfrak{N}}$ is the unitary scattering matrix of problem (3.19). The functions

$$\widetilde{\mathcal{Y}}^{+}_{\mathcal{N},j} := k^{1/2}(\vec{0}, 1, -(i/k)\,\nabla, 0) y^{+}_{\mathfrak{N},j}, \quad j = 1,\dots,\Upsilon_{\mathcal{N}}, \tag{3.43}$$

satisfy the homogeneous problem (2.3) and, by (3.30), admit the asymptotics

$$\widetilde{\mathcal{Y}}^{+}_{\mathcal{N},j} - \mathcal{U}^{+}_{\mathcal{N},j} - \sum_{l=1}^{\Upsilon_{\mathcal{N}}} S^{\mathfrak{N}}_{jl}\mathcal{U}^{-}_{\mathcal{N},l} \in H^1_\alpha(G), \quad j = 1,\dots,\Upsilon_{\mathcal{N}}.$$

By Proposition 2.5, the functions $\widetilde{\mathcal{Y}}^+_{\mathcal{N},j}$ are equal to $\mathcal{Y}^+_{\mathcal{N},j}$, $j=1,\dots,\Upsilon_{\mathcal{N}}$, up to some summands in $\ker\mathcal{L}_\alpha$. Consequently, the block $S^{\mathcal{N}\mathcal{N}}$ of the matrix (3.41) is equal to $S^{\mathfrak{N}}$, while the blocks $S^{\mathcal{N}\mathcal{M}}$ and $S^{\mathcal{N}\mathcal{D}}$ are zero. Arguing similarly, we can prove that $S^{\mathcal{D}\mathcal{D}}=S^{\mathfrak{D}}$ (where $S^{\mathfrak{D}}$ stands for the scattering matrix of problem (3.20)), while the blocks $S^{\mathcal{D}\mathcal{M}}$ and $S^{\mathcal{D}\mathcal{N}}$ are zero. Since the scattering matrix S is unitary, now we see that S is block-diagonal and its block $S^{\mathcal{M}\mathcal{M}}$ is unitary. Therefore, (3.42) follows from (3.40). $\square$

In accordance with (3.42), the principal term of the asymptotics for $\mathcal{Y}^+_{\mathcal{P},j}$ is a wave in $\mathbb{W}_{\mathcal{P}}$. However, in general, this does not mean that the function $\mathcal{Y}^+_{\mathcal{P},j}$ belongs to $E_{\mathcal{P}}(-\alpha)$.

Proposition 3.12. *Assume that k is not an eigenvalue of problem* (2.3)*. Then, for each $\mathcal{P}=\mathcal{M},\mathcal{N}$, and $\mathcal{D}$, the functions $\mathcal{Y}^+_{\mathcal{P},j}$, $j=1,\dots,\Upsilon_{\mathcal{P}}$, with the asymptotics* (3.42) *are unique and belong to $E_{\mathcal{P}}(-\alpha)$.*

Proof. First, we consider the case where $\mathcal{P}=\mathcal{N}$. As was shown in the proof of Proposition 3.11, the functions $\mathcal{Y}^+_{\mathcal{N},j}$ are given by (3.43) and, in view of (3.38), belong to $E_{\mathcal{N}}(-\alpha)$. The case of $\mathcal{P}=\mathcal{D}$ can be studied in the same way. Now we turn to discussing the case where $\mathcal{P}=\mathcal{M}$. Since the waves $\mathcal{U}^{\pm}_{\mathcal{M},l}$ are of the form $(u^1,0,u^2,0)$, the asymptotics (3.42) implies that the components a^1, a^2 of the vector-valued function $\mathcal{Y}^+_{\mathcal{M},j}=:(u^1,a^1,u^2,a^2)$ are in $H^1_\alpha(G)$. On the other hand, $\mathcal{Y}^+_{\mathcal{M},j}$ is a solution to the homogeneous problem (2.3). Hence, by item 1 of Proposition 3.9, the functions a^1 and a^2 satisfy problems (3.19) and (3.20), respectively. By assumption, the space $\ker\mathcal{L}_\alpha$ is trivial. Applying Proposition 3.10 with $\beta=\alpha$, we show that the spaces $\ker\mathcal{L}^{\mathfrak{N}}_\alpha$ and $\ker\mathcal{L}^{\mathfrak{D}}_\alpha$ are trivial. Therefore, $a^1=0$ and $a^2=0$. The first formula in (3.38) shows that the function $\mathcal{Y}^+_{\mathcal{M},j}$ belongs to $E_{\mathcal{M}}(-\alpha)$. $\square$

Now we assume that k is an eigenvalue of problem (2.3). The functions $\mathcal{Y}^+_{\mathcal{P},j}$ subject to the asymptotics (3.40) are defined up to an arbitrary summand in $\ker\mathcal{L}_\alpha$. This allows us to ensure that $\mathcal{Y}^+_{\mathcal{P},j}\in E_{\mathcal{P}}(-\alpha)$.

Proposition 3.13. *Let $\ker\mathcal{L}_\alpha\neq\{0\}$. Then the elements of the basis in $\ker\mathcal{L}_{-\alpha}/\ker\mathcal{L}_\alpha$ described in Proposition* 2.4 *possess representatives $\mathcal{Y}^+_{\mathcal{P},j}$ such that $\mathcal{Y}^+_{\mathcal{P},j}\in E_{\mathcal{P}}(-\alpha)$, $j=1,\dots,\Upsilon_{\mathcal{P}}$, $\mathcal{P}=\mathcal{M},\mathcal{N},\mathcal{D}$.*

Proof. For $\mathcal{P}=\mathcal{N}$, the "correct" choice of representatives was described in (3.43). Indeed, if k is an eigenvalue for problem (3.19), then the functions $y^+_{\mathfrak{N},j}$ are defined up to an arbitrary summand in $\ker\mathcal{L}^{\mathfrak{N}}_\alpha$. Thus, the functions $\widetilde{\mathcal{Y}}^+_{\mathcal{N},j}$ in (3.43) are also defined up to an arbitrary summand in $E_{\mathcal{N}}(\alpha)$, which does not violate the fact that $\widetilde{\mathcal{Y}}^+_{\mathcal{N},j}\in E_{\mathcal{N}}(-\alpha)$. The same is true for $\mathcal{P}=\mathcal{D}$. Now we turn to the case where $\mathcal{P}=\mathcal{M}$. Like in the proof of Proposition 3.12, we establish that the components a^1 and a^2 of $\mathcal{Y}^+_{\mathcal{M},j}=:(u^1,a^1,u^2,a^2)$ belong to $H^1_\alpha(G)$ and satisfy problems (3.19) and (3.20), respectively. In other words, $a^1\in\ker\mathcal{L}^{\mathfrak{N}}_\alpha$ and $a^2\in\ker\mathcal{L}^{\mathfrak{D}}_\alpha$. Let us construct functions $\mathcal{U}_{\mathcal{N}}$ and $\mathcal{U}_{\mathcal{D}}$ in $\ker\mathcal{L}_\alpha$ of the form (3.35) with these a^1 and a^2. The function $\mathcal{Y}^+_{\mathcal{M},j}-\mathcal{U}_{\mathcal{N}}-\mathcal{U}_{\mathcal{D}}$ belongs to $E_{\mathcal{M}}(-\alpha)$ and is a required representative. Such a representative can be defined up to an arbitrary summand from $E_{\mathcal{M}}(\alpha)$ whenever k is an eigenvalue of problem (1.8), (1.9). $\square$

Theorem 3.14. *For each $\mathcal{P}=\mathcal{M},\mathcal{N}$, or $\mathcal{D}$, in the quotient space $E_{\mathcal{P}}(-\alpha)/E_{\mathcal{P}}(\alpha)$ there exists a basis with representatives $\mathcal{Y}^+_{\mathcal{P},j}$, $j=1,\dots,\Upsilon_{\mathcal{P}}$, subject to*

$$\mathcal{Y}^+_{\mathcal{P},j}-\mathcal{U}^+_{\mathcal{P},j}-\sum_{l=1}^{\Upsilon_{\mathcal{P}}}S^{\mathcal{P}}_{jl}\mathcal{U}^-_{\mathcal{P},l}\in H^1_\alpha(G),\quad j=1,\dots,\Upsilon_{\mathcal{P}}. \tag{3.44}$$

Proof. For the role of representatives of the required basis we take the functions $\mathcal{Y}^{+}_{\mathcal{P},j}$, $j = 1,\dots,\Upsilon_{\mathcal{P}}$, constructed in Proposition 3.13. The asymptotics (3.44) follows from Proposition 3.11. From this asymptotics, it follows that the set of elements in $E_{\mathcal{P}}(-\alpha)/E_{\mathcal{P}}(\alpha)$ with representatives $\mathcal{Y}^{+}_{\mathcal{P},j}$, $j = 1,\dots,\Upsilon_{\mathcal{P}}$, is linearly independent. The number of elements in that set coincides with the dimension of $E_{\mathcal{P}}(-\alpha)/E_{\mathcal{P}}(\alpha)$. Therefore, the set is a basis. □

The assertion of Theorem 3.14 for $\mathcal{P} = \mathcal{M}$ deserves separate discussion. In this case, all the vector-valued functions in (3.44) have zero components a^1 and a^2. Crossing out these components, we obtain continuous spectrum eigenfunctions and waves for the original problem (1.8), (1.9). We shall call the matrix $S^{\mathcal{M}}$ the scattering matrix for problem (1.8), (1.9).

3.4. Radiation principle for problem (1.8)**,** (1.9)**.** By the radiation principle we mean a well-posed problem with natural radiation conditions. By employing results of preceding sections, we shall derive the radiation principle for the Maxwell system (Theorem 1.3) from the principle for the elliptic problem (Proposition 2.5).

Problem (1.8), (1.9) is overdetermined and compatibility conditions are necessary for its solvability.

Proposition 3.15.
Let (u^1, u^2) *be a smooth solution of problem* (1.8), (1.9) *with smooth functions* f^j, h^j, *and* g^j, $j = 1, 2$. *Then the compatibility conditions*

$$\begin{aligned} \operatorname{div}(\varepsilon(x) f^1(x)) - ik\, h^2(x) &= 0, \quad x \in G,\\ \operatorname{div}(\mu(x) f^2(x)) + ik\, h^1(x) &= 0, \quad x \in G,\\ i\,(\mu f^2)_\nu(x) + \operatorname{div}_2 g(x) + ik\, g^3(x) &= 0, \quad x \in \partial G, \end{aligned} \tag{3.45}$$

are fulfilled, where $g(x) := (g^1(x), g^2(x))$ *and* div_2 *stands for the divergence on* ∂G.

Proof. Applying the operator $\operatorname{div}\varepsilon$ to the first identity in (1.8), we get $-k \operatorname{div}(\varepsilon u^1) = \operatorname{div}(\varepsilon f^1)$. Since $i \operatorname{div}\varepsilon u^1 = -ih^2$, we arrive at the first formula in (3.45). The second can be established similarly.

Now we verify the third (“boundary”) compatibility condition in (3.45). We multiply the second curl-equation in (1.8) by μ, restrict the resulting identity to ∂G, and then consider the projection of this restriction to the normal direction:

$$-i\,(\operatorname{curl}\, u^1)_\nu - k(\mu u^2)_\nu = (\mu f^2)_\nu. \tag{3.46}$$

To compute $(\operatorname{curl} u^1)_\nu$, we introduce local orthogonal curvilinear coordinates (s_1, s_2, s_3) in a (sufficiently narrow) neighborhood of ∂G, precisely as in the proof of Proposition 3.1. Using (3.7), we see that

$$(\operatorname{curl} u^1)_{s_3} = \big(H_1\, H_2\big)^{-1} \left(\frac{\partial}{\partial s_1}\big(H_2 u^1_{s_2}\big) - \frac{\partial}{\partial s_2}\big(H_1 u^1_{s_1}\big)\right). \tag{3.47}$$

Taking into account conditions (1.9) and the definition of the divergence in curvilinear coordinates, we can rewrite the right-hand side of (3.47) for $s_3 = 0$ in the form

$$-\big(H_1\, H_2\big)^{-1} \left(\frac{\partial}{\partial s_1}\big(H_2 g^1\big) + \frac{\partial}{\partial s_2}\big(H_1 g^2\big)\right) = -\operatorname{div}_2 g, \tag{3.48}$$

where $g := (g^1, g^2)$. Relations (3.46), (3.7), and (3.48) imply the third compatibility condition in (3.45). □

Let us explain conditions (3.45) for (nonsmooth) $\mathcal{F} = (f^1, h^1, f^2, h^2) \in H^0_\alpha(G)$ and $\mathcal{G} = (g^1, g^2, g^3, 0) \in H^{1/2}_\alpha(\partial G)$. The standard arguments (see, e.g., [19]) should be changed somewhat because of the weighted classes. We denote by $H_\beta(\operatorname{div}, G)$ the space of vector-valued functions $v \in H^0_\beta(G; \mathbb{C}^3)$ such that $\operatorname{div} v \in H^0_\beta(G; \mathbb{C})$. Here the divergence is defined by the identity

$$\int_G \langle v, \nabla\phi\rangle\, dx + \int_G \operatorname{div} v \cdot \bar{\phi}\, dx = 0$$

for any $\phi \in C^\infty_c(G)$. The space $H_\beta(\operatorname{div}, G)$ is equipped with the norm

$$\|v; H_\beta(\operatorname{div}, G)\| = \left(\|v; H^0_\beta(G; \mathbb{C}^3)\|^2 + \|\operatorname{div} v; H^0_\beta(G; \mathbb{C})\|^2\right)^{1/2}$$

and the corresponding inner product. The mapping γ_ν given by $\gamma_\nu v = \langle v|_{\partial G}, \nu\rangle$ for smooth vector-valued functions can be extended by continuity to an operator

$$\gamma_\nu \colon H_\beta(\operatorname{div}, G) \to H^{-1/2}_\beta(\partial G; \mathbb{C})$$

(see, e.g., [19]). For $v \in H_\beta(\operatorname{div}, G)$ and $\phi \in H^1_{-\beta}(G; \mathbb{C})$, we have the Green formula

$$\int_G \langle v, \nabla\phi\rangle\, dx + \int_G \operatorname{div} v \cdot \bar{\phi}\, dx = \int_{\partial G} \gamma_\nu(v) \cdot \bar{\phi}\, dS.$$

The gradient operator ∇_2 on ∂G implements a continuous mapping

$$\nabla_2 \,:\, H^{1/2}_\beta(\partial G; \mathbb{C}) \to H^{-1/2}_\beta(\partial G; \mathbb{C}^2). \tag{3.49}$$

We denote by $-\operatorname{div}_2$ the operator

$$-\operatorname{div}_2 \colon H^{1/2}_{-\beta}(\partial G; \mathbb{C}^2) \to H^{-1/2}_{-\beta}(\partial G; \mathbb{C})$$

adjoint to the operator (3.49) with respect to the Green formula

$$\int_{\partial G} \langle v, \nabla_2\psi\rangle\, dS = -\int_{\partial G} \operatorname{div}_2 v \cdot \bar{\psi}\, dS.$$

Now conditions (3.45) can be interpreted as

$$\begin{aligned}
&\varepsilon f^1 \in H_\alpha(\operatorname{div}, G), \quad h^2 \in H^0_\alpha(G; \mathbb{C}), \quad \operatorname{div}(\varepsilon f^1) - ikh^2 = 0 \text{ in } H^0_\alpha(G; \mathbb{C}),\\
&\mu f^2 \in H_\alpha(\operatorname{div}, G), \quad h^1 \in H^0_\alpha(G; \mathbb{C}), \quad \operatorname{div}(\mu f^2) + ikh^1 = 0 \text{ in } H^0_\alpha(G; \mathbb{C}),\\
&g \in H^{1/2}_\alpha(\partial G; \mathbb{C}^2), \quad g^3 \in H^{1/2}_\alpha(\partial G; \mathbb{C}),\\
&i\gamma_\nu(\mu f^2) + \operatorname{div}_2 g + ikg^3 = 0 \text{ in } H^{-1/2}_\alpha(\partial G; \mathbb{C}).
\end{aligned} \tag{3.50}$$

Proposition 3.16. *Let $k > 0$, and let the vector-valued functions $\mathcal{F} = (f^1, h^1, f^2, h^2) \in H^l_\alpha(G)$ and $\mathcal{G} = (g^1, g^2, g^3, 0) \in H^{l+1/2}_\alpha(\partial G)$ satisfy the compatibility conditions* (3.45) *for $l \geq 1$ or* (3.50) *for $l = 0$. Let $\mathcal{U} = (u^1, a^1, u^2, a^2)$ be the solution of the equation $\mathbb{L}\mathcal{U} = \{\mathcal{F}, \mathcal{G}\}$ from Proposition* 2.5 *that satisfies the radiation conditions* (2.16). *Then the component a^1 (a^2) of $\mathcal{U}$ belongs to the space* $\ker \mathcal{L}^{\mathfrak{N}}_\alpha$ ($\ker \mathcal{L}^{\mathfrak{D}}_\alpha$) *of eigenfunctions for problem* (3.19) (*for problem* (3.20)). *Moreover, we have*

$$\mathcal{U} - \sum_{j=1}^{\Upsilon_{\mathcal{M}}} c_{\mathcal{M},j}\mathcal{U}^-_{\mathcal{M},j} \in H^{l+1}_\alpha(G). \tag{3.51}$$

Proof. The components of the vector-valued function $\mathcal{U}$ satisfy the system of equations

$$\begin{aligned}
i\varepsilon^{-1}(x) \operatorname{curl} u^2(x) + i\,\nabla a^2(x) - ku^1(x) &= f^1(x),\\
-i \operatorname{div}\left(\mu(x)u^2(x)\right) - ka^1(x) &= h^1(x),\\
-i\mu^{-1}(x) \operatorname{curl} u^1(x) - i\,\nabla a^1(x) - ku^2(x) &= f^2(x),\\
i \operatorname{div}(\varepsilon(x)u^1(x)) - ka^2(x) &= h^2(x), \qquad x \in G,
\end{aligned} \tag{3.52}$$

and the boundary conditions

$$-\langle u^1(x),\tau_2(x)\rangle = g^1(x), \quad \langle u^1(x),\tau_1(x)\rangle = g^2(x),$$
$$\langle \mu(x)u^2(x),\nu(x)\rangle = g^3(x), \qquad a^2(x)=0, \qquad x\in\partial G. \tag{3.53}$$

First we show that a^2 solves problem (3.20). Multiplying the first curl-equation in (3.52) by $\varepsilon\nabla\zeta$ and the second div-equation by $ik\zeta$, where $\zeta\in C_c^\infty(G)$, we see that

$$i(\varepsilon^{-1}\operatorname{curl}u^2,\varepsilon\nabla\zeta)_G + i(\nabla a^2,\varepsilon\nabla\zeta)_G - k(u^1,\varepsilon\nabla\zeta)_G = (f^1,\varepsilon\nabla\zeta)_G,$$
$$-k(\operatorname{div}(\varepsilon\, u^1),\zeta)_G - ik^2(a^2,\zeta)_G = ik(h^2,\zeta)_G.$$

We integrate by parts all terms in the first line and add the result to the second line, obtaining

$$-i(a^2,(\operatorname{div}\varepsilon\nabla + k^2)\zeta)_G = (-\operatorname{div}(\varepsilon\, f^1)+ikh^2,\zeta)_G.$$

By the compatibility conditions (3.45), the right-hand side vanishes, whence

$$-i(a^2,(\operatorname{div}\varepsilon\nabla+k^2)\zeta)_G = 0$$

for any $\zeta\in C_c^\infty(G)$. In other words, the function $a^2\in H^{l+1}_{-\alpha}$ is a generalized solution of (elliptic) problem (3.20). Therefore, a^2 is in $H^m_{-\alpha}(G)$ for every $m=1,2,\dots$ and satisfies (3.20) in the classical sense.

Turning to the component a^1, we multiply the second curl-equation in (3.52) by $\mu\nabla\eta$ and the second div-equation by $-ik\eta$, where $\eta\in C_c^\infty(\bar G)$. This yields

$$-i(\mu^{-1}\operatorname{curl}u^1,\mu\nabla\eta)_G - i(\nabla a^1,\mu\nabla\eta)_G - k(u^2,\mu\nabla\eta)_G = (f^2,\mu\nabla\eta)_G,$$
$$-k(\operatorname{div}(\mu\, u^2),\eta)_G + ik^2(a^1,\eta)_G = -ik(h^1,\eta)_G. \tag{3.54}$$

We integrate by parts the terms in the first line:

$$(\operatorname{curl}u^1,\ \nabla\eta)_G = (\nu\times u^1_\tau,(\nabla\eta)_\tau)_{\partial G} + (u^1,\ \operatorname{curl}\nabla\eta)_G = (\nu\times u^1_\tau,(\nabla\eta)_\tau)_{\partial G}.$$

Here u^1_τ stands for the tangent component $u^1_\tau := \langle u^1,\tau_1\rangle\tau_1 + \langle u^1,\tau_2\rangle\tau_2$ of the vector u^1 on the boundary ∂G and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $\mathbb{C}^3$; the expression $(\nabla\eta)_\tau$ has a similar meaning. Finally, $\cdot\times\cdot$ is the vector product in $\mathbb{R}^3$. Recalling the boundary conditions (3.53), we find

$$(\nu\times u^1_\tau,(\nabla\eta)_\tau)_{\partial G} = (g,\nabla_2\eta)_{\partial G} = -(\operatorname{div}_2 g,\eta)_{\partial G}, \tag{3.55}$$

where $g=(g^1,g^2)$ and the expressions $\eta\mapsto\nabla_2\eta := (\langle\nabla\eta,\tau_1\rangle,\langle\nabla\eta,\tau_2\rangle)$ and $g\mapsto\operatorname{div}_2 g$, respectively, stand for the gradient and divergence operators on the surface ∂G. Then

$$(\nabla a^1,\mu\nabla\eta)_G = (a^1,(\mu\nabla\eta)_\nu)_{\partial G} - (a^1,\operatorname{div}\mu\nabla\eta)_G,$$
$$(u^2,\mu\nabla\eta)_G = ((\mu u^2)_\nu,\eta)_{\partial G} - (\operatorname{div}(\mu u^2),\eta)_G = (g^3,\eta)_{\partial G} - (\operatorname{div}(\mu u^2),\eta)_G,$$
$$(f^2,\mu\nabla\eta)_G = ((\mu f^2)_\nu,\eta)_{\partial G} - (\operatorname{div}(\mu f^2),\eta)_G.$$

Taking these idetitics into account, we add equations (3.54) and obtain

$$i(a^1,(\operatorname{div}\mu\nabla+k^2)\eta)_G - i(a^1,(\mu\nabla\eta)_\nu)_{\partial G}$$
$$= -i(i(\mu\, f^2)_\nu + \operatorname{div}_2 g + ikg^3,\eta)_{\partial G} - (\operatorname{div}(\mu\, f^2)+ikh^1,\eta)_G.$$

By the compatibility conditions (3.45), this yields

$$(a^1,-(\operatorname{div}\mu\nabla+k^2)\eta)_G + (a^1,(\mu\nabla\eta)_\nu)_{\partial G} = 0 \tag{3.56}$$

for any $\eta\in C_c^\infty(\bar G)$; this equation is still true for any $\eta\in H^2_\alpha(G)$. This means that the couple $\{a^1,a^1|_{\partial G}\}$ belongs to the cokernel $\operatorname{coker}\mathcal{L}^{\mathfrak{N}}_\alpha$ of the operator $\mathcal{L}^{\mathfrak{N}}_\alpha$ in (3.36). Therefore, the function a^1 is in $H^m_{-\alpha}(G)$ for each $m=1,2,\dots$ and satisfies (3.19) (see, e.g., [20, Chapter 2, 5.2, 5.3] and Theorem 5.1.4 in [1]).

Now we discuss the radiation conditions for a^1 and a^2. We rewrite (2.16) in a more detailed form:

$$\mathcal{U} - \sum_{j=1}^{\Upsilon_{\mathcal{M}}} c_{\mathcal{M},j}\mathcal{U}^-_{\mathcal{M},j} - \sum_{j=1}^{\Upsilon_{\mathcal{N}}} c_{\mathcal{N},j}\mathcal{U}^-_{\mathcal{N},j} - \sum_{j=1}^{\Upsilon_{\mathcal{D}}} c_{\mathcal{D},j}\mathcal{U}^-_{\mathcal{D},j} \in H^{l+1}_\alpha(G); \tag{3.57}$$

here $c_{\mathcal{M},j}$, $c_{\mathcal{N},j}$, and $c_{\mathcal{D},j}$ are constant coefficients. From Proposition 3.6 it follows that only the waves $\mathcal{U}^-_{\mathcal{N},j}$, $j = 1,\dots,\Upsilon_{\mathcal{N}}$, ($\mathcal{U}^-_{\mathcal{D},j}$, $j = 1,\dots,\Upsilon_{\mathcal{D}}$) contribute to the asymptotics of a^1 (a^2). Hence, (3.57), (3.30), and (3.31) lead to the relations

$$a^1 - k^{1/2}\sum c_{\mathcal{N},j}u^-_{\mathfrak{N},j} \in H^{l+1}_\alpha(G), \quad a^2 - k^{1/2}\sum c_{\mathcal{D},j}u^-_{\mathfrak{D},j} \in H^{l+1}_\alpha(G), \tag{3.58}$$

where $u^-_{\mathfrak{N},j}$ ($u^-_{\mathfrak{D},j}$) are outgoing waves for problem (3.19) (for problem (3.20)). Thus, the intrinsic radiation conditions (3.57) imply the intrinsic radiation conditions (3.58) for a^1 (a^2) in the sense of problem (3.19) (in the sense of problem (3.20)). By Proposition 2.5, the function a^1 (a^2) belongs to the space $\ker\mathcal{L}^{\mathfrak{N}}_\alpha$ ($\ker\mathcal{L}^{\mathfrak{D}}_\alpha$) of eigenfunctions for problem (3.19) (for problem (3.20)). Consequently, the coefficients $c_{\mathcal{N},j}$ and $c_{\mathcal{D},l}$ in (3.57) vanish, $j = 1,\dots,\Upsilon_{\mathcal{N}}$, $l = 1,\dots,\Upsilon_{\mathcal{D}}$, and (3.57) takes the form (3.51). □

The waves $\mathcal{U}^-_{\mathcal{M},j}$ are of "Maxwell" type, that is, their components a^1 and a^2 are zero. However, relation (3.51) is still not quite satisfactory for our purpose (we want to return to the Maxwell system), because the coefficients $c_{\mathcal{M},j}$ are defined in terms of the elliptic problem. Moreover, if k turns out to be an eigenvalue of the elliptic problem, then, for a solution to exist, the right-hand side must be orthogonal to the eigenfunctions of the elliptic problem corresponding to the eigenvalue k. Finally, in this case, the elliptic problem manifests itself also by the fact that a solution $\mathcal{U}$ is determined up to adding an arbitrary eigenfunction.

If the space $\ker\mathcal{L}_\alpha$ is nontrivial, then, by (3.39), it can be decomposed into the direct sum of subspaces

$$\ker\mathcal{L}_\alpha = E_{\mathcal{M}}(\alpha) \dotplus E_{\nabla}(\alpha), \tag{3.59}$$

where $E_\nabla(\alpha) = E_{\mathcal{N}}(\alpha) \dotplus E_{\mathcal{D}}(\alpha)$. The dimension of the space $E_{\mathcal{M}}(\alpha)$ will be denoted by $\Theta_{\mathcal{M}}$.

We shall also need some additional information about the functions (2.18) occurring in formulas (2.17) for the coefficients (in the asymptotics of the solution from Proposition 2.5). Let us enumerate such functions by two indices: we write $\mathcal{Y}^-_{\mathcal{P},j}$ in accordance with the already introduced new enumeration of the functions $\mathcal{Y}^+_{\mathcal{P},j}$ and the entries of the scattering matrix S (for problem (2.3)). Set

$$\mathcal{Y}^-_{\mathcal{P},j} := \sum_{\mathcal{R}=\mathcal{M},\mathcal{N},\mathcal{D}} \sum_{l=1}^{\Upsilon_{\mathcal{R}}} T^{\mathcal{P}\mathcal{R}}_{jl}\mathcal{Y}^+_{\mathcal{R},l}, \tag{3.60}$$

where $T := S^{-1}$. Since the matrix S is block-diagonal, the same is true for T. Hence, (3.60) takes the form

$$\mathcal{Y}^-_{\mathcal{P},j} := \sum_{l=1}^{\Upsilon_{\mathcal{P}}} T^{\mathcal{P}}_{jl}\mathcal{Y}^+_{\mathcal{P},j} \tag{3.61}$$

with $T^{\mathcal{P}} = (S^{\mathcal{P}})^{-1}$.

Theorem 3.17. *Let $\mathcal{Z}_{\mathcal{M},1},\dots,\mathcal{Z}_{\mathcal{M},\Theta_{\mathcal{M}}}$ be a basis in $E_{\mathcal{M}}(\alpha)$, and let the vector-valued functions $\mathcal{F} = (f^1, h^1, f^2, h^2) \in H^l_\alpha(G;\mathbb{C}^8)$ and $\mathcal{G} = (g^1, g^2, g^3, 0) \in H^{l+1/2}_\alpha(\partial G;\mathbb{C}^4)$ satisfy the compatibility conditions* (3.45) *for $l \geq 1$ or* (3.50) *for $l = 0$, and the orthogonality* *conditions*

$$(\mathcal{F}, \mathcal{Z}_{\mathcal{M},j})_{G,\varpi} + (\mathcal{G}, \mathcal{Q}\mathcal{Z}_{\mathcal{M},j})_{\partial G} = 0, \quad j = 1, \dots, \Theta_{\mathcal{M}}. \tag{3.62}$$

Then the problem

$$\mathcal{A}(x, D_x)\mathcal{U}(x) = \mathcal{F}(x), \quad x \in G, \quad \mathcal{B}(x)\mathcal{U}(x) = \mathcal{G}(x), \quad x \in \partial G, \tag{3.63}$$

admits a solution of the form $\mathcal{U} = (u^1, 0, u^2, 0)$ *subject to the radiation conditions*

$$\mathcal{V} := \mathcal{U} - \sum_{j=1}^{\Upsilon_{\mathcal{M}}} c_{\mathcal{M},j}\mathcal{U}^-_{\mathcal{M},j} \in H^{l+1}_\alpha(G; \mathbb{C}^8). \tag{3.64}$$

Here $c_{\mathcal{M},j} = i(\mathcal{F}, \mathcal{Y}^-_{\mathcal{M},j})_{G,\varpi} + i(\mathcal{G}, \mathcal{Q}\mathcal{Y}^-_{\mathcal{M},j})_{\partial G}$ *and the* $\mathcal{Y}^-_{\mathcal{M},j}$ *are the elements of* $E_{\mathcal{M}}(-\alpha)$ *given by* (3.61) *with* $\mathcal{P} = \mathcal{M}$.

Such a solution $\mathcal{U}$ *is defined up to adding an arbitrary eigenfunction in* $E_{\mathcal{M}}(\alpha)$. *We have*

$$\begin{aligned} &\big\|\mathcal{V}; H^{l+1}_\alpha(G; \mathbb{C}^8)\big\| + \sum_{j=1}^{\Upsilon_{\mathcal{M}}} |c_j| \\ &\qquad \le \operatorname{const}\big(\|\mathcal{F}; H^l_\alpha(G; \mathbb{C}^8)\| + \|\mathcal{G}; H^{l+1/2}_\alpha(\partial G; \mathbb{C}^4)\| + \|\rho_\alpha\mathcal{V}; L_2(G; \mathbb{C}^8)\|\big). \end{aligned} \tag{3.65}$$

A solution $\mathcal{U}_0$ *satisfying the additional conditions* $(\mathcal{U}_0, \mathcal{Z}_{\mathcal{M},j})_{G,\varpi} = 0$, $j = 1, \dots, \Theta_{\mathcal{M}}$, *is unique. For the function* $\mathcal{V}$ *of the form* (3.64) *with* $\mathcal{U}$ *replaced by* $\mathcal{U}_0$, *estimate* (3.65) *is valid with the right-hand side replaced by* $\operatorname{const}(\|\mathcal{F}; H^l_\alpha(G; \mathbb{C}^8)\| + \|\mathcal{G}; H^{l+1/2}_\alpha(\partial G; \mathbb{C}^4)\|)$.

Proof. If $\mathcal{F} = (f^1, h^1, f^2, h^2) \in H^l_\alpha(G)$ and $\mathcal{G} = (g^1, g^2, g^3, 0) \in H^{l+1/2}_\alpha(\partial G)$ meet the compatibility conditions (3.45), then $(\mathcal{F}, \mathcal{Z})_{G,\varpi} + (\mathcal{G}, \mathcal{Q}\mathcal{Z})_{\partial G} = 0$ for every $\mathcal{Z} \in E_\nabla(\alpha)$. Indeed, let

$$\mathcal{Z} = \big((i/k)\nabla a^2, a^1, -(i/k)\nabla a^1, a^2\big).$$

Then

$$(\mathcal{F}, \mathcal{Z})_{G,\varpi} = -(i/k)\big(\varepsilon f^1, \nabla a^2\big)_G + (h^1, a^1)_G + (i/k)\big(\mu f^2, \nabla a^1\big)_G + (h^2, a^2)_G.$$

Integrating by parts the first and third terms on the right, we obtain

$$\begin{aligned} (\mathcal{F}, \mathcal{Z})_{G,\varpi} = {} & \big((i/k)\operatorname{div}(\varepsilon f^1) + h^2, a^2\big)_G \\ & + \big(-(i/k)\operatorname{div}(\mu f^2) + h^1, a^1\big)_G + (i/k)\big((\mu f^2)_\nu, a^1\big)_{\partial G}. \end{aligned}$$

Moreover,

$$(\mathcal{G}, \mathcal{Q}\mathcal{Z})_{\partial G} = (g, -(1/k)\nabla_2 a^1)_{\partial G} + (g^3, -i\, a^1)_{\partial G}.$$

Here $g = (g^1, g^2)$ and ∇_2 is the gradient operator on the surface ∂G:

$$a^1 \mapsto \nabla_2 a^1 := (\langle \nabla a^1, \tau_1\rangle, \langle \nabla a^1, \tau_2\rangle).$$

Recalling (3.55), we see that

$$(\mathcal{G}, \mathcal{Q}\mathcal{Z})_{\partial G} = ((1/k)\operatorname{div}_2 g + ig^3, a^1)_{\partial G}.$$

Taking the compatibility conditions (3.45) into account, we arrive at the identity

$$(\mathcal{F}, \mathcal{Z})_{G,\varpi} + (\mathcal{G}, \mathcal{Q}\mathcal{Z})_{\partial G} = 0 \tag{3.66}$$

for every $\mathcal{Z} \in E_\nabla(\alpha)$. From (3.62), (3.66), and (3.59) it follows that the orthogonality condition $(\mathcal{F}, \mathcal{Z})_{G,\varpi} + (\mathcal{G}, \mathcal{Q}\mathcal{Z})_{\partial G} = 0$ is fulfilled for any $\mathcal{Z}$ in $\ker \mathcal{L}_\alpha$.

By Proposition 2.5, problem (3.63) admits a solution $\mathcal{U}$ subject to the intrinsic radiation conditions; such conditions take the form (3.64) by Proposition 3.16. Employing Proposition 2.5 once again, we have

$$c_{\mathcal{M},j} = i(\mathcal{F}, \mathcal{Y}^-_{\mathcal{M},j})_{G,\varpi} + i(\mathcal{G}, \mathcal{Q}\mathcal{Y}^-_{\mathcal{M},j})_{\partial G}.$$

The solution $\mathcal{U}$ is defined up to an arbitrary summand in $\ker \mathcal{L}_\alpha$. Let us take any special solution $\mathcal{U}'$ and represent the general solution as $\mathcal{U} = \mathcal{U}' + \mathcal{Z}_\nabla + \mathcal{Z}_\mathcal{M}$ with $\mathcal{Z}_\nabla \in E_\nabla(\alpha)$ and $\mathcal{Z}_\mathcal{M} \in E_\mathcal{M}(\alpha)$. Proposition 3.16 implies that $\mathcal{U}' = (u^1, a^1, u^2, a^2)$, where $a^1 \in \ker \mathcal{L}_\alpha^\mathfrak{N}$ and $a^2 \in \ker \mathcal{L}_\alpha^\mathfrak{D}$ are eigenfunctions of problems (3.19) and (3.20), respectively. Set $\mathcal{Z}'_\nabla = (-i/k\nabla a^2, -a^1, i/k\nabla a^1, -a^2) \in E_\nabla(\alpha)$, then $\mathcal{U}' + \mathcal{Z}'_\nabla + \mathcal{Z}_\mathcal{M} = (v^1, 0, v^2, 0)$ for all $\mathcal{Z}_\mathcal{M} \in E_\mathcal{M}(\alpha)$. Finally, we choose $\mathcal{Z}'_\mathcal{M}$ so that the solution $\mathcal{U}_0 := \mathcal{U}' + \mathcal{Z}'_\nabla + \mathcal{Z}'_\mathcal{M}$ satisfies $(\mathcal{U}_0, \mathcal{Z}_{\mathcal{M},j})_{G,\varpi} = 0$ for each $j = 1, \dots, \Theta_\mathcal{M}$. Then the solution $\mathcal{U}_0$ is defined uniquely and satisfies the estimate mentioned in the theorem. $\square$

In order to get a radiation principle directly for problem (1.8), (1.9), we only need to make some changes of notation in Theorem 3.17. Namely, the domain of the operator of problem (3.63) consists of vectors with eight components, whereas the fourth and the eighth components (a^1 and a^2) are equal to zero. Crossing out these zero components, we replace the 8-vectors by the "Maxwell" 6-vectors, arriving at problem (1.8), (1.9) instead of (3.63). To avoid ambiguity, when passing from (3.63) to (1.8), (1.9), we change the notation $\mathcal{U}$, $\mathcal{U}^\pm_{\mathcal{M},j}$, $\mathcal{Y}^\pm_{\mathcal{M},j}$, $\mathcal{Z}_{\mathcal{M},j}$, $S^\mathcal{M}$ for U, $u_j^\pm$, $Y_j^\pm$, Z_j, s, respectively. Now, Theorem 1.3 is a special case of Theorem 3.17 reformulated in accordance with the new notation.

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