Алгебра и анализ
Том 29 (2017), № 2

St. Petersburg Math. J.
Vol. 29 (2018), No. 2, Pages 315–323
http://dx.doi.org/10.1090/spmj/1495
Article electronically published on March 12, 2018

CWIKEL TYPE ESTIMATES FOR THE BORDERED AIRY TRANSFORM

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To the memory of Vladimir Savel'evich Buslaev

Abstract. Compactness conditions as well as estimates for singular values of the bordered Airy transform $f\mathbb{A}g$ in $L_2(\mathbb{R})$ are studied for suitable functions $f(x)$, $g(x)$, $x \in \mathbb{R}$. Sufficient conditions for the operator $f\mathbb{A}g$ to be in the Schatten–von Neumann class $\mathfrak{S}_p$, $p \in (0,2)$, are obtained. In particular, certain conditions ensuring that the operator $f\mathbb{A}g$ is in trace class are given.

Introduction

In the space $L_2(\mathbb{R})$, we consider the unitary integral Airy transformation $\mathbb{A}$ defined by the formula

$$\mathbb{A}u(x) := \int_{\mathbb{R}} \mathrm{Ai}(y-x)u(y)\,dy, \quad u \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}).$$

Here $\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos(\frac{t^3}{3} + zt)\,dt$, $z \in \mathbb{R}$, is the Airy function. We shall be interested in compactness conditions and in estimates for the singular values of the operator $f\mathbb{A}g$, $f, g \in L_{2,\mathrm{loc}}(\mathbb{R})$. In what follows, the conditions for the operator $f\mathbb{A}g$ to belong to the Lorentz classes $\mathfrak{S}_{p,q}$, $p \in (0,2)$, $q \in (0,+\infty]$ will be stated. The results can prove to be useful in the study of the spectrum of the Stark operator $H = -\frac{d^2}{dx^2} + x$ perturbed by a decaying potential.

The conditions of being in the class $\mathfrak{S}_{p,q}$ for $p > 2$ have been studied for a wide range of integral operators (see, e.g., a survey of these results in [1]). These questions naturally arise in the spectral theory of differential operators (the references to relevant papers can be found in [2]). In particular, the conditions for the operator $f\mathbb{A}g$ to belong to the classes $\mathfrak{S}_{p,q}$, $p > 2$, were obtained in [1] (see Theorem 1.1 below). The conditions ensuring that integral operators belong to the classes $\mathfrak{S}_{p,q}$, $p < 2$, have been studied to a lesser extent. Such results with no additional requirements on the smoothness of the operators' kernel were obtained in the papers [3, 4], and [2] for the operator $f(i\nabla)g(x)$ and in the paper [5] for the operator $f(\widehat{H})g(x)$, where $\widehat{H}$ is the Dirac operator. In [6] the conditions of being in the classes $\mathfrak{S}_{p,q}$, $p < 2$, were obtained for the operator $f(\mathcal{H})g(x)$, provided that the self-adjoint and lower bounded operator $\mathcal{H}$ generates a semigroup satisfying the Nash–Aronson estimate (the upper Gaussian estimate). The results of [6] cannot be directly applied to the operator $f\mathbb{A}g = \mathbb{A}f(H)g$ because the Stark operator H is not bounded from below. Nevertheless, after a certain modernization, the method developed in [6] applies to the operator $f\mathbb{A}g$. The present paper is devoted to the results obtained in this way.

2010 *Mathematics Subject Classification.* 47B10.

Key words and phrases. Integral operators, estimates for singular values, classes of compact operators, Airy transform.

Supported by SpBSU (grant no. 11.38.263.2014) and by RFBR (grant no. 14-01-00760_a).

The paper comprises Introduction and three sections. In §1 the relevant statements pertaining to function spaces and classes of compact operators are collected. The conditions for the operator $f\mathbb{A}g$ to belong to the classes $\mathfrak{S}_{p,q}$, $p > 2$, as obtained in [1], are also given. In §2, the main result of the paper (Theorem 2.1) is formulated. The proof of the main result is given in §3. A brief outline of the results of the present paper was published in [7].

In what follows, for a linear operator T acting in a pair of Hilbert spaces the symbol T^* denotes the adjoint operator. For a measurable function $f(x)$, $x \in \mathbb{R}$, the symbol $[f(x)]$ (or sometimes simply f for brevity) denotes the operator of multiplication by the function $f(x)$. The characteristic function of a set Ω is denoted by $\mathbf{1}_\Omega$. Moreover, the following notation is used throughout:

$$x_+ := \begin{cases} x, & x \geq 0, \\ 0, & x < 0; \end{cases} \quad \langle x \rangle := (1+x^2)^{1/2}; \quad \operatorname{int} x := \max\{m \in \mathbb{N} : m \leq x\}, \quad x \in \mathbb{R}.$$

The author expresses his gratitude to Professor T. A. Suslina for useful discussions and help in the preparation of the manuscript.

§1. Preliminary results

1.1. Function spaces and classes of compact operators. Let $(\mathcal{Z}, d\nu)$ be a separable measurable space with a σ-finite measure. Alongside the standard classes $L_p(\mathcal{Z}, d\nu)$, we define the Lorentz classes $L_{p,q}(\mathcal{Z}, d\nu)$, $p \in (0,+\infty)$, $q \in (0, +\infty]$ (see, e.g., [3, 2]). Namely, for a ν-measurable $f\colon \mathcal{Z} \to \mathbb{C}$ put $O_f(s) := \{z \in \mathcal{Z} : |f(z)| > s\}$ and $\nu_f(s) := \nu(O_f(s))$, $s > 0$. The class $L_{p,q}$ is singled out by the requirement that the functional

$$\|f\|_{L_{p,q}} := \begin{cases} \left(q \int_0^{+\infty} s^{q-1} \nu_f^{q/p}(s)\, ds\right)^{1/q}, & 0 < q < +\infty; \\ \sup_{s>0} s\, \nu_f^{1/p}(s), & q = +\infty. \end{cases} \tag{1.1}$$

be finite. The space $L_{p,q}$ is complete with respect to the quasinorm $\|\cdot\|_{L_{p,q}}$; $L_{p,q}$ is separable for $q \in (0, +\infty)$; the space $L_{p,\infty}$ is generally nonseparable and contains the separable subspace

$$L^0_{p,\infty} := \left\{ f \in L_{p,\infty} : \nu_f(s) = o(s^{-p}),\ s \to +0,\ s \to +\infty \right\}.$$

The following relations are noteworthy: $L_{p,p} = L_p$, $\|f\|_{L_{p,p}} = \|f\|_{L_p}$.

For an arbitrary compact operator T acting from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$, we denote by $s_n(T)$, $n \in \mathbb{N}$, the singular values of the operator T (i.e., the consecutive eigenvalues of the operator $(T^*T)^{1/2}$); let $n(s,T) := \#\{n \in \mathbb{N} : s_n(T) > s\}$ stand for the distribution function of the singular values. The following inequality will be used below (see [9, §11.1, Subsection 3]:

$$n(s+t, \mathbb{S}+\mathbb{T}) \leq n(s, \mathbb{S}) + n(t, \mathbb{T}), \quad s,t > 0, \quad \mathbb{S}, \mathbb{T} \in \mathfrak{S}_\infty. \tag{1.2}$$

The class $\mathfrak{S}_{p,q}(\mathfrak{H}_1, \mathfrak{H}_2)$, $p \in (0,+\infty)$, $q \in (0,+\infty]$ (see, e.g., [3]), is singled out by the condition that the functional

$$\|\mathbb{T}\|_{\mathfrak{S}_{p,q}} := \begin{cases} \left(q \int_0^{+\infty} s^{q-1} n^{q/p}(s, \mathbb{T})\, ds\right)^{1/q}, & 0 < q < +\infty; \\ \sup_{s>0} s\, n^{1/p}(s, \mathbb{T}), & q = +\infty. \end{cases} \tag{1.3}$$

is bounded. The space $\mathfrak{S}_{p,q}$ is complete with respect to the quasinorm $\|\cdot\|_{\mathfrak{S}_{p,q}}$; $\mathfrak{S}_{p,q}$ for $q \in (0, +\infty)$ is separable. The space $\mathfrak{S}_{p,\infty}$ is nonseparable and contains the separable subspace

$$\mathfrak{S}^0_{p,\infty} := \left\{ T \in \mathfrak{S}_{p,\infty} : n(s,T) = o(s^{-p}),\ s \to +0 \right\},$$

in which the set of finite-rank operators is dense. The class $\mathfrak{S}_{p,p}$ coincides with the standard Schatten–von Neumann class $\mathfrak{S}_p$. The functional $\|T\|_{\mathfrak{S}_{p,p}}$ coincides with the standard (quasi)norm in $\mathfrak{S}_p$:

$$\|T\|_{\mathfrak{S}_p} = \Big(\sum_{n\in\mathbb{N}} s_n^p(T)\Big)^{1/p}.$$

1.2. The conditions for the operator $f\mathbb{A}g$ to belong to the classes $\mathfrak{S}_{p,q}$, $p > 2$. On the plane $\mathbb{R}^2$ we define the measure $d\nu(x,y) := \mathrm{Ai}^2(y-x)\,dx\,dy$, $(x,y) \in \mathbb{R}^2$. With each pair of functions $f, g \in L_{2,\mathrm{loc}}(\mathbb{R})$, we associate the function of two variables $(f\otimes g)(x,y) := f(x)g(y)$. Theorem 3.2 in [1] implies the following statement.

Theorem 1.1. *Suppose that the condition $f\otimes g \in L_{p,q}(\mathbb{R}^2, d\nu)$ is satisfied for some $p > 2$, $q \in (0,+\infty]$ (or $p = q = 2$). Then $f\mathbb{A}g \in \mathfrak{S}_{p,q}$ and we have*

$$\|f\mathbb{A}g\|_{\mathfrak{S}_{p,q}} \le C(p,q)\|f\otimes g\|_{L_{p,q}}.$$

If, moreover, $q = +\infty$ and $\nu_{f\otimes g}(s) = o(s^{-p})$, $s \to +0$, then $f\mathbb{A}g \in \mathfrak{S}^0_{p,\infty}$.

It is not hard to deduce the following corollary to Theorem 1.1.

Corollary 1.2. *Suppose that, for some $p \ge 2$, the conditions*

$$|f(x)| \le C_1(1+x^2)^{-\alpha/2}, \quad x\in\mathbb{R}, \tag{1.4}$$

$$|g(y)| \le C_2(1+y^2)^{-\beta/2}, \quad y\in\mathbb{R}, \tag{1.5}$$

$$\alpha > 1/2p, \quad \beta > 1/2p, \quad (\alpha+\beta) > 3/2p \tag{1.6}$$

are satisfied. Then $f\mathbb{A}g \in \mathfrak{S}_p$ and we have

$$\|f\mathbb{A}g\|_{\mathfrak{S}_p} \le C(\alpha,\beta,p)C_1C_2. \tag{1.7}$$

§2. The main result

Let $\mathfrak{N}$ denote the following set of rectangles forming a partition of the plane $\mathbb{R}^2$:

- $[m, m+1) \times [m, m+1)$, $m \in \mathbb{Z}$;
- $[m, m+1) \times [-m-1, -m)$, $m \in \mathbb{Z}$;
- $[m+2^l, m+2^{l+1}) \times [m, m+1)$, $m, l \in \mathbb{Z}_+$;
- $[m+2^l, m+2^{l+1}) \times [-m-1, -m)$, $m, l \in \mathbb{Z}_+$;
- $[m, m+1) \times [-m-2^{l+1}, -m-2^l)$, $m, l \in \mathbb{Z}_+$;
- $[-m-1, -m) \times [-m-2^{l+1}, -m-2^l)$, $m, l \in \mathbb{Z}_+$;
- $[-m-2^{l+1}, -m-2^l) \times [-m-1, -m)$, $m, l \in \mathbb{Z}_+$;
- $[-m-2^{l+1}, -m-2^l) \times [m, m+1)$, $m, l \in \mathbb{Z}_+$;
- $[-m-1, -m) \times [m+2^l, m+2^{l+1})$, $m, l \in \mathbb{Z}_+$;
- $[m, m+1) \times [m+2^l, m+2^{l+1})$, $m, l \in \mathbb{Z}_+$.

We define the following two functions on the set $\mathfrak{N}$:

$$\varphi(K) := \begin{cases} \langle z\rangle^{-7/16} e^{-(2/3)z^{3/2}}, & z \ge 0, \\ \langle z\rangle^{-1/2}, & z < 0, \end{cases} \qquad z = c - b, \ \ K = [a,b)\times[c,d) \in \mathfrak{N};$$

$$\psi_\varkappa(K) := \begin{cases} \langle z\rangle^{\varkappa}, & z \ge 0, \\ \langle z\rangle^{1/2}, & z < 0, \end{cases} \qquad z = c - b, \ \ K = [a,b)\times[c,d) \in \mathfrak{N}, \ \ \varkappa > 0.$$

We introduce a measure on the set $\mathfrak{N}$ by the formula $d\nu_\varkappa(K) := \psi_\varkappa(K)d\varrho(K)$, where $d\varrho(K)$ is the counting measure on $\mathfrak{N}$. With each pair of functions $f, g \in L_{2,\mathrm{loc}}(\mathbb{R})$ we associate the sequence

$$\vartheta(f,g) = \big\{\vartheta_K(f,g)\big\}_{K\in\mathfrak{N}}, \quad \vartheta_K(f,g) = \|f\otimes g\|_{L_2(K)}\cdot\varphi(K), \quad K \in \mathfrak{N}.$$

The main result of the present paper is the following claim.

Theorem 2.1. *Assume that the condition $\vartheta(f,g) \in L_{p,q}(\mathfrak{N}, d\nu_\varkappa)$ is satisfied for some $p \in (0,2)$, $q \in (0,+\infty]$, $\varkappa > \frac{5}{8} \cdot \operatorname{int}\left(\frac{2}{p}\right) + \frac{3}{8}$. Then $f\mathbb{A}g \in \mathfrak{S}_{p,q}$ and we have*

$$\|f\mathbb{A}g\|_{\mathfrak{S}_{p,q}} \le C(p,q,\varkappa)\|\vartheta(f,g)\|_{L_{p,q}}.$$

If $\vartheta(f,g) \in L^0_{p,\infty}(\mathfrak{N}, d\nu_\varkappa)$, then $f\mathbb{A}g \in \mathfrak{S}^0_{p,\infty}$.

The proof of Theorem 2.1 will be given in §3. Theorem 2.1 implies the following.

Corollary 2.2. *Let conditions* (1.4)–(1.6) *be satisfied for some $p \in (0,2)$. Then $f\mathbb{A}g \in \mathfrak{S}_p$ and estimate* (1.7) *is true.*

§3. Proof of Theorem 2.1

3.1. Since the rectangles in the set $\mathfrak{N}$ form a partition of the plane $\mathbb{R}^2$, the kernel of the operator $f\mathbb{A}g$ admits the representation

$$\begin{aligned} &f(x)\operatorname{Ai}(y-x)g(y) = \sum_{K \in \mathfrak{N}} T_K(x,y), \\ &T_K(x,y) := \mathbf{1}_{[a,b)}(x) f(x) \operatorname{Ai}(y-x) g(y) \mathbf{1}_{[c,d)}(y), \quad (x,y) \in \mathbb{R}^2, \\ &K = [a,b) \times [c,d) \in \mathfrak{N}. \end{aligned} \tag{3.1}$$

Under the condition $f, g \in L_{2,\mathrm{loc}}(\mathbb{R})$, the kernel $T_K(x,y)$ determines a Hilbert–Schmidt operator $\mathbb{T}_K$, $K \in \mathfrak{N}$.

Proposition 3.1. *For arbitrary $p = \frac{2}{n}$, $n \in \mathbb{N}$, we have $\mathbb{T}_K \in \mathfrak{S}_p$, $K \in \mathfrak{N}$, and*

$$\|\mathbb{T}_K\|_{\mathfrak{S}_p} \le C(p,\varkappa)\vartheta_K(f,g)\psi_\varkappa^{1/p}(K), \quad \varkappa > \frac{5}{8} \cdot \frac{2}{p} - \frac{1}{4}, \quad K \in \mathfrak{N}. \tag{3.2}$$

Proposition 3.1 is proved in Subsection 3.4. For $p = 2$, estimate (3.2) takes the form

$$\|\mathbb{T}_K\|_{\mathfrak{S}_2} \le C(\varkappa)\vartheta_K(f,g)\psi_\varkappa^{1/2}(K), \quad \varkappa > \frac{3}{8}, \quad K \in \mathfrak{N}. \tag{3.3}$$

From the definition of the measure $d\nu_\varkappa$ it follows that $L_{p,q}(\mathfrak{N}, d\nu_\varkappa) \subset L_2(\mathfrak{N}, d\nu_\varkappa)$, $p \in (0,2)$, $q \in (0,+\infty]$, and, hence, under the conditions of Theorem 2.1 we have $\vartheta(f,g) \in L_2(\mathfrak{N}, d\nu_\varkappa)$, $\varkappa > \frac{5}{8} \cdot \operatorname{int}\left(\frac{2}{p}\right) + \frac{3}{8}$. Therefore, by (3.1) and (3.3), the series $\sum_{K \in \mathfrak{N}} \|\mathbb{T}_K\|^2_{\mathfrak{S}_2}$ converges, the series $\sum_{K\in\mathfrak{N}} \mathbb{T}_K = f\mathbb{A}g$ converges in $\mathfrak{S}_2$, and

$$\|f\mathbb{A}g\|^2_{\mathfrak{S}_2} = \sum_{K \in \mathfrak{N}} \|\mathbb{T}_K\|^2_{\mathfrak{S}_2}.$$

The results of [8] (see also [9, §11.5, Subsection 4]) show that the following statement is valid.

Proposition 3.2. *If $T_1, T_2 \in \mathfrak{S}_p$, $p \in (0,1]$, then*

$$\|T_1 + T_2\|^p_{\mathfrak{S}_p} \le \|T_1\|^p_{\mathfrak{S}_p} + \|T_2\|^p_{\mathfrak{S}_p}.$$

Proposition 3.2 immediately yields the next corollary.

Corollary 3.3. *Supose $\{T_n\}_{n\in\mathfrak{N}} \subset \mathfrak{S}_p$, $p \in (0,1]$, and the series $\sum_n \|T_n\|^p_{\mathfrak{S}_p}$ converges. Then the series $\sum_n T_n$ converges in $\mathfrak{S}_p$, and $\|\sum_n T_n\|^p_{\mathfrak{S}_p} \le \sum_n \|T_n\|^p_{\mathfrak{S}_p}$.*

3.2. Proof of Theorem 2.1. Suppose that $\vartheta(f,g) \in L_{p,q}(\mathfrak{N}, d\nu_\varkappa)$ for some $p \in (0,2)$, $q \in (0,+\infty]$, $\varkappa > \frac{5}{8} \cdot \operatorname{int}\left(\frac{2}{p}\right) + \frac{3}{8}$. Picking the largest $p_0 = \frac{2}{n} < p$, $n \in \mathbb{N}$, we observe that $\frac{2}{p_0} = n = \operatorname{int}\left(\frac{2}{p}\right) + 1$, $\varkappa > \frac{5}{8} \cdot \frac{2}{p_0} - \frac{1}{4}$. We split the operator $f\mathbb{A}g$ (see (3.1)) into the sum

$$
\begin{gathered}
f\mathbb{A}g = \mathbb{T}^{(0)}(s) + \mathbb{T}^{(2)}(s), \\
\mathbb{T}^{(0)}(s) = \sum_{\vartheta_K(f,g) > s} \mathbb{T}_K, \quad \mathbb{T}^{(2)}(s) = \sum_{\vartheta_K(f,g) \le s} \mathbb{T}_K, \quad s > 0.
\end{gathered} \tag{3.4}
$$

Proposition 3.1 and Corollary 3.3 now yield the relation $\mathbb{T}^{(0)}(s) \in \mathfrak{S}_{p_0}$, $s > 0$, and the estimate

$$
\begin{gathered}
\|\mathbb{T}^{(0)}(s)\|_{\mathfrak{S}_{p_0}}^{p_0} \le C(p_0,\varkappa) \|\vartheta^{(0)}(f,g,s)\|_{L_{p_0}}^{p_0}, \quad s > 0, \\
\vartheta_K^{(0)}(f,g,s) := \begin{cases} \vartheta_K(f,g) & \text{if } \vartheta_K(f,g) > s, \\ 0 & \text{if } \vartheta_K(f,g) \le s. \end{cases}
\end{gathered} \tag{3.5}
$$

In a similar way, (3.3) yields the relation $\mathbb{T}^{(2)}(s) \in \mathfrak{S}_2$, $s > 0$, and the estimate

$$
\begin{gathered}
\|\mathbb{T}^{(2)}(s)\|_{\mathfrak{S}_2}^{2} \le C(\varkappa) \|\vartheta^{(2)}(f,g,s)\|_{L_2}^{2}, \quad s > 0, \\
\vartheta_K^{(2)}(f,g,s) := \begin{cases} 0 & \text{if } \vartheta_K(f,g) > s, \\ \vartheta_K(f,g) & \text{if } \vartheta_K(f,g) \le s. \end{cases}
\end{gathered} \tag{3.6}
$$

By (1.1) we have

$$
\begin{aligned}
\|\vartheta^{(0)}(f,g,s)\|_{L_{p_0}}^{p_0} &\le (\nu_\varkappa)_\vartheta(s) s^{p_0} + \int_s^{+\infty} (\nu_\varkappa)_\vartheta(\sigma)\, d\sigma^{p_0}, \\
\|\vartheta^{(2)}(f,g,s)\|_{L_2}^{2} &\le \int_0^s (\nu_\varkappa)_\vartheta(\sigma)\, d\,\sigma^2, \quad s > 0.
\end{aligned}
$$

Combined with (3.4)–(3.6) and (1.2), this leads to the inequalities

$$
\begin{aligned}
n(s, f\mathbb{A}g) &\le n(s/2, \mathbb{T}^{(2)}(s)) + n(s/2, \mathbb{T}^{(0)}(s)) \\
&\le 4s^{-2} \|\mathbb{T}^{(2)}(s)\|_{\mathfrak{S}_2}^2 + 2^{p_0} s^{-p_0} \|\mathbb{T}^{(0)}(s)\|_{\mathfrak{S}_{p_0}}^{p_0} \\
&\le C(\varkappa) s^{-2} \int_0^s (\nu_\varkappa)_\vartheta(\sigma)\, d\sigma^2 + C(p_0,\varkappa) s^{-p_0} \left[(\nu_\varkappa)_\vartheta(s) s^{p_0} + \int_s^{+\infty} (\nu_\varkappa)_\vartheta(\sigma)\, d\sigma^{p_0} \right].
\end{aligned} \tag{3.7}
$$

The assertions of Theorem 2.1 for $q = +\infty$ follow easily from (3.7). For any $q \in (0,+\infty)$, relations (1.3) and (3.7) yield the estimate

$$
\begin{aligned}
\|f\mathbb{A}g\|_{\mathfrak{S}_{p,q}}^q \le S_1 \int_0^{+\infty} ds^q \left[s^{-2} \cdot \int_0^s d\sigma^2 (\nu_\varkappa)_\vartheta(\sigma) \right]^{q/p} &+ S_2 \int_0^{+\infty} ds^q (\nu_\varkappa)_\vartheta^{q/p}(s) \\
&+ S_3 \int_0^{+\infty} ds^q \left[s^{-p_0} \cdot \int_s^{+\infty} d\sigma^{p_0} (\nu_\varkappa)_\vartheta(\sigma) \right]^{q/p}.
\end{aligned} \tag{3.8}
$$

Here $S_i = S_i(\varkappa, p, q, p_0)$, $i = 1,2,3$. By applying [6, Lemma 4.2] to the first integral in (3.8) and [6, Lemma 4.1] to the third integral, we get the required assertions of Theorem 2.1. □

3.3. To complete the proof of Theorem 2.1 it remains to check the validity of Proposition 3.1. First, we obtain a number of preliminary estimates for the singular values of the operators $\mathbb{T}_K$, $K \in \mathfrak{N}$.

Proposition 3.4. *For arbitrary* $p = 2/(n+1)$, $n \in \mathbb{N}$, $K = [a,b) \times [c,d) \in \mathfrak{N}$, *and* $t > 0$, *we have* $\mathbb{T}_K \in \mathfrak{S}_p$ *and*

$$\|\mathbb{T}_K\|_{\mathfrak{S}_p} \le C(p,\alpha,\beta,t)\langle d-c\rangle^{\alpha} \max_{x\in[a,b]} \langle x-c\rangle^{-1/4} \|f\otimes g\|_{L_2(K)}, \tag{3.9}$$

$$\|\mathbb{T}_K\|_{\mathfrak{S}_p} \le C(p,\alpha,\beta,t)\langle b-a\rangle^{\alpha} \max_{y\in[c,d]} \langle y-a\rangle^{-1/4} \|f\otimes g\|_{L_2(K)}, \tag{3.10}$$

$$C(p,\alpha,\beta,t) = C(p,\alpha,\beta) e^{nt(b-c)+\frac{n^3t^3}{3}} \langle t^2\rangle^{\beta} \langle \sqrt{t}\rangle^{\beta} t^{-\frac{n}{4}},$$

$$\alpha > \alpha(n) := 1/4 + n/2, \quad \beta > \beta(n) := \frac{n(n+2)}{4}, \quad t > 0.$$

To verify Proposition 3.4, note that operator $\mathbb{T}_K$ admits the decomposition $\mathbb{T}_K = M_0 \cdots M_n$, where the operators M_j are defined by as follows:

$$\begin{aligned}
&M_0 := \mathbf{1}_{[a,b)}[f(p)e^{ntp}]\mathbb{A}[\langle x-c\rangle^{-\alpha_1}],\\
&M_j := [\langle x-c\rangle^{\alpha_j} e^{-(j-1)tx}]\mathbb{A}^*[e^{-tp}]\mathbb{A}[\langle x-c\rangle^{-\alpha_{j+1}} e^{jtx}], \quad j = 1,\dots,n-1,\\
&M_n := [\langle x-c\rangle^{\alpha_n} e^{-(n-1)tx}]\mathbb{A}^*[e^{-tp}]\mathbb{A}[g(y)]\mathbf{1}_{[c,d)},\\
&\alpha_1 > 3/4, \quad \alpha_{j+1} > \alpha_j + 1/2, \quad t > 0.
\end{aligned}$$

The operators M_j, $j = 0,\dots,n$ satisfy the following estimates.

Lemma 3.5. *For any* $\alpha_1 > 3/4$, $\alpha_{j+1} > \alpha_j + 1/2$, $j = 1,\dots,n-1$, *and* $t > 0$, *the operators* M_j, $j = 0,\dots,n$ *belong to the Hilbert–Schmidt class, and, moreover,*

$$\|M_0\|_{\mathfrak{S}_2} \le C(\alpha_1) e^{ntb} \max_{x\in[a,b]} \langle x-c\rangle^{-1/4} \|f\|_{L_2(a,b)}; \tag{3.11a}$$

$$\|M_j\|_{\mathfrak{S}_2} \le C(\alpha_j,\alpha_{j+1},j)\langle t^2\rangle^{\alpha_j}\langle\sqrt{t}\rangle^{\alpha_j} t^{-\frac14} e^{(3j^2-3j+1)t^3/3}, \quad j = 1,\dots,n-1; \tag{3.11b}$$

$$\|M_n\|_{\mathfrak{S}_2} \le C(\alpha_n,n)\langle d-c\rangle^{\alpha_n}\langle t^2\rangle^{\alpha_n}\langle\sqrt{t}\rangle^{\alpha_n} t^{-\frac14} e^{-nct} e^{(3n^2-3n+1)t^3/3} \|g\|_{L_2(c,d)}. \tag{3.11c}$$

The proof of Lemma 3.5 is given in Subsection 3.6. Estimate (3.9) follows from the decomposition $\mathbb{T}_K = M_0 \cdots M_n$, estimates (3.11), and the inequality (see, e.g., [9, §11.5, Subsection 4])

$$\begin{gathered}
\|TS\|_{\mathfrak{S}_r} \le C(p,q)\|T\|_{\mathfrak{S}_p}\|S\|_{\mathfrak{S}_q}, \quad T \in \mathfrak{S}_p, \quad S \in \mathfrak{S}_q,\\
r^{-1} = p^{-1} + q^{-1}, \quad p,q \in (0,+\infty).
\end{gathered} \tag{3.12}$$

Similarly, the operator $\mathbb{T}_K$ admits the decomposition $\mathbb{T}_K = N_n \cdot N_{n-1} \cdots N_0$, where

$$\begin{aligned}
&N_0 := [\langle x-a\rangle^{-\alpha_1}]\mathbb{A}[g(p)e^{-tnp}]\mathbf{1}_{[c,d)},\\
&N_j := [\langle x-a\rangle^{-\alpha_{j+1}} e^{-jtx}]\mathbb{A}[e^{tp}]\mathbb{A}^*[\langle x-a\rangle^{\alpha_j} e^{(j-1)tx}], \quad j = 1,\dots,n-1,\\
&N_n := \mathbf{1}_{[a,b)} f\mathbb{A}[e^{tp}]\mathbb{A}^*[\langle x-a\rangle^{\alpha_n} e^{(n-1)tx}],\\
&\alpha_1 > 3/4, \quad \alpha_{j+1} > \alpha_j + 1/2, \quad t > 0.
\end{aligned}$$

Lemma 3.6. *For any* $\alpha_1 > 3/4$, $\alpha_{j+1} > \alpha_j + 1/2$, $j = 1,\dots,n-1$, *and* $t > 0$, *the operators* N_j, $j = 0,\dots,n$, *belong to the Hilbert–Schmidt class, and*

$$\|N_0\|_{\mathfrak{S}_2} le C(\alpha_1) e^{-ntc} \max_{y\in[c,d]} \langle y-a\rangle^{-1/4} \|g\|_{L_2(c,d)}; \tag{3.13a}$$

$$\|N_j\|_{\mathfrak{S}_2} \le C(\alpha_j,\alpha_{j+1},j)\langle t^2\rangle^{\alpha_j}\langle\sqrt{t}\rangle^{\alpha_j} t^{-\frac14} e^{(3j^2-3j+1)t^3/3} \quad j = 1,\dots,n-1; \tag{3.13b}$$

$$\|N_n\|_{\mathfrak{S}_2} \le C(\alpha_n,n)\langle b-a\rangle^{\alpha_n}\langle t^2\rangle^{\alpha_n}\langle\sqrt{t}\rangle^{\alpha_n} t^{-\frac14} e^{nbt} e^{(3n^2-3n+1)t^3/3} \|f\|_{L_2(c,d)}. \tag{3.13c}$$

Lemma 3.6 is proved in Subsection 3.6. Estimate (3.10) follows from the decomposition $\mathbb{T}_K = N_n \cdot N_{n-1} \cdots N_0$, estimates (3.13), and inequality (3.12). □

3.4. Proof of Proposition 3.1. First, we verify Proposition 3.1 for $p = 2$. From the estimate

$$|\operatorname{Ai}(z)| \le C\langle z\rangle^{-1/4} e^{-\frac{2}{3} z_+^{3/2}}, \quad z \in \mathbb{R},$$

for the Airy function (see, e.g., [10]) and the elementary inequality $\langle a+b\rangle^\theta \le C_\theta \langle a\rangle^\theta \langle b\rangle^{|\theta|}$, $a, b \in \mathbb{R}$, $\theta \in \mathbb{R}$ it follows that

$$\operatorname{Ai}^2(y-x) \le C\langle y-c\rangle^{1/2}\langle x-c\rangle^{-1/2} e^{-\frac{4}{3}(y-x)_+^{3/2}},$$
$$\operatorname{Ai}^2(y-x) \le C\langle x-a\rangle^{1/2}\langle y-a\rangle^{-1/2} e^{-\frac{4}{3}(y-x)_+^{3/2}}.$$

This yields the inequality

$$\|\mathbb{T}_K\|_{\mathfrak{S}_2} \le \Gamma_{1/4}(K)\|f\otimes g\|_{L_2(K)} e^{-\frac{2}{3}(c-b)_+^{3/2}}, \quad K = [a,b)\times[c,d) \in \mathfrak{N}. \tag{3.14}$$

Here and in what follows we use the notation

$$\Gamma_\alpha(K) := \min\big\{\langle b-a\rangle^\alpha \max_{y\in[c,d]}\langle y-a\rangle^{-1/4}, \langle d-c\rangle^\alpha \max_{x\in[a,b]}\langle x-c\rangle^{-1/4}\big\}.$$

It remains to observe that for all $K = [a,b)\times[c,d) \in \mathfrak{N}$ and $\alpha > 0$ we have

$$\Gamma_\alpha(K) \le C(\alpha)\langle b-c\rangle^{-1/4}. \tag{3.15}$$

Inequalities (3.14) and (3.15) imply (3.3).

Now we verify Proposition 3.1 for $p = 2/(n+1)$, $n \in \mathbb{N}$. From (3.9), (3.10) it follows that

$$\begin{gathered}\|\mathbb{T}_K\|_{\mathfrak{S}_p} \le C(p,\alpha,\beta)\Gamma_\alpha(K) e^{nt(b-c)+\frac{n^3t^3}{3}}\langle t^2\rangle^\beta\langle\sqrt{t}\rangle^\beta t^{-n/4}\|f\otimes g\|_{L_2(K)},\\ \alpha > \frac{1}{4}+\frac{n}{2}, \quad \beta > \frac{n(n+2)}{4}, \quad t>0, \quad K=[a,b)\times[c,d)\in\mathfrak{N}.\end{gathered} \tag{3.16}$$

Choosing the parameter

$$t = \begin{cases} n^{-1}(b-c)^{-1}, & b-c \ge 1,\\ n^{-1}\sqrt{c-b}, & b-c \le -1,\\ n^{-1}, & -1 < b-c < 1,\end{cases}$$

in (3.16) and using (3.15), we arrive at (3.2). ☐

3.5. In order to prove Lemmas 3.5 and 3.6, we shall need explicit expressions for the kernels of the integral operators $\mathbb{A}^*[e^{-zp}]\mathbb{A}$, $\mathbb{A}[e^{zp}]\mathbb{A}^*$, $\operatorname{Re} z > 0$. By the Avron–Herbst formula (see [11])

$$\mathbb{A}^*[e^{itp}]\mathbb{A} = e^{i\frac{t^3}{3}}[e^{itx}]\Phi^*[e^{it^2p+ip^2t}]\Phi, \quad t\in\mathbb{R},$$

the integral operator $\mathbb{A}^*[e^{itp}]\mathbb{A}$, $t \in \mathbb{R}$, has the kernel

$$K_0(t,x,y) := \frac{1}{\sqrt{-4\pi i t}} e^{i\frac{t^3}{12} - i\frac{(x-y)^2}{4t} + i\frac{x+y}{2}t}, \quad x,y,t\in\mathbb{R}. \tag{3.17}$$

Extending the form $(\mathbb{A}^*[e^{-zp}]\mathbb{A}u, v)$, $u, v \in C_0^\infty(\mathbb{R})$, analytically from the imaginary line to the right half-plane and using (3.17), we prove that the operator $\mathbb{A}^*[e^{-zp}]\mathbb{A}$, $\operatorname{Re} z > 0$, is an integral operator with the kernel

$$K_1(z,x,y) := \frac{1}{\sqrt{4\pi z}} e^{\frac{z^3}{12} - \frac{(x-y)^2}{4z} - \frac{x+y}{2}z}, \quad x,y\in\mathbb{R}, \quad \operatorname{Re} z > 0. \tag{3.18}$$

On the other hand, the kernel of the operator $\mathbb{A}^*[e^{-zp}]\mathbb{A}$, $\operatorname{Re} z > 0$, can be expressed as follows:

$$K_1(z,x,y) = \int_{\mathbb{R}} \operatorname{Ai}(x-p)e^{-zp}\operatorname{Ai}(y-p)\,dp, \quad x,y\in\mathbb{R}, \ \operatorname{Re} z > 0. \tag{3.19}$$

Finally, the kernel of the operator $\mathbb{A}[e^{zp}]\mathbb{A}^*$, $\operatorname{Re} z > 0$, can be written as

$$K_2(z,x,y) := \int_{\mathbb{R}} \operatorname{Ai}(p-x)e^{zp}\operatorname{Ai}(p-y)\,dp, \quad x,y\in\mathbb{R},\ \operatorname{Re} z>0. \tag{3.20}$$

Comparing (3.18)–(3.20), we see that

$$K_2(z,x,y) = \frac{1}{\sqrt{4\pi z}} e^{\frac{z^3}{12} - \frac{(x-y)^2}{4z} + \frac{x+y}{2}z}, \quad x,y\in\mathbb{R}, \quad \operatorname{Re} z>0. \tag{3.21}$$

3.6. Proof of Lemmas 3.5 and 3.6.

Remark 3.7. For all $\alpha, m \ge 0$ and $t>0$, we have

$$\int_{\mathbb{R}} \langle z\rangle^{2\alpha} e^{-\frac{z^2}{2t} - ztm}\,dz \le C(\alpha)\langle m\rangle^{2\alpha}\langle t^2\rangle^{2\alpha}\langle\sqrt{t}\rangle^{2\alpha} e^{\frac{t^3m^2}{2}} t^{1/2}. \tag{3.22}$$

Proof. Estimate (3.22) will follow if we pass to the new variable $\eta = \frac{z+t^2m}{\sqrt{t}}$ and use the inequalities $\langle\sqrt{t}\eta\rangle^{2\alpha} \le \langle\sqrt{t}\rangle^{2\alpha}\langle\eta\rangle^{2\alpha}$, $\langle t^2 m\rangle^{2\alpha} \le \langle t^2\rangle^{2\alpha}\langle m\rangle^{2\alpha}$. □

Proof of Lemma 3.5. 1) Estimate (3.11a) is checked by an explicit calculation, with the help of the inequalities

$$\operatorname{Ai}^2(y-x) \le C\langle y-x\rangle^{-1/2} \le C\langle x-c\rangle^{-1/2}\langle y-c\rangle^{1/2}.$$

2) From (3.18) it follows that each M_j, $j=1,\dots,n-1$, is an integral operator with the kernel

$$M_j[x,y] := \frac{e^{t^3/12}}{\sqrt{4\pi t}}\langle y-c\rangle^{-\alpha_{j+1}}\langle x-c\rangle^{\alpha_j} e^{-\frac{(x-y)^2}{4t} - \frac{(2j-1)(x-y)}{2}t}, \quad x,y\in\mathbb{R},\ t>0. \tag{3.23}$$

Estimate (3.11b) follows from (3.22), (3.23), and the inequality

$$\langle x-c\rangle^{\alpha_j} \le C\langle y-c\rangle^{\alpha_j}\langle x-y\rangle^{\alpha_j}.$$

3) The proof of estimate (3.11c) is similar to that of estimate (3.11b). □

Lemma 3.6 is proved much as 3.5. In this proof, identity (3.21) is employed instead of (3.18).

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Received 01/NOV/2016

Translated by A. V. KISELEV