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COMPLEX WKB METHOD FOR A DIFFERENCE SCHRÖDINGER EQUATION WITH THE POTENTIAL BEING A TRIGONOMETRIC POLYNOMIAL

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Dedicated to the memory of Vladimir Savel'evich Buslaev

Abstract. The difference Schrödinger equation $\psi(z+h)+\psi(z-h)+v(z)\psi(z) = E\psi(z)$, $z\in\mathbb{C}$, is considered; here h is a positive number, E is the spectral parameter, and v is a trigonometric polynomial. The asymptotics of its entire solutions is studied as $h\to 0$.

§1. Introduction

We consider the difference Schrödinger equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z) = E\psi(z), \quad z\in\mathbb{C}, \tag{1.1}$$

where $h>0$ is a fixed number, $E\in\mathbb{C}$ is the spectral parameter, and v is an analytic function. Our aim is to study the asymptotics of analytic solutions of (1.1) as $h\to 0$.

Note that, formally, $f(z\pm h) = e^{\pm h\frac{d}{dz}}f(z)$, and thus, the small parameter becomes a factor in front of the derivative. Therefore, it turns out to be a quasiclassical parameter.

We remind that, to study the quasiclassical asymptotics of solutions of the differential Schrödinger equation

$$-h^2\frac{d^2\psi}{dz^2}(z)+v(z)\psi(z) = E\psi(z), \quad z\in\mathbb{C}, \tag{1.2}$$

with an analytic potential v, the complex WKB method is employed, see [5]. With its help one constructs analytic solutions having simple asymptotic behavior in certain *canonical* domains in the complex plane.

For the first time, the quasiclassical asymptotics of entire solutions of difference equations in the complex plane were studied in [2]. The authors suggested an analog of the complex WKB method for equation (1.1) with $v(z)=2\cos z$, the Harper equation, well-known in the solid state physics, see, e.g., [3]. They formulated the definition of a canonical domain and proved the main theorem of the method, which establishes the existence of an entire solution having a simple asymptotic behavior in a canonical domains. However, the proof of that theorem turned out to be rather involved and was carried out only for the Harper equation and for two special classes of canonical domain.

In [6], we used some ideas of [4] to treat equation (1.1) with general analytic potentials v and suggested a new, relatively simple way to prove the existence theorem for analytic solutions having a simple asymptotic behavior in a bounded canonical domain.

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In the present paper we turn to the case of unbounded canonical domains. Our analysis strongly depends on the behavior of all the analytic and geometric objects at infinity, and we only consider the potentials v that are trigonometric polynomials,

$$v(z) = \sum_{k=-m}^{n} c_k e^{ikz}, \quad m, n > 0, \quad c_n, c_{-m} \neq 0. \tag{1.3}$$

Difference equations with such potentials arise, for example, in the theory of diffraction, see, e.g., [1], and, as it has already been mentioned, in the solid state physics.

§2. The main theorem of the complex WKB method

Here, for the difference equations under consideration, we formulate the main theorem of the complex WKB method and describe the plan and the leading ideas of its proof. First, we recall an analog of that theorem for the differential equation (1.2).

2.1. The case of the differential equation (1.2)**.** We begin with the definitions of the analytic and geometric objects needed to formulate the main theorem. For simplicity, we assume that v is an entire function.

The multivalued analytic function p defined by the relation

$$p^2(z) + v(z) = E, \quad z \in \mathbb{C},$$

is called the *complex momentum.* We note that the branch points of p are simultaneously its zeros and the turning points for equation (1.2). We say that a set in the complex plane is *regular* if it contains no branch points of p.

Let $\gamma \subset \mathbb{C}$ be a regular oriented simple smooth curve, and let p be a branch of the complex momentum analytic on it. The curve γ is said to be *canonical* with respect to the branch p if the derivative of $\operatorname{Im} \int^z p(z)\,dz$ along γ is positive at any point of γ.

Let $K \subset \mathbb{C}$ be a regular domain, let p be a branch of the complex momentum analytic in K, and let z_* be a point of the boundary of K. We say that the domain K is *canonical* with respect to p if for any $z \in K$ there exists a curve $\gamma \subset K$ canonical with respect to p and going from z to z_*.

Theorem 2.1. *Let $K \subset \mathbb{C}$ be a domain canonical with respect to a branch p of the complex momentum, and let $z_0 \in K$. Then, for sufficiently small h, equation* (1.2) *has entire solutions $\psi_\pm$ such that in K we have the asymptotic representations*

$$\psi_\pm(z) = \frac{e^{\pm \frac{i}{h}\theta(z) + O(h)}}{\sqrt{p(z)}}, \quad \theta(z) = \int_{z_0}^{z} p\,dz, \qquad h \to 0,$$

where the error estimate is locally uniform in $z \in K$.

Note that θ is called the *action.* The further details can be found in [5].

2.2. The case of the difference equation (1.1)**.** We begin with formulating the definitions of the main analytic and geometric objects.

2.2.1. We define the *the complex momentum* p by the relation

$$2\cos p(z) + v(z) = E, \qquad z \in \mathbb{C}. \tag{2.1}$$

Now, p is a multivalued function, and its branch points satisfy the equation $\pm 2 + v(z) = E$. We say that a set is *regular* if in that set we have $E - v(z) \neq \pm 2$.

Properties of the complex momentum are described in Subsection 3.1. For example, in Subsection 3.1.1 we list all its branches analytic in a given regular domain, and in Subsection 3.1.2 we describe the asymptotics of $p(z)$ as $\operatorname{Im} z \to \pm\infty$.

2.2.2. *Canonical curves.* For $z \in \mathbb{C}$ we put $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. A curve $\gamma \subset \mathbb{C}$ is said to be *vertical* if, on γ, z is a piecewise continuously differentiable function of $y = \operatorname{Im} z$.

Let γ be a regular vertical curve, let $z = z(y)$ along γ, let $z_0 \in \gamma$, and let p be a branch of the complex momentum analytic on γ. The curve γ is *canonical* with respect to p if on γ we have

$$\frac{d \operatorname{Im} \theta_0(z(y))}{dy} > 0 \quad \text{and} \quad \frac{d \operatorname{Im} \theta_\pi(z(y))}{dy} < 0, \tag{2.2}$$

where

$$\theta_0(z) = \int_{z_0}^{z} p\, dz \quad \text{and} \quad \theta_\pi(z) = \int_{z_0}^{z} (p - \pi)\, dz. \tag{2.3}$$

At the discontinuity points of the derivative of $z(.)$, these inequalities should be fulfilled for the its left and right derivatives.

In this paper we usually consider infinite canonical curves along which $y = \operatorname{Im} z$ grows from $-\infty$ to ∞. If we consider a finite or semi-infinite canonical curve, we say this explicitly.

For v of the form (1.3), in Subsection 3.2.2 we prove the following.

Lemma 2.1. *Let γ be a curve canonical with respect to a branch p of the complex momentum. For a suitable definition of* $\arg c_{-m}$ *(the definition depends only on p), the distance from $z \in \gamma$ to the strip*

$$\Pi_+(p) = \left\{ z \in \mathbb{C} \,:\, \frac{\arg c_{-m}}{m} < x < \frac{\arg c_{-m} + \pi}{m} \right\}$$

tends to zero as $\operatorname{Im} z \to +\infty$.

Note that a canonical curve γ may go to $+i\infty$ either inside the strip $\Pi_+(p)$, or outside of this strip, approaching its boundary. The behavior of the canonical curves at $-i\infty$ is described similarly, but now the role of $\Pi_+(p)$ is played by the strip of the form $\Pi_-(p) = \left\{ z \in \mathbb{C} \,:\, \frac{-\arg c_n}{n} < x < \frac{-\arg c_n + \pi}{n} \right\}$.

2.2.3. In this paper we define and discuss only the *canonical domains* containing infinite canonical curves.

We say that a domain is *horizontally connected* if, for any two its points with equal imaginary parts, the segment of straight line that connects them is located in this domain. A domain is said to be *horizontally bounded* if the absolute values of the real parts of all its points are bounded by one and the same constant.

Let $K \subset \mathbb{C}$ be a regular, horizontally connected, and horizontally bounded domain, and let p be a branch of the complex momentum analytic in K. We say that K is *canonical* with respect to p if for any $z \in K$ there is a curve $\gamma \subset K$ canonical with respect to p and such that $z \in K$.

For the Harper equation, three examples of canonical domains can be found in [2]. Their horizontal boundedness is related to the geometry of the set of the branch points of the complex momentum. In the present paper, we do not discuss the existence and global geometry of canonical domains, simply including the condition of their horizontal boundedness in the definition.

For v of the form (1.3), in Subsection 3.3.1 we prove the next statement.

Lemma 2.2. *Let K be a domain canonical with respect to a branch p of the complex momentum. Then*:

- *there exist two vertical curves such that the left boundary of the strip $\Pi_+(p)$ is the asymptote for one of them, its right boundary is the asymptote for the second curve, and the domain K is located between these two curves*;

• *K can be extended to a canonical domain containing two vertical curves such that the left boundary of the strip $\Pi_+(p)$ is the asymptote for one of them, and its right boundary is the asymptote for the second curve.*

There is a similar statement concerning canonical domains below the real axis.

Let K be a canonical domain, and let δ be a positive number. The domain

$$K_\delta = \{z \in K \,:\, \operatorname{dist}(z, \partial K) > \delta\}$$

is said to be *admissible.*

2.2.4. Now we formulate the *main theorem* of the complex WKB method.

Theorem 2.2. *Let $K \subset \mathbb{C}$ be a domain canonical with respect to a branch p of the complex momentum. Then, for sufficiently small h, equation* (1.1) *has entire solutions $\psi_\pm$ such that in K we have the asymptotic representations*

$$\psi_\pm(z) = \frac{e^{\pm \frac{i}{h}\theta_0(z)+O(h)}}{\sqrt{\sin p(z)}}, \quad h \to 0, \tag{2.4}$$

with error estimates uniform in $z \in K_\delta$ for any fixed $\delta > 0$.

We note that the error estimates in (2.4) are uniform in the domain K_δ, which can be unbounded for sufficiently small δ.

We note also that the leading terms of the asymptotics of solutions can easily be found via formal substitution of the *Ansatz* $e^{\frac{i}{h}B(z)} \sum_{k=0}^{\infty} h^k A_k(z)$ into equation (1.1). However, the definition of the canonical domain where this asymptotics is valid can be found only in the course of the proof of the theorem.

2.3. The plan and ideas of the proof of Theorem 2.2. The main idea is to reduce the difference equation in question to a singular integral equation on a vertical curve γ. It turns out that, roughly speaking, if γ is a canonical curve, then the norm of the integral operator is small. This enables us to study the equation efficiently.

The way from the difference equation to the singular integral equation can be described as follows. Below we assume that z is in a canonical domain K.

1. First, we transform the second order difference equation (1.1) to a first order difference equation for a vector-valued function $\Psi\colon \mathbb{C} \mapsto \mathbb{C}^2$, obtaining the equation

$$\Psi(z+h) = M(z)\Psi(z), \qquad M(z) = \begin{pmatrix} E - v(z) & -1 \\ 1 & 0 \end{pmatrix}, \qquad z \in \mathbb{C}. \tag{2.5}$$

It is easily seen that Ψ is a solution of (2.5) if and only if $\Psi(z) = \left(\begin{smallmatrix} \psi(z) \\ \psi(z-h) \end{smallmatrix}\right)$, where ψ is a solution of equation (1.1).

2. We note that $\operatorname{tr} M(z) = 2\cos p(z)$, and that the eigenvalues of $M(z)$ are equal to $e^{\pm ip(z)}$. Obviously, in a regular domain these eigenvalues are not equal, and the matrix $M(z)$ can be diagonalized.

Since K is regular, we can choose a matrix function U so that

$$U^{-1}(z)M(z)U(z) = \begin{pmatrix} e^{ip(z)} & 0 \\ 0 & e^{-ip(z)} \end{pmatrix}, \quad z \in K.$$

The function Φ defined by the formula

$$\Phi(z) = U^{-1}(z)\Psi(z), \quad z \in K, \tag{2.6}$$

satisfies the equation

$$\Phi(z+h) = T(z)\Phi(z), \qquad z \in K, \tag{2.7}$$

with the matrix

$$T(z) = U^{-1}(z+h)M(z)U(z).$$

Since h is small, we can hope that this matrix is close to $\begin{pmatrix} e^{ip(z)} & 0 \\ 0 & e^{-ip(z)} \end{pmatrix}$. Since the domain K is unbounded, we need to check that these matrices remain close as z tends to infinity. It turns out that the proof depends on the behavior of $e^{ip(z)}$ as $\operatorname{Im} z \to \pm\infty$. Lemma 3.1 describes all the possible types of the behavior of p as $\operatorname{Im} z \to \pm\infty$, v being a trigonometric polynomial of the form (1.3). First, we prove Theorem 2.2 in the case where $e^{ip(z)}$ exponentially decays both as $\operatorname{Im} z \to +\infty$ and $\operatorname{Im} z \to -\infty$. In this case we choose

$$U(z) = \begin{pmatrix} e^{ip(z)} & 1 \\ 1 & e^{ip(z)} \end{pmatrix} \tag{2.8}$$

and arrive at the following statement.

Lemma 2.3. *Pick $\delta > 0$. Let V_δ be the δ-neighborhood of the branch points. The following asymptotic representation is valid in $K \setminus V_\delta$ as $h \to 0$:*

$$T = \begin{pmatrix} e^{ip - \frac{h}{2}(ip + \ln \sin p)' + O(h^2 e^{2ip})} & O(h) \\ O(he^{2ip}) & e^{-ip - \frac{h}{2}(ip + \ln \sin p)' + O(h^2 e^{2ip})} \end{pmatrix} \tag{2.9}$$

with error estimate uniform in z.

This lemma is proved in §4.

3. To pass from equation (2.7) to an integral equation, we use an argument well-known in the theory of differential equations. We transform (2.7) to the equation

$$\Phi(z+h) - \begin{pmatrix} T_{11}(z) & 0 \\ 0 & T_{22}(z) \end{pmatrix} \Phi(z) = \begin{pmatrix} 0 & T_{12}(z) \\ T_{21}(z) & 0 \end{pmatrix} \Phi(z)$$

and view it as a nonhomogeneous one. The matrix

$$\Phi_0(z) = \begin{pmatrix} e^{\Theta_1(z)} & 0 \\ 0 & e^{\Theta_2(z)} \end{pmatrix}$$

satisfies the homogeneous equation

$$\Phi_0(z+h) - \begin{pmatrix} T_{11}(z) & 0 \\ 0 & T_{22}(z) \end{pmatrix} \Phi_0(z) = 0$$

whenever

$$\Theta_1(z+h) = \ln T_{11}(z) + \Theta_1(z) \quad \text{and} \quad \Theta_2(z+h) = \ln T_{22}(z) + \Theta_2(z). \tag{2.10}$$

For the role of $\ln T_{11}$ and $\ln T_{22}$ we take the exponents from the asymptotic formulas for T_{11} and T_{22} in (2.9). We represent $\Phi(z)$ in the form

$$\Phi(z) = \Phi_0(z)X(z). \tag{2.11}$$

Then X satisfies the equation

$$X(z+h) - X(z) = S(z)X(z) \tag{2.12}$$

where

$$S(z) = \begin{pmatrix} 0 & e^{\Theta_2(z) - \Theta_1(z+h)} T_{12}(z) \\ e^{\Theta_1(z) - \Theta_2(z+h)} T_{21}(z) \end{pmatrix}. \tag{2.13}$$

We arrive at an integral equation by inverting the first order difference operator on the left-hand side of (2.12).

Methods for solving first order difference equations are described in §5, and in §6 we prove the next claim.

Lemma 2.4. *There exist functions Θ_1 and Θ_2 analytic in K that satisfy* (2.10) *if z, $z+h \in K$. As $h \to 0$, these functions and the corresponding matrix-valued function S admit the asymptotic representations*

$$\begin{aligned} \Theta_1(z) &= \frac{i\theta_0(z)}{h} - \frac{\ln \sin p(z)}{2} - ip(z) + O(h), \\ \Theta_2(z) &= -\frac{i\theta_0(z)}{h} - \frac{\ln \sin p(z)}{2} + O(h), \end{aligned} \tag{2.14}$$

and

$$S(z) = \begin{pmatrix} 0 & O(he^{-\frac{2i}{h}\theta_0(z)}) \\ O(he^{\frac{2i}{h}\theta_0(z)}e^{2ip(z)}) & 0 \end{pmatrix}. \tag{2.15}$$

These representations are uniform in z in $K \setminus V_\delta$ for any fixed $\delta > 0$.

4. Equation (2.12) is transformed to the equation

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{L}_+(SX) \tag{2.16}$$

on a vertical curve $\gamma \subset K$. Here, $\mathcal{L}_+$ is the singular integral operator acting on a suitable space of functions defined on γ by the formula

$$\mathcal{L}_+ g(z) = \frac{1}{2ih} \int_\gamma \left(\cot \left[\frac{\pi(\zeta - z - 0)}{h} \right] - i \right) g(\zeta)\, d\zeta. \tag{2.17}$$

Since the matrix $S(z)$ is antidiagonal, from (2.16) it is easy to deduce that the first component of the vector X satisfies the equation

$$X_1 = 1 + \mathcal{L}_+(S_{12}\mathcal{L}_+(S_{21}\, X_1)). \tag{2.18}$$

In view of (2.15), this can be rewritten in the form

$$X_1 = 1 + \mathcal{L}_+(s_{12}\, \mathcal{K}_+(e^{2ip}\, s_{21}\, X_1)), \tag{2.19}$$

where s_{12} and s_{21} are functions analytic in K and satisfying the uniform estimates

$$s_{12}(z),\, s_{21}(z) = O(h) \quad \text{if} \quad z \in K \setminus V_\delta, \tag{2.20}$$

and

$$\mathcal{K}_+ g(z) = e^{-\frac{2i\theta_0(z)}{h}}\, \mathcal{L}_+\left(e^{\frac{2i\theta_0}{h}}\, g\right)(z). \tag{2.21}$$

It turns out that, if the curve γ is canonical, the norm of the operator $\mathcal{K}_+$ is of the order of one, and the norm of the operator applied to X_1 on the right-hand side of (2.19) is small. This observation enables us to construct solutions of (2.19) and is a key observation for the proof of Theorem 2.2.

In §7 we derive the integral equation for X_1, study the operators $\mathcal{K}_+$ and $\mathcal{L}_+$, and construct a solution of the integral equation. In §8 we construct the solution ψ_+ of the difference Schrödinger equation and prove formula (2.4) for it. §9 is devoted to the solution ψ_-. In §10 we briefly discuss the modifications of the proof needed in the case of the canonical domains where $e^{ip(z)}$ does not tend to zero as $|\operatorname{Im} z| \to \infty$.

§3. Complex momentum, canonical curves, and canonical domains

The structure of the canonical domains at infinity and the asymptotics of the complex momentum in them play an important role in the proof of the main theorem. Here we collect the required information on the complex momentum, canonical domains, and canonical curves.

3.1. Properties of the complex momentum.

3.1.1. Let $D \subset \mathbb{C}$ be a regular, simply connected domain, and let p be a branch of the complex momentum analytic in D. From the definition (2.1) of the momentum it follows that all the other its branches analytic in D are of the form

$$\pm p + 2\pi k, \qquad k \in \mathbb{Z},$$

where the sign and the integer k depend on the branch of the momentum.

3.1.2. We remind the reader that $z = x + iy$, $x, y \in \mathbb{R}$. The behavior of the complex momentum for large $|y|$ is described by the following lemma.

Lemma 3.1. *Let $Y > 0$ be sufficiently large. Then, the half-planes $\mathbb{C}_\pm(Y) = \{z \in \mathbb{C} : \pm \operatorname{Im} z > Y\}$ are regular. The branches of the complex momentum analytic in $\mathbb{C}_+(Y)$ and $\mathbb{C}_-(Y)$ admit the representations*

$$p(z) = \pm(\pi - mz - i \ln c_{-m}) + O(e^{-y}), \quad y \to +\infty, \tag{3.1}$$

and

$$p(z) = \pm(\pi + nz - i \ln c_n) + O(e^{y}), \quad y \to -\infty, \tag{3.2}$$

respectively. The signs and the branches of ln *depend on the branches p. The error estimates are uniform in x. The asymptotic formulas for the derivatives of p with respect to z can be obtained by formal differentiation of the above asymptotics.*

Proof. The branch points of the complex momentum satisfy $\pm 2 + v(z) = E$, and for v given by (1.3) the regularity of $\mathbb{C}_+(Y)$ is evident. From (2.1) it follows that, as $y \to +\infty$, the absolute value of one of the exponentials $e^{\pm ip(z)}$ becomes large, and the other one tends to zero. If, for the chosen branch of the complex momentum, the exponential $e^{ip(z)}$ becomes large, we get

$$e^{ip(z)} = -c_{-m} e^{-imz} \left(1 + O(e^{-y})\right), \quad y \to +\infty.$$

Taking the logarithms of both sides and using the observation made in Subsection 3.1.1, we get (3.1). The statement on the asymptotic formulas for the derivatives of p follows from the asymptotics for p and the Cauchy formulas for the derivatives of analytic functions. The case where $\operatorname{Im} z \to +\infty$ and $e^{-ip(z)}$ is large, and the case where $\operatorname{Im} z \to -\infty$ are analyzed similarly. $\square$

Let V_δ be the δ-neighborhood of the branch points. The observation in Subsection 3.1.1 and the last lemma imply the following claim.

Corollary 3.1. *Let D be a regular domain, and let p be a branch of the momentum analytic in D. For any $\delta > 0$,*

- *there exists a constant $C_1 > 0$ such that*

$$|p'(z)| \le C_1, \quad |p''(z)| \le C_1 e^{-|y|}, \qquad z \in D \setminus V_\delta; \tag{3.3}$$

- *there exists a constant $C_2 > 0$ such that*

$$\left|e^{2ip(z)} - 1\right| > C_2, \quad z \in D \setminus V_\delta.$$

3.2. Canonical curves.

3.2.1. *A characteristic property of the canonical curves.*

Lemma 3.2. *Let D be a regular domain, let p be a branch of the complex momentum analytic in D, and let $z \in D$. Consider the straight line that passes through zero and the point $\overline{p(z)}$, and the straight line that passes through zero and the point $\overline{p(z)} - \pi$. Consider the four open sectors (domains) bounded by these lines. Let $S(z)$ denote the sector such that $\operatorname{Im}(p(z)\,u) > 0$ and $\operatorname{Im}((p(z)-\pi)\,u) < 0$ for all $u \in S(z)$. A vertical curve $\gamma \in D$ is canonical with respect to the branch p if and only if, for any $z \in \gamma$, the vector that begins at zero, is parallel to the vector tangent to γ at z, and is oriented upwards belongs to $S(z)$.*

Proof. This follows from the definition of a canonical curve and the relations $d\,\operatorname{Im}\theta_0(z) = \operatorname{Im}(p(z)\,dz)$ and $d\,\operatorname{Im}\theta_\pi(z) = \operatorname{Im}((p(z)-\pi)\,dz)$. □

3.2.2. *Behavior of canonical curves at infinity.* Here we prove Lemma 2.1. Let γ be a curve canonical with respect to an analytic branch p of the complex momentum. Assume that the asymptotics of p as $y \to +\infty$ is described by formula (3.1) with the sign "plus". The other cases are treated similarly.

Let us study γ inside the half-plane $\mathbb{C}_+(Y)$ for a sufficiently large $Y > 0$. This half-plane is regular. We keep the notation p for the analytic continuation of p into $\mathbb{C}_+(Y)$.

From (3.1) it follows that $-\operatorname{Im} p(z) > 0$ for sufficiently large y. Therefore, in $\mathbb{C}_+(Y)$ the vector fields $z \mapsto (\operatorname{Re} p(z), -\operatorname{Im} p(z))$ and $z \mapsto (\operatorname{Re} p(z) - \pi, -\operatorname{Im} p(z))$ are directed upwards.

Note that the domain $S(z)$ described in Lemma 3.2 is bounded by the rays containing the vectors $(\operatorname{Re} p(z), -\operatorname{Im} p(z))$ and $(\operatorname{Re} p(z) - \pi, -\operatorname{Im} p(z))$. This follows directly from the definition of $S(z)$, the fact that these vectors are directed upwards, and the following two observations: (1) if $v = e^{i\alpha}\overline{p(z)}$ with $0 < \alpha < \pi$, then $\operatorname{Im}(p(z)\,v) > 0$; (2) if $v = e^{-i\beta}(\overline{p(z)} - \pi)$ with $0 < \beta < \pi$, then $\operatorname{Im}((p(z)-\pi)\,v) < 0$.

Let $\mathring{z} \in \gamma \cap \mathbb{C}_+(Y)$. We denote by γ_0 and γ_p the integral curves of the vector fields $z \mapsto (\operatorname{Re} p(z), -\operatorname{Im} p(z))$ and $z \mapsto (\operatorname{Re} p(z) - \pi, -\operatorname{Im} p(z))$ that begin at $\mathring{z}$. Note that since in $\mathbb{C}_+(Y)$ these vector fields are directed upwards, the curves γ_0 and γ_π are vertical. Lemma 3.2 and the observations in the preceding paragraph imply that above $\mathring{z}$ the curve γ stays between γ_0 and γ_π.

Consider the curve γ_0. Along it, x and y can be parametrized by a parameter $t \geq 0$ so that

$$\frac{dy}{dt} = -\operatorname{Im} p(x+iy), \quad \frac{dx}{dt} = \operatorname{Re} p(x+iy), \qquad x(0) = \mathring{x}, \quad y(0) = \mathring{y}.$$

Using (3.1), we can rewrite this problem in the form

$$y' = my + \ln|c_{-m}| + O(e^{-y}),\ x' = \pi - mx + \arg c_{-m} + O(e^{-y}), \quad x(0) = \mathring{x},\ y(0) = \mathring{y}.$$

It can easily be shown that, for sufficiently large $\mathring{y}$, the solution of this problem is described by the uniform asymptotic formulas

$$y + \ln|c_{-m}|/m = \left(\mathring{y} + \ln|c_{-m}|/m + O(e^{-\mathring{y}})\right) e^{mt}, \tag{3.4}$$

$$x - (\pi + \arg c_{-m})/m = e^{-mt}\left(\mathring{x} - (\pi + \arg c_{-m})/m + O(e^{-\mathring{y}})\right). \tag{3.5}$$

Omitting elementary details, we only note that the equation for y implies that, for sufficiently large Y and $\mathring{y} > Y$, the function $y(t)$ is monotone increasing so that the error term in this equation is $O(e^{-\mathring{y}})$. We construct the solution to the equation for y as a solution of the linear inhomogeneous equation with $O(e^{-\mathring{y}})$ on the right hand side. After that, we solve the equation for x.

From (3.5) and (3.4) it follows that γ has an asymptote, and that this asymptote is the straight line $x = (\pi + \arg c_{-m})/m$.

Similarly we can check that above $\mathring{y} > Y$ the curve γ_π has an asymptote, and that this asymptote is the straight line $x = \arg c_{-m}/m$. This completes the proof of Lemma 2.1. $\square$

3.3. Canonical domains. Here we prove Lemma 2.2. Below we identify the complex numbers with their radius vectors in a standard way. We use the constructions from Subsection 3.2.2.

Let K be a domain canonical with respect to a branch p of the complex momentum. We only study the case where p is described by formula (3.1) with the sign "plus". All the other cases are analyzed similarly.

We assume that $Y > 0$ is so large that the half-plane $\{\operatorname{Im} z > Y\}$ is regular. The vector field $\overline{p(z)}$ in this half-plane is directed upwards.

We prove the first statement of Lemma 2.2. Since K is a horizontally connected and horizontally bounded domain, the intersection of K and the line $\operatorname{Im} z = Y$ is a finite segment. We denote its ends by z_π and z_0 so that $\operatorname{Re} z_\pi < \operatorname{Re} z_0$, and we denote the segment by (z_π, z_0). Let $a \in \{0, \pi\}$, and let α_a be the integral curve of the vector field $\overline{p(z)} - a$ that goes upwards from z_a. In the proof of Lemma 2.1 it was checked that α_a is a vertical curve having the asymptote $x = (\pi - a + \arg c_{-m})/m$. We remind the reader that for any point of K there is a canonical curve that passes through this point and lies in K. Consider the canonical curves intersecting the segment (z_π, z_0). By Lemma 3.2, above this segment all of them stay between α_π and α_0. This implies the first statement of the lemma. Now we prove the second statement.

Let $0 < a < \pi$, let $\gamma \subset K$ be a curve canonical with respect to p, and let z_Y be the point of intersection of γ and the line $\operatorname{Im} z = Y$. Let α_a be the integral curve of the vector field $\overline{p(z)} - a$ going upwards from z_Y. Arguing as in Subsection 3.2.2, we check that it is vertical and that the line $x = (\pi - a + \arg c_{-m})/m$ is its asymptote.

Along α_a we have $\operatorname{Im} \int_{z_Y}^{z} (p(z) - a)\, dz = 0$, whence

$$\operatorname{Im} \int_{z_Y}^{z} p(z)\, dz = a(y - Y) \quad \text{and} \quad \operatorname{Im} \int_{z_Y}^{z} (p(z) - \pi)\, dz = -(\pi - a)(y - Y). \tag{3.6}$$

Therefore, α_a is a canonical (semi-infinite) curve. The vertical curve σ_a that goes along γ to Z_Y and then along α_a is canonical. Drawing the curves σ_a for all $0 < a < \pi$, we get the desired canonical domain $K \cup \bigcup_{0<a<\pi} \sigma_a$. $\square$

3.3.1. *Strict canonicity.* We need two more definitions.

Let $\gamma \subset \mathbb{C}$ be a vertical curve, and let $z = z(y)$ along it. The curve γ is said to be *strictly canonical* if the derivative dz/dy is bounded.

A strictly vertical canonical curve γ is said to be *strictly canonical* if there exists $\epsilon > 0$ such that, at all its points $z = z(y)$, we have

$$\frac{d \operatorname{Im} \theta_0(z(y))}{dy} > \epsilon \quad \text{and} \quad \frac{d \operatorname{Im} \theta_\pi(z(y))}{dy} < -\epsilon. \tag{3.7}$$

Lemma 3.3. *Let K be a domain canonical with respect to a branch p of the complex momentum. The domain K can be extended to a canonical domain K' such that through any its point one can draw a curve that is strictly canonical with respect to the analytic continuation of p to K' and is contained in K' with some δ-neighborhood of that curve.*

Proof. To verify this lemma, we need to continue the proof of Lemma 2.2. We use the notations introduced in Subsection 3.2.2.

Consider all the curves α_a, $0 < a < \pi$, for each of the points of the segment (z_π, z_0). Obviously, all these curves lie between $\alpha_\pi(z_\pi)$ and $\alpha_0(z_0)$, and, by (3.6), each of them is a strictly canonical curve (semi-infinite).

We transform each of the canonical curves $\gamma \subset K$ into a new canonical curve $\sigma_a(\gamma)$ by replacing the part of γ above the line $\operatorname{Im} z = Y$ with the curve α_a beginning at the point of intersection of γ and the line $\operatorname{Im} z = Y$.

The union of all the curves $\sigma_a(\gamma)$ is a canonical domain containing K. We denote it by $\widetilde{K}$. Thus, we have extended K inside the half-plane $\{\operatorname{Im} z > Y\}$.

Assuming that Y is sufficiently large, we similarly extend $\widetilde{K}$ inside the half-plane $\{\operatorname{Im} z < -Y\}$. But now, the new canonical curves are the curves obtained from $\sigma_a(\gamma)$ constructed at the previous step.

To complete the proof, it suffices to check that each of the canonical curves constructed when extending the domain K is strictly canonical. Let one of them be denoted by Γ.

The part of Γ above the line $\operatorname{Im} z = Y$ is one of the curves α_a with $0 < a < \pi$. As we have already mentioned, this curve is strictly canonical. Similarly, the part of Γ below the line $\operatorname{Im} z = -Y$ is a semi-infinite strictly canonical curve. Finally, along the part of Γ where $|\operatorname{Im} z| \le Y$, we have (3.7) with a positive ϵ (because this part is compact). Therefore, Γ is strictly canonical. The description of its asymptotes as $\operatorname{Im} z \to \pm\infty$ shows that Γ is contained in the extended canonical domain, together with some its δ-neighborhood. This completes the proof of the lemma. □

§4. Beginning of the proof. Transformation of (1.1) to a system of equations with an almost diagonal matrix

4.1. Preliminaries.

4.1.1. Instead of proving Theorem 2.2 for a given canonical domain K, it suffices to prove this theorem for a canonical domain obtained from K by extension in the sense of Lemmas 2.2 and 3.3. So, we assume that K is an already extended canonical domain.

4.1.2. The proof depends on the behavior of $e^{ip(z)}$ in K as $\operatorname{Im} z \to \pm\infty$. In view of Lemma 3.1, there are four possible cases:
(i) $e^{ip(z)} \to 0$ as $\operatorname{Im} z \to \pm\infty$,
(ii) $e^{-ip(z)} \to 0$ as $\operatorname{Im} z \to \pm\infty$,
(iii) $|e^{ip(z)}|$ increases as $\operatorname{Im} z \to +\infty$ and decreases as $\operatorname{Im} z \to -\infty$,
(iv) $|e^{ip(z)}|$ increases as $\operatorname{Im} z \to -\infty$ and decreases as $\operatorname{Im} z \to +\infty$.
First, we prove the theorem in case (i), and then we describe the modifications of the proof needed to treat the other cases.

Below, all the canonical curves are assumed to lie in K, all these curves and the domain K are canonical with respect to one and the same branch p, and we do not consider any other branches of the complex momentum.

We note that, in case (i), Lemma 3.1 implies that for $z \in K$ we have the uniform estimates

$$e^{ip(z)} \asymp e^{-|\operatorname{Im} p(z)|} \asymp \min\{e^{-my}, e^{ny}\}, \qquad y = \operatorname{Im} z. \tag{4.1}$$

4.2. Passage to a system of equations with an almost diagonal matrix. In Subsection 2.3 we described the first two steps of the proof of Theorem 2.2. To complete these steps, we need to justify Lemma 2.3. We do this here.

Proof. It is easily seen that

$$T(z) = \frac{1}{e^{2ip(z+h)} - 1} \begin{pmatrix} e^{ip(z)}\big(e^{ip(z)+ip(z+h)} - 1\big) & e^{-ip(z)}\big(e^{ip(z+h)} - e^{ip(z)}\big) \\ e^{ip(z)}\big(e^{ip(z+h)} - e^{ip(z)}\big) & e^{-ip(z)}\big(e^{ip(z)+ip(z+h)} - 1\big) \end{pmatrix}.$$

We assume that $z \in K \setminus V_\delta$, where V_δ is the δ-neighborhood of the branch points. Using Corollary 3.1, we check that

$$\frac{e^{ip(z)+ip(z+h)}-1}{e^{2ip(z+h)}-1} = 1 + \frac{e^{ip(z)+ip(z+h)}-e^{2ip(z+h)}}{e^{2ip(z+h)}-1}$$
$$= 1 + \frac{e^{2ip+ip'h+O(h^2)}-e^{2ip+2ip'h+O(h^2)}}{e^{2ip+O(h)}-1} = 1 - \frac{ip'he^{2ip}}{e^{2ip}-1} + O(h^2e^{2ip}),$$

where we have used the fact that in K the exponential $e^{ip(z)}$ is bounded. Using the result obtained and the identity

$$\frac{(ip+\ln\sin p)'}{2} = \frac{ip'e^{2ip}}{e^{2ip}-1},$$

we see that the required asymptotic representations for T_{11} and T_{22} are valid. Estimates for T_{12} and T_{21} are obtained by arguments similar to the previous ones. We have

$$\frac{e^{ip(z+h)}-e^{ip(z)}}{e^{2ip(z+h)}-1} = e^{ip(z)}\frac{e^{i(p(z+h)-p(z))}-1}{e^{2ip(z+h)}-1} = O\left(he^{ip}\right). \qquad \square$$

§5. First order difference equations

Here we discuss construction and estimates for analytic solutions to first order difference equations. These results are used systematically in the proof of Theorem 2.2.

In this section, when discussing a vertical curve, we assume that, along it, $\operatorname{Im} z$ grows from $-\infty$ to $+\infty$.

5.1. General construction.

Here we describe a general construction of solutions. Let $D \subset \mathbb{C}$ be a domain. Consider the first order difference equation

$$f(z+h) - f(z) = g(z) \tag{5.1}$$

where $z, z+h \in D$.

Lemma 5.1. *Let $h > 0$, and let D be a horizontally connected domain through any point z of which one can draw a strictly vertical curve $\gamma(z) \subset D$. Let g be a function analytic in D such that $|g(z)| \le C/(1+|\operatorname{Im} z|^2)$ in D. Then the function defined in D by the formula*

$$f(z) = \frac{1}{2ih}\int_{\gamma(z)} \cot\left[\frac{\pi(\zeta - z - 0)}{h}\right] g(\zeta)\, d\zeta \tag{5.2}$$

is analytic in D and satisfies (5.1) *for all z, $z+h \in D$.*

Kindred statements can be found in [2] and [4].

Proof. The integral in (5.2) is well defined thanks to the strict verticality of γ, the estimate for g, and its analyticity. To prove that f is analytic at a point $z_0 \in D$, we replace the curves $\gamma(z)$ (the integration paths in (5.2)) with z sufficiently close to z_0, by a curve that differs from $\gamma(z_0)$ only in a neighborhood of z_0 and goes to the left of z_0. This is possible because the integration contours are strictly vertical and the function g satisfies the estimate prescribed in the lemma.

Finally, equation (5.1) immediately follows from the residue theorem. We omit elementary details. $\square$

5.2. Local estimates of solutions. Here, we describe estimates in a neighborhood of a vertical curve for a solution of equation (5.1).

Let $\gamma \subset \mathbb{C}$ be a strictly vertical curve, and put, for $a > 0$,

$$\Pi_{\gamma,a} = \big\{z \in \mathbb{C} \,:\, \exists \zeta \in \gamma \,:\, \operatorname{Im}\zeta = \operatorname{Im} z \text{ and } |\operatorname{Re}\zeta - \operatorname{Re} z| < ah\big\}. \tag{5.3}$$

For a complex-valued function f defined in $\Pi_{\gamma,a}$, we set

$$\|f\|_{\gamma,a} = \sup_{z\in\Pi_{\gamma,a}} |f(z)|, \tag{5.4}$$

We also use the notation $\rho(z) = 1 + |\operatorname{Im} z|^2$.

Remark 5.1. **1.** The domain $D = \Pi_{\gamma,a}$ satisfies the hypotheses of Lemma (5.2): it is horizontally connected, and for $z \in \Pi_{\gamma,a}$ one can obtain $\gamma(z)$ by translating γ in the direction of the real axis.
2. A function g analytic in $D = \Pi_{\gamma,a}$ and such that $\|\rho\, g\|_{\gamma,a,\rho} < \infty$ satisfies the hypotheses of Lemma (5.2).

Lemma 5.2. *Let $\gamma \subset \mathbb{C}$ be a strictly vertical curve, let a number a satisfy the inequality $0 < a < 1$, let a function g be analytic in $\Pi_{\gamma,a}$, and let $\|\rho\, g\|_{\gamma,a} < \infty$. Then the function f constructed in terms of g for the domain $D = \Pi_{\gamma,a}$ in accordance with Lemma* 5.1 *satisfies the estimate*

$$\|f\|_{\gamma,a} \le C\, h^{-1}\, \|\rho\, g\|_{\gamma,a}. \tag{5.5}$$

Proof. We note that the cotangent function is uniformly bounded outside a fixced neighborhood of the points $\pi\mathbb{Z}$ in the complex plane.

First, we consider the case where the point $z \in \Pi_{\gamma,a}$ is located either on the curve γ or to the right of it. In this case we can and shall describe f by formula (5.2) with the integration path γ.

For all z located between $\gamma + ah/2$ and $\gamma + ah$, the distance between the poles of the integrand and γ is greater than Ch, where C is a positive constant depending on a and γ. Therefore, for all z under consideration and for all $\zeta \in \gamma$, the cotangent in (5.2) is bounded uniformly in z, ζ and h. This observation and the strict verticality of the curve γ imply that

$$|f(z)| \le Ch^{-1}\|\rho g\|_{\gamma,a} \int_{-\infty}^{+\infty} \frac{d\eta}{1+\eta^2} = Ch^{-1}\|\rho g\|_{\gamma,a}.$$

If z is located either between the curves γ and $\gamma + ah/2$ or on one of them, we deform the integration path to $\gamma - ah/2$ and get similar estimates.

Let z be to the left of γ. We note that, for $z \in \gamma$, in the integral in (5.2) we integrate in fact along an arc of an infinitesimally small circle going to the left of z. By the residue theorem, this integral equals the sum of $g(z)$ and an integral along a path going to the right of z. The analysis of the latter integral is similar to that of the integral in (5.2) carried out above.

As a result, our estimates lead to (5.5). □

5.3. Global estimates. In the next section we shall need the following statement.

Proposition 5.1. *Let D be a horizontally bounded, horizontally connected domain, let D' be its horizontally connected subdomain, and let $\gamma \subset D'$ be a strictly vertical curve. Let a function g be analytic in D and satisfy the estimate $|g(z)| \le C/\rho(z)$ in D'.*

Assume that $\Pi_{\gamma,1} \subset D'$. Using formula (5.2), *we construct f for the domain $\Pi_{\gamma,1}$.*

The function f admits analytic continuation to the domain D; after continuation it satisfies equation (5.1) *for all $z, z + h \in D$, and in D' we have $|f(z)| \le C/h$.*

Proof. Let $1/2 < a < 1$. Obviously $\Pi_{\gamma,a} \subset D$. By Lemma 5.2, in $\Pi_{\gamma,a}$ the function f satisfies the estimate $|f(z)| \le C/h$.

The function f satisfies equation (5.1) for all $z, z+h \in \Pi_{\gamma,a}$ and, since $a > 1/2$, this equation itself allows us to continue it analytically to the entire horizontally connected domain D. As a result, it turns out to be a solution in D.

Consider points z located in D' to the right of $\Pi_{\gamma,a}$. Let $\widetilde{z}(z) \in \Pi_{\gamma,a}$ be a point such that $\operatorname{Im} \widetilde{z} = \operatorname{Im} z$ and $\operatorname{Re} z - \operatorname{Re} \widetilde{z} \in h\mathbb{Z}$. Let $N = (z - \widetilde{z})/h$. Using equation (5.1) and the estimates for g in D' and for f in $\Pi_{\gamma,a}$, we get

$$|f(z)| \le \sum_{n=1}^{N} |g(z-nh)| + |f(\widetilde{z})| \le CN + C/h.$$

Since D is horizontally bounded, this implies the desired estimate. The case where z is to the left of $\Pi_{\gamma,a}$ is studied similarly. □

§6. The proof of Lemma 2.4

6.1. The functions $\Theta_{1,2}$. We prove the statement of the lemma concerning the first equation in (2.10). A solution of the second equation is constructed and analyzed similarly. We put

$$\Theta_1(z) = \frac{i}{h}\theta_0(z) - \frac{1}{2}\ln\sin p(z) - ip(z) + f(z). \tag{6.1}$$

From (2.10) it follows that for $z, z+h \in K$ the function f satisfies equation (5.1) with

$$g(z) = \Big(-\frac{i}{h}\theta_0 + ip + \frac{1}{2}\ln\sin p\Big)\Big|_z^{z+h} + \ln T_{11}(z), \quad z \in K. \tag{6.2}$$

We recall that V_δ is the δ-neighborhood of the branch points of p. By using Corollary 3.1, estimate (4.1), and Lemma 2.3, it is easy to check that

$$g(z) = O\big(h^2 e^{-|y|}\big), \quad z \in K \setminus V_\delta, \tag{6.3}$$

for any fixed δ as $h \to 0$.

Recall that K is a canonical domain extended in the sense of Lemma 3.3. Therefore, for each point $z \in K$, inside K there is a strictly canonical and, thus, a strictly vertical curve containing z. Moreover, this curve is contained in K with some its (2δ)-neighborhood.

The last observation implies that $\Pi_{\gamma,1} \subset K$ for sufficiently small h, and that $\Pi_{\gamma,1}$ lies outside V_δ.

With the help of Lemma 5.1, we construct a solution of (5.1) that is analytic in K and satisfies the required estimate in $K \setminus V_\delta$. This completes the proof of formulas (2.14).

6.1.1. *The matrix S.* Using equations (2.10) for $\Theta_{1,2}$, we see that

$$S_{12}(z) = e^{\Theta_2(z)-\Theta_1(z)-\ln T_{11}(z)}\,T_{12}(z), \qquad S_{21}(z) = e^{\Theta_1(z)-\Theta_2(z)-\ln T_{22}(z)}\,T_{12}(z).$$

Now formula (2.15) follows from the asymptotic representations (2.9) for $T(z)$ and (2.14) for $\Theta_{1,2}$. Like in these asymptotic representations, the estimates in (2.15) are uniform in the domain K outside any fixed δ-neighborhood of the branch points. □

§7. Integral equation

First, we derive equation (2.19) formally, without taking care of the convergence of integrals. Then we study this equation rigorously as an equation in a suitable function space. Finally, having estimated the norms of the operators in this equation, we construct its solution.

7.1. Derivation of the integral equation. Equation (2.16) is obtained by formal inversion of the difference operator on the left-hand side of (2.12) with the help of Lemma 5.1. We note that the expression on the right-hand of (2.16) coincides up to a constant term with the solution of the homogeneous equation $f(z+h)-f(z)=0$ suggested by formula (5.2).

Equations (2.18) and (2.19) follow (again, formally) from (2.16) because the matrix $S(z)$ is antidiagonal.

Like in formula (2.15), $O(\,\cdot\,)$ in (2.20) is used to denote estimates uniform in $K\setminus V_\delta$.

7.2. Function spaces. Let $0<a<1$, and let γ be a strictly vertical curve. First, we define the operators $\mathcal{L}_+$ and $\mathcal{K}_+$ by formulas (2.17) and (2.21) on functions that are analytic in $\Pi_{\gamma,a}$ and decay sufficiently fast as $|\operatorname{Im} z|\to\infty$. Since the function $\mathcal{L}_+g$ admits analytic continuation to $\Pi_{\gamma,a}$, we regard $\mathcal{L}_+g$ and $\mathcal{K}_+g$ as functions analytic in $\Pi_{\gamma,a}$ by definition.

Let $b\in\mathbb{R}$, and let $H_{\gamma,a,b}$ be the space of functions analytic in $\Pi_{\gamma,a}$ and having the finite norms

$$\|f\|_{\gamma,a,b}=\|e^{-ibp(z)}f(z)\|_{\gamma,a}.$$

Clearly, $H_{\gamma,a,b}$ is a Banach space.

We denote by C positive constants independent of h.

Proposition 7.1. *Let γ be a strictly vertical curve, and let $b>0$. Then*

$$\|\mathcal{L}_+\|_{H_{\gamma,a,b}\mapsto H_{\gamma,a,0}}\le C/h.$$

Proposition 7.2. *Let γ be a strictly canonical curve. For sufficiently small h, we have*

$$\|\mathcal{K}_+\|_{H_{\gamma,a,b}\to H_{\gamma,a,b-2a}}\le C.$$

The first of these two statements immediately follows from Lemma 5.2, and the second is proved below.

Proof. The proof is divided into several steps. Since γ is a regular vertical curve, we can and do assume that h is so small that the strip $\Pi_{\gamma,a}$ lies outside V_δ, a δ-neighborhood of the branch points.

1. We estimate the kernel of the operator $\mathcal{K}_+$. Let $z,\zeta\in\Pi_{\gamma,a}$. We pick $0<c<1/2$ and assume that ζ is outside the ch-neighborhood of the set $z+h\mathbb{Z}$. Let $y=\operatorname{Im} z$ and $\eta=\operatorname{Im}\zeta$. For $w\in\mathbb{C}$ we denote by $w_\perp$ the point of the curve γ such that $\operatorname{Im} w_\perp=\operatorname{Im} w$. We want to show that

$$\left|e^{\frac{2i}{h}\int_z^\zeta p\,dz}\left(\cot\frac{\pi(\zeta-z)}{h}-i\right)\right|\le Ce^{-\frac{2\varepsilon}{h}|y-\eta|+\frac{2|z-z_\perp|\,|\operatorname{Im} p(z)|}{h}+2\frac{|\zeta-\zeta_\perp|\,|\operatorname{Im} p(\zeta)|}{h}}\tag{7.1}$$

where ϵ is the constant occurring in inequalities (3.7).

Note that if $w\in\mathbb{C}$ is outside a neighborhood of $\pi\mathbb{Z}$ independent of h, then

$$|\cot w-i|\le C\,\min\{1,\,e^{2\operatorname{Im} w}\}.$$

Therefore,

$$\left|e^{\frac{2i}{h}\int_z^\zeta p\,dz}\left(\cot\frac{\pi(\zeta-z)}{h}-i\right)\right|\le C\,\frac{A(z)}{A(\zeta)}\begin{cases}e^{-\frac{2}{h}\operatorname{Im}\int_{z_\perp}^{\zeta_\perp}p\,dz} & \text{if }\ \eta\ge y,\\ e^{-\frac{2}{h}\operatorname{Im}\int_{z_\perp}^{\zeta_\perp}(p-\pi)\,dz} & \text{if }\ \eta\le y,\end{cases}\tag{7.2}$$

where

$$A(w)=\left|e^{\frac{2i}{h}\int_w^{w_\perp}p\,dz}\right|.$$

For $w\in\Pi_{\gamma,a}$, Corollary 3.1 implies that the following uniform estimates are valid for sufficiently small h:

$$|A(w)|\,,\ |A(w)|^{-1}\le Ce^{2|w_\perp-w|\,|\operatorname{Im} p(w)|/h}.\tag{7.3}$$

Now (7.1) follows from (7.2), (3.7), and (7.3).
2. Let $g \in H_{\gamma,a,b}$. Estimates (7.1) and (4.1) show that, for sufficiently small h and $z \in \Pi_{\gamma,a}$, the integral in the definition of $\mathcal{K}_+ g(z)$ converges and yields an analytic function.
3. Let $g \in H_{\gamma,a,b}$. Let us show that

$$|\mathcal{K}_+ g(z)| \leq C \left| e^{i(b-2a)p(z)} \right| \|f\|_{\gamma,a,b} \tag{7.4}$$

for sufficiently small h. First, we assume that z is located between the curves $\gamma + ah/2$ and $\gamma + ah$. From (7.1), the definition of $\|\cdot\|_{\gamma,a,b}$, and (4.1) it follows that, for all z in question we have

$$\begin{aligned}|\mathcal{K}_+ f(z)| &\leq C\, h^{-1} \int_\gamma e^{-2\epsilon|y-\eta|/h + 2a|\,\mathrm{Im}\, p(z)| - b\ \mathrm{Im}\, p(\zeta)}\, |d\zeta|\, \|f\|_{\gamma,a,b} \\ &\leq C\, h^{-1}\, e^{-b\ \mathrm{Im}\, p(z) + 2a|\,\mathrm{Im}\, p(z)|} \int_\gamma e^{-2\epsilon|y-\eta|/h - b\ \mathrm{Im}(p(\zeta)-p(z))}\, |d\zeta|\, \|f\|_{\gamma,a,b}.\end{aligned}$$

Since γ is strictly vertical, $|d\zeta| \leq C\, d\eta$. Using Corollary 3.1, we get $|p(\zeta) - p(z)| \leq C|\eta - y|$ for $\zeta, z \in \Pi_{\gamma,a}$. Furthermore, by (4.1) we have $e^{-b\ \mathrm{Im}\, p(z) + 2a|\,\mathrm{Im}\, p(z)|} \leq C\, e^{-(b-2a)\,\mathrm{Im}\, p(z)} \leq C\, |e^{i(b-2a)p(z)}|$. Therefore,

$$\left|\mathcal{K}_+ f(z)\right| \leq C\, h^{-1} \left| e^{i(b-2a)p(z)} \right| \int_\gamma e^{-2\epsilon|y-\eta|/h + C|y-\eta|}\, d\eta \cdot \|f\|_{\gamma,a,b}. \tag{7.5}$$

This implies (7.4) for sufficiently small h. To prove (7.4) in the case where z lies either between the curves γ and $\gamma + ah/2$ or on them, we start with deforming the integration path in the definition of $\mathcal{L}_+$ to $\gamma - ah/2$, and then argue as in the previous case.

If z is on γ or to the left of it, we use the residue theorem as described at the end of the proof of Lemma 5.2, arriving at a representation for $\mathcal{L}_+ g$ that leads to (7.4), as before.

The claim of Proposition 7.2 immediately follows from estimate (7.4). ☐

7.3. Integral equation. We pick $0 < a < 1$. Let $\gamma \subset K$ be a strictly canonical curve. We view (2.19) as an equation in $H_{\gamma,a,0}$. Denote by $\mathcal{O}$ the linear operator applied to X_1 on the right-hand side of (2.19). Propositions 7.1 and 7.2 and estimates (2.20) imply that for sufficiently small h we have

$$\begin{aligned}\|\mathcal{O}\, X_1\|_{\gamma,a,0} &= \|\mathcal{L}_+ \left(s_{12}\, \mathcal{K}_+ \left(e^{2ip}\, s_{21}\, X_1 \right) \right) \|_{\gamma,a,0} \\ &\leq C\, \|\mathcal{K}_+ \left(e^{2ip}\, s_{21}\, X_1 \right) \|_{\gamma,a,2(1-a)} \\ &\leq C \|e^{2ip}\, s_{21}\, X_1\|_{\gamma,a,2} = C\, \|s_{21} X_1\|_{\gamma,a,0} \\ &\leq Ch \|X_1\|_{\gamma,a,0}.\end{aligned}$$

As a direct consequence, we get the following.

Proposition 7.3. *For sufficiently small h, equation (2.18) in $H_{\gamma,a,0}$ has a unique solution X_1. We have*

$$X_1(z) = 1 + O(h) \tag{7.6}$$

where the $H_{\gamma,a,0}$-norm of the error term is $O(h)$.

§8. Solution of the difference Schrödinger equation

Here we construct the solution ψ_+ of equation (1.1) described in Theorem 2.2.

8.1. Constructing the solution ψ_+. Let $\gamma \in K$ be a strictly canonical curve, and let $1/2 < a < 1$.

Proposition 8.1. *For sufficiently small h, equation* (1.1) *has an entire solution ψ_+. In the strip $\Pi_{\gamma,a}$ the solution ψ_+ admits the asymptotic representation* (2.4)*.*

Proof. **1.** Let X_1 be a solution to (2.19) described in Proposition 7.3. We define a function X_2 in $\Pi_{\gamma,a}$ by the formula

$$X_2 = \mathcal{L}_+\big(e^{\frac{2i\theta_0}{h}} e^{2ip} s_{21} X_1\big). \tag{8.1}$$

Since

$$\mathcal{L}_+\big(e^{\frac{2i\theta_0}{h}} e^{2ip} s_{21} X_1\big) = e^{\frac{2i\theta_0}{h}} \mathcal{K}_+\big(e^{2ip} s_{21} X_1\big),$$

Proposition 7.2 implies that the function X_2 is well defined and satisfies the estimate

$$\|e^{-\frac{2i\theta_0}{h}} X_2\|_{\gamma,a,2(1-a)} \leq \|e^{2ip} s_{21} X_1\|_{\gamma,a,2} \leq \|s_{21} X_1\|_{\gamma,a,0} \leq Ch. \tag{8.2}$$

2. Obviously, equation (2.19) can be written in the form

$$X_1 = 1 + \mathcal{L}_+\big(s_{12} e^{-\frac{2i\theta_0}{h}} X_2\big). \tag{8.3}$$

Let X be the vector with the components X_1 and X_2. From (8.1) and (8.3) it follows that X satisfies equation (2.16) in $\Pi_{\gamma,a}$.

3. Computing the difference $\mathcal{L}_+(SX)(z+h) - \mathcal{L}_+(SX)(z)$ with the help of the residue theorem, we check that, also, X satisfies equation (2.12) for $z+h, z \in \Pi_{\gamma,a}$. Using formulas (2.11) and (2.6), we finally construct a solution Ψ of the difference equation (2.5) in terms of X.

4. By construction, the solution Ψ is analytic in $\Pi_{\gamma,a}$. Since $a > 1/2$, and the matrix-valued function M in (2.5) is entire, Ψ can be analytically continued to the right of γ via the equation $\Psi(z+h) = M(z)\Psi(z)$ itself. The function Ψ continues analytically to the left of γ by the formula $\Psi(z) = (M(z))^{-1}\Psi(z+h)$. As a result, Ψ becomes an entire solution of (2.5).

5. As it was already mentioned in part 1 of Subsection 2.3, the first component of the vector Ψ satisfies equation (1.1). We denote it by ψ_+. At the third step of the proof, we constructed Ψ in terms of X. It is easy to check that

$$\psi_+ = e^{\Theta_1 + ip}\,\big(X_1 + e^{\Theta_2 - \Theta_1 - ip} X_2\big).$$

This and the asymptotics for X_1, X_2, Θ_1 and Θ_2, see (7.6), (8.2) and (2.14), show that in the strip $\Pi_{\gamma,a}$ we have

$$\psi_+ = e^{\frac{i\theta_0}{h} - \frac{\ln \sin p}{2} + O(h)}\big(X_1 + e^{-\frac{2i\theta_0}{h} + O(h)} X_2\big) = e^{\frac{i\theta_0}{h} - \frac{\ln \sin p}{2} + O(h)},$$

and this implies the statement on the asymptotics of ψ_+. $\square$

8.2. Asymptotics of ψ_+ in the canonical domain. Now we study the asymptotics of ψ_+ in the canonical domain K.

Proposition 8.2. *In the domain K, the solution ψ_+ admits the asymptotic representation* (2.4)*. It is locally uniform in z.*

Proof. Let $z_0 \in K$. It suffices to show that there exists a neighborhood $V_0 \subset K$ of the point z_0 independent of h and such that in V_0 ψ_+ admits the asymptotic representation (2.4), and that the error estimate in this representation is uniform in $z \in V_0$. The proof splits into four steps.

1. Let $\gamma' \subset K$ be a strictly canonical curve that coincides with γ outside a compact set. The solution ψ_+ admits the asymptotics (2.4) in the strip $\Pi_{\gamma',a}$ (with an estimate for the $H_{\gamma',a,0}$-norm the error term). Indeed, both Ψ_+ and the corresponding vector-valued function X can be analytically continued to the domain K. Deforming in equation (2.19) the integration path γ inside K to γ', we easily show that X is a solution of that equation in $H_{\gamma',a,0}$. This implies the claim.
2. Recall that we parametrize the points z of a vertical curve by y, the imaginary part of z. Let a function $y \mapsto z(y)$ correspond to a strictly canonical curve. The definition of a strictly canonical curve implies that if to this function we add a compactly supported perturbation sufficiently small in C^1-topology, we get a function that still parametrizes a strictly canonical curve.
3. Let $z_0 \in K$, and let $\gamma_0 \subset K$ be a strictly canonical curve containing z_0. The observation made at step 2 shows that we may assume that z_0 is an inner point of a simply connected domain bounded by two canonical curves γ_1 and γ_2 that coincide with γ_0 outside a neighborhood of z_0.
4. On the boundary of V_0, as well as on the curves $\gamma_{1,2}$, ψ_+ admits the uniform asymptotic representation (2.4). Therefore, on the boundary of V_0 the function $f\colon z \mapsto \sqrt{\sin p(z)}e^{-\frac{i}{h}\int_{z_0}^{z} p(z)\,dz}\,\psi_+(z)$ equals $1+O(h)$. Inside V_0 this function is analytic. Therefore, by the maximum principle, in V_0 we have $f(z) = 1 + O(h)$ with a uniform error estimate. Therefore, in V_0 ψ_+ admits the uniform asymptotic representation (2.4). $\square$

Lemma 8.1. *Asymptotic representation* (2.4) *for ψ_+ is uniform in z in any admissible subdomain K_δ.*

Proof. **1.** By Proposition 8.2, it suffices to prove that, for sufficiently large $Y_0 > 0$, the asymptotics (2.4) for ψ_+ is uniform in the domains $K_\delta^{\pm}(Y_0) = \{z \in K_\delta \,:\, \pm \operatorname{Im} z > Y_0\}$. We prove this for $K_\delta^{+}(Y_0)$. For the second domain the proof is similar.
2. Let $\gamma \subset K$ be a strictly canonical curve, and let $Y > 0$ be sufficiently large. Like in the proof of Lemma 2.2, we consider the canonical curve obtained from γ by replacing its part located above the line $\operatorname{Im} z = Y$ with the integral curve of the vector field $\overline{p(z)} - a$ with $0 < a < \pi$. We denote this canonical curve by $\sigma_a(\gamma)$. The analysis carried out in the proof of Lemma 2.2 implies the following statements.
(a) All the curves $\sigma_a(\gamma)$ with $0 < a < \pi$ are strictly canonical.
(b) Let $I \subset (0, \pi)$ be a closed interval. There are positive constants ϵ and C such that condition (3.7) is satisfied along each curve $\sigma_a(\gamma)$ with $a \in I$, and that $|dz/dy| \le C$ along any of these curves.
(c) One can choose I so that for sufficiently large Y_0 the family of curves described in (b) covers $K_\delta^{+}(Y_0)$.
3. The proofs of Propositions 7.1 and 7.2 show that the estimates of the norms of the operators $\mathcal{L}_+$ and $\mathcal{K}_+$ are uniform with respect to the integration path from the family of canonical curves such that, on any of these curves, condition (3.7) is satisfied with one and the same ϵ and the derivative dz/dy is bounded by one and the same constant.

The fact that, for sufficiently large Y_0, the asymptotics (2.4) of the solution ψ_+ is uniform in z in the domain $K_\delta^{+}(Y_0)$ follows from the observations made at steps 3 and 2. $\square$

§9. Construction of the second solution

The solution ψ_- of equation (1.1) described in Theorem 2.2 is constructed almost as the solution ψ_+. The main difference is related to the derivation of the integral equation.

Now, on a strictly canonical curve γ, instead of (2.16) we consider the equation

$$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{L}_-(SX), \tag{9.1}$$

where $\mathcal{L}_-$ acts on functions analytic in $\Pi_{\gamma,a}$ and decaying sufficiently fast at infinity by the formula

$$\mathcal{L}_- f(z) = \frac{1}{2ih} \int_\gamma \left(\cot\left[\frac{\pi(\zeta - z - 0)}{h} \right] + i \right) f(\zeta)\, d\zeta.$$

Then instead of (2.18) we study the equation for the second component X_2 of the vector X. The analysis is similar to that carried out above, and we omit the further details.

§10. Construction of solutions in domains of other types

In Subsection (4.1.2) we singled out four groups of canonical domains in terms of the behavior of $e^{ip(z)}$ as $\operatorname{Im} z \to \pm\infty$. For these groups, the ways to prove the main theorem slightly differ. Above, we described the proof for the first group (case (i)). Now, consider a canonical domain of the second group (case (ii)). All the calculations and estimations are carried out almost as in case (i), but the momentum p should be changed by $-p$ and the operators $\mathcal{L}_\pm$ by $\mathcal{L}_\mp$. For instance, at the first step we use the matrix $U(z) = \left(\begin{smallmatrix} e^{-ip(z)} & 1 \\ 1 & e^{-ip(z)} \end{smallmatrix} \right)$ in place of (2.8).

In cases (iii) and (iv), $|e^{ip(z)}|$ grows into one direction and decays in the other. Consider case (iii), where $|e^{ip(z)}| \to \infty$ as $\operatorname{Im} z \to +\infty$ and $e^{ip(z)} \to 0$ as $\operatorname{Im} z \to -\infty$. Case (iv) is treated similarly. We begin the proof as in case (i). For example, we choose $U(z) = \left(\begin{smallmatrix} e^{ip(z)} & 1 \\ 1 & e^{ip(z)} \end{smallmatrix} \right)$. The asymptotic representation for the matrix T for $y = \operatorname{Im} z \leq 0$ is still described by formula (2.9). But for $y \geq 0$ we get

$$T = \begin{pmatrix} e^{ip - \frac{h}{2}(ip + \ln\sin p)' + O\left(h^2 e^{-y}\right)} & O(he^{-2ip(z)}) \\ O(h) & e^{-ip - \frac{h}{2}(ip + \ln\sin p)' + O\left(h^2 e^{-y}\right)} \end{pmatrix}. \tag{10.1}$$

The estimates for the error terms in the formulas for the diagonal elements, and the estimates for the antidiagonal terms became different. The error terms in the diagonal elements still decay exponentially at infinity, and therefore, the asymptotic formulas (2.14) for $\Theta_{1,2}$ remain valid. Because of new estimates for the antidiagonal terms of T, equation (2.19) is replaced by an equation of the form

$$X_1 = 1 + \mathcal{L}_+ \left(\widetilde{s}_{12}\, \mathcal{K}_+ \left(\widetilde{s}_{21}\, X_1 \right) \right),$$

where $\widetilde{s}_{12}$ and $\widetilde{s}_{21}$ are functions such that in K we have

$$|\widetilde{s}_{12}(z)| \leq Ch \min\{|e^{-2ip(z)}|, 1\}, \qquad |\widetilde{s}_{21}(z)| \leq Ch \min\{|e^{2ip}|, 1\}.$$

Let γ be a strictly canonical curve, and let $0 < a < 1$. Arguing as in the proof of Proposition 7.2, we check that, for $z \in \Pi_{\gamma,a}$,

$$|\widetilde{s}_{12}(z)\, \mathcal{K}_+ \left(\widetilde{s}_{21}\, f \right)(z)| \leq Ch^2 e^{-2(1-a)|\operatorname{Im} p(z)|} \|f\|_{\gamma,a,0}.$$

This estimate and Proposition 7.1 still allow us to prove Proposition 7.3. The further modifications of the proof look natural and we do not discuss them.

References

[1] V. M. Babich, M. A. Lyalinov, and V. E. Grikurov, *Diffraction theory. The Sommerfeld–Malyuzhinets technique*, Alpha Sci. Ser. Wave Phenom., Alpha Sci. Ltd, Oxford, 2008.

[2] V. S. Buslaev and A. A. Fedotov, *The complex WKB method for the Harper equation*, Algebra i Analiz **6** (1994), no. 3, 59–83; English transl., St. Petersburg Math. J. **6** (1995), no. 3, 495–517. MR1301830

[3] J. P. Guillement, B. Helffer, and P. Treton, *Walk inside Hofstadter's butterfly*, J. Phys. France **50** (1989), 2019–2058.

[4] A. Fedotov and F. Klopp, *A complex WKB method for adiabatic problems*, Asymptot. Anal. **27** (2001), no. 3–4, 219–264. MR1858917

[5] M. V. Fedorchuk, *Asymptotic methods for linear ordinary differential equations*, Nauka, Moscow, 1983. MR732787; English transl., Springer-Verlag, Berlin, 1993. MR1295032 (95m:34091)

[6] A. A. Fedotov and E. V. Schetka, *A complex WKB method for difference equations in bounded domains*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **438** (2015), 236–254; English transl., J. Math. Sci. (N.Y.) **224** (2017), no. 1, 157–169. MR3501077

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