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ABSOLUTE CONTINUITY OF THE SPECTRUM OF TWO-DIMENSIONAL SCHRÖDINGER OPERATOR WITH PARTIALLY PERIODIC COEFFICIENTS

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To the memory of V. S. Buslaev

Abstract. On the plane, the operator $-\mathrm{div}(g(x)\nabla\,\cdot\,) + V(x)$ is considered. The absolute continuity of its spectrum is proved under the assumption that each coefficient is the sum of a $\mathbb{Z}^2$-periodic term and a summand that is periodic in one of the variables and decays superexponentially with respect to the other variable.

§1. Introduction

In the space $L_2(\mathbb{R}^n)$ we consider the selfadjoint operator

$$H = -\,\mathrm{div}(g(x)\nabla\,\cdot\,) + V(x), \tag{1.1}$$

where g and V are real scalar functions. It is the Schrödinger operator if $g \equiv 1$, and it is an acoustic operator if $V \equiv 0$. Below we assume that each coefficient can be represented as a sum of two terms

$$g = g_0 + g_1, \qquad V = V_0 + V_1, \tag{1.2}$$

where the functions g_0 and V_0 are periodic with respect to all variables (periodic background), the functions g_1 and V_1 are periodic with respect to some variables, and decay very fast in all other variables. The operator (1.1) with such coefficients describes what is called a "soft waveguide".

It is well known that the spectrum of H has a band-zone structure if its coefficients are periodic with respect to a nondegenerate lattice in $\mathbb{R}^n$, i.e., $g_1 \equiv V_1 \equiv 0$. Moreover, the spectrum has no singular continuous component, and either it is absolutely continuous, or it contains an absolute continuous component and some eigenvalues of infinite multiplicity (degenerate bands). There is a large number of papers devoted to the proof of the absolute continuity of the spectrum of various operators with periodic coefficients (see the original work of Thomas [11], and also [1, 10] and the references therein).

In the case of partially periodic coefficients it is also natural to assume the absence of eigenvalues, both from the physical and mathematical point of view. However, there are a few rigorous results in this direction, see, e.g., [2, 3]. For the operator (1.1), (1.2), only the following two results are known.

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Theorem 1.1 ([5])**.** *In* $L_2(\mathbb{R}^{m+d})$*, consider the operator* (1.1) *with the coefficients* (1.2)*, and assume that* $g_0 \equiv \mathrm{const} > 0$*,* $V_0 \equiv 0$*. Next, assume that*

- *the functions* g_1*,* V_1 *are periodic with respect to the last* d *variables,*

$$g_1(x, y+l) = g_1(x, y), \quad V_1(x, y+l) = V_1(x, y), \quad \forall\, x \in \mathbb{R}^m,\ y \in \mathbb{R}^d,\ l \in \mathbb{Z}^d;$$

- *the functions* $g_1(x,y)e^{a|x|}, \Delta g_1(x,y)e^{a|x|}, V_1(x,y)e^{a|x|}$ *are bounded in* $\mathbb{R}^{m+d}$ *for all* $a \in \mathbb{R}$*, i.e.,* g_1 *and* V_1 *decay in the first* m *variables superexponentially;*
- $g(x,y) \ge c_0 > 0$*.*

Then the spectrum of H *is absolutely continuous.*

Theorem 1.2 ([8])**.** *Consider the selfadjoint operator* $H = -\frac{1}{\varepsilon(x)}\Delta$ *in the Hilbert space* $L_2(\mathbb{R}^2, \varepsilon)$ *with the weight* $\varepsilon(x)dx$*. Assume that the function* $\varepsilon \in L_\infty(\mathbb{R}^2)$*,* $\varepsilon(x) \ge c_0 > 0$*, can be represented as a sum* $\varepsilon = \varepsilon_0 + \varepsilon_1$*, where* ε_0 *is periodic with respect to the lattice* $\mathbb{Z}^2$*, and* ε_1 *is* 1*-periodic in* x_2*, and* $\operatorname{supp}\varepsilon_1 \subset (0,1)\times\mathbb{R}$*. Then the operator* H *has no eigenvalues.*

The proof of Theorem 1.2 can be carried over without changes to the case of the operator

$$H = -\Delta + V, \qquad V = V_0 + V_1,$$

where the potential V_0 is periodic with respect to $\mathbb{Z}^2$, the potential V_1 is 1-periodic in x_2, and

$$\operatorname{supp} V_1 \subset (0,1)\times\mathbb{R}$$

(see the Remark at the end of the Introduction in [8]). On the other hand, the proof of Theorem 1.2 does not work if the operator involves both a potential and a nontrivial metric.

Thus, in Theorem 1.1 the space dimension is arbitrary, and the coefficients decay fast in the nonperiodic variables (this condition is more realistic than being identically zero outside a compact support as in Theorem 1.2). But there is no periodic background. On the contrary, in Theorem 1.2 there is a periodic background, but there is a dimension restriction $n = 2$, and the question about the singular continuous spectrum was not studied. In the present paper, we combine the approaches of [5] and [8]. We formulate our main result.

Theorem 1.3. *In* $L_2(\mathbb{R}^2)$ *we consider the operator* (1.1) *on the domain* $\operatorname{Dom} H = W_2^2(\mathbb{R}^2)$ *with the coefficients* (1.2)*. Assume that* g_0, g_1, V_0, V_1 *are real scalar bounded functions on the plane,* $g_j, V_j \in L_\infty(\mathbb{R}^2)$*, satisfying the following conditions*:

- g_0 *and* V_0 *are periodic with respect to* $\mathbb{Z}^2$*;*
- $g_0, g_1 \in W^2_\infty(\mathbb{R}^2)$*;*
- $g_1(x_1, x_2+1) = g_1(x_1, x_2)$*,* $V_1(x_1, x_2+1) = V_1(x_1, x_2)$ *a.e.* $x \in \mathbb{R}^2$*;*
- *the functions* $g_1(x)e^{a|x_1|}, \Delta g_1(x)e^{a|x_1|}, V_1(x)e^{a|x_1|}$ *are bounded in* $\mathbb{R}^2$ *for all* $a \in \mathbb{R}$*;*
- $g(x) \ge c_0 > 0$ *for all* $x \in \mathbb{R}^2$*.*

Then the spectrum of the operator H *is absolutely continuous.*

Remark 1.4. The assumptions on the coefficients g imply also that $g_0(x) \ge c_0 > 0$ for all $x \in \mathbb{R}^2$.

We list the disadvantages of Theorem 1.3:

a) it also has the restriction $n = 2$;

b) the lattice of periods is exactly $\mathbb{Z}^2$. The result can easily be generalized to the case of rectangular lattices, but we are unable to prove it for skew-angular lattices.

Let us briefly outline the proof of Theorem 1.3. We make the Floquet–Bloch transformation in the periodic variable x_2, and consider the family of operators $H(k_2)$ in the strip

(§2). The resolvent of the corresponding free operator ($g_1 \equiv V_1 \equiv 0$) can be represented by the integral

$$R_0(k_2, \lambda) = \int_0^{2\pi} A(k_1, k_2, \lambda)\, dk_1.$$

Here A is a meromorphic function with values in bounded operators in a weighted space L_2 in the strip; λ is the spectral parameter, $\operatorname{Im} \lambda > 0$. Then we transform the contour of integration using the residue theorem, and obtain an analytic continuation of the free resolvent through the spectrum into the region $\operatorname{Im} \lambda < 0$ (into the "nonphysical sheet"), see §6. Next, we construct an analytic continuation of the resolvent of the full operator $H(k_2)$ by perturbation theory (§7). This turns out to suffice for the proof of the absolute continuity of the initial operator H (§8).

We use the following standard notation: $B_r(\lambda) = \{\mu \in \mathbb{C} : |\lambda - \mu| < r\}$ is the disk in the complex plane centered at λ and of radius r; $\operatorname{Hol}(\mathcal{O})$ is the set of analytic functions in the domain $\mathcal{O}$; $\sigma(A)$ is the spectrum of the operator A; $B(X, Y)$ is the space of bounded operators from the space X to the space Y; $B(X) = B(X, X)$.

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§2. Floquet–Bloch transform

2.1. Floquet–Bloch transform in the x_2 variable. We denote by S the following strip on the plane:

$$S = \{x \in \mathbb{R}^2 \,:\, x_2 \in (0,1)\} = \mathbb{R} \times (0,1).$$

For real a, we introduce the following function spaces in the strip:

$$L_{p,a} = \{f : e^{a\sqrt{x_1^2+1}} f \in L_p(S)\},$$

$$\|f\|_{L_{p,a}}^p = \int_S e^{pa\sqrt{x_1^2+1}} |f(x)|^p\, dx \quad \text{for} \quad p < \infty, \qquad \|f\|_{L_{\infty,a}} = \operatorname{ess\,sup} e^{a\sqrt{x_1^2+1}} |f(x)|,$$

and

$$H_a^2 = \{f : e^{a\sqrt{x_1^2+1}} f \in H^2(S)\},$$

$$\|f\|_{H_a^2}^2 = \int_S \left(\left| \nabla^2 (e^{a\sqrt{x_1^2+1}} f(x)) \right|^2 + e^{2a\sqrt{x_1^2+1}} |f(x)|^2 \right) dx,$$

where $H^2 \equiv W_2^2$ is the standard Sobolev space. We introduce the Floquet–Bloch transformation in x_2 by the formulas

$$U_2 \colon L_2(\mathbb{R}^2) \to L_2(S \times (0, 2\pi)) = \int_0^{2\pi} \oplus L_2(S)\, dk_2,$$

$$(U_2 f)(x, k_2) = \frac{1}{\sqrt{2\pi}} \sum_{n_2 \in \mathbb{Z}} e^{-ik_2(x_2+n_2)} f(x_1, x_2 + n_2).$$

It is easily seen that the operator U_2 is unitary, and

$$U_2 H U_2^* = \int_0^{2\pi} \oplus H(k_2)\, dk_2.$$

Here

$$H(k_2) = -\partial_1(g(x)\partial_1 \,\cdot\,) - (\partial_2 + ik_2)(g(x)(\partial_2 + ik_2) \,\cdot\,) + V(x)$$

is the operator in $L_2(S)$ defined on the domain

$$\operatorname{Dom} H(k_2) = H^2_{\mathrm{per}}(S) \equiv \{f \in H^2(S) \,:\, f(x_1, 0) = f(x_1, 1), \partial_2 f(x_1, 0) = \partial_2 f(x_1, 1)\}.$$

The following lemma is straightforward (see also [5, Lemma 5.1] and [1, formula (6.13)]).

Lemma 2.1. *Let* $\lambda \in \mathbb{C}$. *Then*

$$H(k_2) - \lambda = g(x)^{1/2}(-\Delta_{k_2} + W(\lambda))g(x)^{1/2},$$

where

$$\Delta_{k_2} = \partial_1^2 + (\partial_2 + ik_2)^2$$

is the operator in $L_2(S)$ *defined on the domain* $\operatorname{Dom}\Delta_{k_2} = H^2_{\mathrm{per}}(S)$, *and*

$$W(\lambda) = \frac{1}{g}\left(\frac{\Delta g}{2} - \frac{|\nabla g|^2}{4g} + V - \lambda\right).$$

Remark 2.2. Taking into account the identity $\frac{1}{g} = \frac{1}{g_0} - \frac{g_1}{g_0(g_0+g_1)}$, and the implication

$$f_0 \in L_\infty(S),\ f_1 \in L_{\infty,a} \quad \Longrightarrow \quad (f_0 f_1) \in L_{\infty,a},$$

we see that the function $W(\lambda)$ can be written as

$$W(\lambda) = W_0(\lambda) + W_1(\lambda),$$

where

$$W_0(\lambda) = \frac{1}{g_0}\left(\frac{\Delta g_0}{2} - \frac{|\nabla g_0|^2}{4g_0} + V_0 - \lambda\right) =: W_{00} - \frac{\lambda}{g_0}, \tag{2.1}$$

the function $W_0(\lambda) \in L_\infty(\mathbb{R}^2)$ is periodic with respect to $\mathbb{Z}^2$, W_{00} does not depend on λ,

$$W_1(\lambda) = W(\lambda) - W_0(\lambda) =: W_{10} + \frac{\lambda g_1}{g_0(g_0+g_1)},$$

the function $W_1(\lambda)$ is periodic in x_2, $W_1(\lambda) \in L_{\infty,a}$ for all $a \in \mathbb{R}$, and W_{10} does not depend on λ.

Remark 2.3. If $g(x) \equiv 1$, then

$$W(\lambda) = V - \lambda, \quad W_0(\lambda) = V_0 - \lambda, \quad W_1(\lambda) = V_1.$$

2.2. Floquet–Bloch transform in the x_1 variable. Denoting by Ω the cell of the lattice,

$$\Omega = (0,1) \times (0,1),$$

we introduce the operators

$$U_1 \colon L_2(S) \to L_2(\Omega \times (0,2\pi)) = \int_0^{2\pi} \oplus L_2(\Omega)\, dk_1, \quad U_1(k_1) \colon L_2(S) \to L_2(\Omega),$$

$$(U_1 f)(x,k_1) = (U_1(k_1)f)(x_1) = \frac{1}{\sqrt{2\pi}} \sum_{n_1 \in \mathbb{Z}} e^{-ik_1(x_1+n_1)} f(x_1+n_1, x_2).$$

It is easy to check that the operator U_1 is unitary, while $U_1(k_1)$ for each k_1 is an unbounded operator defined on a domain dense in $L_2(S)$. We need also the operator

$$(U_1(k_1)^\dagger g)(x) = \frac{1}{\sqrt{2\pi}} e^{ik_1x_1}(\Pi g)(x),$$

where Π is the operator that extends the functions defined in a cell Ω to functions 1-periodic in x_1 and defined in the strip S.

Lemma 2.4. *Let* $k_1 \in \mathbb{C}$, $a > |\operatorname{Im} k_1|$. *Then the operator* $U_1(k_1)$ *is bounded as an operator from* $L_{2,a}$ *to* $L_2(\Omega)$, *and the operator* $U_1(k_1)^\dagger$ *is bounded as an operator from* $L_2(\Omega)$ *to* $L_{2,-a}$ *and as an operator from* $H^2_{\mathrm{per}}(\Omega)$ *to* H^2_{-a}. *Here*

$$\begin{aligned} H^2_{\mathrm{per}}(\Omega) = \big\{ f \in H^2(\Omega) \ :\ & f(x_1,0) = f(x_1,1), \partial_2 f(x_1,0) = \partial_2 f(x_1,1), \\ & f(0,x_2) = f(1,x_2), \partial_1 f(0,x_2) = \partial_1 f(1,x_2) \big\}. \end{aligned} \tag{2.2}$$

Moreover,

$$\|U_1(k_1)\|_{L_{2,a}\to L_2(\Omega)} \le c_1, \quad \|U_1(k_1)^\dagger\|_{L_2(\Omega)\to L_{2,-a}} \le c_1, \tag{2.3}$$

where the constant c_1 depends only on the difference $(a - |\operatorname{Im} k_1|)$.

Proof. Let $f \in L_{2,a}$. By the triangle inequality for the L_2-norm, we have

$$\begin{aligned}
\|U_1(k_1)f\|_{L_2(\Omega)} &\le \sum_{n_1\in\mathbb{Z}} \left(\int_\Omega \left|e^{-ik_1(x_1+n_1)} f(x_1+n_1,x_2)\right|^2 dx_1\,dx_2 \right)^{1/2} \\
&\le \sum_{n_1\in\mathbb{Z}} \max_{x_1\in[0,1]} e^{(|\operatorname{Im} k_1|-a)|x_1+n_1|} \left(\int_\Omega \left|e^{a|x_1+n_1|} f(x_1+n_1,x_2)\right|^2 dx_1\,dx_2 \right)^{1/2} \\
&\le \sum_{n_1\in\mathbb{Z}} e^{-(a-|\operatorname{Im} k_1|)\min(|n_1|,|n_1+1|)} \|f\|_{L_{2,a}}.
\end{aligned}$$

Thus, the first estimate (2.3) is fulfilled with

$$c_1 = \sum_{n_1\in\mathbb{Z}} e^{-(a-|\operatorname{Im} k_1|)\min(|n_1|,|n_1+1|)}.$$

The other claims of the lemma can be proved in the same way. □

As usual, the transformation U_1 realizes a decomposition of the operator Δ_{k_2} into the direct integral

$$U_1 \Delta_{k_2} U_1^* = \int_0^{2\pi} \oplus \Delta_k \, dk_1,$$

where

$$(U_1^* g)(x) = \int_0^{2\pi} (U_1(k_1)^\dagger g)(x)\, dk_1,$$

and Δ_k is the operator in $L_2(\Omega)$ acting by the formula

$$\Delta_k = (\partial_1 + ik_1)^2 + (\partial_2 + ik_2)^2$$

on the domain $\operatorname{Dom} \Delta_k = H^2_{\mathrm{per}}(\Omega)$, see (2.2).

Lemma 2.5. *If $k \in \mathbb{R}^2$ and $\lambda \notin \mathbb{R}$, then the following operators exist: the bounded inverse operator $(-\Delta_k + W_0(\lambda))^{-1}$ in $L_2(\Omega)$ and the bounded inverse operators $(-\Delta_{k_2} + W_0(\lambda))^{-1}$ and $(-\Delta_{k_2} + W(\lambda))^{-1}$ in $L_2(S)$.*

Proof. The operator $(-\Delta_k + W_{00})$ is selfadjoint in $L_2(\Omega)$ for $k \in \mathbb{R}^2$. Therefore,

$$\operatorname{Im}\big((-\Delta_k + W_{00} - \lambda g_0^{-1})f, f\big)_{L_2(\Omega)} = -\operatorname{Im}\lambda (g_0^{-1} f, f)_{L_2(\Omega)},$$

whence

$$\big\|(-\Delta_k + W_0(\lambda))f\big\|_{L_2(\Omega)} \ge |\operatorname{Im}\lambda|\, \|g_0\|^{-1}_{L_\infty} \|f\|_{L_2(\Omega)} \quad \forall f \in H^2_{\mathrm{per}}(\Omega).$$

Thus, the operator $(-\Delta_k + W_0(\lambda))$ is invertible. The other claims of the lemma can be proved similarly. □

Lemma 2.6. *Let $k_2 \in \mathbb{R}$, $\lambda \notin \mathbb{R}$, $f \in L_{2,a}$, $a > 0$. Then*

$$(-\Delta_{k_2} + W_0(\lambda))^{-1} f = \int_0^{2\pi} U_1(k_1)^\dagger (-\Delta_k + W_0(\lambda))^{-1} U_1(k_1) f \, dk_1.$$

Proof. Denote $h = (-\Delta_{k_2} + W_0(\lambda))^{-1} f \in L_2(S) \cap H^2_{\mathrm{loc}}(S)$. It is clear that

$$(-\Delta_k + W_0(\lambda)) U_1 h = U_1(-\Delta_{k_2} + W_0(\lambda))h,$$

and therefore,

$$U_1 h = (-\Delta_k + W_0(\lambda))^{-1} U_1 f, \quad \text{and} \quad h = U_1^*(-\Delta_k + W_0(\lambda))^{-1} U_1 f. \qquad \square$$

§3. Estimation of the free resolvent

The content of this and the next two sections is very similar to the papers [8] and [4]. However, the constructions we need are not literally the same as the constructions in [8] and [4]. So, we give it in detail for the reader's convenience.

We fix numbers $\lambda_0 \in \mathbb{R}$ and $\delta \in (0, \pi/4)$.

3.1. The sets Σ and G. The operator $-\Delta_k$ with periodic boundary conditions has eigenfunctions

$$e^{im_1x_1+im_2x_2}, \quad m_1, m_2 \in 2\pi\mathbb{Z},$$

and the eigenvalues

$$h_m(k) = (m_1+k_1)^2 + (m_2+k_2)^2 = q_m^+(k)q_m^-(k),$$

where

$$q_m^\pm(k) = m_1 + k_1 \pm i(m_2 + k_2).$$

In the complex plane of the parameter k_2, we fix the point

$$k_2^* = \frac{\pi}{2} + i\left(\frac{\pi}{2} + l_*\right), \tag{3.1}$$

where

$$l_* \in 2\pi\mathbb{Z}, \qquad l_* \geq 2\left(\|W_{00}\|_{L_\infty} + (|\lambda_0|+1)c_0^{-1}\right), \tag{3.2}$$

c_0 is the constant from Remark 1.4, and W_{00} is the function from Remark 2.2.

We introduce the following set Σ in the plane of the parameter k_1:

$$\begin{aligned} \Sigma &= \left\{k_1 \in \mathbb{C} \,:\, h_m(k_1, k_2^*) = 0 \ \text{ for some } \ m \in (2\pi\mathbb{Z})^2\right\} \\ &= \left\{k_1 \in \mathbb{C} \,:\, \operatorname{Re} k_1 = -m_1 \pm (l_* + \pi/2), \operatorname{Im} k_1 = \mp(m_2 + \pi/2)\right\}_{m_1, m_2 \in 2\pi\mathbb{Z}}. \end{aligned} \tag{3.3}$$

Note that different pairs (m_1, m_2) give different values of k_1. The set Σ is contained in a countable union of horizontal lines $\operatorname{Im} k_1 \equiv \frac{\pi}{2} \pmod{\pi\mathbb{Z}}$. On each such line it is a sequence of equally spaced points, with the spacings 2π. Also, we introduce the set G in the plane of the parameter k_1:

$$G = (\mathbb{R} + i\pi\mathbb{Z}) \cup \bigcup_{z\in\Sigma}\left(z + \pi + i\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right). \tag{3.4}$$

It is a "brick wall"; each "brick" is a rectangle of size $(2\pi \times \pi)$ centered at one point of Σ. By construction,

$$\operatorname{dist}(G, \Sigma) = \frac{\pi}{2}. \tag{3.5}$$

3.2. Estimates of the symbol. The following lemma is an analog of [8, Lemma 5] and [4, Lemma 5.3].

Lemma 3.1. *Let $k = (k_1, k_2^*)$, $k_1 \in G$. Then $|h_m(k)| \geq l_*$.*

Proof. Since $q_m^+(k) - q_m^-(k) = 2i(m_2 + k_2^*)$, we have

$$|\operatorname{Re} q_m^+(k) - \operatorname{Re} q_m^-(k)| = \pi + 2l_* > 2l_*.$$

So, $\max_\pm |\operatorname{Re} q_m^\pm(k)| > l_*$. Next, $|q_m^\pm(k)| = |k_1 - z|$ for some $z \in \Sigma$. Thus, $\min_\pm |q_m^\pm(k)| \geq \frac{\pi}{2}$ for $k_1 \in G$ due to (3.5). Therefore,

$$|h_m(k)| = |q_m^+(k)||q_m^-(k)| \geq \frac{\pi l_*}{2} > l_*. \qquad \square$$

We fix a number τ,

$$\tau \in 2\pi\mathbb{Z}, \qquad \tau \geq 2\delta^{-1}\left(\|W_{00}\|_{L_\infty} + (|\lambda_0|+1)c_0^{-1}\right). \tag{3.6}$$

Lemma 3.2. *Let $k = (k_1, k_2)$ be such that $\operatorname{Im} k_1 = \pm\tau$, $\operatorname{Re} k_2 \in [\delta, 2\pi - \delta]$. Then $|h_m(k)| \geq \delta\tau$.*

Proof. Since $\operatorname{Im} q_m^\pm(k) = \pm \operatorname{Re} k_2 \pmod{2\pi\mathbb{Z}}$, we have

$$\min_\pm |q_m^\pm(k)| \geq \min_\pm |\operatorname{Im} q_m^\pm(k)| \geq \delta.$$

Next, $|\operatorname{Im} q_m^+(k) + \operatorname{Im} q_m^-(k)| = 2|\operatorname{Im} k_1| = 2\tau$, and

$$\max_\pm |q_m^\pm(k)| \geq \max_\pm |\operatorname{Im} q_m^\pm(k)| \geq \tau.$$

Thus, $|h_m(k)| \geq \delta\tau$. □

3.3. Estimates for the norm $(-\Delta_k + W_0(\lambda))^{-1}$. The following lemma is an analog of [8, Corollary 1] and [4, Theorem 4.2].

Lemma 3.3. *Suppose $\lambda \in B_1(\lambda_0)$, $k = (k_1, k_2^*)$, $k_1 \in G$, let k_2^* be defined by (3.1), and let $\mu \in [0,1]$. Then there exists an inverse operator $(-\Delta_k + \mu W_0(\lambda))^{-1}$ in $L_2(\Omega)$, and*

$$\left\|(-\Delta_k + \mu W_0(\lambda))^{-1}\right\| \leq 2 l_*^{-1}.$$

Proof. By Lemma 3.1, $\|\Delta_k^{-1}\| \leq l_*^{-1}$. By the definition (2.1) of the function $W_0(\lambda)$, we have

$$\|W_0(\lambda)\| \leq \|W_{00}\|_{L_\infty} + (|\lambda_0| + 1)c_0^{-1}, \tag{3.7}$$

whence $\|\mu W_0(\lambda)\Delta_k^{-1}\| \leq 1/2$, where we have taken the condition (3.2) into account. So, the operator $(-\Delta_k + \mu W_0(\lambda))$ is invertible in $L_2(\Omega)$ and

$$\left\|(-\Delta_k + \mu W_0(\lambda))^{-1}\right\| = \left\|(-\Delta_k)^{-1}(I - \mu W_0(\lambda)\Delta_k^{-1})^{-1}\right\| \leq 2\|\Delta_k^{-1}\| \leq 2 l_*^{-1}.$$

Here we have used the Neumann series arguments for the first inequality. □

The next lemma is an analog of [8, Theorem 2] and [4, Corollary 5.5].

Lemma 3.4. *Suppose $\lambda \in B_1(\lambda_0)$, $\operatorname{Im} k_1 = \pm\tau$, and $\operatorname{Re} k_2 \in [\delta, 2\pi - \delta]$. Then there exists a bounded inverse operator $(-\Delta_k + W_0(\lambda))^{-1}$ in $L_2(\Omega)$.*

Proof. By Lemma 3.2, the inverse Δ_k^{-1} exists, $\|\Delta_k^{-1}\| \leq (\delta\tau)^{-1}$. Inequalities (3.6) and (3.7) yield the estimate $\|W_0(\lambda)\Delta_k^{-1}\| \leq 1/2$, which implies the invertibility of the operator $(-\Delta_k + W_0(\lambda))$ and the estimate

$$\left\|(-\Delta_k + W_0(\lambda))^{-1}\right\| \leq 2\|\Delta_k^{-1}\| \leq 2(\delta\tau)^{-1}.$$ □

§4. The operator $T_\mu(k_2, \lambda)$

4.1. The operators T_μ. We denote

$$H^1_{\mathrm{per}}(\Omega) = \{f \in H^1(\Omega) : f(x_1, 0) = f(x_1, 1), f(0, x_2) = f(1, x_2)\},$$

and the notation $H^2_{\mathrm{per}}(\Omega)$ has a similar meaning, see (2.2). As in [8], for $\mu \in [0,1]$ in the space $H^1_{\mathrm{per}}(\Omega) \oplus L_2(\Omega)$ we introduce the operator

$$T_\mu(k_2, \lambda) = \begin{pmatrix} 0 & I \\ \Delta_{k_2} - \mu W_0(\lambda) & 2i\partial_1 \end{pmatrix}$$

on the domain

$$\operatorname{Dom} T_\mu(k_2, \lambda) = H^2_{\mathrm{per}}(\Omega) \oplus H^1_{\mathrm{per}}(\Omega).$$

The following lemma is elementary, see [4, Proposition 3.1].

Lemma 4.1. *The spectrum of the operator $T_\mu(k_2, \lambda)$ is discrecte and (2π)-periodic. Moreover, $k_1 \in \sigma(T_\mu(k_2, \lambda))$ if and only if $0 \in \sigma(-\Delta_k + \mu W_0(\lambda))$.*

Recall that the sets Σ and G are defined by formulas (3.3) and (3.4). The next lemma is an analog of [8, Lemma 3] and [4, Theorem 4.3].

Lemma 4.2. *Let $\lambda \in B_1(\lambda_0)$. Then each "brick" of the set G, i.e., the set*

$$Q_z = \{k_1 \in \mathbb{C} : |\operatorname{Re}(k_1 - z)| < \pi, |\operatorname{Im}(k_1 - z)| < \pi/2\}, \quad z \in \Sigma, \tag{4.1}$$

contains exactly one eigenvalue of the operator $T_1(k_2^*, \lambda)$, where k_2^* is defined by (3.1).

Proof. We prove this claim for all operators $T_\mu(k_2^*, \lambda)$, $\mu \in [0, 1]$. For $\mu = 0$ the spectrum of $T_0(k_2^*, \lambda)$ coincides with the set Σ by construction. The corresponding vectors $\binom{e^{imx}}{k_1 e^{imx}}$ form a basis in the space $L_2(\Omega) \oplus L_2(\Omega)$, so that the spectrum of the operator $T_0(k_2^*, \lambda)$ is simple.

For a given point $z \in \Sigma$, consider the Riesz projection

$$\frac{1}{2\pi i} \oint_{\partial Q_z} (T_\mu(k_2^*, \lambda) - \varkappa)^{-1} \, d\varkappa,$$

where the contour of integration is the boundary of the corresponding "brick". For $\mu \in [0, 1]$ the eigenvalues of $T_\mu(k_2^*, \lambda)$ cannot cross $\partial Q_z \subset G$ by Lemma 3.3. Therefore, the rank of this projection is constant for $\mu \in [0, 1]$, and equals 1 (see, e.g., [9, Chapter IV, §3.5]). □

We denote the strip between the lines $\operatorname{Im} k_1 = \pm\tau$ by

$$D = \{k_1 \in \mathbb{C} : |\operatorname{Im} k_1| < \tau\}, \tag{4.2}$$

where the parameter τ is as defined in (3.6).

Corollary 4.3. *The spectrum of the operator $T_1(k_2^*, \lambda)$, $\lambda \in B_1(\lambda_0)$, in the strip D is a (2π)-periodic set. It consists of exactly $N = \frac{2\tau}{\pi}$ simple eigenvalues modulo 2π.*

The following lemma is elementary.

Lemma 4.4. *Let $k_1 \notin \sigma(T_1(k_2, \lambda))$. Then there exists a positive number ε such that*

$$B_\varepsilon(k_1) \cap \sigma(T_1(\widetilde{k}_2, \widetilde{\lambda})) = \varnothing \quad \text{if} \quad \widetilde{k}_2 \in B_\varepsilon(k_2), \ \widetilde{\lambda} \in B_\varepsilon(\lambda).$$

4.2. Degenerate eigenvalues.

Lemma 4.5. *Let $T(\xi_1, \ldots, \xi_m)$ be an analytic function defined in a domain $\mathcal{O} \subset \mathbb{C}^m$ with values in the operators in a finite-dimensional space,*

$$T(\vec{\xi}) \in B(\mathcal{H}), \quad \vec{\xi} \in \mathcal{O}, \qquad \dim \mathcal{H} = L < \infty.$$

Then a function $F \in \operatorname{Hol}(\mathcal{O})$ such that the operator $T(\vec{\xi})$ has (at least one) degenerate eigenvalue exists if and only if $F(\vec{\xi}) = 0$.

Proof. We consider the monic polynomial

$$p(\varkappa) = (-1)^L \det(T(\vec{\xi}) - \varkappa) = \varkappa^L + a_{L-1}(\vec{\xi})\varkappa^{L-1} + \cdots + a_1(\vec{\xi})\varkappa + a_0(\vec{\xi}).$$

The coefficients $a_0, \ldots, a_{L-1}$ are analytic functions of $\vec{\xi}$, $a_j \in \operatorname{Hol}(\mathcal{O})$, $j = 0, \ldots, L-1$. The discriminant $\Delta(p)$ is well defined for any monic polynomial. It is well known (see, e.g., [12]) that $\Delta(p)$ is a polynomial in the coefficients $a_0, \ldots, a_{L-1}$ of the initial polynomial. The polynomial p has roots of multiplicity greater than one if and only if $\Delta(p) = 0$. Thus, we can put $F(\vec{\xi}) = \Delta(p)$. □

We introduce the following closed set M in the plane of the parameter k_2:

$$M = [2\delta, 2\pi - 2\delta] \cup \left\{\frac{\pi}{2} + il\right\}_{l \in [0, l^*]}, \tag{4.3}$$

where l^* is defined by (3.2).

Theorem 4.6. *Let $k_2^0 \in M$. Then there exists a number $\varepsilon \in (0,\delta)$ and a function $F \in \mathrm{Hol}(B_\varepsilon(k_2^0) \times B_\varepsilon(\lambda_0))$ with the following properties. For $(k_2;\lambda) \in B_\varepsilon(k_2^0) \times B_\varepsilon(\lambda_0)$ the operator $T_1(k_2,\lambda)$ has* (*at least one*) *degenerate eigenvalue in the strip D* (*see* (4.2)) *if and only if $F(k_2,\lambda) = 0$.*

Proof. Clearly, there is a number $\rho \in \mathbb{R}$ such that $\operatorname{Re} k_1 - \rho \notin 2\pi\mathbb{Z}$ for all $k_1 \in \sigma(T_1(k_2^0,\lambda_0)) \cap D$. Lemma 4.4 implies the existence of $\varepsilon_1 \in (0,\delta)$ such that if $k_1 \in \sigma(T_1(k_2,\lambda)) \cap D$, then $\operatorname{Re} k_1 - \rho \notin 2\pi\mathbb{Z}$ for all $k_2 \in B_{\varepsilon_1}(k_2^0)$, $\lambda \in B_{\varepsilon_1}(\lambda_0)$.

Consider the Riesz projections

$$P(k_2,\lambda) = \frac{1}{2\pi i} \oint (T_1(k_2,\lambda) - \varkappa)^{-1}\, d\varkappa,$$

where the contour of integration is the rectangle

$$\{\rho + i\eta\}_{\eta\in[-\tau,\tau]} \cup \{\rho + 2\pi + i\eta\}_{\eta\in[-\tau,\tau]} \cup \{r + i\tau\}_{r\in[\rho,\rho+2\pi]} \cup \{r - i\tau\}_{r\in[\rho,\rho+2\pi]}.$$

The projector P is analytic in the two variables $(k_2;\lambda) \in B_{\varepsilon_1}(k_2^0) \times B_{\varepsilon_1}(\lambda_0)$, and its rank is constant by Lemmas 3.4 and 4.1. There exists a number $\varepsilon \in (0,\varepsilon_1)$ such that

$$\|P(k_2,\lambda) - P(k_2^0,\lambda_0)\| < 1 \quad \text{for} \quad |k_2 - k_2^0| \le \varepsilon,\ |\lambda - \lambda_0| \le \varepsilon. \tag{4.4}$$

Next, we consider the operator-valued function

$$U(k_2,\lambda) = P(k_2,\lambda)P(k_2^0,\lambda_0) + (I - P(k_2,\lambda))(I - P(k_2^0,\lambda_0)).$$

It is analytic in the two variables $(k_2;\lambda)$. By (4.4) the operators U are invertible, the inverse operators are bounded, and $U(k_2,\lambda)^{-1}$ is also an analytic operator-valued function,

$$U^{-1} \in \mathrm{Hol}(B_\varepsilon(k_2^0) \times B_\varepsilon(\lambda_0), B(L_2(\Omega)))$$

(see [9, Chapter I §4.6, Chapter II §4.2 and Chapter VII §1.3]). Moreover,

$$P(k_2,\lambda) = U(k_2,\lambda)P(k_2^0,\lambda_0)U(k_2,\lambda)^{-1}.$$

Now, we consider the operator

$$\check{T}(k_2,\lambda) = U(k_2,\lambda)^{-1}T_1(k_2,\lambda)U(k_2,\lambda).$$

We have $\check{T} \in \mathrm{Hol}(B_\varepsilon(k_2^0) \times B_\varepsilon(\lambda_0), B(L_2(\Omega)))$ and

$$\check{T}(k_2,\lambda)P(k_2^0,\lambda_0) = P(k_2^0,\lambda_0)\check{T}(k_2,\lambda).$$

The family $\check{T}(k_2,\lambda)\big|_{\mathrm{Ran}(P(k_2^0,\lambda_0))}$ is an analytic operator-valued function in a finite-dimensional space. The operators $\check{T}(k_2,\lambda)\big|_{\mathrm{Ran}(P(k_2^0,\lambda_0))}$ have the same eigenvalues (taking multiplicity into account) as the operators $T_1(k_2,\lambda)\big|_{\mathrm{Ran}(P(k_2,\lambda))}$. Now, a reference to Lemma 4.5 completes the proof. □

§5. The set $\mathcal{N}(\Gamma)$

5.1. The functions $F_j(k_2,\lambda)$. The set M (see (4.3)) is compact. Therefore, Theorem 4.6 shows that there exist

- a finite set of points $k_{2,j} \in M$, $j = 1,\dots,J$;
- numbers $\varepsilon_j \in (0,\delta)$;
- neighbourhoods $\mathcal{O}_j = B_{\varepsilon_j}(k_{2,j}) \times B_\varepsilon(\lambda_0)$, where

$$\varepsilon = \min(\varepsilon_1,\dots,\varepsilon_J) \in (0,\delta); \tag{5.1}$$

- and analytic functions of two variables $F_j \in \mathrm{Hol}(\mathcal{O}_j)$,

such that $M \subset \bigcup_{j=1}^J B_{\varepsilon_j}(k_{2,j})$, and for $(k_2;\lambda) \in \mathcal{O}_j$ the operator $T_1(k_2,\lambda)$ has degenerate eigenvalues in the strip D if and only if $F_j(k_2,\lambda) = 0$.

Lemma 5.1. *For all $\lambda \in B_\varepsilon(\lambda_0)$ and $j = 1, \ldots, J$ there exists $k_2 \in B_{\varepsilon_j}(k_{2,j})$ such that $F_j(k_2, \lambda) \neq 0$.*

Proof. Assume the claim is false. Suppose $j \in \{1, \ldots, J\}$, $\lambda \in B_\varepsilon(\lambda_0)$, and $F_j(k_2, \lambda) = 0$ for all $k_2 \in B_{\varepsilon_j}(k_{2,j})$. Then the operator $T_1(k_2, \lambda)$ has a degenerate eigenvalue in the strip D for all $k_2 \in B_{\varepsilon_j}(k_{2,j})$. If $B_{\varepsilon_j}(k_{2,j}) \cap B_{\varepsilon_l}(k_{2,l}) \neq \varnothing$, then

$$F_l(k_2, \lambda) = 0, \qquad k_2 \in B_{\varepsilon_j}(k_{2,j}) \cap B_{\varepsilon_l}(k_{2,l}),$$

and

$$F_l(k_2, \lambda) = 0, \qquad k_2 \in B_{\varepsilon_l}(k_{2,l}), \tag{5.2}$$

by the analyticity of the function F_l. Since the set M is connected, (5.2) is fulfilled for all l. Thus, the operator $T_1(k_2^*, \lambda)$ has also a degenerate eigenvalue in D. But this fact contradicts Lemma 4.2. □

5.2. Set $\mathcal{N}(\Gamma)$. Now, we fix a pair

$$(\widetilde{k}_2; \widetilde{\lambda}) \in (2\delta, 2\pi - 2\delta) \times (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$$

satisfying the condition

$$F_j(\widetilde{k}_2, \widetilde{\lambda}) \neq 0, \quad \text{where } j \text{ is such that } \widetilde{k}_2 \in B_{\varepsilon_j}(k_{2,j}). \tag{5.3}$$

Obviously, on the k_2-plane there exists a continuous path $\Gamma(t)$,

$$\Gamma\colon [0,1] \to \{k_2 \in \mathbb{C} \,:\, \operatorname{Re} k_2 \in (\delta, 2\pi - \delta)\},$$

such that

- $\Gamma(0) = \widetilde{k}_2$, $\Gamma(1) = k_2^*$, where k_2^* is defined in (3.1);
- $\Gamma([0,1]) \subset \bigcup_{j=1}^J B_{\varepsilon_j}(k_{2,j})$;
- $F_j(\Gamma(t), \widetilde{\lambda}) \neq 0$ for all $t \in [0,1]$, where j is such that $\Gamma(t) \in B_{\varepsilon_j}(k_{2,j})$.

Next, we choose an open neighborhood $\mathcal{N}(\Gamma)$ of the path Γ in the k_2-plane, and choose a number $\varepsilon_0 \in (0, \varepsilon]$ such that

$$\Gamma([0,1]) \cup B_{\varepsilon_0}(\widetilde{k}_2) \subset \mathcal{N}(\Gamma) \subset \bigcup_{j=1}^J B_{\varepsilon_j}(k_{2,j}) \subset \{k_2 \in \mathbb{C} \,:\, \operatorname{Re} k_2 \in (\delta, 2\pi - \delta)\}$$

and

$$F_j(k_2, \lambda) \neq 0, \quad \forall k_2 \in \mathcal{N}(\Gamma), \forall \lambda \in B_{\varepsilon_0}(\widetilde{\lambda}),$$

where j is such that $k_2 \in B_{\varepsilon_j}(k_{2,j})$. Now, Corollary 4.3, Lemma 3.4, and the definition of the functions F_j imply that the spectrum of the operator $T_1(k_2, \lambda)$ in the strip D consists of $N = \frac{2\tau}{\pi}$ simple eigenvalues repeated infinitely many times with period 2π, for all $k_2 \in \mathcal{N}(\Gamma)$, $\lambda \in B_{\varepsilon_0}(\widetilde{\lambda})$. Thus, we get N analytic functions

$$\{q_n(k_2, \lambda)\}_{n=1}^N, \qquad q_n \in \operatorname{Hol}\big(\mathcal{N}(\Gamma) \times B_{\varepsilon_0}(\widetilde{\lambda})\big).$$

These functions describe the spectrum of $T_1(k_2, \lambda)$ in D modulo 2π,

$$k_1 \in \sigma(T_1(k_2, \lambda)), \quad |\operatorname{Im} k_1| \le \tau \;\Leftrightarrow\; k_1 - q_n(k_2, \lambda) \in 2\pi\mathbb{Z}$$

for some unique $n = 1, \ldots, N$. Next, Lemma 2.5 and Lemma 4.1 show that for $k_2 \in \mathbb{R}$ and $\lambda \notin \mathbb{R}$ the spectrum of $T_1(k_2, \lambda)$ contains no real eigenvalues, i.e., $q_n(k_2, \lambda) \notin \mathbb{R}$. Therefore, we can renumber the functions q_n as follows:

$$\{q_n\}_{n=1}^N = \{q_n^-\}_{n=1}^{N_-} \cup \{q_n^+\}_{n=1}^{N_+}, \qquad N_+ + N_- = N,$$

where

$$\pm \operatorname{Im} q_j^\pm(k_2, \lambda) > 0,\ j = 1, \ldots, N_\pm, \text{ for } k_2 \in (\widetilde{k}_2 - \varepsilon_0, \widetilde{k}_2 + \varepsilon_0),\ \lambda \in B_{\varepsilon_0}(\widetilde{\lambda}),\ \operatorname{Im} \lambda > 0. \tag{5.4}$$

Here it does not matter what pair $(k_2;\lambda) \in (\widetilde{k}_2-\varepsilon_0,\widetilde{k}_2+\varepsilon_0)\times B_{\varepsilon_0}(\widetilde{\lambda})$, $\operatorname{Im}\lambda>0$, we choose, because (5.4) is fulfilled for all such pairs simultaneously.

§6. Analytic continuation of the free resolvent

Suppose that $(\widetilde{k}_2;\widetilde{\lambda}) \in (2\delta, 2\pi-2\delta)\times(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$ satisfies (5.4), and let $\mathcal{N}(\Gamma)$ be the set constructed in the preceding section with these $(\widetilde{k}_2;\widetilde{\lambda})$.

6.1. The operators $A(k_1,k_2,\lambda)$**.** We introduce the operators

$$A(k_1,k_2,\lambda) = U_1(k_1)^\dagger(-\Delta_k+W_0(\lambda))^{-1}U_1(k_1),$$

where

$$k_1\in\overline{D},\ k_2\in\mathcal{N}(\Gamma),\ \lambda\in B_{\varepsilon_0}(\widetilde{\lambda})\ \text{ and }\ k_1\not\equiv q_n(k_2,\lambda)(\operatorname{mod}2\pi\mathbb{Z}),\ \ n=1,\dots,N,$$

and the operators $U_1(k_1)$, $U_1(k_1)^\dagger$ are as defined in §2.

Lemma 6.1. *Let $a>\tau$ with τ as defined in* (3.6)*. Then*:
a) *$A(k_1,k_2,\lambda)\colon L_{2,a}\to H^2_{-a}$ is a bounded operator*;
b) *for $k_1\in G\cap\overline{D}$ and $k_2=k_2^*$, we have*

$$\|A(k_1,k_2^*,\lambda)\|_{L_{2,a}\to L_{2,-a}} \le 2c_1^2 l_*^{-1},$$

where the constant c_1 is the same as in Lemma 2.4;
c) *the operator-valued function A is analytic in the three variables in*

$$D\times\mathcal{N}(\Gamma)\times B_{\varepsilon_0}(\lambda_0)\setminus\bigcup_{n=1}^{N}\{(k_1,k_2,\lambda)\ :\ k_1\equiv q_n(k_2,\lambda)(\operatorname{mod}2\pi\mathbb{Z})\}.$$

Proof. a) This follows form Lemma 2.4 and the fact that the operator

$$(-\Delta_k+W_0(\lambda))^{-1}\colon L_2(\Omega)\to H^2_{\mathrm{per}}(\Omega)$$

is bounded whenever it exists.

b) By the definition of $A(k_1,k_2,\lambda)$ we have

$$\|A(k_1,k_2^*,\lambda)\|_{L_{2,a}\to L_{2,-a}}$$
$$\le \|U_1(k_1)^\dagger\|_{L_2(\Omega)\to L_{2,-a}}\|(-\Delta_k+W_0(\lambda))^{-1}\|_{L_2(\Omega)\to L_2(\Omega)}\|U_1(k_1)\|_{L_{2,a}\to L_2(\Omega)} \le 2c_1^2 l_*^{-1},$$

due to Lemmas 2.4 and 3.3.

c) This is clear. □

Lemma 6.2. *The operator-valued function A is (2π)-periodic in k_1,*

$$A(k_1+2\pi,k_2,\lambda)=A(k_1,k_2,\lambda).$$

Proof. Let $f\in L_{2,a}$, $a>\tau$. Put

$$h(k,x)=\big(\Pi(-\Delta_k+W_0(\lambda))^{-1}U_1(k_1)f\big)(x).$$

Recall that here Π is the operator of periodic extension of functions defined in Ω to the entire strip S. Thus, the function h is defined for $x\in S$ and is 1-periodic in x_1. We have

$$(-\Delta_k+W_0(\lambda))h(k,x)=\frac{1}{\sqrt{2\pi}}\sum_{n_1\in\mathbb{Z}}e^{-ik_1(x_1+n_1)}f(x_1+n_1,x_2).$$

The function $e^{-2\pi i x_1}h(k,x)$ is also 1-periodic in x_1 and

$$\big(-\Delta_{k+2\pi e_1}+W_0(\lambda)\big)\big(e^{-2\pi i x_1}h(k,x)\big)=e^{-2\pi i x_1}(-\Delta_k+W_0(\lambda))h(k,x)$$
$$=\frac{1}{\sqrt{2\pi}}\sum_{n_1\in\mathbb{Z}}e^{-i(k_1+2\pi)(x_1+n_1)}f(x_1+n_1,x_2)=\big(U_1(k_1+2\pi)f\big)(x).$$

Therefore,

$$(-\Delta_{k+2\pi e_1} + W_0(\lambda))^{-1} U_1(k_1 + 2\pi) f = e^{-2\pi i x_1} h,$$

and

$$\big(A(k_1 + 2\pi, k_2, \lambda) f\big)(x) = \frac{1}{\sqrt{2\pi}} e^{i(k_1+2\pi)x_1} e^{-2\pi i x_1} h(k, x) = \big(A(k_1, k_2, \lambda) f\big)(x). \qquad \square$$

6.2. The operators $R_0(k_2, \lambda)$. For $k_2 \in \mathcal{N}(\Gamma)$ and $\lambda \in B_{\varepsilon_0}(\widetilde{\lambda})$, we introduce the operators

$$R_0(k_2, \lambda) = \int_{[0,2\pi]+i\tau} A(k_1, k_2, \lambda)\, dk_1 + 2\pi i \sum_{j=1}^{N_+} \operatorname{res}_{q_j^+(k_2,\lambda)} A(\,\cdot\,, k_2, \lambda)$$

in the strip S. Here, as usual,

$$2\pi i \operatorname{res}_{q_j^+(k_2,\lambda)} A(\,\cdot\,, k_2, \lambda) = \oint_{\Gamma_j} A(k_1, k_2, \lambda)\, dk_1,$$

Γ_j is a rectifiable contour that contains $q_j^+(k_2, \lambda)$ and does not contain any other eigenvalues of the operator $T_1(k_2, \lambda)$.

Lemma 6.3. *Let $a > \tau$. Then*:

a) $R_0 \in \mathrm{Hol}\big(\mathcal{N}(\Gamma) \times B_{\varepsilon_0}(\widetilde{\lambda}), B(L_{2,a}, H^2_{-a})\big)$;

b) *for $k_2 = k_2^*$ we have*

$$\|R_0(k_2^*, \lambda)\|_{L_{2,a} \to L_{2,-a}} \le c_2 l_*^{-1}, \quad \lambda \in B_{\varepsilon_0}(\widetilde{\lambda}),$$

where $c_2 = 4c_1^2(6\tau + \pi)$ with c_1 as defined in Lemma 2.4.

Proof. a) This follows from Lemma 6.1.

b) By Lemma 4.2, each pole $q_j^+(k_2^*, \lambda)$ belongs to some "brick" Q_j^+ (see (4.1)), and there are no other poles therein. So, we can rewrite the expression for $R_0(k_2^*, \lambda)$ as

$$R_0(k_2^*, \lambda) = \int_{[0,2\pi]+i\tau} A(k_1, k_2^*, \lambda)\, dk_1 + \sum_{j=1}^{N_+} \oint_{\partial Q_j^+} A(k_1, k_2^*, \lambda)\, dk_1.$$

Now, the inclusions $[0, 2\pi] + i\tau \subset G$, $\partial Q_j^+ \subset G$ and Lemma 6.1 b) imply

$$\|R_0(k_2^*, \lambda)\|_{L_{2,a} \to L_{2,-a}} \le (2\pi + 6\pi N_+)\, 2c_1^2 l_*^{-1} \le c_2 l_*^{-1}. \qquad \square$$

Lemma 6.4. *If $k_2 \in (\widetilde{k}_2 - \varepsilon_0, \widetilde{k}_2 + \varepsilon_0)$, $\lambda \in B_{\varepsilon_0}(\widetilde{\lambda})$, $\mathrm{Im}\,\lambda > 0$, and $a > \tau$, then*

$$R_0(k_2, \lambda) f = (-\Delta_{k_2} + W_0(\lambda))^{-1} f, \quad f \in L_{2,a}.$$

Proof. The operator-valued function A is (2π)-periodic in k_1 by Lemma 6.2. Therefore, by the definition of the functions q_j^+ we have

$$R_0(k_2, \lambda) = \int_0^{2\pi} A(k_1, k_2, \lambda)\, dk_1$$

under the above conditions on k_2 and λ. Referring to Lemma 2.6, we complete the proof. $\square$

§7. Analytic continuation of the full resolvent

In the rest of the paper we follow essentially the exposition in [5]. The following fact is well known.

Lemma 7.1. *Let $\mathcal{O}$ be a domain in $\mathbb{C}^p$, and let $\vec{z}_0 \in \mathcal{O}$. Let $M \in \mathrm{Hol}(\mathcal{O}, B(\mathcal{H}))$ be an analytic function with values in the compact operators in a Hilbert space $\mathcal{H}$. Then there is a neighborhood $\mathcal{O}_0$ of the point $\vec{z}_0$ and an analytic function $h \in \mathrm{Hol}(\mathcal{O}_0)$ such that for $\vec{z} \in \mathcal{O}_0$ the operator $(I + M(\vec{z}))$ is invertible if and only if $h(\vec{z}) \neq 0$.*

Everywhere below we assume that conditions (3.2) and (3.6) are fulfilled, $a > \tau$, and moreover,

$$l_* > c_2 \sup_{\lambda \in B_1(\lambda_0)} \|W_1(\lambda)\|_{L_{\infty,2a}}, \tag{7.1}$$

where c_2 is the constant defined in Lemma 6.3. The next lemma is an analog of [5, Theorem 4.1].

Lemma 7.2. *Suppose that $(\widetilde{k}_2; \widetilde{\lambda})$ satisfies condition* (5.3). *Then there is a number $\varepsilon_* \in (0, \varepsilon_0]$ and a function $h \in \mathrm{Hol}(B_{\varepsilon_*}(\widetilde{k}_2) \times B_{\varepsilon_*}(\widetilde{\lambda}))$ such that*

1) *for $(k_2; \lambda) \in B_{\varepsilon_*}(\widetilde{k}_2) \times B_{\varepsilon_*}(\widetilde{\lambda})$ the operator $(I + W_1(\lambda) R_0(k_2, \lambda))$ has bounded inverse in $L_{2,a}$ if and only if $h(k_2, \lambda) \neq 0$;*

2) *for all $\lambda \in B_{\varepsilon_*}(\widetilde{\lambda})$,*

$$\text{there is } \; k_2 \in B_{\varepsilon_*}(\widetilde{k}_2) \; \text{ with } \; h(k_2, \lambda) \neq 0. \tag{7.2}$$

Proof. By the assumption (see Remark 2.2), we have $W_1(\lambda) \in L_{\infty,b}$, $b > 2a$. Therefore, the operator of multiplication by $W_1(\lambda)$ is bounded as an operator from $L_{2,-a}$ to $L_{2,a}$, and is compact as an operator from H^2_{-a} to $L_{2,a}$. Therefore, by Lemma 6.3 a), the operator $W_1(\lambda) R_0(k_2, \lambda)$ is compact in $L_{2,a}$, and is an analytic operator-valued function of $(k_2; \lambda) \in \mathcal{N}(\Gamma) \times B_{\varepsilon_0}(\widetilde{\lambda})$. Furthermore, Lemma 6.3 b) shows that

$$\|W_1(\lambda) R_0(k_2^*, \lambda)\|_{L_{2,a} \to L_{2,a}} \leq c_2 l_*^{-1} \|W_1\|_{L_{\infty,2a}} < 1$$

by the assumption (7.1). Therefore, for all $\lambda \in B_{\varepsilon_0}(\widetilde{\lambda})$ the inverse operator

$$(I + W_1(\lambda) R_0(k_2^*, \lambda))^{-1}$$

exists in $L_{2,a}$. Now, the analytic Fredholm alternative implies that for all $\lambda \in B_{\varepsilon_0}(\widetilde{\lambda})$ there is a point $k_2 \in \mathcal{N}(\Gamma)$ arbitrarily close to $\widetilde{k}_2$ and such that the operator $(I + W_1(\lambda) R_0(k_2, \lambda))$ is also invertible in $L_{2,a}$. The reference to Lemma 7.1 with

$$\vec{z} = (k_2; \lambda), \quad \vec{z}_0 = (\widetilde{k}_2; \widetilde{\lambda}), \quad \mathcal{H} = L_{2,a} \;\text{ and }\; M(\vec{z}) = W_1(\lambda) R_0(k_2, \lambda)$$

completes the proof. □

The next theorem is an analog of [5, Theorem 5.1].

Theorem 7.3. *Under the assumptions of Lemma* 7.2, *there is an operator-valued function R with the following properties*:

1) $R \in \mathrm{Hol}\big(B_{\varepsilon_*}(\widetilde{k}_2) \times B_{\varepsilon_*}(\widetilde{\lambda}) \setminus \{(k_2; \lambda) : h(k_2, \lambda) = 0\}; B(L_{2,a}, H^2_{-a})\big)$;

2) *for $k_2 \in (\widetilde{k}_2 - \varepsilon_*, \widetilde{k}_2 + \varepsilon_*)$, $|\lambda - \widetilde{\lambda}| < \varepsilon_*$, $\operatorname{Im} \lambda > 0$, $h(k_2, \lambda) \neq 0$, we have*

$$R(k_2, \lambda) f = (H(k_2) - \lambda)^{-1} f, \quad f \in L_{2,a}. \tag{7.3}$$

Proof. We put

$$R(k_2, \lambda) = g(x)^{-1/2} R_0(k_2, \lambda) \big(I + W_1(\lambda) R_0(k_2, \lambda)\big)^{-1} g(x)^{-1/2}.$$

Clearly, the first property is fulfilled. Suppose that

$$f \in L_{2,a}, \quad k_2 \in (\widetilde{k}_2 - \varepsilon_*, \widetilde{k}_2 + \varepsilon_*), \quad |\lambda - \widetilde{\lambda}| < \varepsilon_*, \quad \operatorname{Im} \lambda > 0, \quad h(k_2, \lambda) \neq 0.$$

Then

$$\big(I + W_1(\lambda) R_0(k_2, \lambda)\big)^{-1} g(x)^{-1/2} f \in L_{2,a}.$$

Denote

$$u = R_0(k_2, \lambda)\big(I + W_1(\lambda) R_0(k_2, \lambda)\big)^{-1} g(x)^{-1/2} f \in H^2_{-a}.$$

Applying Lemma 2.1, we get

$$(H(k_2) - \lambda) R(k_2, \lambda) f = (H(k_2) - \lambda)(g^{-1/2} u) = g^{1/2}\big(-\Delta_{k_2} + W_0(\lambda) + W_1(\lambda)\big) u.$$

For real k_2 and nonreal λ we have

$$\begin{aligned}\big(-\Delta_{k_2} + W_0(\lambda) + W_1(\lambda)\big) u = {} & \big(I + W_1(\lambda) R_0(k_2, \lambda)\big)^{-1} (g(x)^{-1/2} f) \\ & + W_1(\lambda) R_0(k_2, \lambda)\big(I + W_1(\lambda) R_0(k_2, \lambda)\big)^{-1} (g(x)^{-1/2} f) = g^{-1/2} f\end{aligned}$$

by Lemma 6.4. Thus,

$$(H(k_2) - \lambda) R(k_2, \lambda) f = f. \tag{7.4}$$

For $\operatorname{Im} \lambda > 0$ the operators $(-\Delta_{k_2} + W_0(\lambda))^{-1}$ and $(H(k_2) - \lambda)^{-1}$ are well defined and bounded in $L_2(S)$ by Lemma 2.5. This fact together with Lemma 6.4 yields $u \in L_2(S)$, and therefore $R(k_2, \lambda) f \in L_2(S)$. Now, (7.4) implies (7.3). □

§8. Proof of Theorem 1.3

We need the following fact of theory of functions, see [7, Theorem A] and [6, Lemma 3].

Lemma 8.1. *Let F be a real-analytic function in the rectangle $U \times I$, where U and I are intervals of the real axis. Suppose $\Lambda \subset I$,* $\operatorname{meas} \Lambda = 0$. *Assume that for all $\lambda \in \Lambda$ there is $k \in U$ such that $F(k, \lambda) \neq 0$. Then*

$$\operatorname{meas} \{k \in U \,:\, F(k, \lambda) = 0 \ \textit{ for some } \ \lambda \in \Lambda\} = 0.$$

Now, we assume that $\Lambda \subset (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ with ε as defined in (5.1), and that $\operatorname{meas} \Lambda = 0$. We need to prove that $E_H(\Lambda) = 0$. Consider the set

$$K_0 = \{k_2 \in (2\delta, 2\pi - 2\delta) \,:\, F_j(k_2, \lambda) = 0 \ \text{ for some } \ j = 1, \dots, J, \text{ and some } \ \lambda \in \Lambda\}.$$

Lemma 5.1 and Lemma 8.1 show that $\operatorname{meas} K_0 = 0$.

The following fact is also well known. If there is an open covering of a set in a Euclidean space, then one can choose a countable subcovering of this set. Therefore, Lemma 7.2 and Theorem 7.3 imply that the set of $(k_2; \lambda)$ for which the corresponding $F_j(k_2, \lambda) \neq 0$ can be represented as the following countable union:

$$\begin{aligned}&\big\{(k_2; \lambda) \in (2\delta, 2\pi - 2\delta) \times (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \,:\, F_j(k_2, \lambda) \neq 0\big\} \\ &\qquad = \bigcup_{m=1}^{\infty} (k_2^{(m)} - \varepsilon_m, k_2^{(m)} + \varepsilon_m) \times (\lambda_m - \varepsilon_m, \lambda_m + \varepsilon_m).\end{aligned}$$

Here for any $m \in \mathbb{N}$ we can find

- a function h_m defined and analytic in a complex neighborhood of the set

  $$\overline{B_{\varepsilon_m}(k_2^{(m)}) \times B_{\varepsilon_m}(\lambda_m)},$$

  and such that

  $$\text{for every } \ \lambda \in B_{\varepsilon_m}(\lambda_m) \ \text{ there exists } \ k_2 \in B_{\varepsilon_m}(k_2^{(m)}) \text{with } \ h_m(k_2, \lambda) \neq 0;$$

- a $B(L_{2,a}, L_{2,-a})$-valued function R_m defined and analytic in
$$B_{\varepsilon_m}(k_2^{(m)}) \times B_{\varepsilon_m}(\lambda_m) \setminus \{(k_2;\lambda) \,:\, h_m(k_2,\lambda)=0\}$$
and satisfying (7.3).

Put
$$K_1 = \{k_2 \in (2\delta, 2\pi - 2\delta) \,:\, h_m(k_2,\lambda) = 0 \ \text{ for some } \ m \in \mathbb{N}, \text{ and some } \ \lambda \in \Lambda\}\,.$$
We have meas $K_1 = 0$ again by Lemma 8.1.

For $k_2 \notin K_0$ and $m \in \mathbb{N}$ we put
$$\Lambda_m(k_2) = \begin{cases} \varnothing & \text{if } \ k_2 \notin (k_2^{(m)} - \varepsilon_m, k_2^{(m)} + \varepsilon_m), \\ \Lambda \cap B_{\varepsilon_m}(\lambda_m) & \text{if } \ k_2 \in (k_2^{(m)} - \varepsilon_m, k_2^{(m)} + \varepsilon_m). \end{cases}$$
Clearly, we have
$$\Lambda_m(k_2) \subset (\lambda_m - \varepsilon_m, \lambda_m + \varepsilon_m), \qquad \operatorname{meas} \Lambda_m(k_2) = 0, \tag{8.1}$$
and
$$\Lambda = \bigcup_{m=1}^{\infty} \Lambda_m(k_2) \quad \forall\, k_2 \in (2\delta, 2\pi - 2\delta) \setminus K_0. \tag{8.2}$$
We shall use the following lemma, see [13, Chapter I, §4, Proposition 2] and formula (18) after it.

Lemma 8.2. *Let B be a selfadjoint operator in a Hilbert space $\mathcal{H}$, and let $\mathcal{D}$ be a dense subset in $\mathcal{H}$, $\gamma > 0$. Assume that*
$$\sup_{\substack{a \le \lambda \le b, \\ 0 < \nu < \gamma}} \left| \left((B - (\lambda + i\nu))^{-1} f, f \right) \right| < \infty, \quad f \in \mathcal{D}.$$
Then the spectrum of the operator B on the segment $[a,b]$ is absolutely continuous.

Lemma 8.3. *Let $k_2 \in (2\delta, 2\pi - 2\delta) \setminus (K_0 \cup K_1)$, and let $m \in \mathbb{N}$. Then*
$$E_{H(k_2)}(\Lambda_m(k_2)) = 0.$$
Proof. The claim is trivial if $\Lambda_m(k_2) = \varnothing$. Suppose $\Lambda_m(k_2) \neq \varnothing$. Since $k_2 \notin K_1$, we have
$$h_m(k_2,\lambda) \neq 0 \quad \text{for} \quad \lambda \in \Lambda_m(k_2). \tag{8.3}$$
Therefore, $h_m(k_2,\,\cdot\,)$ has at most finitely many zeros on $[\lambda_m - \varepsilon_m, \lambda_m + \varepsilon_m]$. We denote these zeros by $\mu_1, \dots, \mu_n$. Consider a segment $[\alpha,\beta] \subset [\lambda_m - \varepsilon_m, \lambda_m + \varepsilon_m]$ such that $\mu_j \notin [\alpha,\beta]$. There is a $B(L_{2,a}, L_{2,-a})$-valued function $R_m(k_2,\,\cdot\,)$ defined and analytic in a complex neighborhood of the segment $[\alpha,\beta]$ and such that
$$R_m(k_2,\lambda) f = (H(k_2) - \lambda)^{-1} f, \quad \forall\, f \in L_{2,a}, \quad \operatorname{Re}\lambda \in [\alpha,\beta], \ \ 0 < \operatorname{Im}\lambda < \gamma,$$
for some $\gamma > 0$. Therefore,
$$\left| \left((H(k_2) - \lambda)^{-1} f, f \right) \right| \le \| R_m(k_2,\lambda) \|_{L_{2,a} \to L_{2,-a}} \| f \|^2_{L_{2,a}},$$
and
$$\sup_{\substack{\alpha \le \operatorname{Re}\lambda \le \beta, \\ 0 < \operatorname{Im}\lambda < \gamma}} \left| \left((H(k_2) - \lambda)^{-1} f, f \right) \right| < \infty, \quad f \in L_{2,a}.$$
Now we can apply Lemma 8.2 with $\mathcal{H} = L_2(S)$, $\mathcal{D} = L_{2,a}$. Thus, the spectrum of the operator $H(k_2)$ on the segment $[\alpha,\beta]$ is absolutely continuous. We have meas $\Lambda_m(k_2) = 0$ by (8.1) and $\mu_j \notin \Lambda_m(k_2)$ by (8.3). Therefore, $E_{H(k_2)}(\Lambda_m(k_2)) = 0$. □

Now, (8.2) implies

$$E_{H(k_2)}(\Lambda) = 0 \quad \text{for} \quad k_2 \notin K_0 \cup K_1$$

and

$$\int_{2\delta}^{2\pi-2\delta} E_{H(k_2)}(\Lambda)\, dk_2 = \int_{(2\delta,2\pi-2\delta)\setminus(K_0\cup K_1)} E_{H(k_2)}(\Lambda)\, dk_2 = 0.$$

For all $\lambda_0 \in \mathbb{R}$ and $\delta \in (0, \pi/4)$ this identity is valid for all sets $\Lambda \subset (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ with $\operatorname{meas} \Lambda = 0$. Here ε does not depend on Λ. Therefore,

$$\int_{2\delta}^{2\pi-2\delta} E_{H(k_2)}(\Lambda)\, dk_2 = 0 \quad \text{for all} \quad \Lambda \subset \mathbb{R},\ \operatorname{meas} \Lambda = 0.$$

Since δ is an arbitrary number in $(0, \pi/4)$, we get

$$E_H(\Lambda) = \int_{[0,2\pi]} E_{H(k_2)}(\Lambda)\, dk_2 = \int_{(0,2\pi)} E_{H(k_2)}(\Lambda)\, dk_2 = 0 \quad \text{for all} \quad \Lambda \subset \mathbb{R},\ \operatorname{meas} \Lambda = 0.$$

The proof of Theorem 1.3 is complete.

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