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ON THE RIEMANN–SIEGEL FORMULA FOR THE DERIVATIVES OF THE HARDY FUNCTION

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ABSTRACT. Analogs are obtained of the asymptotic Riemann–Siegel formulas for the first and second order derivatives of the Hardy function $Z(t)$ and the Riemann zeta function on the critical line.

§1. INTRODUCTION

We start with some notation necessary in what follows. Let $t > 0$, and let $\vartheta(t)$ be the increment of an arbitrary continuous branch of the function $\arg\left\{\pi^{-s/2}\Gamma(s/2)\right\}$ along the segment with endpoints $s = 0.5$ and $s = 0.5 + it$. Next, let $Z(t) = e^{i\vartheta(t)}\zeta(0.5 + it)$ be the Hardy function.

In the so-called "discrete" theory of the zeta function, the subject of study is sums of the form $\sum_n f(z_n)$, where f is a function related to $\zeta(s)$ or $Z(t)$, and $\{z_n\}$ is an unbounded sequence on the complex plane.

As a rule, the role of $\{z_n\}$ is played by what is called the Gram points t_n. For $n \geq 0$, the value t_n is defined to be a unique solution of the transcendental equation $\vartheta(t_n) = n\pi$ satisfying $\vartheta'(t_n) > 0$.

The first results of the discrete theory of $\zeta(s)$ were Titchmarsh's asymptotics (estimates) for the sums

$$\sum_{n\leq N}\zeta(0.5 + it_n), \quad \sum_{n\leq N} Z(t_n), \quad \sum_{n\leq N} Z(t_n)Z(t_{n+1}). \tag{1}$$

Titchmarsh used these results to give a new proof of Hardy's theorem saying that $\zeta(s)$ has infinitely many zeros on the critical line. Afterwords, the sums (1) and similar objects became of interest on their own.

Since the quantities t_n are smooth functions of the positive real parameter n, in the study of sums of type (1) we may use the classical formula of partial summation

$$\begin{aligned}\sum_{a<n\leq b} F(n) &= \int_a^b F(x)\,dx + \varrho(b)F(b) - \varrho(a)F(a) - \int_a^b \varrho(x)F'(x)\,dx \\ &= \int_a^b F(x)\,dx + \varrho(b)F(b) - \varrho(a)F(a) - \sigma(b)F'(b) + \sigma(a)F'(a) + \int_a^b \sigma(x)F''(x)\,dx,\end{aligned} \tag{2}$$

where $\varrho(x) = 0.5 - \{x\}$, $\sigma(x) = \int_0^x \varrho(u)\,du$, and the role of $F(x)$ is played by a function like $f(t_x)$, say, $\zeta(0.5+it_x)$, $Z(t_x)$, $Z(t_x)Z(t_{x+1})$, an so on. This approach leads to substantial refinement of formulas for the sums (1) compared to the results known earlier, see [2–7].

However, since formula (2) involves the first and second order derivatives of $F(x)$, to solve such problems we need to have fairly precise expressions for the derivatives

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$\zeta'(0.5+it)$, $\zeta''(0.5+it)$ of the Hardy function, similar to the well-known Riemann–Siegel formula (see [8]). This formula looks like this:

$$Z(t) = 2\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\cos\bigl(\vartheta(t)-t\ln n\bigr) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\mathcal{W}_0 + O\left(\left(\frac{25N}{4et}\right)^{\frac{N}{6}} e^{\sqrt{2\pi N}}t^{-1/4}\right), \tag{3}$$

where $m = \bigl[\sqrt{t/(2\pi)}\bigr]$, N is an arbitrary integer such that $1 \le N \le 10^{-8}t$, the constant in O is absolute, and, for any fixed $j \ge 0$, the function $\mathcal{W}_0 = \mathcal{W}_0(N;t)$ admits an expansion of the form

$$H_0(t) + \left(\frac{t}{2\pi}\right)^{-1/2}H_1(t) + \cdots + \left(\frac{t}{2\pi}\right)^{-j/2}H_j(t) + O_j\bigl(t^{-(j+1)/2}\bigr). \tag{4}$$

The coefficients $H_0(t)$, $H_1(t), \ldots$ in this expansion are linear combinations of the values of the derivatives of the function

$$\Phi(x) = \frac{\cos\pi(\frac{1}{2}x^2 - x - \frac{1}{8})}{\cos\pi x}$$

at the point $x = 2\bigl\{\sqrt{t/(2\pi)}\bigr\}$. The exact form of $H_j(t)$ for $j = 0, 1, \ldots, 14$ was presented in Berry's paper [9]. The first analogs of formula (3) of the form

$$Z^{(k)}(t) = 2\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\cos\left(\frac{\pi k}{2} + \vartheta(t) - t\ln n\right) + O\bigl(t^{-1/4}(1.5\ln t)^{k+1}\bigr)$$

were obtained by Karatsuba in [10] for $k \ge 1$ fixed and by Lavrik [11] for k growing together with t.

Our goal in the present paper is to give analogs of (3) for the functions $\zeta'(0.5+it)$, $\zeta''(0.5+it)$, $Z'(t)$, and $Z''(t)$ and to find the first terms of the expansions (4) in an explicit form. The next theorems are our main results.

Theorem 1. *Let ε be an arbitrarily small fixed number, and suppose that*

$$\lambda = e^{-1}+\varepsilon, \quad t \ge t_0(\varepsilon) > 0, \quad m = \bigl[\sqrt{t/(2\pi)}\bigr], \quad \alpha = \bigl\{\sqrt{t/(2\pi)}\bigr\}.$$

Next, let $\delta = \delta(\varepsilon) > 0$ be a sufficiently small positive constant, and let N be an arbitrary integer such that $1 \le N \le \delta t$. Finally, put

$$D = D(N,t;\varepsilon) = \left(\frac{\lambda N}{t}\right)^{\frac{N}{6}} e^{0.5\sqrt{\pi N}}\left(\frac{t}{2\pi}\right)^{-1/4}.$$

Then the following identities are valid:

$$\zeta(0.5+it) = \sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}} + e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}} + (-1)^{m-1}e^{-i\vartheta(t)}\left(\frac{t}{2\pi}\right)^{-1/4}W_0 + \theta_0 D_0,$$

$$\zeta'(0.5+it) = -\sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}}\ln n - e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}}\bigl(2\vartheta'(t)-\ln n\bigr) + (-1)^{m-1}e^{-i\vartheta(t)}\left(\frac{t}{2\pi}\right)^{-1/4}W_1 + \theta_1 D_1,$$

$$\zeta''(0.5+it) = \sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}}(\ln n)^2 + e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}}\bigl((2\vartheta'(t)-\ln n)^2 + 2i\vartheta''(t)\bigr) + (-1)^{m-1}e^{-i\vartheta(t)}\left(\frac{t}{2\pi}\right)^{-1/4}W_2 + \theta_2 D_2,$$

where $D_k = (1.5\ln t)^k CD$, $C = 0.567\,852\,948\ldots$, $|\theta_k| \le 1$, *and for any fixed* j *each of the function* $W_k = W_k(N;t)$ *admits expansion of the form* (4) *the first coefficients of which are given by formulas* (33) (*for* $k=1$) *and* (35), (36) (*for* $k=2$).

Theorem 2. *Under the conditions of Theorem* 1, *we have*

$$Z(t) = 2\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\cos\left(\vartheta(t) - t\ln n\right) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\mathcal{W}_0 + \theta_0\mathcal{D}_0,$$

$$Z'(t) = -2\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\left(\vartheta'(t) - \ln n\right)\sin\left(\vartheta(t) - t\ln n\right) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\mathcal{W}_1 + \theta_1\mathcal{D}_1,$$

$$\begin{aligned} Z''(t) = &-2\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\left(\vartheta'(t) - \ln n\right)^2\cos\left(\vartheta(t) - t\ln n\right) \\ &- 2\vartheta''(t)\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\sin\left(\vartheta(t) - t\ln n\right) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\mathcal{W}_2 + \theta_2\mathcal{D}_2, \end{aligned}$$

where $\mathcal{D}_k = (\ln t)^k CD$, $|\theta_k| \le 1$, *and for any fixed* j *each of the functions* $\mathcal{W}_k = \mathcal{W}_k(N;t)$ *admits expansion of the form* (4) *the first coefficients of which are given by formulas* (38) (*for* $k=1$) *and* (39) (*for* $k=2$).

The next formulas for $Z'(t)$ and $Z''(t)$ seem most adjusted for practical application.

Corollary. *As* $t \to +\infty$, *we have*

$$\begin{aligned} Z'(t) = &-\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\left(\ln\frac{t}{2\pi n^2}\right)\sin\left(\vartheta(t) - t\ln n\right) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-3/4}\frac{\Phi(2\alpha)}{2\pi} \\ &+ (-1)^m\left(\frac{t}{2\pi}\right)^{-5/4}\left(\frac{\Phi(2\alpha)}{8\pi} + \frac{\Phi^{(4)}(2\alpha)}{24\pi^3}\right) + O\left(t^{-7/4}\right), \end{aligned}$$

$$\begin{aligned} Z''(t) = &-\frac{1}{2}\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\left(\ln\frac{t}{2\pi n^2}\right)^2\cos\left(\vartheta(t) - t\ln n\right) \\ &- \frac{1}{t}\sum_{n=1}^{m}\frac{1}{\sqrt{n}}\sin\left(\vartheta(t) - t\ln n\right) + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-5/4}\frac{\Phi^{(2)}(2\alpha)}{4\pi^2} \\ &+ (-1)^m\left(\frac{t}{2\pi}\right)^{-7/4}\left(\frac{\Phi^{(1)}(2\alpha)}{4\pi^2} + \frac{\Phi^{(5)}(2\alpha)}{48\pi^4}\right) + O\left(t^{-9/4}\right). \end{aligned}$$

Notation. Everywhere in the sequel, $s = \sigma + it$ is a complex number, $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function,

$$\chi(s) = \pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)\Gamma^{-1}\left(\frac{s}{2}\right), \quad m = \left[\sqrt{\frac{t}{2\pi}}\right], \quad \alpha = \left\{\sqrt{\frac{t}{2\pi}}\right\},$$

and $\theta, \theta_0, \theta_1, \ldots$ are complex numbers whose module does not exceed 1; they may take different values in different relations.

§2. Auxiliary statements

In this section we collect auxiliary statements of technical nature.

Lemma 1. *Let* $s = \frac{1}{2} + it$, $t > 0$. *Then*

$$\psi(s) = 2\vartheta'(t) + \ln 2\pi + \frac{\pi}{2}\tan\frac{\pi s}{2}, \quad \psi'(s) = -2i\vartheta''(t) + \left(\frac{\pi}{2}\sec\frac{\pi s}{2}\right)^2.$$

Proof. The doubling and complement formulas for the gamma function imply the identity

$$\Gamma\left(\frac{1}{2}+it\right) = \frac{\sqrt{2\pi}\,2^{it}e^{-\pi t/2+\pi i/4}}{1-ie^{-\pi t}}\,\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}. \tag{5}$$

Taking the logarithms and then differentiating, from (5) we deduce the relation

$$\psi\left(\frac{1}{2}+it\right) = \operatorname{Re}\psi\left(\frac{1}{4}+\frac{it}{2}\right) + \ln 2 + \frac{\pi}{2}\,\tan\left(\frac{\pi}{4}+\frac{\pi it}{2}\right). \tag{6}$$

Next, the definitions of $\vartheta(t)$ and $\chi(s)$ show that

$$\chi\left(\frac{1}{2}+it\right) = \pi^{it}\,\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)} = e^{-2i\vartheta(t)}, \tag{7}$$

whence

$$2\vartheta'(t) = \operatorname{Re}\psi\left(\frac{1}{4}+\frac{it}{2}\right) - \ln\pi. \tag{8}$$

Comparing (6) and (8), we arrive at the first claim of the lemma. The second is obtained then by differentiation. □

Lemma 2. *Let*

$$\Delta(t) = \vartheta(t) - \frac{t}{2}\ln\frac{t}{2\pi e} + \frac{\pi}{8}.$$

Then the following expansions are valid as $t\to+\infty$:

$$\Delta(t) \sim \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}(2^{2n-1}-1)}{2^{2n}\cdot 2n(2n-1)}\,\frac{B_{2n}}{t^{2n-1}}, \tag{9}$$

$$\Delta'(t) \sim \sum_{n=1}^{+\infty}\frac{(-1)^{n}(2^{2n-1}-1)}{2^{2n}(2n-1)}\,\frac{B_{2n}}{t^{2n}}, \tag{10}$$

$$\Delta''(t) \sim \sum_{n=1}^{+\infty}(-1)^{n-1}(2^{2n-1}-1)\,\frac{B_{2n}}{2^{2n}t^{2n-1}}, \tag{11}$$

where the B_ν *are the Bernoulli numbers,* $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $B_3=B_5=B_7=\cdots=0$.

Proof. Relations (9)–(11) follow from Lemma 1 and the classical expansions for $\ln\Gamma(s)$ and $\psi(s)$ (see, e.g., [12, §1.4, 1.6]). □

Corollary. *For sufficiently large* t *we have*

$$\left|\psi\left(\frac{1}{2}+it\right)\right| < \ln t + \frac{\pi}{2} + \frac{1}{t^2}, \qquad \left|\psi'\left(\frac{1}{2}+it\right)\right| < \frac{1}{t} + \frac{1}{t^3}.$$

Lemma 3. *For the function* $h(s)=h(\sigma+it)$ *defined by the formula*

$$h(s) = (-1)^{m-1}ie^{-\pi i/8}e^{-\pi is/2}\Gamma(1-s)(2\pi t)^{(\sigma-1)/2}\left(\frac{2\pi t}{e}\right)^{it/2},$$

we have

$$h\left(\frac{1}{2}+it\right) = (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\frac{e^{-i\vartheta(t)-i\Delta(t)}}{1-ie^{-\pi t}}.$$

Proof. The desired identity follows from (5) and the relations

$$\left(\frac{t}{2\pi e}\right)^{it/2} = \exp\left(\frac{it}{2}\ln\frac{t}{2\pi e}\right) = e^{i(\vartheta(t)+\Delta(t)+\pi/8)}. \qquad \square$$

Corollary. *For sufficiently large $t > 0$ we have* $\left|h\left(\frac{1}{2}+it\right)\right| < (t/(2\pi))^{-1/4}$.

Let $\mathcal{C}$ be a contour in the complex plane that goes from infinity along the positive part of the real axis, then bypasses the coordinate origin counterclockwise, and then returns to infinity so that the points $2\pi r$, where the r are integers with $|r| \le m$, are kept inside the contour, while the points $2\pi r$ with $|r| > m$ are left outside of it.

We put

$$J_k(s) = \int_{\mathcal{C}} \frac{w^{s-1}e^{-mw}(\ln w)^k}{e^w - 1}\,dw, \quad I_k(s) = \frac{J_k(s)}{\Gamma(s)(e^{2\pi i s}-1)}, \quad k = 0,1,2,$$

where $w^{s-1} = e^{(s-1)\ln w}$, and the branch of $\ln w$ is chosen so that $0 < \operatorname{Im}\ln w < 2\pi$ for all $w \in \mathcal{C}$.

Lemma 4. *For any $s \ne 1$ we have*

$$(12)\quad \zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s} + \chi(s)\sum_{n=1}^{m}\frac{1}{n^{1-s}} + Q_0(s),$$

$$(13)\quad \zeta'(s) = -\sum_{n=1}^{m}\frac{\ln n}{n^s} + \chi(s)\sum_{n=1}^{m}\frac{1}{n^{1-s}}\left(\ln 2\pi n - \psi(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}\right) + Q_1(s) + R_1(s),$$

$$(14)\quad \zeta''(s) = \sum_{n=1}^{m}\frac{(\ln n)^2}{n^s} + \chi(s)\sum_{n=1}^{m}\frac{1}{n^{1-s}}\left(\left(\ln 2\pi n - \psi(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}\right)^2 - \psi'(s) + \left(\frac{\pi}{2}\sec\frac{\pi s}{2}\right)^2\right) + Q_2(s) + R_2(s),$$

where $Q_0(s) = I_0(s)$, $Q_1(s) = I_1(s) - \psi(s)I_0(s)$,

$$Q_2(s) = I_2(s) - 2\psi(s)I_1(s) + (\psi^2(s) - \psi'(s))I_0(s),$$

$$R_1(s) = \frac{2\pi i Q_0(s)}{e^{-2\pi i s}-1}, \quad R_2(s) = \frac{4\pi i Q_1(s)}{e^{-2\pi i s}-1} + \frac{4\pi^2 Q_0(s)}{e^{-2\pi i s}-1}\cdot\frac{1+e^{2\pi i s}}{1-e^{2\pi i s}}.$$

Proof. The derivation of formula (12) is presented in [13, §7, Chapter IV]. Differentiating (12) and observing that

$$J_k'(s) = J_{k+1}(s), \quad \chi'(s) = \chi(s)\left(\ln 2\pi - \psi(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}\right),$$

we get

$$\zeta'(s) = -\sum_{n=1}^{m}\frac{\ln n}{n^s} + \chi(s)\sum_{n=1}^{m}\frac{\ln n}{n^{1-s}} + \chi(s)\left(\ln 2\pi - \psi(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}\right)\sum_{n=1}^{m}\frac{1}{n^{1-s}} + \frac{J_1(s) - \psi(s)J_0(s)}{\Gamma(s)(e^{2\pi i s}-1)} - \frac{2\pi i J_0(s)e^{2\pi i s}}{\Gamma(s)(e^{2\pi i s}-1)^2},$$

which implies (13). Similarly, we differentiate (13) to obtain the identity

$$\zeta''(s) = \sum_{n=1}^{m}\frac{(\ln n)^2}{n^s} + \chi(s)\sum_{n=1}^{m}\frac{1}{n^{1-s}}\left(\left(\ln 2\pi n - \psi(s) + \frac{\pi}{2}\tan\frac{\pi s}{2}\right)^2 - \psi'(s) + \left(\frac{\pi}{2}\sec\frac{\pi s}{2}\right)^2\right) + \frac{d}{ds}\left(\frac{J_1(s) - \psi(s)J_0(s)}{\Gamma(s)(e^{2\pi i s}-1)}\right) - \frac{d}{ds}\left(\frac{2\pi i J_0(s)e^{2\pi i s}}{\Gamma(s)(e^{2\pi i s}-1)^2}\right).$$

The contribution of the last two sumands is equal to

$$\frac{J_2(s) - 2\psi(s)J_1(s) + (\psi^2(s) - \psi'(s))J_0(s)}{\Gamma(s)(e^{2\pi is} - 1)} - \frac{4\pi ie^{2\pi is}}{\Gamma(s)(e^{2\pi is} - 1)^2}\big(J_1(s) - \psi(s)J_0(s) + \pi iJ_0(s)\big) + \frac{8\pi^2e^{2\pi is}J_0(s)}{\Gamma(s)(e^{2\pi is} - 1)^3}$$

and coincides with

$$Q_2(s) - \frac{4\pi ie^{2\pi is}Q_1(s)}{e^{2\pi is} - 1} + \frac{4\pi^2e^{2\pi is}Q_0(s)}{e^{2\pi is} - 1} + \frac{8\pi^2e^{2\pi is}Q_0(s)}{(e^{2\pi is} - 1)^2},$$

which yields (14). The lemma is proved. □

§3. The main lemma

In this section we deduce formulas for the integrals $I_k(s)$; these formulas will be used in the proof of Theorems 1 and 2. The expansion of $I_0(s)$ was obtained, in essence, by Riemann; detailed calculations were published by Siegel in [8]. The formula for $I_0(s)$ is presented here because it results in a sharper error term in the formula for $Z(t)$, compared with Siegel's (compare (3) with the expression for $Z(t)$ in Theorem 2).

Our way to deduce formulas for $I_k(s)$ is similar to that suggested by Siegel (see also [13, §7, 8, Chapter IV]). A minor modification of arguments allows us to avoid analyzing several cases separately and treat uniformly the situations where the number $\alpha = \{\sqrt{t/(2\pi)}\}$ is close to 0 or to 1, or, on the contrary, is far from those points.

Lemma 5. *Let $0 < \varepsilon < 0.5$ be an arbitrary fixed number, and let $\lambda = e^{-1} + \varepsilon$. Then there exist constants $t_0 = t_0(\varepsilon)$ and $\delta = \delta(\varepsilon) > 0$ such that for every $s = \sigma + it$, $0 \le \sigma \le 1$, $t \ge t_0$, and every integer N with $1 \le N \le \delta t$, we have*

$$I_k(s) = h(s)S_k(N;\sigma,t) \,+\, R_k(N;\sigma,t), \quad k = 0, 1, 2,$$

where

$$S_k(N;\sigma,t) = \sum_{n=0}^{N-1} F_n \sum_{0\le\nu\le n/2} \frac{n!\,i^{\nu-n}}{\nu!(n-2\nu)!\,2^n}\left(\frac{2}{\pi}\right)^{\frac{n}{2}-\nu}\Phi^{(n-2\nu)}(2\alpha),$$

the F_n are some coefficients depending on n, k, σ, and t, and

$$|R_k(N;\sigma,t)| \le 2^{-k}C\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}}\left(\frac{t}{2\pi}\right)^{-\frac{\sigma}{2}} e^{0.5\sqrt{\pi N}}(\ln t)^k.$$

Proof. We put $\eta = \sqrt{2\pi t}$, so that $\eta = 2\pi(m+\alpha)$, and pick a number $c = c(\varepsilon)$, $0 < c < 0.5$, the value of which will be chosen later. The role of $\mathcal{C}$ will be played by the contour that consists of the ray $\mathcal{C}_1$ going from $+\infty$ to the point $i\eta + (1+i)c\eta$ in parallel with the real axis, the segment $\mathcal{C}_2$ connecting the points $i\eta + (1+i)c\eta$ and $i\eta - (1+i)c\eta$, the vertical segments $\mathcal{C}_3$ with the endpoints $i\eta - (1+i)c\eta$ and $-\eta c - 2\pi i(m+0.5)$, and the ray $\mathcal{C}_4$ going from the point $-\eta c - 2\pi i(m+0.5)$ to $+\infty$ in parallel with the real axis. In case the number $\sqrt{t/(2\pi)}$ is an integer, we modify $\mathcal{C}_2$, bypassing the poin $i\eta = 2\pi im$ along a semicircle of infinitely small radius. Accordingly, the integral $J_k(s)$ becomes the sum of the corresponding integrals $j_{k,r}(s) = j_r(s)$, $r = 1, \dots, 4$.

Putting $\rho = |w|$, $\varphi = \arg w$, we see that, everywhere on $\mathcal{C}_1$,

$$0 < \varphi \le \frac{\pi}{2} - \arctan\frac{c}{1+c},$$
$$|\ln w| \le \ln\rho + \frac{\pi}{2} = \frac{1}{2}\ln\big(u^2 + (1+c)^2\eta^2\big) + \frac{\pi}{2} < \ln u + 3,$$

where $u = \operatorname{Re} w$. Repeating the arguments in [13, §7, Chapter IV] word for word, we find

$$\left|\frac{w^{s-1}e^{-mw}}{e^w-1}(\ln w)^k\right| < \frac{(\ln u+3)^k}{e^u-1}(2\pi t)^{0.5(\sigma-1)}\, e^{-(\pi/2+\delta_1)t},$$
$$\delta_1 = c - \arctan\frac{c}{1+c} > 0.$$

Consequently,

$$|j_1(s)| \le (2\pi t)^{0.5(\sigma-1)}\, e^{-(\pi/2+\delta_1)t}\int_{c\eta}^{+\infty}\frac{(\ln u+3)^k\,du}{e^u-1} < (2\pi t)^{0.5(\sigma-1)}\, e^{-(\pi/2+\delta_1)t}.$$

Similarly, observing that $|\ln w| < 0.5(\ln t + 3\pi)$ on $\mathcal{C}_2$ and using the estimate

$$\left|\frac{w^{s-1}e^{-mw}}{e^w-1}\right| < \frac{(2\pi t)^{0.5(\sigma-1)}}{e^u-1}\, e^{-(\pi/2+\delta_2)t}, \quad \delta_2 = \arctan\frac{c}{1-c} - c > 0,$$

established in [13, §7, Chapater IV], we obtain

$$\begin{aligned}|j_3(s)| &< \big((1-c)\eta + 2\pi(m+0.5)\big)(2\pi t)^{0.5(\sigma-1)}\,\frac{(\ln t+3\pi)^k}{1-e^{-c\eta}}\, e^{-(\pi/2+\delta_2)t}\\ &< (2\pi t)^{\sigma/2}(\ln t+3\pi)^k e^{-(\pi/2+\delta_2)t}.\end{aligned}$$

Finally, we use the fact that, for $w = \rho e^{i\varphi} = u - 2\pi i(m+0.5)$ running over the ray $\mathcal{C}_4$, we have

$$\begin{aligned}&\rho \ge 2\pi(m+0.5) \ge \eta - \pi,\\ &|\ln w| \le 0.5\ln\big(u^2 + 4\pi^2(m+0.5)^2\big) + 2\pi < 0.5\ln(u^2+6.3t) + 2\pi.\end{aligned}$$

This yields

$$|j_4(s)| < 0.5(\eta-\pi)^{0.5(\sigma-1)}e^{-5\pi t/4}\int_{-c\eta}^{+\infty}\frac{e^{-mu}}{e^u+1}\big(0.5\ln(u^2+6.3t)+2\pi\big)^k\,du.$$

The contributions to the last integral of the intervals $-c\eta \le u \le 0$, $0 \le u \le c\eta$, and $u \ge c\eta$ do not exceed, respectively, the quantities

$$(\ln t+9)^k\int_0^{c\eta} e^{mu}\,du = (\ln t+9)^k\,\frac{e^{c\eta}}{m},$$
$$(\ln t+9)^k\int_0^{c\eta} e^{-(m+1)u}\,du < \frac{(\ln t+9)^k}{m+1},$$
$$\int_{c\eta}^{+\infty} e^{-(m+1)u}(2\ln u+9)^k\,du < (2\ln c\eta+9)^k e^{-c\eta}\int_{c\eta}^{+\infty} e^{-mu}\,du < \frac{(\ln t+9)^k}{m+1}\, e^{-(m+1)c\eta}.$$

Therefore,

$$(15)\qquad \begin{aligned}&|j_4| < (2\pi t)^{0.5(\sigma-1)}(\ln t+9)^k e^{-3\pi t/2},\\ &\sum_{r=1,3,4}|j_r| < 2(2\pi t)^{0.5(\sigma-1)}(\ln t)^k e^{-(\pi/2+\delta_0)t},\end{aligned}$$

where $\delta_0 > 0$ is the smallest of the numbers δ_1 and δ_2.

Putting $z = (w - i\eta)/(i\sqrt{2\pi})$, on $\mathcal{C}_2$ we get (see [13, §8, Chapter IV])

$$w^{s-1} = (i\eta)^{s-1}\exp\left(iz\sqrt{t} - \frac{iz^2}{2}\right)A(z),$$

where

$$A(z) = \exp\left((s-1)\ln\left(1+\frac{z}{\sqrt{t}}\right) - iz\sqrt{t} + \frac{iz^2}{2}\right).$$

Next, we put $B(z) = A(z)\ln w$, $C(z) = B(z)\ln w$ and denote by A_n the Taylor coefficients of $A(z)$. Let $A_N(z) = \sum_{n=N}^{+\infty} A_n z^n$, and let B_n, $B_N(z)$, C_n, and $C_N(z)$ be defined similarly. Then

$$\begin{gathered}
A_0 = 1, \quad A_1 = \frac{\sigma-1}{\sqrt{t}}, \quad A_2 = \frac{(\sigma-1)(\sigma-2)}{2t}, \\
A_{n+1} = \frac{(\sigma-1-n)A_n + iA_{n-2}}{(n+1)\sqrt{t}} \quad \text{for} \quad n \ge 2, \\
B_n = \sum_{k=0}^{n} \ell_k A_{n-k}, \quad C_n = \sum_{k=0}^{n} \ell_k B_{n-k} \quad \text{for} \quad n \ge 0,
\end{gathered} \tag{16}$$

where $\ell_0 = \ln i\eta = \ln\sqrt{2\pi t} + \pi i/2$, $\ell_k = (-1)^{k-1}/k$, $k \ge 1$.

For what follows, we need estimates of the quantities $|A_N(z)|$, $|B_N(z)|$, $|C_N(z)|$ for all N satisfying $1 \le N \le \delta t$, δ being a sufficiently small constant. Let $0 < \beta < 1$ be fixed, and let v be an arbitrary number with $|v| \le \beta\sqrt{t}$. Then

$$\begin{aligned}
\ln A(v) &= (s-1)\ln\left(1 + \frac{v}{\sqrt{t}}\right) - iv\sqrt{t} + \frac{iv^2}{2} \\
&= (\sigma-1)\left(\frac{v}{\sqrt{t}} - \frac{v^2}{2t}\right) + (s-1)\sum_{k=3}^{+\infty} \frac{(-1)^{k-1}}{k}\left(\frac{v}{\sqrt{t}}\right)^k,
\end{aligned}$$

whence

$$\begin{aligned}
\ln|A(v)| = \operatorname{Re}\ln A(v) &\le (1-\sigma)\left(\frac{|v|}{\sqrt{t}} + \frac{|v|^2}{2t}\right) + |s-1|\sum_{k=3}^{+\infty}\frac{1}{k}\left(\frac{|v|}{\sqrt{t}}\right)^k \\
&\le (1-\sigma)\big(\beta + 0.5\beta^2\big) + \sqrt{t^2 + (1-\sigma)^2}\left(\frac{|v|}{\sqrt{t}}\right)^3 \sum_{k=0}^{+\infty}\frac{\beta^k}{k+3}.
\end{aligned}$$

Denoting the last sum by $\gamma = \gamma(\beta)$, so that

$$\gamma = \frac{1}{\beta^3}\left(\ln\frac{1}{1-\beta} - \beta - \frac{\beta^2}{2}\right), \tag{17}$$

we get

$$\begin{aligned}
\ln|A(v)| &\le (1-\sigma)\big(\beta + 0.5\beta^2\big) + \frac{\gamma|v|^3}{\sqrt{t}}\left(1 + \frac{(1-\sigma)^2}{t^2}\right) \\
&\le (1-\sigma)\big(\beta + 0.5\beta^2\big) + \frac{\gamma|v|^3}{\sqrt{t}} + (1-\sigma)^2\frac{\gamma\beta^3}{t}.
\end{aligned}$$

Next, let $|v| = R \le \beta\sqrt{t}$, and let $|z| \le \kappa R$, where κ is a constant with $0 < \kappa < 1$. Using estimate (5) and

$$A_N(z) = \frac{z^N}{2\pi i}\int_{|v|=R}\frac{A(v)\,dv}{v^N(v-z)},$$

we get

$$|A_N(z)| \le \frac{|z|^N}{R^N}\exp\left((1-\sigma)\big(\beta + 0.5\beta^2\big) + (1-\sigma)^2\frac{\gamma\beta^3}{t} + \frac{\gamma R^3}{\sqrt{t}}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R\,d\varphi}{|Re^{i\varphi} - z|}.$$

Putting $\varkappa = |z|/R$, we easily check that the last integral coincides with

$$\frac{4}{1-\varkappa}\mathbf{K}\left(-\frac{4\varkappa}{(1-\varkappa)^2}\right), \quad \text{where} \quad \mathbf{K}(v) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - v\sin^2\varphi}}$$

is the elliptic integral of first kind. Since the function

$$\frac{4}{1-\varkappa}\,\mathbf{K}\biggl(-\frac{4\varkappa}{(1-\varkappa)^2}\biggr)$$

is monotone increasing on the segment $0 \le \varkappa \le \kappa \le 1$, for z under consideration we have

$$\begin{aligned} |A_N(z)| &\le c_0(1+o(1))\frac{|z|^N}{R^N}\,\exp\left(\frac{\gamma R^3}{\sqrt{t}}\right), \\ c_0 &= \frac{2}{\pi}\,\frac{1}{1-\kappa}\,\mathbf{K}\biggl(-\frac{4\kappa}{(1-\kappa)^2}\biggr)\exp\bigl((1-\sigma)(\beta+0.5\beta^2)\bigr). \end{aligned} \tag{18}$$

The minimum over R of the expression $R^{-N}\exp\bigl(\gamma R^3/\sqrt{t}\bigr)$, equal to

$$\left(\frac{e}{R^3}\right)^{\frac{N}{3}} = \left(\frac{3e\gamma}{N\sqrt{t}}\right)^{\frac{N}{3}},$$

is attained at $R = R_0 = (N\sqrt{t}/(3\gamma))^{1/3}$. Consequently, for $|z| \le Y = \kappa R_0$, from (18) we obtain the inequality

$$|A_N(z)| \le c_0(1+o(1))|z|^N\left(\frac{3e\gamma}{N\sqrt{t}}\right)^{\frac{N}{3}}. \tag{19}$$

In the case where $|z| > Y$, for sufficiently large t and $R = |z|/\kappa$, relation (18) implies

$$|A_N(z)| \le c_0\kappa^N(1+o(1))\exp\left(\frac{\gamma|z|}{\kappa^3\sqrt{t}}\,|z|^2\right).$$

For all w lying on the curve $\mathcal{C}_2$ and the corresponding z we have $|z|/\sqrt{t} \le c\sqrt{2}$; therefore,

$$|A_N(z)| \le c_0\exp\left(\frac{c\gamma\sqrt{2}}{\kappa^3}\,|z|^2\right).$$

Assume that c is so small that $c\gamma\sqrt{2}/\kappa^3 < 0.5$. Denoting $\omega = 0.5(0.5 - c\gamma\sqrt{2}/\kappa^3)$, for $|z| > Y$ we get

$$\begin{aligned} |A_N(z)| &\le c_0\exp\bigl((0.5-2\omega)|z|^2\bigr) \\ &= c_0\exp\bigl(-\omega Y^2\bigr)\exp\bigl((0.5-\omega)|z|^2\bigr) < 0.01\exp\bigl((0.5-\omega)|z|^2\bigr). \end{aligned} \tag{20}$$

Observe that

$$\left|\ln\left(1+\frac{v}{\sqrt{t}}\right)\right| \le -\ln\left(1-\frac{|v|}{\sqrt{t}}\right) \le -\ln(1-\beta)$$

whenever $|v| \le \beta\sqrt{t}$. It follows that if w and v satisfy $v = (w-i\eta)/(i\sqrt{2\pi})$, then

$$|\ln w| = \left|\ln i\eta + \ln\left(1+\frac{v}{\sqrt{t}}\right)\right| \le \frac{1}{2}\,\ln(2\pi t) + \frac{\pi}{2} + \ln\frac{1}{1-\beta}.$$

Consequently,

$$|B(v)| < \frac{1}{2}\,(1+o(1))(\ln t)|A(v)|, \quad |C(v)| < \frac{1}{4}\,(1+o(1))(\ln t)^2|A(v)|.$$

Let in what follows $F(z)$ denote any of the functions A, B, C, and let

$$F(z) = \sum_{n=0}^{+\infty} F_n z^n = \sum_{n=0}^{N} F_n z^n \;+\; F_N(z). \tag{21}$$

In the case where $F = B, C$ we can estimate $F_N(z)$ like it was done above; combining the resulting inequalities with (19) and (20), we get

$$(22)\qquad |F_N(z)| \le \begin{cases} 2^{-k}c_0\big(1+o(1)\big)(\ln t)^k|z|^N\big(\frac{3e\gamma}{N\sqrt{t}}\big)^{\frac{N}{3}} & \text{if } \ |z|\le Y,\\ 0.01\exp\big((0.5-\omega)|z|^2\big) & \text{if } \ Y<|z|\le c\sqrt{2t}. \end{cases}$$

Next, we represent the integral $j_{2,r}(s) = j_2(s)$ in the form

$$(i\eta)^{s-1}\int_{\mathcal{C}_2}\frac{F(z)e^{G(w)}}{e^w-1}\,dw,$$

where

$$G(w) = iz\sqrt{t} - \frac{iz^2}{2} - mw = (w-i\eta)\sqrt{\frac{t}{2\pi}} + \frac{i}{4\pi}\,(w-i\eta)^2 - mw.$$

Let $f(s)$ and $g(s)$ denote the contributions to the last integral of the first and second terms on the right in (21), and let $w = i\eta + (1+i)\xi = \xi + i(\xi+\eta)$, where $-c\eta \le \xi \le c\eta$. Then

$$\mathrm{Re}\ G(w) = -\frac{\xi^2}{2\pi} + \xi\sqrt{\frac{t}{2\pi}} - \xi m = -\frac{\xi^2}{2\pi} + \alpha\xi,$$

so that

$$(23)\qquad |g(s)| \le \sqrt{2}\int_{-c\eta}^{c\eta}\exp\left(-\frac{\xi^2}{2\pi} + \alpha\xi\right)d\xi.$$

Since for w and ξ under consideration we have

$$e^w = e^{i\eta+(1+i)\xi} = e^{2\pi i(m+\alpha)+(1+i)\xi} = e^{\xi+i\varphi},$$

where we have put $\varphi = \xi + 2\pi\alpha$, it follows that

$$|e^w-1| = \big(e^{\xi} - 2\cos\varphi + e^{-\xi}\big)^{1/2} = 2e^{\xi/2}\left(\sinh^2\frac{\xi}{2} + \sin^2\frac{\varphi}{2}\right)^{1/2} \ge 2e^{\xi/2}\left|\sinh\frac{\xi}{2}\right|,$$

whence

$$\frac{|\xi|}{|e^w-1|} \le \frac{|\xi|e^{-\xi/2}}{|\sinh(\xi/2)|} \le e^{-\xi/2}$$

on the entire integration interval in (23). Therefore, for w, ξ in question and for $z = (w-i\eta)/(i\sqrt{2\pi}) = (1-i)\xi/\sqrt{2\pi}$ we have

$$\left|\frac{F(z)}{e^w-1}\right| = \left|\frac{F(z)}{\xi}\,\frac{\xi}{e^w-1}\right| \le \frac{|F(z)|}{|\xi|}\,e^{-\xi/2} = \frac{|F(z)|}{|z|}\,\frac{e^{-\xi/2}}{\sqrt{\pi}}.$$

Now, let $g_1(s)$ and $g_2(s)$ denote the contributions to $g(s)$ made by the intervals $|\xi| \le Y_1 = Y\sqrt{\pi}$ and $Y_1 \le |\xi| \le c\eta$, respectively. For $|\xi| \le Y_1$ we use the first estimate in (22), obtaining

$$|g_1(s)| \le 2^{0.5-k}c_0\big(1+o(1)\big)(\ln t)^k\left(\frac{3e\gamma}{N\sqrt{t}}\right)^{\frac{N}{3}}\int_{-Y_1}^{Y_1}\exp\left(-\frac{\xi^2}{2\pi} + (\alpha-0.5)\xi\right)d\frac{\xi}{\sqrt{\pi}}.$$

Obviously, the last integral does not exceed the quantity

$$2\int_0^{+\infty} e^{-v^2/2}\cosh\big((\alpha-0.5)v\sqrt{\pi}\big)v^{N-1}\,dv \le 2D_N,$$

$$D_N = \int_0^{+\infty} e^{-v^2/2}\cosh\big(0.5v\sqrt{\pi}\big)v^{N-1}\,dv.$$

Direct calculations show that, for $1 \le N < 10$, the ratio between D_N and the quantity $E_N = (N/e)^{N/2} e^{0.5\sqrt{\pi N}}$ attains its largest value, equal to

$$e_0 = \sqrt{\frac{\pi}{2}} \exp\left(\frac{1}{2}\left(1 - \frac{\sqrt{\pi}}{2}\right)^2\right) = 1.26145\ldots,$$

at $N = 1$, so that $D_N \le e_0 E_N$ for $1 \le N < 10$. For $N \ge 10$, we represent D_N by the sum

$$\left(\int_0^{\sqrt{N}} + \int_{\sqrt{N}}^{+\infty}\right) e^{-v^2/2} \cosh\big(0.5v\sqrt{\pi}\,\big) v^{N-1}\, dv = D'_N + D''_N.$$

Then

$$D'_N \le \cosh\big(\sqrt{\pi N}/2\big) \int_0^{+\infty} e^{-v^2/2} v^{N-1}\, dv = 2^{N/2-1}\,\Gamma\big(N/2\big) \cosh\big(\sqrt{\pi N}/2\big).$$

Next, we have

$$\begin{aligned} D''_N &\le \frac{1}{2} \int_{\sqrt{N}}^{+\infty} e^{-v^2/2} \big(1 + e^{0.5v\sqrt{\pi}}\big) v^{N-1}\, dv \\ &\le 2^{N/2-2}\,\Gamma\big(N/2\big) \cosh\big(\sqrt{\pi N}/2\big) + \frac{1}{2} \int_{\sqrt{N}}^{+\infty} e^{-v^2/2 + v\sqrt{\pi}/2} v^{N-1}\, dv. \end{aligned}$$

Putting $v = u + \sqrt{N}$ in the last integral, we see that it is not greater than

$$\begin{aligned} &\int_0^{+\infty} \exp\left(-\frac{u^2}{2} - u\sqrt{N} - \frac{N}{2} + \frac{u\sqrt{\pi}}{2} + \frac{\sqrt{\pi N}}{2}\right) \frac{(u + \sqrt{N})^N}{u + \sqrt{N}}\, du \\ &\quad \le \frac{1}{\sqrt{N}} \left(\frac{N}{e}\right)^{N/2} \int_0^{+\infty} \exp\left(-\frac{u^2}{2} + \frac{u\sqrt{\pi}}{2} - u\sqrt{N}\right) \left(1 + \frac{u}{\sqrt{N}}\right)^N du \\ &\quad \le \frac{1}{\sqrt{N}} \left(\frac{N}{e}\right)^{N/2} \int_0^{+\infty} \exp\left(-\frac{u^2}{2} + \frac{u\sqrt{\pi}}{2} - u\sqrt{N} + u\sqrt{N}\right) du \\ &\quad < \frac{3.02}{\sqrt{N}} \left(\frac{N}{e}\right)^{N/2} < \left(\frac{N}{e}\right)^{N/2}. \end{aligned}$$

Consequently,

$$\begin{aligned} D''_N &\le \left(\frac{N}{e}\right)^{N/2} \left(\frac{1}{2} + \frac{1}{4}\left(\frac{2e}{N}\right)^{\frac{N}{2}} \Gamma\big(N/2\big)\right) < \frac{3}{5}\left(\frac{N}{e}\right)^{N/2}, \\ D_N &\le 2^{N/2-1}\,\Gamma\big(N/2\big) \cosh\big(\sqrt{\pi N}/2\big) + \frac{3}{5}\left(\frac{N}{e}\right)^{N/2} < \frac{2}{5} E_N. \end{aligned}$$

Thus, the estimate $D_N \le e_0 E_N$ is valid for all $N \ge 1$. Therefore,

$$\begin{aligned} |g_1(s)| &\le 2^{1.5-k} c_0 e_0 \big(1 + o(1)\big) (\ln t)^k \left(\frac{3e\gamma}{N\sqrt{t}}\right)^{\frac{N}{3}} \cdot \left(\frac{N}{e}\right)^{\frac{N}{2}} e^{0.5\sqrt{\pi N}} \\ &= 2^{1.5-k} c_0 e_0 \big(1 + o(1)\big) (\ln t)^k \left(\frac{\lambda N}{t}\right)^{\frac{N}{6}} e^{0.5\sqrt{\pi N}}, \quad \lambda = \frac{(3\gamma)^2}{e}. \end{aligned}$$

Turning to estimation of $|g_2(s)|$, we note that

$$|e^w - 1| \ge \begin{cases} 1 - e^{-|\xi|} & \text{if } \xi \le -Y_1, \\ e^{\xi} - 1 & \text{if } \xi \ge Y_1. \end{cases}$$

The second estimate in (22) yields

$$\begin{aligned}|g_2(s)| &\le 0.01\sqrt{2}\int_{Y_1\le|\xi|\le c\eta}\exp\left(-\frac{\xi^2}{2\pi}+\alpha\xi+\left(\frac{1}{2}-\omega\right)\frac{\xi^2}{\pi}\right)\frac{d\xi}{|e^w-1|}\\ &\le 0.01\sqrt{2}\left(\int_{Y_1}^{+\infty}\exp\left(-\frac{\omega\xi^2}{\pi}+\alpha\xi\right)\frac{d\xi}{e^{\xi}-1}+\int_{Y_1}^{+\infty}\exp\left(-\frac{\omega\xi^2}{\pi}-\alpha\xi\right)\frac{d\xi}{1-e^{-\xi}}\right)\\ &< 0.03\int_{Y_1}^{+\infty}\exp\left(-\frac{\omega\xi^2}{\pi}\right)d\xi<\frac{\pi}{2\omega Y_1}\exp\left(-\omega Y^2\right)\\ &=\frac{\sqrt{\pi}(3\gamma)^{1/3}}{2\kappa\omega}\,t^{-1/6}\exp\left(-\omega\kappa^2\left(\frac{N\sqrt{t}}{3\gamma}\right)^{2/3}\right).\end{aligned}$$

It is not hard to check that the last factor does not exceed $(\lambda N/t)^{N/6}$ for $1\le N\le\delta t$ it δ is sufficiently small. Indeed, the inequality

$$\exp\left(-\omega\kappa^2\left(\frac{N\sqrt{t}}{3\gamma}\right)^{2/3}\right)\le\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}} \tag{24}$$

is equivalent to the inequality $\ln y\le\tau y$ in which $y=(t/(\lambda N))^{1/3}$, $\tau=2\omega\kappa^2e^{-1/3}$. Put y_0 to be the largest root of the equation $\ln y=\tau y$ if $\tau\le e^{-1}$ and $y=1$ otherwise. Then for (24) to be true it suffices to require that $y\ge y_0$, or in other words, $N\le(\lambda y_0^3)^{-1}t$. Assuming in what follows that $\delta\le(\lambda y_0^3)^{-1}$, we get

$$|g_1(s)|\le t^{-1/7}\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}},\quad |g(s)|<2^{1.5-k}c_0e_0\big(1+o(1)\big)(\ln t)^k\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}}e^{0.5\sqrt{\pi N}}.$$

We pass to the calculation of $f(s)$. First, we obtain estimates for the coefficients A_n, B_n, and C_n for $1\le n\le N$. Putting $z=(2/(3e\gamma))^{1/3}$ in the identity $|A_n|=|A_n(z)-A_{n+1}(z)|\cdot|z|^{-n}$ ans using the first estimate in (22), we see that

$$\begin{aligned}|A_n| &\le c_0\big(1+o(1)\big)\left(\left(\frac{3e\gamma}{n\sqrt{t}}\right)^{\frac{n}{3}}+\left(\frac{3e\gamma}{(n+1)\sqrt{t}}\right)^{\frac{n+1}{3}}\right)\\ &= c_0\big(1+o(1)\big)\left(\frac{3e\gamma}{n\sqrt{t}}\right)^{\frac{n}{3}}\left(1+|z|\left(1+\frac{1}{n}\right)^{-\frac{n}{3}}\left(\frac{3e\gamma}{(n+1)\sqrt{t}}\right)^{1/3}\right)\\ &\le c_0\big(1+o(1)\big)\left(\frac{3e\gamma}{n\sqrt{t}}\right)^{\frac{n}{3}}\left(1+t^{-1/6}\right)=c_0\big(1+o(1)\big)\left(\frac{3e\gamma}{n\sqrt{t}}\right)^{\frac{n}{3}}.\end{aligned}$$

The quantities $|B_n|$ and $|C_n|$ are estimated similarly. Writing estimates uniformly, we get

$$|F_n|\le 2^{-k}c_0\big(1+o(1)\big)\left(\frac{3e\gamma}{n\sqrt{t}}\right)^{\frac{n}{3}}(\ln t)^k. \tag{25}$$

Next, we have

$$f(s)=\sum_{n=0}^{N-1}F_ni^{-n}(2\pi)^{-n/2}j(n),\quad j(n)=\int_{C_2}\frac{e^{G(w)}}{e^w-1}(w-i\eta)^n\,dw.$$

It we replace $j(n)$ with the integral $J(n)$ that differs from $j(n)$ by changing the segment $\mathcal{C}_2$ for the infinite line containing it, the absolute value of the error will not exceed

$$\sqrt{2}\int_{c\eta}^{+\infty}\exp\left(-\frac{\xi^2}{2\pi}+\alpha\xi\right)\frac{(\xi\sqrt{2})^n}{e^{\xi}-1}\,d\xi+\sqrt{2}\int_{c\eta}^{+\infty}\exp\left(-\frac{\xi^2}{2\pi}-\alpha\xi\right)\frac{(\xi\sqrt{2})^n}{1-e^{-\xi}}\,d\xi$$
$$<3\int_{c\eta}^{+\infty}\exp\left(-\frac{\xi^2}{2\pi}\right)(\xi\sqrt{2})^n\,d\xi=3\sqrt{\pi}(2\pi)^{n/2}\int_{0.5\sqrt{t}}^{+\infty}e^{-v^2/4}\cdot e^{-v^2/4}v^n\,dv.$$

For all n with $1\le n\le N-1$, the function $e^{-v^2/4}v^n$ is monotone decreasing for $v\ge 0.5\sqrt{t}$. Therefore, the error in question is at most

$$3\sqrt{\pi}(2\pi)^{n/2}(0.5\sqrt{t})^n e^{-t/16}\int_{0.5\sqrt{t}}^{+\infty}e^{-v^2/4}\,dv<6\sqrt{\pi}(2\pi)^{n/2}(0.5\sqrt{t})^{n-1}e^{-t/8}.$$

By (25), the contribution to $f(s)$ of these quantities is not greater than

$$\begin{aligned}&\frac{12\sqrt{\pi}}{\sqrt{t}}\,e^{-t/8}\sum_{n=0}^{N-1}|F_n|\big(0.5\sqrt{t}\big)^n\\ &\qquad\le\frac{12\sqrt{\pi}}{\sqrt{t}}\,e^{-t/8}\left(|F_0|+2^{-k}c_0\big(1+o(1)\big)(\ln t)^k\sum_{n=1}^{N-1}\left(\frac{3e\gamma t}{8n}\right)^{\frac{n}{3}}\right).\end{aligned}\tag{26}$$

Assuming that $N\le 3\gamma t/8$, we conclude that the maximal value of $(3e\gamma t/(8n))^{n/3}$ on the segment in question is at most $(3e\gamma t/(8N))^{N/3}$. Since $|F_0|<2^{-k}(\ln 2\pi t+\pi)^k$, the right-hand side of (26) is dominated by the quantity

$$\frac{12\sqrt{\pi}}{\sqrt{t}}\,e^{-t/8}(\ln t)^k\,2^{-k}c_0N\left(\frac{3e\gamma t}{8N}\right)^{\frac{N}{3}}.\tag{27}$$

Since

$$\frac{N}{3}\ln\left(\frac{3e\gamma t}{8N}\right)\le\frac{\gamma t}{8},$$

it follows that (27) does not exceed $12\sqrt{\pi/t}\,(\ln t)^k\,2^{-k}c_0N\,e^{-(1-\gamma)t/8}<e^{-(1-\gamma)t/9}$, provided $0<\gamma<1$ and t is sufficiently large. Let y_1 be equal to the largest root of the equation $(\ln y)/y=3\lambda(1-\gamma)/5$ if its right-hand side does not exceed $1/e$ and to 1 otherwise. Then, assuming that $N\le t/(\lambda y_1)$, we obtain the inequalities

$$\frac{\lambda N}{t}\ln\frac{t}{\lambda N}\le\frac{3\lambda}{5}(1-\gamma),\quad \ln\frac{t}{\lambda N}\le\frac{6t}{10N}(1-\gamma),\quad \left(\frac{t}{\lambda N}\right)^{\frac{N}{6}}\le e^{(1-\gamma)t/10}.$$

Consequently,

$$f(s)=\sum_{n=0}^{+\infty}F_n i^{-n}(2\pi)^{-n/2}J(n)+\theta\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}}e^{-(1-\gamma)t/90}.$$

Now we invoke the expression for $J(n)$ obtained in [13, §8, Chapter IV]:

$$J(n)=(-1)^{m-1}\,2\pi e^{-it/2-5\pi i/8}\sum_{0\le\nu\le n/2}\frac{n!}{(n-2\nu)!\nu!}\left(\frac{\pi i}{2}\right)^{\nu}\Phi^{(n-2\nu)}(2\alpha).$$

Observing that $\left|(i\eta)^{s-1}\right|=(2\pi t)^{(\sigma-1)/2}e^{-\pi t/2}$, we find

$$\begin{aligned}j_2(s)&=(-1)^{m-1}(i\eta)^{s-1}\,2\pi e^{-it/2-5\pi i/8}S_k(N;\sigma,t)\\&\qquad+2^{1.5-k}\theta c_0e_0\big(1+o(1)\big)(\ln t)^k\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}}e^{0.5\sqrt{\pi N}}\,(2\pi)^{(\sigma-1)/2}e^{-\pi t/2},\end{aligned}$$

where

$$S_k(N;\sigma,t) = \sum_{n=0}^{N-1} F_n \sum_{0\le\nu\le n/2} \frac{n!\, i^{\nu-n}}{(n-2\nu)!\,\nu!\,2^n}\left(\frac{2}{\pi}\right)^{n/2-\nu}\Phi^{(n-2\nu)}(2\alpha).$$

Using (15), we get

$$\begin{aligned}J_k(s) = \sum_{r=1}^{4} j_{k,r}(s) \;&=\; (-1)^{m-1}(i\eta)^{s-1}\,2\pi e^{-it/2-5\pi i/8}S_k(N;\sigma,t)\\ &+ 2^{1.5-k}\theta c_0 e_0\big(1\,+\,o(1)\big)(\ln t)^k\left(\frac{\lambda N}{t}\right)^{\frac{N}{6}} e^{0.5\sqrt{\pi N}}\,(2\pi)^{(\sigma-1)/2}e^{-\,\pi t/2}.\end{aligned}$$

Since

$$\Gamma(s)(e^{2\pi is}-1) = \frac{2\pi i e^{\pi is}}{\Gamma(1-s)},$$

we have $I_k(s) = h(s)S_k(N;\sigma,t) + R_k(N;\sigma,t)$, where

$$\begin{aligned}h(s) &= i^{-1}\,e^{-\,\pi is}\Gamma(1-s)(-1)^{m-1}e^{\pi i(s-1)/2}(2\pi t)^{(s-1)/2}e^{-it/2-5\pi i/8}\\ &= ie^{-\pi is/2}\Gamma(1-s)(-1)^{m-1}(2\pi t)^{(s-1)/2}e^{-it/2-5\pi i/8}\\ &= (-1)^{m-1}ie^{-\pi i/8}e^{-\pi is/2}\Gamma(1-s)(2\pi t)^{(\sigma-1)/2}\left(\frac{2\pi t}{e}\right)^{it/2},\end{aligned}$$

and the quantity $R_k = R_k(N;\sigma,t)$ obeys the inequalities

$$|R_k| \le \frac{2^{\,0.5-k}}{\pi}\,c_0e_0\,\big(1+o(1)\big)(\ln t)^k\left(\frac{\lambda N}{t}\right)^{N/6} e^{0.5\sqrt{\pi N}}\left(\frac{t}{2\pi}\right)^{\sigma/2}.$$

Now we choose the parameters c, β, and κ. When we deduced (21), it was assumed that $\kappa^3 > 2\sqrt{2}c\gamma$. Also, the estimate (19) of $|A_N(z)|$ was obtained under the assumption that $|z| \le \kappa R$, $R \le \beta\sqrt{t}$, i.e., under the condition $|z| \le \kappa\beta\sqrt{t}$. Then this estimate was used for z running over $\mathcal{C}_2$, i.e., for $|z| \le c\sqrt{2t}$. To have right to apply the estimate of $|A_N(z)|$, it suffices to require that $c\sqrt{2} \le \kappa\beta$. Finally, everywhere above it was assumed that $0 < \beta$, $\kappa < 1$.

Let $c \to 0$. We put $\kappa = c^{1/3}$ and $\beta = 1.5\kappa^2$. Then $0 < \beta$, $\kappa < 1$ if c is sufficiently small. Also, it is obvious that $\beta\kappa = 1.5\kappa^3 = 1.5c > c\sqrt{2}$. Identity (17) implies that

$$2\sqrt{2}\,c\gamma = 2\sqrt{2}\,c\left(\frac{1}{3}+\frac{3}{8}\,c^{2/3}+\frac{9}{20}\,c^{4/3}+\ldots\right) = \frac{2\sqrt{2}}{3}\,c + O\big(c^{5/3}\big) < c = \kappa^3.$$

Since β, $\kappa \to 0$ as $c \to 0$, we have

$$\frac{1}{1-\kappa}\mathbf{K}\left(-\frac{4\kappa}{(1-\kappa)^2}\right) = \mathbf{K}(0) + o(1) = \frac{\pi}{2} + o(1), \quad e^{(1-\sigma)(\beta+0.5\beta^2)} = 1 + o(1),$$

whence $c_0 = 1 + o(1)$ and

$$\frac{2^{0.5-k}}{\pi}\,c_0e_0 = \frac{2^{-k}}{\sqrt{\pi}}\,\exp\left(\frac{1}{2}\left(1-\frac{\sqrt{\pi}}{2}\right)^2\right) = C\,2^{-k}.$$

Finally, $\lambda = (3\gamma)^2e^{-1} = e^{-1} + o(1)$. Therefore, given ε, we can choose c so small that

$$|R_k(N;\sigma,t)| \le 2^{-k}C\left(\frac{(1+\varepsilon)N}{et}\right)^{\frac{N}{6}} e^{0.5\sqrt{\pi N}}(\ln t)^k\left(\frac{t}{2\pi}\right)^{-\,\sigma/2}.$$

for $t \ge t_0(\varepsilon)$.

The lemma is proved. □

Lemma 6. *For sufficiently large t we have $\left|Q_k\left(\frac{1}{2}+it\right)\right| < c_k\left(t/(2\pi)\right)^{-1/4}(\ln t)^k$, where $c_0 = 1$, $c_1 = 1.4$, $c_2 = 2.1$.*

Proof. The definition of the quantities $Q_k = Q_k\left(\frac{1}{2}+it\right)$ shows that

$$Q_0 = hS_0 + R_0, \quad Q_1 = h(S_1 - \psi S_0) + R_1 - \psi R_0,$$
$$Q_2 = h(S_2 - 2\psi S_1 + (\psi^2 - \psi')S_0) + R_2 - 2\psi R_1 + (\psi^2 - \psi')R_0,$$

where, for brevity, we have put

$$h = h\left(\tfrac{1}{2}+it\right), \quad S_k = S_k\left(N; \tfrac{1}{2}, t\right), \quad R_k = R_k\left(N; \tfrac{1}{2}, t\right), \quad \psi^{(k)} = \psi^{(k)}\left(\tfrac{1}{2}+it\right).$$

We set $N = 1$, $\varepsilon = 1/3$ and observe that, in this case,

$$S_0 = \Phi(2\alpha), \quad S_1 = B_0\Phi(2\alpha) = \frac{1}{2}\left(\ln 2\pi t + \pi i\right)\Phi(2\alpha),$$
$$S_2 = C_0\Phi(2\alpha) = \frac{1}{4}\left(\ln 2\pi t + \pi i\right)^2\Phi(2\alpha).$$

Since for $0 \le x \le 2$ we have $0.382 < \Phi(x) < 0.924$, application of the preceding lemma yields the estimates

$$|Q_0| < \left(\frac{t}{2\pi}\right)^{-1/4}\left(0.924 + C(2t)^{-1/6}e^{0.5\sqrt{\pi}}\right) < \left(\frac{t}{2\pi}\right)^{-1/4},$$

$$|Q_1| < \left(\frac{t}{2\pi}\right)^{-1/4}\left(\frac{1}{2}\ln 2\pi t + \frac{\pi}{2} + \ln t + \frac{\pi}{2} + \Delta'(t)\right)\left(0.924 + C(2t)^{-1/6}e^{0.5\sqrt{\pi}}\right)$$
$$< 1.4\left(\frac{t}{2\pi}\right)^{-1/4}(\ln t),$$

$$|Q_2| < \left(\frac{t}{2\pi}\right)^{-1/4}\left(\left(\frac{1}{2}\ln 2\pi t + \frac{\pi}{2}\right)^2 + 2\left(\frac{1}{2}\ln 2\pi t + \frac{\pi}{2}\right)\left(\ln t + \frac{\pi}{2} + \Delta'(t)\right)\right.$$
$$\left. + \left(\ln t + \frac{\pi}{2} + \Delta'(t)\right)^2 + \frac{2}{t}\right)\left(0.924 + C(2t)^{-1/6}e^{0.5\sqrt{\pi}}\right)$$
$$< 2.1\left(\frac{t}{2\pi}\right)^{-1/4}(\ln t)^2.$$

This proves Lemma 6. □

Corollary. *Let $m = \left[\sqrt{t/(2\pi)}\right]$. Then for all sufficiently large t we have*

$$\zeta'\left(\tfrac{1}{2}+it\right) = -\sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}}\ln n - e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}}\left(\ln\frac{t}{2\pi n} + 2\Delta'(t)\right) + Q_1\left(\tfrac{1}{2}+it\right) + \theta_1 e^{-2\pi t}, \tag{28}$$

$$\zeta''\left(\tfrac{1}{2}+it\right) = -\sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}}(\ln n)^2 - e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}}\left(\left(\ln\frac{t}{2\pi n} + 2\Delta'(t)\right)^2 + 2i\vartheta''(t)\right) + Q_2\left(\tfrac{1}{2}+it\right) + \theta_2 e^{-2\pi t}. \tag{29}$$

Proof. We obtain these formulas by putting $s = \frac{1}{2} + it$ in (13) and (14) and applying Lemma 1, together with the estimates of Lemma 6. □

§4. Proof of Theorems 1 and 2

Put $s = \frac{1}{2} + it$ in formula (12). The definition of $Q_0(s)$ and Lemmas 3 and 5 show that

$$\zeta\big(\tfrac{1}{2} + it\big) = \sum_{n=1}^{m} \frac{n^{-it}}{\sqrt{n}} + e^{-2i\vartheta(t)} \sum_{n=1}^{m} \frac{n^{it}}{\sqrt{n}} + \frac{(-1)^{m-1}}{1 - ie^{-\pi t}} \left(\frac{t}{2\pi}\right)^{-1/4} e^{-2i\vartheta(t)}\, e^{-i\Delta(t)} S_0 + \theta\, C\, D(\varepsilon),$$

where we have denoted

$$S_k = S_k\big(N; \tfrac{1}{2}, t\big), \quad D(\varepsilon) = D(\varepsilon; N, t) = \left(\frac{(1+\varepsilon)N}{et}\right)^{\frac{N}{6}} e^{0.5\sqrt{\pi N}} \left(\frac{t}{2\pi}\right)^{-1/4}.$$

As has already been mentioned, the first terms of the expansion of

$$W_0 = \frac{e^{-i\Delta(t)} S_0}{1 - ie^{-\pi t}}$$

are well known, so that we do not present them here.

By Lemma 5 and the corollary to Lemma 2, we have

$$(30) \quad \big|R_1 - \psi R_0\big| \le CD(\varepsilon/2)\left(\frac{1}{2}\ln t + \ln t + \frac{\pi}{2} + \Delta'(t)\right) < \frac{3}{2}\, CD(\varepsilon/2)\big(1 + o(1)\big)(\ln t),$$

where, as before, $\psi = \psi\big(\frac{1}{2} + it\big)$, $R_k = R_k\big(N; \frac{1}{2}, t\big)$. When calculating the quantity $Q_1\big(\frac{1}{2} + it\big) = h(S_1 - \psi S_0) + R_1 - \psi R_0$, the replacement of the denominator $1 - ie^{-\pi t}$ in the expression for $h\big(\frac{1}{2} + it\big)$ (see Lemma 3) produces the error $ie^{-\pi t}\big(Q_1\big(\frac{1}{2} + it\big) - (R_1 - \psi R_0)\big)$, which, by Lemma 6 and (30), does not exceed $1.5(t/(2\pi))^{-1/4}(\ln t)e^{-\pi t}$ in the absolute value. Thus, for sufficiently large t, (28) implies the identity

$$\zeta'\big(\tfrac{1}{2} + it\big) = -\sum_{n=1}^{m} \frac{n^{-it}}{\sqrt{n}} \ln n - e^{-2i\vartheta(t)} \sum_{n=1}^{m} \frac{n^{it}}{\sqrt{n}} \left(\ln \frac{t}{2\pi n} + 2\Delta'(t)\right) + (-1)^{m-1} \left(\frac{t}{2\pi}\right)^{-1/4} e^{-i\vartheta(t)} W_1(N; t) + 1.5\theta C D(\varepsilon)(\ln t),$$

where $W_1 = W_1(N; t) = e^{-i\Delta(t)}(S_1 - \psi S_0)$. We transform $W_1(N; t)$ as follows. First, we write

$$(31) \qquad S_2 - \psi S_0 = \sum_{n=0}^{N-1} v_n \sum_{0 \le \nu \le n/2} \frac{n!\, i^{\nu - n}}{\nu!(n - 2\nu)!2^n} \left(\frac{2}{\pi}\right)^{n/2 - \nu} \Phi^{(n-2\nu)}(2\alpha),$$

where $v_n = B_n - \psi A_n$. The v_n can be found with the help of the recurrence relations (14). However, their calculation can be simplifies somewhat. For this, we put $v(z) = \sum_{n=1}^{+\infty} v_n z^z$. Then $v(z) = B(z) - \psi A(z) = A(z)u(z)$, where $u(z) = \ln w - \psi$. By Lemma 1,

$$u(z) = \ln\left(1 + \frac{z}{\sqrt{t}}\right) - \mathcal{L} - 2\Delta'(t), \quad \mathcal{L} = \frac{1}{2}\ln\frac{t}{2\pi} + \delta(t), \quad \delta(t) = \frac{\pi e^{-\pi t}}{1 + ie^{-\pi t}}.$$

Consequently, $v(z) = b(z) - (\mathcal{L} + 2\Delta'(t))a(z)$, where

$$a(z) = A(z), \quad b(z) = a(z)\ln\big(1 + z/\sqrt{t}\big).$$

Denoting by a_n, b_n the Taylor coefficients of $a(z)$, $b(z)$, we get

$$(32)\quad a_n = A_n,\ \ b_n = \sum_{k=1}^{n-1} a_k \ell_{n-k}, \quad \ell_k = \frac{(-1)^{k-1}}{k}\left(\frac{1}{\sqrt{k}}\right)^k, \quad v_n = b_n - (\mathcal{L} + 2\delta'(t))a_n.$$

The last identity implies, in particular, that the coefficients v_n are linear functions of $\mathcal{L}$. Computing v_n via (32), plugging the result in (31), and, finally, multiplying by the partial sums of the series

$$e^{-i\Delta(t)} = 1 - \frac{it}{2^4\cdot 3} - \frac{t^2}{2^9\cdot 3^2} - \frac{4027it^3}{2^{13}\cdot 3^4\cdot 5} + \dots,$$

we arrive at the expansion (4) with

$$(33)\quad \begin{aligned} H_0 &= -\mathcal{L}\,\Phi_0, \quad H_1 = \mathcal{L}\,\frac{\Phi_3}{2^2\cdot 3\pi^2} - i\,\frac{\Phi_1}{2\pi},\\ H_2 &= -\mathcal{L}\left(\frac{\Phi_2}{2^4\cdot \pi^2} + \frac{\Phi_6}{2^5\cdot 3^2\pi^4}\right) + i\left(\frac{\Phi_0}{2^3\cdot\pi} + \frac{\Phi_4}{2^3\cdot 3\pi^3}\right),\\ H_3 &= \mathcal{L}\left(\frac{\Phi_1}{2^5\cdot\pi^2} + \frac{\Phi_5}{2^3\cdot 3\cdot 5\pi^4} + \frac{\Phi_9}{2^7\cdot 3^4\pi^6}\right) - i\left(\frac{\Phi_3}{2^4\cdot\pi^3} + \frac{\Phi_7}{2^6\cdot 3^2\pi^5}\right),\\ H_4 &= -\,\mathcal{L}\left(\frac{\Phi_0}{2^7\cdot\pi^2} + \frac{19\Phi_4}{2^9\cdot 3\pi^4} + \frac{11\Phi_8}{2^9\cdot 3^2\cdot 5\pi^6} + \frac{\Phi_{12}}{2^{11}\cdot 3^5\pi^8}\right) + \frac{\Phi_0}{2^6\cdot 3\pi^2}\\ &\quad + i\left(\frac{7\Phi_2}{2^7\cdot\pi^3} + \frac{73\Phi_6}{2^8\cdot 3^2\cdot 5\pi^5} + \frac{\Phi_{10}}{2^8\cdot 3^4\pi^7}\right),\\ H_5 &= \mathcal{L}\left(\frac{5\Phi_3}{2^7\cdot 3\pi^4} + \frac{17\cdot 53\Phi_7}{2^{11}\cdot 3^2\cdot 5\cdot 7\pi^6} + \frac{7\Phi_{11}}{2^{10}\cdot 3^4\cdot 5\pi^8} + \frac{\Phi_{15}}{2^{13}\cdot 3^6\cdot 5\pi^{10}}\right)\\ &\quad - \frac{\Phi_3}{2^8\cdot 3^2\pi^4} - i\left(\frac{\Phi_1}{2^5\cdot\pi^3} + \frac{3\cdot 23\Phi_5}{2^{10}\cdot 5\pi^5} + \frac{67\Phi_9}{2^9\cdot 3^4\cdot 5\pi^7} + \frac{\Phi_{13}}{2^{12}\cdot 3^5\pi^9}\right), \end{aligned}$$

and so on (here and in the sequel $\Phi_j = \Phi^{(j)}(2\alpha)$).

Similarly, by Lemma 5 and the corollary to Lemma 2, we have

$$(34)\qquad \bigl|R_2 - 2\psi R_1 + (\psi^2 - \psi')R_0\bigr| < \left(\frac{3}{2}\right)^2 CD(\varepsilon/2)\bigl(1 + o(1)\bigr)(\ln t)^2.$$

Also, when calculating the quantity

$$Q_2\bigl(\tfrac12 + it\bigr) = h(S_2 - 2\psi S_1 + (\psi^2 - \psi')S_0) + R_2 - 2\psi R_1 + (\psi^2 - \psi')R_0$$

the replacement of the denominator $1 - ie^{-\pi t}$ in the expression for $h\bigl(\tfrac12 + it\bigr)$ (see Lemma 3) produces the error

$$ie^{-\pi t}\bigl(Q_2\bigl(\tfrac12 + it\bigr) - \bigl(R_2 - 2\psi R_1 + (\psi^2 - \psi')R_0\bigr)\bigr),$$

which, by Lemma 6 and estimate (34), does not exceed $2.2(t/(2\pi))^{-1/4}(\ln t)^2 e^{-\pi t}$. Thus, from (29) we obtain the indentity

$$\begin{aligned}\zeta''\bigl(\tfrac12 + it\bigr) = -\sum_{n=1}^{m}\frac{n^{-it}}{\sqrt{n}}(\ln n)^2 - e^{-2i\vartheta(t)}\sum_{n=1}^{m}\frac{n^{it}}{\sqrt{n}}\left(\left(\ln\frac{t}{2\pi n} + 2\Delta'(t)\right)^2 + 2i\vartheta''(t)\right)\\ + (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4} e^{-i\vartheta(t)} W_2(N;t) + \left(\frac{3}{2}\right)^2 \theta CD(\varepsilon)(\ln t)^2,\end{aligned}$$

where $W_2 = W_2(N;t) = e^{-i\Delta(t)}\bigl(S_2 - 2\psi S_1 + (\psi^2 - \psi')S_0\bigr)$. Repeating the above arguments word for word, we arrive at the relation

$$S_2 - 2\psi S_1 + (\psi^2 - \psi')S_0 = \sum_{n=0}^{N-1} w_n \sum_{0\le\nu\le n/2} \frac{n!i^{\nu-n}}{\nu!(n-2\nu)!2^n}\left(\frac{2}{\pi}\right)^{n/2-\nu}\Phi^{(n-2\nu)}(2\alpha),$$

where the coefficients w_n are defined by the formulas

$$w_n = c_n - 2(\mathcal{L} + 2\Delta'(t))b_n + \bigl((\mathcal{L}+2\Delta'(t))^2 + 2i(\vartheta''(t) - \varepsilon(t))\bigr)a_n,$$
$$c_n = \sum_{k=1}^{n-1} b_k\ell_{n-k}, \quad \varepsilon(t) = \frac{\pi^2 e^{-\pi t}}{(1+ie^{-\pi t})^2}.$$

The calculation of w_n leads to the expansion (4) with

$$H_k(t) = F_k(t) + iG_k(t), \quad k = 0,1,2,\ldots, \tag{35}$$

and

$$\begin{aligned}
&F_0 = \mathcal{L}^2\Phi_0, \quad G_0 = 0; \quad F_1 = -\mathcal{L}^2\,\frac{\Phi_3}{2^2\cdot 3\pi^2}, \quad G_1 = \mathcal{L}\,\frac{\Phi_1}{\pi};\\
&F_2 = \mathcal{L}^2\left(\frac{\Phi_2}{2^4\pi^2} + \frac{\Phi_6}{2^5\cdot 3^2\pi^4}\right) - \frac{\Phi_2}{2^2\pi^2},\\
&G_2 = -\mathcal{L}\left(\frac{\Phi_0}{2^2\pi} + \frac{\Phi_4}{2^2\cdot 3\pi^3}\right) - \frac{3\Phi_0}{2^2\pi};\\
&F_3 = -\mathcal{L}^2\left(\frac{\Phi_1}{2^5\pi^2} + \frac{\Phi_5}{2^3\cdot 3\cdot 5\pi^4} + \frac{\Phi_9}{2^7\cdot 3^4\pi^6}\right) + \frac{\Phi_1}{2^2\pi^2} + \frac{\Phi_5}{2^4\cdot 3\pi^4},\\
&G_3 = \mathcal{L}\left(\frac{\Phi_3}{2^3\pi^3} + \frac{\Phi_7}{2^5\cdot 3^2\pi^5}\right) + \frac{\Phi_1}{2^2\pi^2} + \frac{\Phi_5}{2^4\cdot 3\pi^4};\\
&F_4 = \mathcal{L}^2\left(\frac{\Phi_0}{2^7\pi^2} + \frac{19\Phi_4}{2^9\cdot 3\pi^4} + \frac{11\Phi_8}{2^9\cdot 3^2\cdot 5\pi^6} + \frac{\Phi_{12}}{2^{11}\cdot 3^5\pi^8}\right)\\
&\qquad\qquad - \left(\frac{5\Phi_0}{2^6\pi^2} + \frac{11\Phi_4}{2^6\cdot 3\pi^4} + \frac{\Phi_8}{2^7\cdot 3^2\pi^6}\right),\\
&G_4 = -\mathcal{L}\left(\frac{7\Phi_2}{2^6\pi^3} + \frac{73\Phi_6}{2^7\cdot 3^2\cdot 5\pi^5} + \frac{\Phi_{10}}{2^7\cdot 3^4\pi^7}\right) - \left(\frac{3\Phi_2}{2^6\pi^3} + \frac{\Phi_6}{2^7\cdot 3\pi^5}\right)
\end{aligned} \tag{36}$$

and so on. Theorem 1 is proved.

We turn to the proof of the formulas for $Z'(t)$, $Z''(t)$ in Theorem 2. Differentiating the relation

$$Z(t) = e^{i\vartheta(t)}\zeta\bigl(\tfrac{1}{2}+it\bigr),$$

we see that

$$Z'(t) = i\bigl(\vartheta'(t)e^{i\vartheta(t)}\zeta\bigl(\tfrac{1}{2}+it\bigr) + e^{i\vartheta(t)}\zeta'\bigl(\tfrac{1}{2}+it\bigr)\bigr). \tag{37}$$

Now we use identities (12), (28) and Lemma 1 to show that ($s = \frac{1}{2}+it$)

$$Z'(t) = \sum_{n=1}^{m}\frac{f_n(s)}{n^s} + \sum_{n=1}^{m}\frac{g_n(s)}{n^{1-s}} + q(s) + \theta e^{-2\pi t},$$

where

$$\begin{aligned} f_n(s) &= ie^{i\vartheta(t)}\big(\vartheta'(t)-\ln n\big),\\ g_n(s) &= ie^{i\vartheta(t)}\chi(s)\left(\vartheta'(t)+\ln 2\pi n-\psi(s)+\frac{\pi}{2}\tan\frac{\pi s}{2}\right),\\ q(s) &= ie^{i\vartheta(t)}\big(\vartheta'(t)Q_0(s)+Q_1(s)\big). \end{aligned}$$

Next, we have

$$\begin{aligned} f_n(s) &= ie^{i\vartheta(t)}\left(\frac{1}{2}\ln\frac{t}{2\pi}-\ln n+\Delta'(t)\right) = ie^{i\vartheta(t)}\left(\frac{1}{2}\ln\frac{t}{2\pi n^2}+\Delta'(t)\right),\\ g_n(s) &= ie^{-i\vartheta(t)}\big(-\vartheta'(t)+\ln n\big) = -ie^{-i\vartheta(t)}\left(\frac{1}{2}\ln\frac{t}{2\pi n^2}+\Delta'(t)\right) = \bar{f}_n(s). \end{aligned}$$

Finally, using Lemma 5, we get

$$\begin{aligned} q(s) &= ie^{i\vartheta(t)}\big(I_1(s)+(\vartheta'-\psi)I_0(s)\big)\\ &= ie^{i\vartheta(t)}h\big(S_1-(\vartheta'-\psi)S_0\big)+ie^{i\vartheta(t)}h\big(R_1-(\vartheta'-\psi)R_0\big), \end{aligned}$$

where $\vartheta'=\vartheta'(t)$. We reshape the expression $S_1-(\vartheta'-\psi)S_0$ like it was done above, obtaining

$$S_1-(\vartheta'-\psi)S_0 = \sum_{n=0}^{N-1} v_n \sum_{0\le\nu\le n/2}\frac{n!\,i^{\nu-n}}{\nu!(n-2\nu)!2^n}\left(\frac{2}{\pi}\right)^{n/2-\nu}\Phi^{(n-2\nu)}(2\alpha),$$

$$q(s) = (-1)^{m-1}\left(\frac{t}{2\pi}\right)^{-1/4}\mathcal{W}_1+\theta CD(\varepsilon/2)\ln t,$$

where

$$\begin{aligned} \mathcal{W}_1 = \mathcal{W}_1(N;t) &= \frac{e^{-i\Delta(t)}}{1-ie^{-\pi t}}\sum_{n=0}^{N-1} v_n \sum_{0\le\nu\le n/2}\frac{n!\,i^{\nu-n}}{\nu!(n-2\nu)!2^n}\left(\frac{2}{\pi}\right)^{n/2-\nu}\Phi^{(n-2\nu)}(2\alpha),\\ v_n &= B_n+\big(\vartheta'-\psi\big)A_n. \end{aligned}$$

Putting $v(z)=\sum_{n=1}^{+\infty} v_n z^n$, we see that

$$\begin{aligned} v(z) &= A(z)u(z), \quad u(z) = \ln w+\vartheta-\psi = \ln\left(1+\frac{z}{\sqrt{t}}\right)-\Delta'(t)-\delta(t),\\ u_n &= b_n-\big(\Delta'(t)+\delta(t)\big)a_n, \end{aligned}$$

with a_n, b_n, and $\delta(t)$ as before. These relations show that the expressions for u_n do not involve the quantities $\mathcal{L}=\ln\big(t/(2\pi)\big)$. Calculating, we arrive at the expansion (4) with the coefficients

$$\begin{aligned} &H_0 = 0, \quad H_1 = \frac{\Phi_1}{2\pi}, \quad H_2 = -\frac{\Phi_0}{2^3\pi}-\frac{\Phi_4}{2^3\cdot 3\pi^3}, \quad H_3 = \frac{\Phi_3}{2^4\pi^3}+\frac{\Phi_7}{2^6\cdot 3^2\pi^5},\\ &H_4 = -\frac{7\Phi_2}{2^7\pi^3}-\frac{73\,\Phi_6}{2^8\cdot 3^2\cdot 5\pi^5}-\frac{\Phi_{10}}{2^8\cdot 3^4\pi^7},\\ &H_5 = \frac{\Phi_1}{2^5\pi^3}+\frac{3\cdot 23\,\Phi_5}{2^{10}\cdot 5\pi^5}+\frac{67\,\Phi_9}{2^9\cdot 3^4\cdot 5\pi^7}+\frac{\Phi_{13}}{2^{12}\cdot 3^5\pi^9},\\ &H_6 = -\frac{3^2\,\Phi_0}{2^{10}\pi^3}-\frac{251\,\Phi_4}{2^{12}\cdot 3\pi^5}-\frac{11\cdot 313\,\Phi_8}{2^{14}\cdot 3^5\pi^7}-\frac{71\,\Phi_{12}}{2^{14}\cdot 3^4\cdot 5\pi^9}-\frac{\Phi_{16}}{2^{14}\cdot 3^6\cdot 5\cdot 7\pi^{11}} \end{aligned} \tag{38}$$

and so on.

Next, differentiating (37) results in the identity

$$Z''(t) = \big(i\vartheta''(t)-(\vartheta'(t))^2\big)e^{i\vartheta(t)}\zeta\big(\tfrac{1}{2}+it\big)-2\vartheta'(t)e^{i\vartheta(t)}\zeta'\big(\tfrac{1}{2}+it\big)-e^{i\vartheta(t)}\zeta''\big(\tfrac{1}{2}+it\big).$$

For $s=\frac{1}{2}+it$, from (12), (28), and Lemma 1 we obtain the formulas

$$\begin{aligned}
Z''(t) &= \sum_{n=1}^{m}\frac{f_n(s)}{n^s}+\sum_{n=1}^{m}\frac{g_n(s)}{n^{1-s}}+q(s)+\theta e^{-2\pi t},\\
f_n(s) &= e^{i\vartheta(t)}\bigl(i\vartheta''(t)-(\vartheta'(t))^2-2\vartheta'(t)\ln n-(\ln n)^2\bigr)\\
&= e^{i\vartheta(t)}\biggl(i\vartheta''(t)-\Bigl(\frac{1}{2}\ln\frac{t}{2\pi n^2}+\Delta'(t)\Bigr)^2\biggr),\\
g_n(s) &= e^{i\vartheta(t)}\chi(s)\biggl(i\vartheta''(t)-(\vartheta'(t))^2-2\vartheta'(t)\Bigl(\ln 2\pi n-\psi(s)+\frac{\pi}{2}\tan\frac{\pi s}{2}\Bigr)\\
&\qquad -\Bigl(\ln 2\pi n-\psi(s)+\frac{\pi}{2}\tan\frac{\pi s}{2}\Bigr)^2-\psi'(s)+\Bigl(\frac{\pi}{2}\sec\frac{\pi s}{2}\Bigr)^2\biggr)\\
&= -e^{-i\vartheta(t)}\biggl(i\vartheta''(t)+\Bigl(\frac{1}{2}\ln\frac{t}{2\pi n^2}+\Delta'(t)\Bigr)^2\biggr),\\
q(s) &= e^{i\vartheta(t)}\bigl((i\vartheta''(t)-(\vartheta'(t))^2)Q_0(s)-2\vartheta'(t)Q_1(s)\;-\;Q_2(s)\bigr)\\
&= e^{i\vartheta(t)}\bigl(I_0(s)(\psi'+i\vartheta''-(\psi-\vartheta')^2)+2I_1(s)(\psi-\vartheta')\;-\;I_2\bigr).
\end{aligned}$$

Lemmas 3 and 5 yield a representation of the form

$$q(s)=(-1)^{m-1}\Bigl(\frac{t}{2\pi}\Bigr)^{-1/4}\mathcal{W}_2+\theta CD(\varepsilon/2)(\ln t)^2,$$

where

$$\mathcal{W}_2=\mathcal{W}_2(N;t)=\frac{e^{-i\Delta(t)}}{1-ie^{-\pi t}}\sum_{n=0}^{N-1}u_n\sum_{0\le\nu\le n/2}\frac{n!\,i^{\nu-n}}{\nu!(n-2\nu)!2^n}\Bigl(\frac{2}{\pi}\Bigr)^{n/2-\nu}\Phi^{(n-2\nu)}(2\alpha),$$

$$u_n=\bigl(i(\vartheta''(t)-\varepsilon(t))-(\Delta'(t)+\delta(t))^2\bigr)a_n+2\bigl(\Delta'(t)+\delta(t)\bigr)b_n-c_n,$$

with a_n, b_n, c_n, $\delta(t)$, $\varepsilon(t)$ as before. Like in the case of $Z'(t)$, the expressions for the u_n do not involve $\mathcal{L}$. Calculations result in the expansion (4) with the coefficients

$$\begin{aligned}
&H_0=0,\quad H_1=0,\quad H_2=\frac{\Phi_2}{2^2\pi^2},\quad H_3=-\frac{\Phi_1}{2^2\pi^2}-\frac{\Phi_5}{2^4\cdot 3\pi^4},\\
&H_4=\frac{5\Phi_0}{2^6\pi^2}+\frac{11\Phi_4}{2^6\cdot 3\pi^4}+\frac{\Phi_8}{2^7\cdot 3^2\pi^6},\\
&H_5=-\frac{3\cdot 7\Phi_3}{2^8\pi^4}-\frac{3\,\Phi_7}{2^7\cdot 5\pi^6}-\frac{\Phi_{11}}{2^9\cdot 3^4\pi^8},\\
&H_6=\frac{79\,\Phi_2}{2^{10}\pi^4}+\frac{71\,\Phi_6}{2^{10}\cdot 5\pi^6}+\frac{179\,\Phi_{10}}{2^{11}\cdot 3^4\cdot 5\pi^8}+\frac{\Phi_{14}}{2^{13}\cdot 3^5\pi^{10}}
\end{aligned}\tag{39}$$

and so on. Theorem 2 is proved. □

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