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REPRESENTATION OF FUNCTIONS IN AN INVARIANT SUBSPACE WITH ALMOST REAL SPECTRUM

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Abstract. The invariant subspaces with almost real spectrum are studied. By a method based on the Leont′ev interpolation function, a criterion of the fundamental principle is obtained for these spaces. This criterion only consists of a simple geometric condition on the local distribution of the points of the spectrum with multiplicities. A complete characterization is given for the space of coefficients of the series that represent functions in an invariant subspace.

§1. Introduction

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of complex numbers λ_k and their multiplicities n_k. We assume that $|\lambda_k|$ is strictly monotone increasing and $|\lambda_k| \to \infty$ as $k \to \infty$. We say that the sequence Λ is *almost real* if $\operatorname{Re}\lambda_k > 0$ and $\operatorname{Im}\lambda_k/\operatorname{Re}\lambda_k \to 0$ as $k \to \infty$.

Let W be a nontrivial closed subspace of the space $H(D)$ of analytic functions in a convex domain $D \subset \mathbb{C}$ (endowed with the topology of uniform convergence on compact subsets of D), and let W be invariant under the operator of differentiation. Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be the multiple spectrum of this operator on W, and let $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$ be the family of its eigenfunctions and generalized eigenfunctions on W. The paper is devoted to the study of invariant subspaces with almost real spectrum Λ.

As partial cases of invariant subspaces, we mention the spaces of solutions of linear homogeneous differential equations, difference or differential-difference equations with constant coefficients of finite or infinite order, as well as more general convolution equations and systems of convolution equations.

The basic problem in the theory of invariant subspaces is to represent an arbitrary function in W in terms of elements of the system $\mathcal{E}(\Lambda)$. Depending on the structure of such a representation, one can distinguish several different problems. The most difficult situation arises when we consider the "weakest" case. This can be described as the problem of spectral synthesis, i.e., approximation of all functions in W by linear combinations of elements of $\mathcal{E}(\Lambda)$. A criterion for the possibility spectral synthesis was obtained by I. F. Krasichkov-Ternovckiĭ in the paper [1] for an arbitrary invariant subspace on a convex domain. It is formulated in terms of the stability of the annihilator submodule for the subspace W (this is the set of all functions of exponential type that are the Laplace transforms of functionals annihilating W in the space $H^*(D)$ strongly dual to $H(D)$). In the paper [2], this result was applied to solve the problem of spectral synthesis in some particular cases. For example, it was proved that each space of solutions of a homogeneous convolution equation in a convex domain admits spectral synthesis. Moreover, it was established that an invariant subspace on an unbounded convex domain always admits spectral synthesis.

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If W admits spectral synthesis, a natural desire is to "improve" the approximation. Definitely, most desirable is the representation of any function $g \in W$ by a "pure" series

$$g(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n \exp(\lambda_k z), \quad z \in D, \tag{1.1}$$

converging uniformly on the compact subsets of D. This problem is called the problem of fundamental principle. The name comes from a partial case of an invariant subspace, the space of solutions of a linear homogeneous differential equation with constant coefficients. It is known that each solution of such an equation is a linear combination of elementary solutions, i.e., the exponential monomials $z^n \exp(\lambda_k z)$ whose exponents are the zeros (possibly multiple) of the characteristic polynomial. The existence of such a representation is called the Euler fundamental principle.

Via the Laplace transform, the fundamental principle problem can be reduced to a dual problem of multiple interpolation in the space of entire functions of exponential type. The analysis of these two problems, first conducted independently of each other, has a long history. Its basic steps were described in the papers [3] and [4]. In [4], the author solved the fundamental principle problem for the invariant subspaces that admit spectral synthesis, and the interpolation problem for an arbitrary convex domain $D \subset \mathbb{C}$ under the restriction ($m_D(\Lambda) = 0$): $n_{k(j)}/|\lambda_{k(j)}| \to 0$ as $j \to \infty$ for any subsequence $\{\lambda_{k(j)}\}$ "accumulating" to the direction where the support function H_D of the domain D is bounded (i.e., $\lambda_{k(j)}/|\lambda_{k(j)}| \to \xi$ and $H_D(\xi) < +\infty$). In the paper [5], this restriction was lifted for a bounded domain. This means that the criterion for the fundamental principle was found for an invariant subspace in a bounded convex domain D. It consists of two conditions. The first is related to a local distribution of points in the spectrum and means, in a sense, that they are "separated away" from each other (the condensation index S_Λ equals 0, see the next section). The second condition corresponds to the global distribution of the λ_k. This is a condition of compatibility of a sequence Λ and the domain D. It is required that Λ be a part of the zero set of an entire function of exponential type and regular growth whose conjugate diagram coincides with the closure of D. By the well-known result of Levin [6], this is equivalent to the fact that Λ is a subset of a special regularly distributed set.

If the condition $S_\Lambda = 0$ fails, it is impossible to represent all functions $g \in W$ by a series of type (1.1). In this connection the following problem arises naturally: to represent g by the series (1.1) "with parentheses":

$$g(z) = \sum_{m=1}^{\infty} \Bigg(\sum_{\lambda_k \in U_m} \sum_{n=0}^{n_k-1} d_{k,n} z^n \exp(\lambda_k z) \Bigg). \tag{1.2}$$

The monograph [7] by A. F. Leont′ev is devoted to the study of the problems mentioned above (see also [22]). Many results of that author and his predecessors were presented there.

The desire to "improve" representation (1.2) leads to a basis problem in an invariant subspace, which is formulated as follows. Under what conditions can one construct a partition $U = \{U_m\}_{m=1}^{\infty}$ of the sequence Λ in groups U_m and choose in these groups some fixed linear combinations $e_{m,j}$, $j = 1, \dots, N_m$, of elements of $\mathcal{E}(\Lambda)$ so that the family of exponential polynomials $\mathcal{E}(\Lambda, U) = \{e_{m,j}\}$ is a basis in W? A series of additional questions arises in the case where a basis as above exists. How can one realize the partition U, and is it possible to describe all admissible partitions? How can one compose linear combinations inside each group and is it possible to describe all admissible combinations? How small can one make the diameter of the groups U_m? Finally, how can one describe the space of coefficients of the series with respect to the system $\mathcal{E}(\Lambda, U)$? In the case of a bounded convex domain D, all these questions were answered in the papers [8–11]. In particular, a criterion for the existence of a basis was found for a subspace W constructed by a partition U in a relatively small groups U_m, i.e., groups whose diameters and the number of elements are infinitely small (in comparison with the modulus of their elements) as $m \to \infty$. This criterion consists of one condition: compatibility of the sequence Λ with the domain D.

Therefore, the problem of representation of functions in an invariant subspace is completely investigated in the case of a bounded convex domain. The situation is much worse in the case of unbounded convex domains. Indeed, the fundamental principle criterion of the paper [4], as well as the majority of results of the carlier papers, involve the following condition. The existence is required of an entire function with zeros at the points λ_k of multiplicities at least n_k, or of a family of such functions whose growth is close to regular and depends on the domain D. The question about conditions on Λ and D that ensure the existence of such a family of functions remains open. The problem of constructing such a family is quite difficult. For bounded domains this problem was solved with the help of the result by Levin mentioned above. For unbounded domains only two cases were investigated: D is the plane or a half-plane.

A complete solution of the representation problem for invariant subspaces of entire functions was obtained in the paper [12]. It was proved that in each subspace of this sort there exists a basis consisting of linear combinations of eigenfunctions and generalized eigenfunctions. The linear combinations are formed within groups of exponents with arbitrarily small relative diameter. A criterion for the existence of a basis generated by groups of zero relative diameter (i.e., relatively small groups) was also found. As a special case, this gives a solution of the fundamental principal problem under the above assumptions.

The invariant subspaces on a half-plane were mostly investigated in the case of a simple positive spectrum with a density (see [7]). In this case, the problem of fundamental principle can be stated as a problem of closedness of the set of sums of the Dirichlet series. This problem was solved completely in the paper [13] for an arbitrary convex domain D. The solution was found in terms of simple geometric characteristics of the sequence Λ and the domain D. It contained an essentially new idea. It turns out that in the case of a vertical half-plane, for the validity of the fundamental principle we need to require neither the measurability of the sequence Λ, nor the finiteness of its maximum density (despite the fact that the support function of a half-plane is bounded in the positive direction). In this case, a necessary and sufficient condition is the vanishing of the characteristics S_Λ.

The present paper is devoted to the study of invariant subspaces with almost real spectrum. At first sight, it seems that this case is very similar to that of positive spectrum studied in [13]. However, these two cases differ principally from each other. First, in the present paper we allow the points λ_k to be multiple. The most important fact is that, in contrast with the paper [4], we do not impose any additional restrictions on the multiplicity. In comparison with the case of a simple spectrum, the investigation becomes more difficult. Moreover, unlike the case of positive spectrum, the results obtained for invariant subspaces with spectrum “accumulating” to the positive half-axis may be applied for the study of invariant subspaces with arbitrary spectrum. This fact is demonstrated in the present paper. Furthermore, the appearance of this paper was motivated by the necessity to remove the only restriction in [4] ($m_D = 0$) on the way of the complete solution of the fundamental principal problem for arbitrary invariant subspaces.

The paper consists of four sections. §2 is devoted to the construction of a special family of entire functions of exponential type (Theorems 2.2, 2.4, 2.5). In the next section, we use that family and the Leont′ev interpolation function to obtain (Theorems 3.1 and 3.3) a representation of functions in invariant subspaces with almost real spectrum on half-planes with a vertical boundary (and located to the left of the boundary). Note that in the paper [4] (as well as in many other earlier papers), the solution of the dual interpolation problem was applied for this purpose. But the duality of the representation and interpolation problems was established in [4] under the additional restriction on the multiplicities of the points $\lambda_k : m_D(\Lambda) = 0$. It is not known whether duality still occurs without this restriction. But even if duality is preserved without this additional restriction, it is still not clear how to solve the interpolation problem. For that reason, the paper [4] contains a restriction on Λ. In the present paper, we use the interpolation function to obtain the fundamental principle criterion (Theorem 3.5) in the case under consideration without any additional restrictions on multiplicities. This criterion involves only a simple geometric condition on the local distribution of points (with multiplicities) of the sequence $\Lambda : S_\Lambda = 0$. Also, we give a complete characterization of the space of coefficients of the series (1.1) representing functions in an invariant subspace. As a result, it is shown that the restrictions imposed on the multiplicities in the paper [4] are technical in the case of an invariant subspace with almost real spectrum in a half-plane with vertical boundary. Later in §4 we explain that, in essence, this situation is the only model case when the restriction $m_D(\Lambda) = 0$ is technical.

In §3, we consider all cases of invariant subspaces with almost real spectrum on an unbounded convex domains D such that the positive half-axis is either inside the set $\mathcal{J}(D)$ where the support function H_D of the domain D is not bounded, or on its boundary (but not necessarily in the set itself). Via one new and several known results related to the solution of a different problem (simultaneous analytic continuation of functions in invariant subspaces), these cases reduce to two situations. Namely, to the case of a half-plane treated before (Theorem 3.5) and to the case where all functions in an invariant subspace are entire (this case was investigated in the paper [4]). The case of any unbounded convex domain D "similar" to a vertical half-plane reduces to the case of a half-plane (Theorem 3.8). This includes the situation when the boundary of D contains a vertical ray, or a vertical line is its asymptote. For such a domain, the positive half-line belongs to the boundary of the set $\mathcal{J}(D)$, but at the same time the function H_D is bounded on this half-axis. In Theorem 3.9, for the first time for one of the classes of invariant subspaces, a simple criterion is obtained for each function in this subspace to be an entire function, without any assumptions about the existence of any families of entire functions. On that basis, with the help of Theorem 5.1 in [4], a fundamental principle criterion is obtained (Theorem 3.10) for invariant subspaces with almost real spectrum in an arbitrary convex domain for which the positive half-axis belongs to the set $\mathcal{J}(D)$ (including the case where it belongs to the boundary of the domain $\mathcal{J}(D)$).

In the last section, the remaining case is studied, where the support function H_D is bounded in a neighborhood of the positive half-axis. It is proved (Theorem 4.1) that a necessary condition for the validity of the fundamental principle in an invariant subspace with almost real spectrum is the relation $m(\Lambda) = \lim_{j\to\infty} n_k/|\lambda_k| = 0$. On that basis in Theorem 4.2 we obtain a necessary condition for the validity of the fundamental principle in arbitrary invariant subspaces. This result contains the main result of the paper [5] (Theorem 1) as a special case. In particular, Theorem 4.2 means that the condition $m_D(\Lambda) = 0$ is a necessary condition for the validity of the fundamental principal in arbitrary invariant subspaces and any convex domains that are not "similar" to a half-plane (i.e., their boundary contains no rays and does not admit any asymptotes). Due to Theorem 4.2, the only restriction in Theorem 5.1 of the paper [4] "narrows" substantially. This fact is reflected in Theorem 4.3, giving a criterion for the validity of the fundamental principle under the following restriction, which is considerably weaker than that in [4]: $\lim_{j\to\infty} n_{k(j)}/|\lambda_{k(j)}| = 0$ for any subsequince $\{\lambda_{k(j)}\}$ such that $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ converges to a point $\xi \in \partial\mathcal{J}(D) \setminus \mathcal{J}(D)$ (there can be at most two such points).

Due to Theorem 4.1, the last result of the paper, Theorem 4.4, (and only it) is a complete analog of the corresponding result (Theorem 4) of the paper [13]. It represents a simple criterion for the validity of the fundamental principle for invariant subspaces with almost real spectrum in the case where the function H_D is bounded in a neighborhood of the positive half-axis.

Finally, it should be noted that, in this paper, the problem of the removal of the "technical" restriction remaining in Theorem 4.3 is basically solved. Some principal steps to its solution are demonstrated in the proofs of Theorems 2.2, 2.4, and 3.3. However, a complete solution of this problem is technically difficult, requiring a separate study.

§2. Construction of special entire functions

In this section, we construct entire functions of exponential type that vanish on an almost real sequence Λ and have growth close to regular.

Let $B(z,r)$ and $S(z,r)$ be the open disk and circle centered at a point z and of radius r. Denote by $n(z,r,\Lambda)$ the number of points λ_k (with multiplicities n_k) belonging to the closure $\overline{B(z,r)}$, and by $\overline{n}(\Lambda)$ the upper density of the sequence Λ:

$$\overline{n}(\Lambda) = \lim_{r\to\infty} \frac{n(0,r,\Lambda)}{r}.$$

Let f be an entire function. We say that f is of exponential type if for some $A, B \geq 0$ we have $\ln|f(\lambda)| \leq A + B|\lambda|$, $\lambda \in \mathbb{C}$. The function

$$h_f(\lambda) = \limsup_{t\to\infty} \frac{\ln|f(t\lambda)|}{t}, \quad \lambda \in \mathbb{C},$$

is called the indicator of f. This function is convex and positive homogeneous of order one, because it coincides with the support function of some compact set called the indicator diagram of f (see, e.g., [15, Chapter I, §5, Theorem 5.4]).

Let $d > 0$. As in [16], we say that a sequence $\{\varsigma_l\}$ is asymptotically d-close to $\{\xi_l\}$ if $\limsup_{l\to\infty} |\varsigma_l - \xi_l|/|\xi_l| \leq d$.

We also say that a collection of disks $E = \bigcup B_i$ is centered with a sequence $\{\xi_l\}$ if every point ξ_l belongs to at least one disk B_i, and each disk B_i contains at least one point ξ_l. Recall that the number

$$\rho_E = \limsup_{r\to\infty} \frac{1}{r} \sum_{|z_i|<r} \rho_i$$

is called the linear density of the set $E = \bigcup B(z_i, \rho_i)$.

The next statement is a refinement of the theorem that estimates from below the modulus of an entire function of exponential type (see [17, Chapter I, §1, Theorem 1.1.9]).

Lemma 2.1. *Let f be an entire function, $f(0) \neq 0$, satisfying the estimate*

$$\ln|f(\lambda)| \leq A + B|\lambda|, \quad \lambda \in \mathbb{C},$$

for some $A, B > 0$. Then for each $\beta \in (0,1)$ there exists a set of disks $E(\beta)$ centered with the zero set of f, having linear density of at most β, and such that

$$\ln|f(\lambda)| \geq \ln|f(0)| - b(\beta)(A + 12B|\lambda|), \quad \lambda \in \mathbb{C} \setminus E(\beta),$$

where $b(\beta) = 3 + \ln(48/\beta)$.

Proof. Let $\beta \in (0,1)$ and $m \geq 1$. In accordance with the theorem estimating from below the modulus of a function analytic in a disk (see [15, Chapter I, §4, Theorem 4.2]), we have

$$\ln|f(\lambda)| \geq |f(0)| - (2 + \ln(48e/\beta))(A + Be2^{m+1})$$

in the disk $B(0, 2^m)$ but outside of exceptional disks with the total sum of radii equal to $\beta 2^m/8$. Note that this theorem is based on the theorem of Cartan (see [15, Chapter I, §4, Theorem 4.1]) about a lower estimate for a polynomial. The proof of that theorem shows that the number of exceptional disks is finite, and each contains at least one zero of the function f. Let $B(z_{j,m}, r_{j,m})$, $j = 1, \dots, j(m)$ be the subset of all exceptional disks that intersect the annulus $K_m = B(0, 2^m) \setminus B(0, 2^{m-1})$. Then $\sum_j r_{j,m} \leq \beta 2^m/8$. Since $2^{m+1} \leq 4|\lambda|$ for $\lambda \in K_m$, we have

$$\ln|f(\lambda)| \geq \ln|f(0)| - b(\beta)(A + 12B|\lambda|), \quad \lambda \in K_m \setminus \bigcup_j B(z_{j,m}, r_{j,m}).$$

Therefore, setting $E(\beta) = B(0,R) \bigcup_{j,m} B(z_{j,m}, r_{j,m})$, where $R \geq 1$, we obtain the desired inequality. The set $E(\beta)$ is centered with the zero set of f. Indeed, each element of the last set obviously belongs to $E(\beta)$. Moreover, every disk $B(z_{j,m}, r_{j,m})$ contains at least one zero of f. If $R > 0$ is sufficiently large, the same is true for the disk $B(0,R)$.

It remains to prove that the linear density of $E(\beta)$ does not exceed β. Let $r > 0$ and $|z_{j,m}| < r$. Then, in accordance with our choice of the disks $B(z_{j,m}, r_{j,m})$, we have $r + (\beta 2^m)/8 > 2^{m-1}$. Hence, we obtain $4r > 2^m$. Denote by $m(r)$ the maximum of all numbers m such that $4r > 2^m$. We have

$$\sum_{|z_{j,m}|<r} r_{j,m} \leq \sum_{m=1}^{m(r)} \sum_{j=1}^{j(m)} r_{j,m} \leq \sum_{m=1}^{m(r)} \frac{\beta 2^m}{8} = \sum_{m=1}^{m(r)} \frac{\beta 2^{m(r)}}{2^{m(r)-m} 8} \leq \sum_{m=1}^{m(r)} \frac{\beta r}{2^{m(r)+1-m}} \leq \beta r.$$

Thus, $\rho_{E(\beta)} \leq \beta$. The lemma is proved. □

Let $\delta \in (0,1)$. We set $\Gamma(\delta) = \{t\lambda : \lambda \in B(1,\delta), t \in \mathbb{R}\}$.

Theorem 2.2. *Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^\infty$ be an almost real sequence such that $\bar{n}(\Lambda) < +\infty$. Then for any $\varepsilon > 0$ and any $\delta \in (0,1)$, there exist $\gamma \in (0,1)$, an entire function of exponential type f, and a strictly monotone increasing sequence of positive numbers $\{t_i\}_{i=1}^\infty$ such that $t_{i+1} \leq (1+\delta)t_i$, $i \geq 1$, $t_i \to \infty$ as $i \to \infty$, f has zeros at the points λ_k of multiplicities at least n_k, and the following inequalities hold true*:

$$\left|\ln|f(\lambda)| - \frac{\pi|\operatorname{Im}\lambda|}{\gamma}\right| \leq \varepsilon|\lambda|, \tag{2.1}$$
$$\lambda \in (\mathbb{C} \setminus (\Gamma(\delta) \cup B(0,t_1))) \cup \left(\bigcup_{i=1}^\infty S(0,t_i)\right),$$

$$h_f(\lambda) \leq \pi|\operatorname{Im}\lambda|/\gamma + \varepsilon|\lambda|, \quad \lambda \in \mathbb{C}. \tag{2.2}$$

Proof. We fix $\varepsilon, \delta > 0$. If necessary, one may assume that ε are δ are arbitrarily small. To define our desired function f, we need to construct its zero set. This will be done in 6 steps.

1) At the first step we replace Λ by the sequence of positive numbers $\Lambda^1 = \{\xi_l\}$, consisting of the real parts $\operatorname{Re}\lambda_k$ of the elements of Λ. We assume that the number of each $\operatorname{Re}\lambda_k$ belonging to Λ^1 coincides with the sum of multiplicities of the points λ_j with real part $\operatorname{Re}\lambda_k$. Consider the functions

$$L(\lambda) = \prod_{k=1}^\infty \left(1 - \frac{\lambda^2}{\lambda_k^2}\right)^{n_k}, \quad L_1(\lambda) = \prod_{l=1}^\infty \left(1 - \frac{\lambda^2}{\xi_l^2}\right).$$

Obviously, both Λ^1 and Λ have finite upper density. This implies (see, e.g., [17, Chapter. I, §1, Theorem 1.1.5]) that $L(\lambda)$ and $L_1(\lambda)$ are entire functions of exponential type. We compare their behavior. For this purpose, we use the results of [16].

Since Λ^1 has finite upper density, the number $C = \sup_r n(0,r,\Lambda^1)/r$ is finite. Let $\Lambda(\widetilde{\Lambda}^1)$ be the union of $\Lambda(\Lambda^1)$ and the set $-\Lambda(-\Lambda^1)$ symmetric to $\Lambda(\Lambda^1)$ about the origin. By assumption, Λ is almost real. Thus, Λ is asymptotically d-close to Λ^1 for any $d > 0$. Then by Theorem B in [16] (since the zeros of L and L_1 are symmetric about the origin), for any $d \in (0,1/2)$ we can find $C_1 > 0$ (depending only on C) and a union of disks $E^1(d) = \bigcup B(y_i, q_i)$ such that $E^1(d)$ is centered with $\widetilde{\Lambda} \cup \widetilde{\Lambda}^1$, has linear density of at most $\sqrt[4]{d}$, and

$$\big|\ln|L(\lambda)| - \ln|L_1(\lambda)|\big| \le C_1\sqrt{d}|\lambda|, \quad \lambda \in \mathbb{C} \setminus E^1(d). \tag{2.3}$$

2) At the second step we split Λ^1 in groups belonging to special semiopen intervals of the real line. We fix $\gamma \in (0, 1/4C)$, $\gamma < 1$. Using induction, we construct groups $\Lambda^1_p(\gamma)$, $p \ge 1$, so that $\Lambda^1 = \bigcup_p \Lambda^1_p(\gamma)$. Let $p = 1$, and let $l_1(\gamma) \ge 0$ be the smallest integer for which the semiopen interval $(l_1(\gamma)\gamma, (l_1(\gamma)+1)\gamma]$ contains at least one point of Λ^1. Now we choose the smallest natural number $m_1(\gamma) > l_1(\gamma)$ such that $(l_1(\gamma)\gamma, m_1(\gamma)\gamma]$ contains at most $m_1(\gamma) - l_1(\gamma)$ elements of the sequence Λ^1. This number exists because, by the definition of C and our choice of γ, for sufficiently large m we have $n(0, m\gamma, \Lambda^1) \le Cm\gamma \le m(1 - (l_1(\gamma))/m) = m - l_1(\gamma)$. We define $\Lambda^1_1(\gamma)$ as the set of elements of Λ^1 lying on $(l_1(\gamma)\gamma, m_1(\gamma)\gamma]$. Suppose that we have constructed the groups $\Lambda^1_p(\gamma) \subset (l_p(\gamma)\gamma, m_p(\gamma)\gamma]$, $p = 1, \dots, j$. Let $l_{j+1}(\gamma) \ge m_j(\gamma)$ be the smallest natural number such that $(l_{j+1}(\gamma)\gamma, (l_{j+1}(\gamma)+1)\gamma]$ contains points of Λ^1. As above, take the smallest natural number $m_{j+1}(\gamma) > l_{j+1}(\gamma)\gamma$ such that

$$n(0, m_{j+1}(\gamma)\gamma, \Lambda^1) - n(0, l_{j+1}(\gamma)\gamma, \Lambda^1) \le m_{j+1}(\gamma) - l_{j+1}(\gamma). \tag{2.4}$$

We define $\Lambda^1_{j+1}(\gamma)$ as the set of all elements of Λ^1 belonging to the semiopen interval $(l_{j+1}(\gamma)\gamma, m_{j+1}(\gamma)\gamma]$. Thus, we have split Λ^1 in the groups $\Lambda^1_p(\gamma)$, $p \ge 1$.

By the definition of $m_p(\gamma)$, we have $n(0, (m_p(\gamma)-1)\gamma, \Lambda^1) - n(0, l_p(\gamma)\gamma, \Lambda^1) > m_p(\gamma) - 1 - l_p(\gamma)$. Therefore,

$$n(0, m_p(\gamma)\gamma, \Lambda^1) - n(0, l_p(\gamma)\gamma, \Lambda^1) \ge m_p(\gamma) - l_p(\gamma).$$

The last inequality together with (2.4) yields

$$n(0, m_p(\gamma)\gamma, \Lambda^1) - n(0, l_p(\gamma)\gamma, \Lambda^1) = m_p(\gamma) - l_p(\gamma), \quad p \ge 1. \tag{2.5}$$

3) At the third step we construct an auxiliary sequence $\Lambda^2(\gamma)$ whose elements lie on the semiopen intervals $(l_p(\gamma)\gamma, m_p(\gamma)\gamma]$ in a regular way. We put $\Lambda^2(\gamma) = \bigcup_p \Lambda^2_p(\gamma)$, where $\Lambda^2_p(\gamma) = \{(l_p(\gamma)+1)\gamma, \dots, m_p(\gamma)\gamma\}$, $p \ge 1$. Then

$$n(0, m_p(\gamma)\gamma, \Lambda^2(\gamma)) - n(0, l_p(\gamma)\gamma, \Lambda^2(\gamma)) = m_p(\gamma) - l_p(\gamma), \quad p \ge 1. \tag{2.6}$$

We define the function

$$L_2(\lambda, \gamma) = \prod_{\eta_j \in \Lambda^2(\gamma)} \left(1 - \frac{\lambda^2}{\eta_j^2}\right).$$

Let us compare the zero sets of $L_2(\lambda,\gamma)$ and $L_1(\lambda)$. By (2.5) and (2.6), there is a one-to-one correspondence between these sets, so that with each zero ξ_j of $L_1(\lambda)$ lying in the semiopen interval $(l_p(\gamma)\gamma, m_p(\gamma)\gamma]$ (or symmetric to it with respect to the origin) we associate the zero η_j of $L_2(\lambda,\gamma)$ lying in the same interval. Hence,

$$\frac{|\xi_j - \eta_j|}{\xi_j} \le \frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma}{l_p(\gamma)\gamma} = \frac{m_p(\gamma) - l_p(\gamma)}{l_p(\gamma)}.$$

By (2.5) and the definition of C, we have

$$m_p(\gamma) - l_p(\gamma) = n(0, m_p(\gamma)\gamma, \Lambda^1) - n(0, l_p(\gamma)\gamma, \Lambda^1) \le n(0, m_p(\gamma)\gamma, \Lambda^1) \le Cm_p(\gamma)\gamma.$$

Hence, $l_p(\gamma) \ge m_p(\gamma)(1 - C\gamma)$. It follows that

$$\frac{|\xi_j - \eta_j|}{\xi_j} \le \frac{m_p(\gamma) - l_p(\gamma)}{l_p(\gamma)} \le \frac{Cm_p(\gamma)\gamma}{m_p(\gamma)(1 - C\gamma)} = \frac{C\gamma}{1 - C\gamma} \le 2C\gamma =: d(\gamma). \tag{2.7}$$

Thus, $\widetilde{\Lambda}^2(\gamma) = \Lambda^2(\gamma) \cup (-\Lambda^2(\gamma))$ is asymptotically $d(\gamma)$-close to $\widetilde{\Lambda}^1$. Then, by Theorem B in [16] (taking the symmetry of $\widetilde{\Lambda}^1$, $\widetilde{\Lambda}^2(\gamma)$ with respect to the origin into account), we find $C_2 > 0$ (depending only on C) and a union of disks $E^2(\gamma) = \bigcup B_i^2(\gamma)$ such that $E^2(\gamma)$ is centered with $\widetilde{\Lambda}^1 \cup \widetilde{\Lambda}^2(\gamma)$, the set $E^2(\gamma)$ has linear density of at most $\sqrt[4]{d(\gamma)}$, and

$$\big| \ln|L_1(\lambda)| - \ln|L_2(\lambda, \gamma)| \big| \le C_2\sqrt{d(\gamma)}|\lambda|, \quad \lambda \in \mathbb{C} \setminus E^2(\gamma). \tag{2.8}$$

4) At the fourth step we improve, in comparison with (2.8), the upper estimates for $\ln|L_1(\lambda)|$ along the real line. Since $\widetilde{\Lambda}^1$, $\widetilde{\Lambda}^2(\gamma)$ are symmetric with respect to the origin, the statement preceding Theorem 4 in [16] (formula (3.8)) shows that

$$\big| \ln|L_1(\lambda)| - \ln|L_2(\lambda, \gamma)| + I(\lambda, \gamma) \big| \le C_3 d(\gamma)|\lambda|, \quad |\lambda| \ge R, \tag{2.9}$$

where the constants C_3, R depend only on C, and

$$I(\lambda, \gamma) = \int_0^1 \frac{n(\lambda, t|\lambda|, \widetilde{\Lambda}^1) - n(\lambda, t|\lambda|, \widetilde{\Lambda}^2(\gamma))}{t(1+t)}\, dt.$$

Let $\widetilde{\delta} > 0$. We want to find upper estimates for $-I(\lambda, \gamma)$ on the real line outside of the disks $B(\eta_j, \widetilde{\delta}\gamma)$, $\eta_j \in \widetilde{\Lambda}^2(\gamma)$. In view of the symmetry of the sets $\widetilde{\Lambda}^1$ and $\widetilde{\Lambda}^2(\gamma)$, it suffices to do this for the positive half-line of $\mathbb{R}$. Suppose $\lambda > 0$ and $|\lambda - \eta_j| \ge \gamma/4$ for all $\eta_j \in \Lambda^2(\gamma)$. For convenience, we set $m_0(\gamma) = 0$. Let $p(0) \ge 0$ be the greatest integer such that $\lambda > m_p(0)(\gamma)\gamma$, and let $p(1)$ be the smallest natural number satisfying $\lambda < l_p(1)(\gamma)\gamma$. We split Λ^1 in three parts:

$$\Lambda^1(\gamma, 0) = \bigcup_{p=1}^{p(0)} \Lambda_p^1(\gamma), \quad \Lambda^1(\gamma, 2) = \bigcup_{p \ge p(1)} \Lambda_p^1(\gamma),$$
$$\Lambda^1(\gamma, 1) = \Lambda^1 \setminus (\Lambda^1(\gamma, 0) \cup \Lambda^1(\gamma, 2)).$$

If $p(0) = 0$, then $\Lambda^1(\gamma, 0)$ is empty. In the case where $p(0) = p(1) - 1$, $\Lambda^1(\gamma, 1)$ is empty. Otherwise, $\Lambda^1(\gamma, 1) = \Lambda^1_{p(0)+1}(\gamma) = \Lambda^1_{p(1)-1}(\gamma)$. Similarly, we define $\Lambda^2(\gamma, 0)$, $\Lambda^2(\gamma, 1)$, and $\Lambda^2(\gamma, 2)$. Using this notation, we obtain

$$-I(\lambda, \gamma) = \sum_{i=0}^{2} \int_0^\lambda \frac{n(\lambda, y, \Lambda^2(\gamma, i) - n(\lambda, y, \Lambda^1(\gamma, i))}{y(1 + y/\lambda)}\, dy. \tag{2.10}$$

If $\Lambda^1(\gamma, i)$ is empty, then the corresponding term of the sum is missing. Let us estimate each of them from above. Since Λ^1 and $\Lambda^2(\gamma)$ belong to the intervals $(l_p(\gamma)\gamma, m_p(\gamma)\gamma]$, $p \ge 1$, by (2.5) and (2.6) we have $n(\lambda, y, \Lambda^2(\gamma, 2)) - n(\lambda, y, \Lambda^1(\gamma, 2)) = 0$, $\lambda + y \notin (l_p(\gamma)\gamma, m_p(\gamma)\gamma]$. Let $\lambda + y \in (l_p(\gamma)\gamma, m_p(\gamma)\gamma]$, $p \ge p(1)$. We choose the smallest natural number $s(y)$ such that $\lambda + y \in (l_p(\gamma)\gamma, s(y)\gamma]$. Let $s(y) = l_p(\gamma) + 1$. If $\lambda + y = s(y)\gamma$, then $n(l_p(\gamma)\gamma, \lambda + y - l_p(\gamma)\gamma, \Lambda_p^2(\gamma)) = 1$ and the definition of $l_p(\gamma)$ implies that $n(l_p(\gamma)\gamma, \lambda + y - l_p(\gamma)\gamma, \Lambda_p^1(\gamma)) \ge 1$. Otherwise, $n(l_p(\gamma)\gamma, \lambda + y - l_p(\gamma)\gamma, \Lambda_p^2(\gamma)) = 0$. Suppose that $s(y) > l_p(\gamma) + 1$. Then, by the definitions of $\Lambda_p^2(\gamma)$ and $m_p(\gamma)$, we have $n(l_p(\gamma)\gamma, \lambda + y - l_p(\gamma)\gamma, \Lambda_p^2(\gamma)) \le s(y) - l_p(\gamma)$ and $n(l_p(\gamma)\gamma, \lambda + y - l_p(\gamma)\gamma, \Lambda_p^1(\gamma)) \ge$

$n(l_p(\gamma)\gamma, (s(y) - l_p(\gamma) - 1)\gamma, \Lambda_p^1(\gamma)) > s(y) - l_p(\gamma) - 1 \geq s(y) - l_p(\gamma)$. Hence, the integrand in the third term in (2.10) is nonpositive. Therefore,

$$\int_0^\lambda \frac{n(\lambda, y, \Lambda^2(\gamma, 2)) - n(\lambda, y, \Lambda^1(\gamma, 2))}{y(1 + y/\lambda)}\, dy \leq 0. \tag{2.11}$$

Now we estimate the second term in (2.10). If $\Lambda^2(\gamma, 1) \neq \varnothing$, then $\Lambda^2(\gamma, 1) = \Lambda^2_{p(0)+1}(\gamma)$ and $\lambda \in (l_{p(0)+1}(\gamma)\gamma, m_{p(0)+1}(\gamma)\gamma]$. We set

$$y(\lambda) = \max\{\lambda - l_{p(0)+1}(\gamma)\gamma, m_{p(0)+1}(\gamma)\gamma - \lambda\}.$$

Taking into account the inequality $|\lambda - \eta_j| \geq \widetilde{\delta}\gamma$ for all $\eta_j \in \Lambda^2(\gamma)$ and the definition of $\Lambda^1_{p(0)+1}(\gamma)$, we have

$$\begin{aligned}
&\int_0^\lambda \frac{n(\lambda, y, \Lambda^2(\gamma, 1)) - n(\lambda, y, \Lambda^1(\gamma, 1))}{y(1 + y/\lambda)}\, dy \\
&\leq \int_{\widetilde{\delta}\gamma}^{y(\lambda)} \frac{n(\lambda, y, \Lambda^2_{p(0)+1}(\gamma))\, dy}{y(1 + y/\lambda)} \leq \int_{\widetilde{\delta}\gamma}^{y(\lambda)} \frac{2y/\gamma + 2}{y(1 + y/\lambda)}\, dy \\
&\leq 2a(\lambda, \gamma, \widetilde{\delta}) + \frac{2\lambda \ln(1 + y(\lambda)/\lambda)}{\gamma} \leq 2a(\lambda, \gamma, \widetilde{\delta}) + \frac{2y(\lambda)}{\gamma},
\end{aligned} \tag{2.12}$$

where

$$a(\lambda, \gamma, \widetilde{\delta}) = \int_{\widetilde{\delta}\gamma}^\lambda \frac{dy}{y(1 + y/\lambda)} = \ln \frac{\lambda}{\widetilde{\delta}\gamma} - \ln \frac{2\lambda}{\widetilde{\delta}\gamma + \gamma} = \ln \frac{\widetilde{\delta}\gamma + \lambda}{2\widetilde{\delta}\gamma}.$$

To estimate the first term in (2.10), we observe that, as above,

$$n(\lambda, y, \Lambda^2(\gamma, 0)) - n(\lambda, y, \Lambda^1(\gamma, 0)) = 0, \quad \lambda - y \notin (l_p(\gamma)\gamma, m_p(\gamma)\gamma].$$

Suppose $\lambda - y \in (l_p(\gamma)\gamma, m_p(\gamma)\gamma]$, $p \leq p(0)$. We set $y_p = y + \lambda - m_p(\gamma)\gamma$. From the definition of $\Lambda^2_p(\gamma)$ we deduce that

$$\begin{aligned}
&\int_0^\lambda \frac{n(\lambda, y, \Lambda^2(\gamma, 0)) - n(\lambda, y, \Lambda^1(\gamma, 0))}{y(1 + y/\lambda)}\, dy \\
&\leq \sum_{p=1}^{p(0)} \int_0^{(m_p(\gamma) - l_p(\gamma))\gamma} \frac{n(m_p(\gamma)\gamma, y, \Lambda^2_p(\gamma))}{y_p(1 + y_p/\lambda)}\, dy \\
&\leq \sum_{p=1}^{p(0)} \int_{\lambda - m_p(\gamma)\gamma}^{\lambda - l_p(\gamma)\gamma} \frac{y_p - \lambda + m_p(\gamma)\gamma + \gamma}{\gamma y_p(1 + y_p/\lambda)}\, dy_p \\
&\leq \sum_{p=1}^{p(0)} \int_{\lambda - m_p(\gamma)\gamma}^{\lambda - l_p(\gamma)\gamma} \frac{dy_p}{\gamma(1 + y_p/\lambda)}\, dy_p + \sum_{p=1}^{p(0)} \int_{\lambda - m_p(\gamma)\gamma}^{\lambda - l_p(\gamma)\gamma} \frac{m_p(\gamma)\gamma - \lambda}{\gamma y_p(1 + y_p/\lambda)}\, dy_p + a(\lambda, \gamma, \widetilde{\delta}) \\
&= \sum_{p=1}^{p(0)} \frac{\lambda}{\gamma} \ln\left(\frac{2\lambda - l_p(\gamma)\gamma}{2\lambda - m_p(\gamma)\gamma}\right) \\
&\quad + \sum_{p=1}^{p(0)} \frac{m_p(\gamma)\gamma - \lambda}{\gamma} \left(\ln\left(\frac{\lambda - l_p(\gamma)\gamma}{\lambda - m_p(\gamma)\gamma}\right) - \ln\left(\frac{2\lambda - l_p(\gamma)\gamma}{2\lambda - m_p(\gamma)\gamma}\right)\right) + a(\lambda, \gamma, \widetilde{\delta}) \\
&= a(\lambda, \gamma, \widetilde{\delta}) + \sum_{p=1}^{p(0)} \frac{2\lambda - m_p(\gamma)\gamma}{\gamma} \ln\left(1 + \frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma}{2\lambda - m_p(\gamma)\gamma}\right) \\
&\quad - \sum_{p=1}^{p(0)} \frac{\lambda - m_p(\gamma)\gamma}{\gamma} \ln\left(1 + \frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma}{\lambda - m_p(\gamma)\gamma}\right) \\
&= \sum_{p=1}^{p(0)} a_p(\lambda, \gamma) + a(\lambda, \gamma, \widetilde{\delta}).
\end{aligned} \tag{2.13}$$

Consequently, from (2.5) it follows that

$$\begin{aligned}\int_0^\lambda &\frac{n(\lambda, y, \Lambda^2(\gamma,0)-n(\lambda,y,\Lambda^1(\gamma,0))}{y(1+y/\lambda)} \\ &\le \sum_{p=1}^{p(0)} \frac{2\lambda - m_p(\gamma)\gamma}{\gamma} \ln\left(1+\frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma}{2\lambda - m_p(\gamma)\gamma}\right) + a(\lambda,\gamma,\widetilde{\delta}) \\ &\le \sum_{p=1}^{p(0)} (m_p(\gamma)-l_p(\gamma)) + a(\lambda,\gamma,\widetilde{\delta}) \\ &\le n(0,\lambda,\Lambda^1) + a(\lambda,\gamma,\widetilde{\delta}) \\ &\le C\lambda + a(\lambda,\gamma,\widetilde{\delta}).\end{aligned}$$

Using (2.10)–(2.12), we get

$$-I(\lambda,\gamma) \le C\lambda + 2\left(m_{p(0)+1}(\gamma) - l_{p(0)+1}(\gamma)\right) + 3a(\lambda,\gamma,\widetilde{\delta}).$$

In view of (2.5), the inequality $m_{p(0)+1}(\gamma) - l_{p(0)+1}(\gamma) \le Cm_{p(0)+1}(\gamma)\gamma$ is true. Hence, $\lambda \ge l_{p(0)+1}(\gamma)\gamma \ge m_{p(0)+1}(\gamma)(1-C\gamma)\gamma$. Therefore,

$$-I(\lambda,\gamma) \le C\lambda + 4C\lambda(1-C\gamma)^{-1} + 3a(\lambda,\gamma,\widetilde{\delta}) \le 7C\lambda + 3a(\lambda,\gamma,\widetilde{\delta}). \tag{2.14}$$

Now we are going to obtain a more general estimate of $I(\lambda,\gamma)$ outside a special set of intervals. Fixing $\varepsilon_1 > 0$, we denote by $P(\varepsilon_1)$ the collection of all indices p such that $m_p(\gamma) - l_p(\gamma) > \varepsilon_1 m_p(\gamma)\gamma$. Put

$$E(\gamma,\varepsilon_1) = \bigcup_{p\in P(\varepsilon_1)} \left(l_p(\gamma)\gamma, m_p(\gamma)\gamma(1+4Cd(\gamma)\varepsilon^{-1})\right).$$

Let $\lambda \notin E(\gamma,\varepsilon_1)$. If to the left of λ there are points in $E(\gamma,\varepsilon_1)$, then one can find the greatest index $p(2) \in P(\varepsilon_1)$ such that $p(2) \le p(0)$. Otherwise, we set $p(2) = 0$. Let $p(2) \ne 0$. Taking (2.7) into account, we obtain

$$\begin{aligned}\sum_{p=1}^{p(2)} a_p(\lambda,\gamma) &\le \sum_{p=1}^{p(2)} \frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma)^2}{2\gamma(\lambda - m_p(\gamma)\gamma)} \le \varepsilon \sum_{p=1}^{p(2)} \frac{m_p(\gamma)-l_p(\gamma))^2}{8Cd(\gamma)m_p(\gamma)} \\ &\le \varepsilon \sum_{p=1}^{p(2)} \frac{m_p(\gamma)-l_p(\gamma)}{8C} \le \varepsilon n(0,\lambda,\Lambda^1)(8C)^{-1} \le \varepsilon\lambda/8.\end{aligned} \tag{2.15}$$

Starting with $p(s) < p(0)$, we choose the greatest indices $p(s) > \cdots > p(3) \ge p(2)$ such that

$$\begin{aligned}&\sum_{p=p(s)+1}^{p(0)} (m_p(\gamma)-l_p(\gamma)) > \varepsilon_1\lambda, \\ &\sum_{p=p(j-1)+1}^{p(j)} (m_p(\gamma)-l_p(\gamma)) > \varepsilon_1\lambda, \quad j=4,\dots,s, \\ &\sum_{p=p(2)+1}^{p(3)} (m_p(\gamma)-l_p(\gamma)) \le \varepsilon_1\lambda.\end{aligned} \tag{2.16}$$

If $p(3) = p(2)$ or $s = 3$, the situation simplifies. We consider the general case. We have

$$\lambda - m_i(\gamma)\gamma \ge \sum_{p=i+1}^{p(0)} (m_p(\gamma)-l_p(\gamma))\,\gamma, \quad i < p(0).$$

Using (2.16), the definition of $p(2)$, and the fact that $p(j)$ is maximal, we obtain

$$\begin{aligned}\sum_{p=p(j-1)+1}^{p(j)} a_p(\lambda,\gamma) &\le \sum_{p=p(j-1)+1}^{p(j)} \frac{\gamma(m_p(\gamma)-l_p(\gamma))^2}{2(\lambda-m_p(\gamma)\gamma)} \\ &\le \sum_{p=p(j-1)+1}^{p(j)} \frac{(m_p(\gamma)-l_p(\gamma))^2}{2\varepsilon_1\lambda(s+1-j)} \le \sum_{p=p(j-1)+1}^{p(j)} \frac{m_p(\gamma)-l_p(\gamma)}{2(s+1-j)} \\ &\le \frac{\varepsilon_1\lambda}{s+1-j}, \quad j=3,\dots,s.\end{aligned} \tag{2.17}$$

Moreover,

$$\sum_{p=p(s)+1}^{p(0)} a_p(\lambda,\gamma) \le \sum_{p=p(s)+1}^{p(0)} (m_p(\gamma)-l_p(\gamma)) \le 2\varepsilon_1\lambda. \tag{2.18}$$

Since $\lambda \notin E(\gamma,\varepsilon_1)$, we have $m_{p(0)+1}(\gamma) - l_{p(0)+1}(\gamma) \le \varepsilon_1 m_{p(0)+1}$. Then, applying (2.12) and the inequalities $\lambda > l_{p(0)+1}(\gamma)\gamma \ge m_{p(0)+1}(\gamma)(1-C\gamma)\gamma$, we see that

$$\int_0^\lambda \frac{n(\lambda,y,\Lambda^2(\gamma,1)) - n(\lambda,y,\Lambda^1(\gamma,1))}{y(1+y/\lambda)}\,dy \le 2\varepsilon_1\lambda + 2a(\lambda,\gamma,\widetilde{\delta}). \tag{2.19}$$

Thus, from (2.10), (2.11), (2.13), (2.15), and (2.17)–(2.19) it follows that

$$-I(\lambda,\gamma) \le \varepsilon\lambda/8 + \sum_{j=3}^{s} \frac{\varepsilon_1\lambda}{s+1-j} + 8\varepsilon_1\lambda + 3a(\lambda,\gamma,\widetilde{\delta}),$$
$$\lambda \notin E(\gamma,\varepsilon_1), |\lambda-\eta_j| \ge \widetilde{\delta}\gamma, \eta_j \in \Lambda^2(\gamma).$$

By (2.19) and (2.5), we have

$$(s-2)\varepsilon_1\lambda \le \sum_{p=p(3)+1}^{p(0)} (m_p(\gamma)-l_p(\gamma)) \le n(0,\lambda,\Lambda^1) \le C\lambda.$$

The Euler formula shows that one can find an absolute constant $B>0$ such that

$$\sum_{j=3}^{s} \frac{1}{s+1-j} \le \ln(s-2) + B - 8.$$

Thus, the following inequality is valid:

$$\begin{aligned} -I(\lambda,\gamma) &\le \varepsilon\lambda/8 + \varepsilon_1\lambda\left(\ln\frac{C}{\varepsilon_1} + B\right) + 3a(\lambda,\gamma,\widetilde{\delta}), \\ &\lambda \notin E(\gamma,\epsilon_1), \quad |\lambda-\eta_j| \ge \widetilde{\delta}\gamma, \quad \eta_j \in \Lambda^2(\gamma). \end{aligned} \tag{2.20}$$

5) At the fifth step we slightly “correct” the function $L_1(\lambda)$ in order to reduce the possible “splashes” of $\ln|L_1(\lambda)|$ on the set $E(\gamma,\varepsilon_1)$. For this, we need to complete the sequence Λ^1 in a special way. First, we observe that

$$E(\gamma,\varepsilon_1) \subset \bigcup_{p\in P(\varepsilon_1)} (m_p(\gamma)\gamma(1-\Theta(\gamma)), m_p(\gamma)\gamma(1+\Theta(\gamma)))\,, \tag{2.21}$$

where $\Theta(\gamma) = 8C^2\gamma\varepsilon^{-1}$. Here, it is assumed that $2C\varepsilon^{-1} \ge 1$. Suppose that $\alpha>0$ and $\beta(p,\alpha,\gamma)$ is the integral part of $\alpha m_p(\gamma)\gamma$. Denote by $\Lambda^3(\gamma,\alpha,\varepsilon_1)$ the sequence consisting of all points $m_p(\gamma)\gamma$, $p \in P(\varepsilon_1)$, such that each of them appears in the sequence exactly $\beta(p,\alpha,\gamma)$ times. Consider the function

$$L_3(\lambda,\alpha,\gamma,\varepsilon_1) = \prod_{p\in P(\varepsilon_1)} \left(1 - \left(\frac{\lambda}{m_p(\gamma)\gamma}\right)^2\right)^{\beta(p,\alpha,\gamma)}.$$

Let $p \in P(\varepsilon_1)$. By (2.5) and the definition of $P(\varepsilon_1)$, we have

$$Cm_p(\gamma)\gamma \geq n(0, m_p(\gamma)\gamma, \Lambda^1) \geq \sum_{j\leq p, j\in P(\varepsilon_1)} \varepsilon_1 m_j(\gamma)\gamma.$$

It follows that

$$n(0, m_p(\gamma)\gamma, \Lambda^3(\gamma,\alpha,\varepsilon_1)) = \sum_{j\leq p, j\in P(\varepsilon_1)} \beta(j,\alpha,\gamma) \leq \sum_{j\leq p, j\in P(\varepsilon_1)} \alpha m_j(\gamma)\gamma \leq \frac{\alpha C m_p(\gamma)\gamma}{\varepsilon_1}.$$

Therefore,

$$\limsup_{r\to\infty} \frac{n(0,r,\Lambda^3(\gamma,\alpha,\varepsilon_1))}{r} = \limsup_{r\to\infty} \frac{n(0,m_p(\gamma)\gamma,\Lambda^3(\gamma,\alpha,\varepsilon_1))}{m_p(\gamma)\gamma} \leq \frac{\alpha C}{\varepsilon_1} = B(\alpha,\varepsilon_1).$$

Hence (see [15, Chapter II, §2, Lemma 2.2]), for some $A(\alpha,\varepsilon_1) > 0$ we have

$$\ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| \leq A(\alpha,\varepsilon_1) + 2\pi B(\alpha,\varepsilon_1)|\lambda|, \quad \lambda\in\mathbb{C}. \tag{2.22}$$

Let $\beta \in (0,1)$. By Lemma 2.1, we also have the estimate

$$\begin{aligned} \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| \geq \ln|L_3(0,\alpha,\gamma,\varepsilon_1)| - b(\beta)(A(\alpha,\varepsilon_1) + 24\pi B(\alpha,\varepsilon_1)|\lambda|),& \\ \lambda \in E^3(\beta),& \end{aligned} \tag{2.23}$$

where the set $E^3(\beta) = \bigcup B_i^3$ is centered with the zero set of the function L_3 and has linear density of at most β.

For $p \in P(\varepsilon_1)$, consider the function

$$g_p(\lambda) = L_3(\lambda,\alpha,\gamma,\varepsilon_1)(1 - \lambda/m_p(\gamma)\gamma)^{-\beta(p,\alpha,\gamma)}.$$

This is an entire function that also satisfies estimate (2.22) on the circle $S(m_p(\gamma)\gamma, m_p(\gamma)\gamma)$. Using some simple estimates and the maximal modulus principle, we obtain

$$\ln|g_p(\lambda)| \leq A(\alpha,\varepsilon_1) + 8\pi B(\alpha,\varepsilon_1)|\lambda|, \quad \lambda\in B(m_p(\gamma)\gamma, m_p(\gamma)\gamma/2).$$

Let $\gamma > 0$ be such that $\Theta(\gamma) < 1/2$. Then for all $\lambda \in B(m_p(\gamma)\gamma, \Theta(\gamma)m_p(\gamma))$ we have

$$\begin{aligned} \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| &= \ln|g_p(\lambda)| + \beta(p,\alpha,\gamma)\ln|(1-\lambda)/(m_p(\gamma)\gamma)| \\ &\leq A(\alpha,\varepsilon_1) + 8\pi B(\alpha,\varepsilon_1)|\lambda| + (\alpha m_p(\gamma)\gamma - 1)\ln\Theta(\gamma) \\ &\leq A(\alpha,\varepsilon_1) + 8\pi B(\alpha,\varepsilon_1)|\lambda| + (\alpha|\lambda|/2 - 1)\ln\Theta(\gamma). \end{aligned}$$

Hence, taking (2.9), (2.14), and (2.21) into account, we obtain

$$\begin{aligned} &\ln|L_1(\lambda)| + \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| \\ &\qquad \leq \ln|L_2(\lambda,\gamma)| + A(\alpha,\varepsilon_1,\gamma) + B(\alpha,\varepsilon_1,\gamma)\lambda + 3a(\lambda,\gamma,\tilde{\delta}), \\ &\qquad \lambda \geq R, \quad |\lambda-\eta_j| \geq \tilde{\delta}\gamma, \quad \eta_j\in\Lambda^2(\gamma), \quad \lambda\in E(\gamma,\varepsilon_1), \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} A(\alpha,\varepsilon_1,\gamma) &= A(\alpha,\varepsilon_1) - \ln\Theta(\gamma), \\ B(\alpha,\varepsilon_1,\gamma) &= 8\pi B(\alpha,\varepsilon_1) + 7C + C_3 d(\gamma) + (\alpha\ln\Theta(\gamma)/2. \end{aligned}$$

Moreover, by (2.9), (2.20), and (2.22), the following inequality is valid:

$$\begin{aligned} &\ln|L_1(\lambda)| + \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| \\ &\quad\le \ln|L_2(\lambda,\gamma)| + A(\alpha,\varepsilon_1) + \widetilde{B}(\alpha,\varepsilon_1,\gamma)\lambda + 3a(\lambda,\gamma,\widetilde{\delta}), \\ &\quad \lambda\ge R,\quad |\lambda-\eta_j|\ge\widetilde{\delta}\gamma,\quad \eta_j\in\Lambda^2(\gamma),\quad \lambda\notin E(\gamma,\varepsilon_1), \end{aligned} \tag{2.25}$$

where $\widetilde{B}(\alpha,\varepsilon_1,\gamma)=\varepsilon/8+\varepsilon_1(\ln(C/\varepsilon_1)+B)+C_3d(\gamma)+2\pi B(\alpha,\varepsilon_1)$.

6) At our last step we construct the desired function f. First, we define two auxiliary functions. We put $\Lambda^4(\gamma)=\bigcup_p\Lambda^4_p(\gamma)$, where $\Lambda^4_p(\gamma)=\varnothing$ if $m_p(\gamma)=l_{p+1}(\gamma)$, and $\Lambda^4_p(\gamma)=\{(m_p(\gamma)+1/2)\gamma,\dots,(l_{p+1}(\gamma)-1/2)\gamma\}$ otherwise. We also put $\Lambda^5(\gamma)=\Lambda^2(\gamma)\cup\Lambda^4(\gamma)$, $\Lambda^6(\gamma)=\Lambda^1\cup\Lambda^3(\gamma,\alpha,\varepsilon_1)\cup\Lambda^4(\gamma)$ and

$$L_5(\lambda,\gamma)=\prod_{\eta_j\in\Lambda^5(\gamma)}\left(1-\frac{\lambda^2}{\eta_j^2}\right),\quad \widetilde{f}(\lambda)=\prod_{\eta_j\in\Lambda^6(\gamma)}\left(1-\frac{\lambda^2}{\eta_j^2}\right).$$

It is not difficult to check that $\Lambda^5(\gamma)$ has density $1/\gamma$ $(n(0,r,\Lambda^4(\gamma))/r\to1/\gamma)$ and is regular (with distances between elements of at least $\gamma/2$). Hence (see [17, Chapter I, §2, Theorem 1.2.9] and [6, Chapter II, §1, Theorem 5]), for each $\widetilde{\delta}>0$ and some $r(\widetilde{\delta})>1$, we have

$$\big|\ln|L_5(\lambda,\gamma)|-\pi|\operatorname{Im}\lambda|/\gamma\big|\le\varepsilon|\lambda|/8,\quad \lambda\in\mathbb{C}\setminus\big(B(0,r(\widetilde{\delta}))\cup E^5(\widetilde{\delta})\big), \tag{2.26}$$

where $E^5(\widetilde{\delta})=\bigcup_{\mp\eta_j\in\Lambda^5(\gamma)}B(\eta_j,\widetilde{\delta}\gamma)$. Sequentially, we fix $\varepsilon_1,\beta,\alpha,\gamma>0$ such that

$$\varepsilon_1(\ln(C/\varepsilon_1)+B)<\varepsilon/8, \tag{2.27}$$

$$\beta<\delta/72, \tag{2.28}$$

$$\pi B(\alpha,\varepsilon_1)<\varepsilon/8,\quad 24b(\beta)\pi B(\alpha,\varepsilon_1)<\varepsilon/8, \tag{2.29}$$

$$\begin{aligned} &\gamma<1/4C,\quad \Theta(\gamma)<1/2,\quad 7C+(\alpha\ln\Theta(\gamma)/2<0, \\ &C_2\sqrt{d(\gamma)}<\varepsilon/8,\quad C_3d(\gamma)<\varepsilon/8,\quad \sqrt[4]{d(\gamma)}<\delta/72. \end{aligned} \tag{2.30}$$

Then applying (2.24)–(2.27), (2.29), and (2.30), we see that

$$\ln|\widetilde{f}(\lambda)|\le A(\alpha,\varepsilon_1,\gamma)+\varepsilon\lambda/2+\varepsilon/32+3a(\lambda,\gamma,\widetilde{\delta})\le 5\varepsilon\lambda/8 \tag{2.31}$$

for all $\lambda\notin E^5(\widetilde{\delta})$, $\lambda\ge R(\alpha,\varepsilon_1,\gamma,\widetilde{\delta})\ge\max\{R,r(\widetilde{\delta})\}$. It follows that

$$h_{\widetilde{f}}(\lambda)\le 2\varepsilon\lambda/3,\quad \lambda\ge0. \tag{2.32}$$

Indeed, if this is not the case, then the Bernstein theorem (see [6, Chapter I, §18, Theorem 31]) allows us to find $\tau>0$ and a sequence $0<r_n\to\infty$ such that on each interval $(r_n,(1+\tau)r_n)$ the inequality

$$\ln|\widetilde{f}(\lambda)|>5\varepsilon\lambda/8 \tag{2.33}$$

is fulfilled everywhere except possibly a set of measure at most $\tau r_n/2$. On the other hand, the measure of the set $E^5(\widetilde{\delta})\cap(r_n,(1+\tau)r_n)$ does not exceed $2\widetilde{\delta}\gamma(\tau r_n/\gamma+1)$. Hence, for $\widetilde{\delta}<1/4$ and for any sufficiently large n, on to the interval $(r_n,(1+\tau)r_n)$ we can find a point λ for which (2.31) and (2.33) are valid. We get a contradiction. Therefore, (2.32) is true.

Finally, we define the function f. We put $\Lambda^7(\gamma)=\Lambda^3(\gamma,\alpha,\varepsilon_1)\cup\Lambda^4(\gamma)$ and

$$f(\lambda)=L(\lambda)\prod_{\eta_j\in\Lambda^7(\gamma)}\left(1-\frac{\lambda^2}{\eta_j^2}\right)$$

(assuming that n_k equal multipliers correspond to each $\lambda_k\in\Lambda$). As above, f is an entire function of exponential type. At the points λ_k it has zeros of multiplicity at least

n_k. Let $\varepsilon' > 0$. Since the function $h_{\widetilde{f}}$ is continuous, one can find $\tau \in (0,1)$ such that $h_{\widetilde{f}}(\lambda) \le h_{\widetilde{f}}(1) + \varepsilon'$, $\lambda \in B(1,\tau)$. Then (see [6, Chapter I, §18, Therem 28]) for some $R' > 0$ we have

$$\ln|\widetilde{f}(t\lambda)| \le h_{\widetilde{f}}(t) + 2\varepsilon' t, \quad t \ge R', \quad \lambda \in B(1,\tau). \tag{2.34}$$

By (2.3),

$$\ln|f(\lambda)| - \ln|\widetilde{f}(\lambda)| \le C_1\sqrt{d}|\lambda|, \quad \lambda \in \mathbb{C} \setminus E^1(d), \tag{2.35}$$

for any $d \in (0,1/2)$, where $E^1(d) = \bigcup B(y_i,q_i)$ has liner density of at most $\sqrt[4]{d}$. We choose d such that $C_1\sqrt{d} \le \varepsilon'$ and $\sqrt[4]{d} \le \tau/6$. Then for some $r' > 0$ and any $r \ge r'$ we have $\sum_{|y_i|<r} q_i \le \tau r/5$.

Suppose $r \ge r'$ and $B(y_i,q_i)$ intersects $B(0,r)$. If $|y_i| \ge r$, then, as above, $q_i \le \tau|y_i|/5$. Thus, $(1-\tau/5)|y_i| < r$. It follows that $|y_i| < 5r/4$. Therefore, the sum of the radii of all disks $B(y_i,q_i)$ that intersect $B(0,r)$ is not greater than $\tau r/4$.

Let $\lambda \ge r'$. As it has been proved, the sum of the diameters of the disks $B(y_i,q_i)$ intersecting $B(\lambda,\tau\lambda)$ does not exceed $\tau(1+\tau)\lambda/2 < \tau\lambda$. Thus, we can find $\tau' \in (0,\tau\lambda)$ such that the circle $S(\lambda,\tau')$ does not intersect the set $E^1(d)$. Then using (2.34) and (2.35) and recalling our choice of d and the maximum of the modulus principle, we obtain $\ln|f(\lambda)| \le \varepsilon'(1+\tau)\lambda + h_{\widetilde{f}}(\lambda) + 2\varepsilon'\lambda$. Since $\varepsilon' > 0$ is arbitrary, now from (2.32) we deduce that

$$h_f(\lambda) \le 2\varepsilon\lambda/3, \quad \lambda \ge 0. \tag{2.36}$$

Now we choose $d \in (0,1/2)$ and $\widetilde{\delta} > 0$ so that $C_1\sqrt{d} < \varepsilon/8$, $\sqrt[4]{d} < \delta/72$, and the linear density of $E^5(\widetilde{\delta})$ is at most $\delta/72$. Put

$$E = E^1(d) \cup E^2(\gamma) \cup E^3(\beta) \cup E^5(\widetilde{\delta}).$$

(Obviously, we may assume that E is symmetric about the origin.) Then by (2.28) and (2.30), the inequality $p_E \le \delta/18$ is true. Thus, as above, we can find $r'' > 0$ such that for $r \ge r''$, the sum of the diameters of all disks in E with nonempty intersection with $B(0,(1+\delta)r)$ is strictly less than $\delta r/3$. Let $r_0 > r''$. Then for each $i \ge 1$, there exists a number t_i in the interval $((1+\delta/3)^{i-1}r_0,\ (1+\delta/3)^i r_0)$ for which the circle $S(0,t_i)$ does not intersect E. By our choice, we have $t_{i+1} > t_i > (1+\delta/3)^{i-1}r_0 \to \infty$ as $i \to \infty$, and $t_{i+1} < (1+\delta/3)^{i+1}r_0 < (1+\delta/3)^2 t_i < (1+\delta)t_i$. It remains to verify (2.1) and (2.2).

By our choice of d and inequalities (2.3), (2.8), (2.22), (2.23), (2.26), and (2.29), we can find $\widetilde{r} \ge r(\widetilde{\delta})$ such that

$$|\ln|f(\lambda)| - \pi|\operatorname{Im}\lambda|/\gamma)| \le 2\varepsilon|\lambda|/3, \quad \lambda \in \mathbb{C} \setminus (B(0,\widetilde{r}) \cup E). \tag{2.37}$$

Let us show that the part of E lying in the exterior of a sufficiently large ball centered at the origin, is contained in $\Gamma(\delta)$. We may assume that E is symmetric about the origin because all functions in our proof are even. Hence, it suffices to consider the part of E in the right half-plane. It is centered with the set $\Lambda'' = \Lambda \cup \Lambda^1 \cup \Lambda^2(\gamma) \cup \Lambda^3(\gamma,\alpha,\varepsilon_1) \cup \Lambda^4(\gamma)$. All components of that set except for the first lie in the positive half-axis. Since Λ is almost real, there exists $\widetilde{r}_1 \ge \max\{\widetilde{r}, r''\}$ such that $|\operatorname{Im}\lambda_k| < (\delta\operatorname{Re}\lambda_k)/6$ for all k satisfying $|\lambda_k| \ge \widetilde{r}_1$. Let $B(y,q)$ be a disk that is included in E and contains a point z' with modulus of at least $(1+\delta)\widetilde{r}_1$. Then $B(y,q) \subset \mathbb{C} \setminus B(0,\widetilde{r}_1)$ (otherwise, as has been proved above, $2q < (\delta\widetilde{r}_1)/3$, and so $B(y,q) \subset B(0,(1+\delta)\widetilde{r}_1)$. Hence, by our choice of r'', we have $q < \delta|y|/6$. There exists a point $\varsigma \in \Lambda''$ belonging to the disk $B(y,q)$. Since $\varsigma \in B(y,q)$, we have $|\varsigma| \ge \widetilde{r}_1$ and $|\varsigma| \ge (1-\delta/6)|y|$. By our choice of $\widetilde{r}_1$, we have

$|\operatorname{Im}\varsigma| < \delta \operatorname{Re}\varsigma/6$. Hence, for any $\eta \in B(y,q)$, we obtain

$$\begin{aligned}|\eta - \operatorname{Re}\varsigma| &\le |\eta - y| + |y-\varsigma| + |\varsigma - \operatorname{Re}\varsigma| \\ &\le \delta|y|/6 + \delta|y|/6 + \delta\operatorname{Re}\varsigma/6 \le \delta|\varsigma|/(3(1-\delta/6)) + \delta\operatorname{Re}\varsigma/6 \\ &\le \delta(1+\delta/6)\operatorname{Re}\varsigma/(3(1-\delta/6)) + \delta\operatorname{Re}\varsigma/6 < \delta\operatorname{Re}\varsigma.\end{aligned}$$

It follows that $E \setminus B(0,(1+\delta)\widetilde{r}_1) \subset \Gamma(\delta)$. Let $r_0 > (1+\delta)\widetilde{r}_1$ be fixed. Then (2.37) is valid for all points of the circles $S(0,t_i)$, $i \ge 1$. This completes the proof of (2.1).

To prove (2.2), consider the function $h(\lambda) = h_f(\lambda) - 2\varepsilon\operatorname{Re}\lambda/3 - 2\varepsilon\operatorname{Im}\lambda/3 - \pi\operatorname{Im}\lambda/\gamma$. The imaginary axis does not pass through $\Gamma(\delta)$. Thus, by (2.36) and (2.37), we have $h(t) \le 0$, $h(it) \le 0$, $t \ge 0$. Since the function $h(\lambda)$ is convex, we have

$$h_f(\lambda) \le \pi|\operatorname{Im}\lambda|/\gamma + 2\varepsilon\operatorname{Re}\lambda/3 + 2\varepsilon\operatorname{Im}\lambda/3 \le \pi|\operatorname{Im}\lambda|/\gamma + \varepsilon|\lambda|$$

for all points in the first quadrant. A similar inequality is valid for the points in the fourth quadrant, and so everywhere on the plane because f is even. The theorem is proved. □

Remarks. 1. The statement of Theorem 31 in Chapter I of the book [6] (Bernstein's theorem) mentioned above contains an inaccuracy. Specifically, $\varepsilon, \delta > 0$ are assumed to be arbitrary numbers. In fact, $\varepsilon > 0$ is allowed to be any number, while $\delta > 0$ only exists and depends on ε. The theorem was proved under such assumptions. We used it above in this formulation. Note that in the English version [23] of the book [6] the theorem of Bernstein is formulated consistently.

2. The idea of the proof of Theorem 2.2 (steps 2, 3, 6) is borrowed from the paper [13] (Lemma 9). For the steps 4, 5 we use the method described in the proof of Theorem 8.3 in [2].

3. Suppose that in the statement of Theorem 2.2 yet another condition is imposed:

$$M(\Lambda) = \lim_{\delta\to 0}\limsup_{\lambda\to\infty}\frac{n(\lambda,\delta|\lambda|,\Lambda)}{|\lambda|} = \lim_{\delta\to 0}\limsup_{k\to\infty}\frac{n(\lambda_k,\delta|\lambda_k|,\Lambda)}{|\lambda_k|} = 0.$$

Then the function $\ln|L_1(\lambda)|$ has no "splashes" on the set $E(\gamma,\varepsilon_1)$. Therefore, we do not need step 5, and the desired function f can be defined by the product $L(\lambda)L_4(\lambda)$, where L_4 is constructed from $\Lambda^4(\gamma)$ in the same way as L_2 was constructed from $\Lambda^2(\gamma)$. Indeed, let

$$\lambda \in \left(l_p(\gamma)\gamma, m_p(\gamma)\gamma(1+4Cd(\gamma)\varepsilon^{-1})\right), \quad p \in P(\varepsilon_1).$$

By (2.12), the second term of (2.10) does not exceed $2a(\lambda,\gamma,\widetilde{\delta}) + 2d(\gamma)\lambda$.

Let $\widetilde{p}$ be the smallest index satisfying $m_{\widetilde{p}}(\gamma)\gamma \le (1 - 4Cd(\gamma)\varepsilon^{-1})\lambda$. Then for sufficiently small γ and sufficiently large $\lambda > 0$, we have

$$\begin{aligned}\sum_{p=\widetilde{p}+1}^{p(0)} a_p(\lambda,\gamma) &\le \sum_{p=\widetilde{p}+1}^{p(0)} \frac{2\lambda - m_p(\gamma)\gamma}{\gamma}\ln\left(1 + \frac{m_p(\gamma)\gamma - l_p(\gamma)\gamma}{2\lambda - m_p(\gamma)\gamma}\right) \\ &\le \sum_{p=\widetilde{p}+1}^{p(0)} (m_p(\gamma) - l_p(\gamma)) \le n(\lambda,(4C\varepsilon^{-1}+1)d(\gamma)\lambda,\Lambda) \le \varepsilon\lambda/8.\end{aligned}$$

Estimation of the remaining sum $\sum_{p=1}^{\widetilde{p}} a_p(\lambda,\gamma)$ is done in the same way as in (2.15). Hence, in this case the required upper estimate of $\ln|L_1(\lambda)|$ is obtained without constructing the additional function L_3, which makes the calculations considerably simpler.

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. As in the paper [4], we set

$$q_\Lambda(\lambda,w,\delta) = \prod_{\lambda_k\in B(w,\delta|w|)}\left(\frac{\lambda-\lambda_k}{3\delta|\lambda_k|}\right)^{n_k}.$$

In the case where the disk $B(w,\delta|w|)$ does not contain any of λ_k, we put $q_\Lambda(\lambda,w,\delta)\equiv 1$. The modulus of $q_\Lambda(\lambda,w,\delta)$ can be interpreted as a characteristic for the concentration of points $\lambda_k\in B(w,\delta|w|)$ near λ. The meaning of the quantity $\ln|q_\Lambda(\lambda,w,\delta)|/|w|$ is similar to the logarithm of the geometric mean (or the arithmetic mean of the logarithms) of the normalized distances from $\lambda_k\in B(w,\delta|w|)$ to λ. If $\delta\in(0,1)$, then the modulus of each factor q_Λ in the disk $B(w,\delta|w|)$ is estimated from above by the number $2(3(1-\delta))^{-1}$. Hence, for $\delta\in(0,1/3)$, it is less than or equal to 1. Moreover, if $\delta_1\le\delta_2$ and $B(w_1,\delta_1|w_1|)\subset B(w_2,\delta_2|w_2|)$, then the number of the factors $q_\Lambda(z,w_1,\delta_1)$ is not greater than the number of the factors $q_\Lambda(z,w_2,\delta_2)$, and the modulus of each factor $q_\Lambda(z,w_1,\delta_1)$ is at least the modulus of the corresponding factor $q_\Lambda(z,w_2,\delta_2)$. Hence, $|q_\Lambda(z,w_1,\delta_1)|\ge|q_\Lambda(z,w_2,\delta_2)|$, $z\in B(w_2,\delta_2|w_2|)$. We introduce the function

$$q^m_\Lambda(\lambda,\delta)=\prod_{\substack{\lambda_k\in B(\lambda_m,\delta|\lambda_m|)\\ k\ne m}}\left(\frac{\lambda-\lambda_k}{3\delta|\lambda_k|}\right)^{n_k},\quad m\ge 1.$$

If the disk $B(\lambda_m,\delta|\lambda_m|)$ contains no points λ_k with $k\ne m$, then $q^m_\Lambda(z,\delta)\equiv 1$. We set (see [4])

$$S_\Lambda=\lim_{\delta\to 0}\liminf_{m\to\infty}\ln|q^m_\Lambda(\lambda_m,\delta)|/|\lambda_m|.$$

The first limit (as $\delta\to 0$) exists, because, by our previous observation, the function under the limit sign is monotone nondecreasing as $\delta\to 0$. Moreover, this function is nonpositive. Consequently, $S_\Lambda\le 0$. The meaning of S_Λ is similar to that of the classical Bernstein index of condensation (see, e.g., [15, Chapter II, §5, Subsection 2]), but is applicable (unlike the Bernstein condensation index) to any complex sequence (not only a measurable positive sequence or a complex sequence of zero density). Note also that the coefficient 3 in the definition of q_Λ is chosen for convenience (see the remark to Theorem 5.1 in [4]). It provides the inequality $S_\Lambda\le 0$.

The fact that $S_\Lambda=0$ means that the points λ_k are separated away from each other in a sense. The character of this sparseness is specified in Lemma 2.3 in [12]. We formulate that statement in a partial case and in a form convenient to us. We need the following notation.

Let $\Theta\in(0,1)$. For any $k\ge 1$, we denote by β_k the minimum of all distances from λ_k to the points λ_m, $m\ne k$. We fix $k\ge 1$. If $n_k\le\beta_k/2$, we put $\gamma_k(\Theta)=\Theta n_k$. Otherwise, we define $\gamma_k(\Theta)=(\Theta\beta_k)/2$.

It is not difficult to check that $\lim_{k\to\infty}n_k/|\lambda_k\le\bar n(\Lambda)$. Therefore, by Lemma 2.3 in [12] (in view of the definition of $\gamma_{m,l}$ at the beginning of its proof), the following statement if true.

Lemma 2.3. *Let a sequence $\Lambda=\{\lambda_k,n_k\}$ have finite upper density, and let $S_\Lambda=0$. Then for every $\varepsilon>0$ and $\Theta\in(0,1)$, there exists $R>0$ and $\delta\in(0,1/3)$ such that for all $|w|\ge R$ and $k\ge 1$ we have*

$$\ln|q_\Lambda(\lambda,w,\delta)|\ge-\varepsilon|\lambda|,\quad \lambda\in B(\lambda_k,\gamma_k(1))\setminus B(\lambda_k,\gamma_k(\Theta))\cap B(w,\delta|w|).$$

In what follows by a contour we mean a simple closed continuous rectifiable curve.

Theorem 2.4. *Suppose $\Lambda=\{\lambda_k,n_k\}$ is an almost real sequence, $\bar n(\Lambda)<+\infty$, and $S_\Lambda=0$. Then for any $\varepsilon_0>0$ and $\delta_0\in(0,1/3)$, there exists $\gamma>0$, an entire function of exponential type f, an index k_0, and numbers $r_k\in(0,\delta_0|\lambda_k|)$, $k\ge k_0$, such that:*

1) *f has zeros at the points λ_k, $k\ge 1$, of multiplicities at least n_k;*

2) *for all $k\ge k_0$, in the disk $B(\lambda_k,r_k)$ there are no points belonging to Λ and different from λ_k;*

3) $\ln|f(\lambda)|\ge\pi|\operatorname{Im}\lambda|/\gamma-\varepsilon_0|\lambda|$, $\lambda\in S(\lambda_k,r_k)$, $k\ge k_0$;

4) $h_f(\lambda)\le\pi|\operatorname{Im}\lambda|/\gamma+\varepsilon_0|\lambda|$, $\lambda\in\mathbb{C}$.

Proof. We fix $\varepsilon_0 > 0$ and $\delta_0 \in (0, 1/3)$, and let $\varepsilon \in (0, \varepsilon_0)$. By assumption, we have $S_\Lambda = 0$. Then Lemma 2.3 shows that we can find k_1 and $\delta_1 \in (0, \delta_0)$ such that

$$\ln|q_\Lambda(\lambda, \lambda_k, \delta_1)| \geq -\varepsilon|\lambda|, \quad \lambda \in B(\lambda_k, \gamma_k(1)) \setminus B(\lambda_k, \gamma_k(1/4)), \tag{2.38}$$

whenever $\gamma_k(1) < \delta_1|\lambda_k|$ é $k \geq k_1$.

Suppose that $\varepsilon_1 > 0$ satisfies (2.27), where the constants C, B are the same as in Theorem 2.2. We choose $s > 2$ such that $\varepsilon_1 \ln(2/s) < -2\varepsilon$ and fix $\delta \in (0, \delta_1/s)$. Let f be a function constructed for the numbers $\varepsilon_1, \beta, \alpha, \gamma > 0$, satisfying (2.27)–(2.30) and yet another condition: $\Theta(\gamma) < \delta_1/2s$. Then statements 1) and 4) are true. It remains to determine an index k_0 and numbers $r_k \in (0, \delta_0|\lambda_k|)$, $k \geq k_0$, such that 2) and 3) are valid.

As in Theorem 2.2, we can find $k_0 \geq k_1$ such that for $k \geq k_0$ the sum of the diameters of all circles from E having a nonempty intersection with $B(0, (1+\delta)|\lambda_k|)$ is strictly less than $\delta r/3$. Taking k_0 larger if necessary, we may assume that $|\lambda_k| \geq (1-\delta_1)^{-1}\widetilde{r}$, $k \geq k_0$, where $\widetilde{r}$ is as in (2.37).

Fixing $k \geq k_0$, we consider two different cases.

1. There are no points of Λ belonging to the disk $B(\lambda_k, (\delta_1|\lambda_k|)/s)$ different from λ_k. By our choice of k_0, we can find $r_k \in (0, \delta|\lambda_k|) \subset (0, \delta_0|\lambda_k|)$ such that (2.37) is valid for all points belonging to the circle $S(\lambda_k, r_k)$. Thus, in this case, taking our choice of δ into account, we obtain 2) and 3).

2. There is a point of Λ belonging to the disk $B(\lambda_k, (\delta_1|\lambda_k|)/s)$ that differs from λ_k. Then $\gamma_k(1) < (\delta_1|\lambda_k|)/2s$. As in (2.37), we obtain

$$\ln|f(\lambda)| - \ln|L_5(\lambda, \gamma)| \geq -\varepsilon|\lambda|, \quad \lambda \in \mathbb{C} \setminus (B(0, \widetilde{r}) \cup E^1(d) \cup E^2(\gamma) \cup E^3(\beta)). \tag{2.39}$$

Moreover, (2.26) allows us to assume that

$$\ln|L_5(\lambda, \gamma)| - \pi|\operatorname{Im}\lambda|/\gamma \geq -\varepsilon|\lambda|, \quad \lambda \in \mathbb{C} \setminus (B(0, \widetilde{r}) \cup E^5(\widetilde{\delta})). \tag{2.40}$$

We recall that the numbers d and $\widetilde{\delta}$ were introduced in Theorem 2.2 before the definition of the set E. By the choice of $\widetilde{\delta}$, we have $\widetilde{\delta}\gamma < \gamma/8 < 1/8$.

As above, we find $\widetilde{r}_k \in (2\delta_1|\lambda_k|)/3s, \delta_1|\lambda_k|/s)$ such that the circle $S(\lambda_k, \widetilde{r}_k)$ does not intersect the exceptional set in (2.39). Then

$$\ln|f(\lambda)| - \ln|L_5(\lambda, \gamma)| \geq -\varepsilon|\lambda|, \quad \lambda \in S(\lambda_k, \widetilde{r}_k).$$

Since Λ is an almost real sequence, we can choose k_0 to be sufficiently large and assume that $|\lambda| \leq 2\operatorname{Re}\lambda$, $\lambda \in B(\lambda_k, (\delta_1|\lambda_k|)/s)$. Hence, using the definitions of the functions f and L_5, we obtain

$$\ln|L(\lambda)| + \ln|L_3(\lambda, \alpha, \gamma, \varepsilon_1)| \geq \ln|L_2(\lambda, \gamma)| - 2\varepsilon\operatorname{Re}\lambda, \quad \lambda \in S(\lambda_k, \widetilde{r}_k). \tag{2.41}$$

Moreover, we may assume that $\gamma_k(1) + 2 < (\delta_1|\lambda_k|)/2s + 2 < (2\delta_1|\lambda_k|)/3s$. Then $\ln|\lambda - \eta_i| \geq 0$, $\lambda \in S(\lambda_k, \widetilde{r}_k)$, $i \in I(k)$. Here, $I(k)$ is the set of all indices i for which the disk $B(\eta_i, \widetilde{\delta}\gamma)$ centered at the zero $\eta_j \in \Lambda^2(\gamma)$ of the function L_2 intersects $B(\lambda_k, \gamma_k(1))$. By (2.41),

$$\begin{gathered}\ln|L(\lambda)| + \ln|L_3(\lambda, \alpha, \gamma, \varepsilon_1)| \geq \ln|L_2(\lambda, \gamma)| - \ln|\lambda - \eta_i| - 2\varepsilon\operatorname{Re}\lambda, \\ \lambda \in S(\lambda_k, \widetilde{r}_k), \quad i \in I(k).\end{gathered} \tag{2.42}$$

The disks $B(\eta_i, \widetilde{\delta}\gamma)$, $\eta_j \in \Lambda^5(\gamma)$ are pairwise disjoint because $\widetilde{\delta}\gamma < \gamma/8$. Hence, (2.40) implies the estimate

$$\ln|L_5(\lambda, \gamma)| - \ln|\lambda - \eta_i| \geq \pi|\operatorname{Im}\lambda|/\gamma - 2\varepsilon\operatorname{Re}\lambda, \quad \lambda \in S(\eta_i, \widetilde{\delta}\gamma), \quad i \in I(k).$$

The function on the left-hand side of the last inequality is harmonic in the disk $B(\eta_i, \widetilde{\delta}\gamma)$. Therefore,

$$\ln|L_5(\lambda,\gamma)| - \ln|\lambda-\eta_i| \ge \pi|\operatorname{Im}\lambda|/\gamma - 2\varepsilon\operatorname{Re}\lambda, \quad \lambda\in\overline{B(\eta_i,\widetilde{\delta}\gamma)}, \quad i\in I(k). \tag{2.43}$$

Suppose that the disk $B(\lambda_k, \delta_1|\lambda_k|/s)$ contains a zero $m_p(\gamma)\gamma$, $p\in P(\varepsilon_1)$, of the function L_3. Then all elements of the group $\Lambda_p^1(\gamma)$ belong to $B(\lambda_k, 7\delta_1|\lambda_k|/4s)$, because $\Theta(\gamma) < \delta_1/2s$. The number of such elements is at least $\varepsilon_1 m_p(\gamma)\gamma$. Since Λ is an almost real sequence, the definition of Λ^1 shows that we can assume that the disk $B(\lambda_k, 2\delta_1|\lambda_k|/s)$ contains at least $\varepsilon_1 m_p(\gamma)\gamma$ points of λ_m with multiplicities.

If $\lambda, \lambda_m \in B(\lambda_k, 2\delta_1|\lambda_k|/s) \subset B(\lambda_k, \delta_1|\lambda_k|)$, then

$$\left|\frac{\lambda-\lambda_m}{3\delta_1|\lambda_m|}\right| \le \frac{4\delta_1|\lambda_k|}{3s\delta_1|\lambda_m|} \le \frac{4|\lambda_k|}{3s(1-2\delta_1/s)|\lambda_k|} \le 2/s.$$

Taking our choice of s into account, we obtain

$$\ln|q_\Lambda(\lambda,\lambda_k,\delta_1)| \le \varepsilon_1 m_p(\gamma)\gamma\ln(2/s) < -2\varepsilon(1-\delta_1/s)(1+2\delta_1/s)^{-1}|\lambda| \le -5\varepsilon|\lambda|/4,$$

for all $\lambda\in B(\lambda_k, 2\delta_1|\lambda_k|/s)$. This contradicts (2.38) because $\gamma_k(1) < (\delta_1|\lambda_k|)/2s$. Therefore, there are no zeros of L_3 in the disk $B(\lambda_k, \delta_1|\lambda_k|/s)$. This implies that the function $\ln|L(\lambda)| + \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| - \ln|q_\Lambda(\lambda,\lambda_k,\delta_1)|$ is harmonic in that disk. By the inequality $|q_\Lambda(\lambda,\lambda_k,\delta_1)| < 1$, $\lambda\in B(\lambda_k,\delta_1|\lambda_k|)$, in view of (2.41) and (2.42) we have

$$\begin{gathered}\ln|L(\lambda)| + \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| - \ln|q_\Lambda(\lambda,\lambda_k,\delta_1)| \ge \ln|L_2(\lambda,\gamma)| - 2\varepsilon\operatorname{Re}\lambda, \\ \lambda\in B(\lambda_k,\widetilde{r}_k),\end{gathered} \tag{2.44}$$

$$\begin{gathered}\ln|L(\lambda)| + \ln|L_3(\lambda,\alpha,\gamma,\varepsilon_1)| - \ln|q_\Lambda(\lambda,\lambda_k,\delta_1)| \\ \ge \ln|L_2(\lambda,\gamma)| - \ln|\lambda-\eta_i| - 2\varepsilon\operatorname{Re}\lambda, \quad \lambda\in B(\lambda_k,\widetilde{r}_k), \quad i\in I(k).\end{gathered} \tag{2.45}$$

Combining estimates (2.38), (2.40) and (2.43)–(2.45), we obtain

$$\ln|f(\lambda)| \ge \pi|\operatorname{Im}\lambda|/\gamma - 5\varepsilon|\lambda|, \quad \lambda\in B(\lambda_k,\gamma_k(1)) \setminus \big(B(\lambda_k,\gamma_k(1/4)) \cup E^4(\widetilde{\delta})\big), \tag{2.46}$$

where $E^4(\widetilde{\delta}) = \bigcup_{\eta_j\in\Lambda^4(\gamma)} B(\eta_j,\widetilde{\delta}\gamma)$. Let $\varepsilon\in(0,\varepsilon_0/5)$. By the definition of $\gamma_k(1)$, there are no points of the sequence Λ in the disk $B(\lambda_k,\gamma_k(1))$ different from λ_k. Hence, by (2.46), it remains to find $r_k\in[\gamma_k(1/4),\gamma_k(1)) \subset (0,\delta_0|\lambda_k|)$ such that the circle $S(\lambda_k,r_k)$ does not intersect the set $E^4(\widetilde{\delta})$.

The definitions of the sequences Λ^1 and $\Lambda^4(\gamma)$ imply that the distance from λ_k to any point $\eta_j\in\Lambda^4(\gamma)$ is at least $\gamma/2$. If $\gamma_k(1/4)\le\gamma/4$, then the circle $S(\lambda_k,\gamma_k(1/4))$ does not intersect $E^4(\widetilde{\delta})$. In this case, we put $r_k=\gamma_k(1/4)$.

Let $\gamma_k(1/4) > \gamma/4$. We know that $B(\lambda_k,\gamma_k(1))$ intersects at most $\gamma_k(1)/\gamma+1$ disks in $E^4(\widetilde{\delta})$. The sum of the diameters of these disks does not exceed the number

$$(\gamma_k(1)/\gamma+1)\gamma/4 \le \gamma_k(1)/4+\gamma/4 \le \gamma_k(1)/4+\gamma_k(1/4) = \gamma_k(1)/2.$$

Therefore, there is a circle $S(\lambda_k,r_k)$ in the annulus $B(\lambda_k,\gamma_k(1)) \setminus B(\lambda_k,\gamma_k(1/4))$ that does not intersect the set $E^4(\widetilde{\delta})$. The theorem is proved. □

Remarks. 1. Suppose that in the statement of Theorem 2.4 we have yet another condition: $\limsup_{k\to\infty} n_k/|\lambda_k| = 0$. Together with the condition $S_\Lambda = 0$, this gives (see [12, Lemma 2.2]) the relation $M(\Lambda)=0$. Then, as it was mentioned in Remark 3 to Theorem 2.2, we can define the desired function f as the product $L(\lambda)L_4(\lambda)$.

2. It is not difficult to check that the conjugate diagram K (see [15, Chapter I, §5, Section 2]) of the function f constructed in Theorems 2.2 and 2.4 contains the origin. Indeed, the inequality $h_f(\lambda)\ge 0$, $\lambda\in\mathbb{C}$, follows from the symmetry of the zeros of f, and

by the Polya theorem (see [15, Chapter I, §5, Theorem 5.4]), we have $h_f(\lambda) = H_K(\lambda)$, $\lambda \in \mathbb{C}$. Here,

$$H_K(\lambda) = \sup_{z\in K} \operatorname{Re}(\lambda z), \quad \lambda \in \mathbb{C},$$

is the support function of the compact convex set K (more precisely, the complex conjugate of K).

Let $\Lambda = \{\lambda_k, n_k\}$. The number

$$n_0(\Lambda) = \limsup_{\delta\to 0} \limsup_{r\to\infty} \frac{n(0, r, \Lambda) - n(0, (1-\delta)r, \Lambda)}{\delta r}$$

is called the maximal density of Λ.

For $\mathcal{R} \subset \mathbb{C}$, we denote by $\mathcal{R}^\delta$, $\delta > 0$, the union of the disks $B(\lambda, \delta|\lambda|)$, $\lambda \in \mathcal{R}$.

Theorem 2.5. *Suppose $\Lambda = \{\lambda_k, n_k\}$ is an almost real sequence, $\bar{n}(\Lambda), n_0(\Lambda) < +\infty$, and $S_\Lambda = 0$. Then there exists an entire function of exponential type f satisfying the following conditions*:

1) *f has zeros at the points λ_k, $k \geq 1$, of multiplicities at least n_k*;

2) *$h_f(\lambda) \leq \pi n_0(\Lambda)|\operatorname{Im}\lambda|$, $\lambda \in \mathbb{C}$*;

3) *for $\varepsilon > 0$, $\delta \in (0, 1/3)$, there exists $T > 0$ such that $\lambda_k \in \mathcal{R}^\delta$ whenever $|\lambda_k| \geq T$, where $\mathcal{R} = \{\lambda \in \mathbb{C} : \ln|f(\lambda)| \geq \pi n_0(\Lambda)|\operatorname{Im}\lambda| - \varepsilon|\lambda|\}$.*

Proof. Let $\Lambda^1 = \{\xi_l\}$ be a sequence consisting of the moduli $|\lambda_k|$ of the elements of Λ. Note that each number $|\lambda_k|$ occurs in Λ^1 exactly as many times as is the sum of the multiplicities of the points λ_j with modulus $|\lambda_k|$. Then we have $n_0(\Lambda) = n_0(\Lambda^1)$. Consider the functions

$$L(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_k^2}\right)^{n_k}, \quad L_1(\lambda) = \prod_{l=1}^{\infty} \left(1 - \frac{\lambda^2}{\xi_l^2}\right).$$

As in the proof of Theorem 2.2, for any $d \in (0, 1/2)$, we can find $C_1 > 0$ (not depending on d) and a union of disks $E^1(d)$ such that $E^1(d)$ is centered with the zero sets of L and L_1, has linear density at most $\sqrt[4]{d}$, and

$$|\ln|L(\lambda)| - \ln|L_1(\lambda)|| \leq C_1\sqrt{d}|\lambda|, \quad \lambda \in \mathbb{C} \setminus E^1(d). \tag{2.47}$$

By Polya's theorem (see, e.g., [13, Lemma 5]), there exists a sequence of positive numbers Λ^2 such that $\Lambda^3 = \Lambda^1 \cup \Lambda^2$ has density $n_0(\Lambda^1)$. Then (see [17, Chapter I, §2, Theorem 1.2.9] and [6, Chapter II, §1, Theorem 5]) the function

$$L_3(\lambda) = \prod_{\eta_j\in\Lambda^3} \left(1 - \frac{\lambda^2}{\eta_j^2}\right)$$

satisfies the estimate

$$|\ln|L_3(\lambda)| - \pi n_0(\Lambda)|\operatorname{Im}\lambda|| \leq \varepsilon(\lambda)|\lambda|, \quad \lambda \in \mathbb{C} \setminus E^3, \tag{2.48}$$

where $\varepsilon(\lambda) \to 0$ as $|\lambda| \to \infty$, and E^3 is a union of disks of zero linear density. We set

$$f(\lambda) = L(\lambda) \prod_{\eta_j\in\Lambda^2} \left(1 - \frac{\lambda^2}{\eta_j^2}\right).$$

By the construction of the function f, statement 1) of the theorem is true. We prove 3). Fixing $\varepsilon > 0$ and $\delta \in (0, 1/3)$, we determine $d \in (0, 1/2)$ from the conditions $\sqrt[4]{d} \leq \delta/18$ and $C_1\sqrt{d} \leq \varepsilon/2$. As in Theorem 2.2, we find $T_1 > 0$ such that the following statement is valid. The sum of the diameters of all disks from $E^1(d) \cup E^3$ whose intersection with $B(0, (1+\delta)r)$, $r \geq T_1$, is nonempty, is strictly less than $\delta r/3$. Now we choose $T > T_1$ such that $|\operatorname{Im}\lambda_k| < (\delta \operatorname{Re}\lambda_k)/3$, $\operatorname{Re}\lambda_k \geq T_1$ provided that $|\lambda_k| \geq T$, and $\varepsilon(\lambda) < \varepsilon/2$, $|\lambda| \geq (1-\delta)T$.

Let $|\lambda_k| \geq T$. Then there exists $\tau_k \in (0, 1/3)$ such that the circle $S(\operatorname{Re}\lambda_k, \tau_k\delta\operatorname{Re}\lambda_k)$ does not intersect the set $E^1(d) \cup E^3$. By (2.47) and (2.48), we have

$$|\ln|f(\lambda)| - \pi n_0(\Lambda)|\operatorname{Im}\lambda|| \leq \varepsilon|\lambda|, \quad \lambda \in S(\operatorname{Re}\lambda_k, \tau_k\delta\operatorname{Re}\lambda_k). \tag{2.49}$$

Hence, $S(\operatorname{Re}\lambda_k, \tau_k\delta\operatorname{Re}\lambda_k) \subset \mathcal{R}$. Let $\lambda \in S(\operatorname{Re}\lambda_k, \tau_k\delta\operatorname{Re}\lambda_k)$. Taking into account our choice of T, we obtain

$$|\lambda_k - \lambda| \leq |\lambda_k - \operatorname{Re}\lambda_k| + |\operatorname{Re}\lambda_k - \lambda| \leq (2\delta\operatorname{Re}\lambda_k)/3 \leq \delta|\lambda|,$$

i.e., $\lambda_k \in \mathcal{R}^\delta$.

Finally, we prove 2). Let $|\lambda| = 1$ and $\widetilde{\varepsilon} > 0$. We choose $\widetilde{\varepsilon} \in (0, 1/3)$ such that

$$\pi n_0(\Lambda)|\operatorname{Im} w| \leq \pi n_0(\Lambda)|\operatorname{Im}\lambda| + \widetilde{\varepsilon}, \quad w \in B(\lambda, \widetilde{\delta}).$$

As in the proof of (2.49), we find $\widetilde{T} > 0$ such that

$$|\ln|f(\xi)| - \pi n_0(\Lambda)|\operatorname{Im}\xi|| \leq \widetilde{\varepsilon}|\xi|, \quad \xi \in S(t\lambda, \tau(t)\delta t), \quad t \geq \widetilde{T},$$

where $\tau(t)$ is a number in the interval $(0, 1/3)$. Therefore, the previous argument and the maximum modulus principle show that

$$\ln|f(t\lambda)| \leq \pi n_0(\Lambda)t|\operatorname{Im}\lambda| + 3\widetilde{\varepsilon}t, \quad t \geq \widetilde{T}.$$

Since $\widetilde{\varepsilon} > 0$ is arbitrary, this gives us statement 2). The theorem is proved. □

§3. Representation of functions in an invariant subspace on the half-plane

In this section, we use Theorems 2.2 and 2.4 to establish the fundamental principle for invariant subspaces with almost real spectrum on the half-plane and on some other unbounded domains. The method of the Leont′ev interpolation function is applied. We need the following auxiliary facts.

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^\infty$, let $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$, and let D be a convex domain. Denote by $W(\Lambda, D)$ the closure of the linear span of the system $\mathcal{E}(\Lambda)$ in the space $H(D)$.

Suppose that there is an entire function f of exponential type that has zeros of multiplicities at least n_k at the points λ_k and whose conjugate diagram is included in D (i.e., $h_f(\lambda) < H_D(\lambda)$, $\lambda \neq 0$). Then there exists (see [17, Chapter IV, §1, Subsection 2]) a system of functionals $\Xi(\Lambda, D) = \{\mu_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$ defined on the space $H^*(D)$ and biorthogonal to $\mathcal{E}(\Lambda)$ such that $\mu_{k,n}(z^l \exp(\lambda_j z)) = 1$ if $j = k, l = n$ and $\mu_{k,n}(z^l \exp(\lambda_j z)) = 0$ otherwise. This system is constructed with the help of the function f and represents a part of the system $\Xi(\widetilde{\Lambda}, D)$ biorthogonal to $\mathcal{E}(\widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is the multiple zero set of f. Suppose that the series (1.2) converges uniformly on all compact subsets of the domain D. Since the functionals $\mu_{k,n}$ are continuous and linear, we obtain $d_{k,n} = \mu_{k,n}(g)$, $k \geq 1$, $n = 0, \ldots, n_k - 1$. Therefore, if the function f does exist, then its representation by the series (1.2) is unique. In this case the coefficients of such a representation are calculated via the biorthogonal system of functionals.

Let D be a convex domain, let $g \in H(D)$, $\alpha \in \mathbb{C}$, and let f be an entire function of exponential type such that the conjugate diagram K of f contains the origin, and the shift $K(\alpha) = K + \alpha$ belongs to D ($K(\alpha)$ is the conjugate diagram of $f(\lambda)\exp(\alpha\lambda)$). The function

$$\omega_f(\lambda, \alpha, g) = \exp(-\alpha\lambda)\frac{1}{2\pi i}\int_\Omega \gamma(\xi)\left(\int_0^\xi g(\xi + \alpha - \eta)\exp(\lambda\eta)\, d\eta\right) d\xi$$

is called the interpolation function of g (see [7, Chapter I, §2, Subsection 1]). Here, Ω is the closed rectifiable contour surrounding the compact set K and lying in the domain

$D-\alpha$, and $\gamma(\xi)$ is the function associated with f in the sense of Borel (see [15, Chapter I, §5]). We list some properties of $\omega_f(\lambda,\alpha,g)$ and $\Xi(\Lambda,D)$.

1. [7, Chapter I, §2, Theorem 1.2.5]. Let Ω be the boundary of a convex neighborhood of a compact set K, and let $\Omega(\alpha)=\Omega+\alpha\subset D$. For any $\varepsilon>0$ there exists $A(\varepsilon)>0$ such that

$$|\omega_f(\lambda,\alpha,g)|\le A(\varepsilon)\exp\left(h_f(\lambda)+\varepsilon|\lambda|-\operatorname{Re}(\alpha\lambda)\right)\max_{z\in\Omega(\alpha)}|g(z)|,\quad \lambda\in\mathbb{C}.$$

2. Suppose $\widetilde{g}\in W(\Lambda,D)$ and $d_{k,n}=\mu_{k,n}(\widetilde{g})$, $k\ge1$, $n=0,\dots,n_k-1$. Then

$$\frac{1}{2\pi i}\int_{S_k}\frac{\omega_f(\lambda,\alpha,\widetilde{g})}{f(\lambda)}\exp(\lambda z)\,d\lambda=\sum_{n=0}^{n_k-1}d_{k,n}z^n\exp(\lambda_k z),\quad k\ge1,$$

where S_k is a circle surrounding no zeros of f different from λ_k. Moreover, if λ' is a zero of f that differs from λ_k, $k\ge1$, and S' is a circle surrounding a single zero λ' of f, then

$$\frac{1}{2\pi i}\int_{S'}\frac{\omega_f(\lambda,\alpha,\widetilde{g})}{f(\lambda)}\exp(\lambda z)\,d\lambda=0.$$

Indeed, let $\widetilde{g}$ be the limit of the sequence

$$P_l(z)=\sum_{k=1}^{l}\sum_{n=0}^{n_k-1}d_{k,n}^{l}z^n\exp(\lambda_k z),\quad l\ge1,$$

uniformly convergent on all compact subsets of D. This sequence exists because $\widetilde{g}\in W(\Lambda,D)$ and we may assume that some of $d_{k,n}^l$ are zeros. By Theorem 1.2.4 in §2 of Chapter I in [7], we have

$$\frac{1}{2\pi i}\int_{S_k}\frac{\omega_f(\lambda,\alpha,P_l)}{f(\lambda)}\exp(\lambda z)\,d\lambda=\sum_{n=0}^{n_k-1}d_{k,n}^{l}z^n\exp(\lambda_k z),\quad k=1,\dots,l,$$
$$\frac{1}{2\pi i}\int_{S_k}\frac{\omega_f(\lambda,\alpha,P_l)}{f(\lambda)}\exp(\lambda z)\,d\lambda=0,\quad k>l,$$
$$\frac{1}{2\pi i}\int_{S'}\frac{\omega_f(\lambda,\alpha,P_l)}{f(\lambda)}\exp(\lambda z)\,d\lambda=0.$$

As above, the relations

$$d_{k,n}=\mu_{k,n}(\widetilde{g})=\lim_{l\to\infty}\mu_{k,n}(P_l)=\lim_{l\to\infty}d_{k,n}^{l},\quad k\ge1,\quad n=0,\dots,n_k-1,\tag{3.1}$$

are valid (for $k>l$, we set $d_{k,n}^l=0$). The estimate in property 1 implies that $\omega_f(\lambda,\alpha,P_l)\to\omega_f(\lambda,\alpha,\widetilde{g})$ as $l\to\infty$ uniformly on any compact subset of the plane. These gives us the required identities.

3. Let $\widetilde{g}\in W(\Lambda,D)$, and let $d_{k,n}=\mu_{k,n}(\widetilde{g})$, $k\ge1$, $n=0,\dots,n_k-1$. Suppose that the series (1.2) converges uniformly on the compact subsets of D. Then $\widetilde{g}\equiv g$.

Indeed, if $\mu'\in\Xi(\widetilde{\Lambda},D)\setminus\Xi(\Lambda,D)$, then $\mu'(g)=\mu'(\widetilde{g})=\mu'(P_l)$. From (3.1) and the uniqueness theorem (see [7, Chapter II, §1, Theorem 2.1.2]) we obtain the desired identity.

Let $a\in\mathbb{R}$. We put $\Pi(a)=\{z\in\mathbb{C}:\operatorname{Re}z<a\}$.

Theorem 3.1. *Let $a\in\mathbb{R}$, and let W be a closed nontrivial invariant subspace of $H(\Pi(a))$ with almost real spectrum $\Lambda=\{\lambda_k,n_k\}_{k=1}^{\infty}$. Then for any compact convex set $K\subset\Pi(a)$ and any $\beta\in(0,1)$, there is a strictly monotone increasing sequence of positive numbers*

$\{t_l\}_{l=1}^{\infty}$ such that $t_{l+1} \le (1+\beta)t_l$, $l \ge 1$, $t_l \to \infty$ as $l \to \infty$, and each function $g \in W$ is represented by the following series:

$$g(z) = \sum_{l=1}^{\infty} \Bigg(\sum_{t_{l-1} < |\lambda_k| < t_l} \sum_{n=0}^{n_k - 1} d_{k,n} z^n \exp(\lambda_k z) \Bigg), \quad z \in K, \tag{3.2}$$

where $t_0 = 0$ (if for some l there are no points λ_k such that $t_{l-1} < |\lambda_k| < t_l$, then the corresponding term of the series is missing). Furthermore, there exists a compact convex set $K' \subset \Pi(a)$ and $C > 0$ (not depending on $g \in W$) such that

$$\sum_{l=1}^{\infty} \max_{z \in K} \Bigg| \sum_{t_{l-1} < |\lambda_k| < t_l} \sum_{n=0}^{n_k - 1} d_{k,n} z^n \exp(\lambda_k z) \Bigg| \le C \max_{z \in K'} |g(z)|. \tag{3.3}$$

Proof. Since W is a closed nontrivial subspace, there is a functional $\mu \in H^*(\Pi(a))$ that annihilates W. Its Laplace transform $F(\lambda) = \mu(\exp(\lambda z))$ is an entire function of exponential type. It is not difficult to show that Λ is a part of the multiple zero set of F. As a consequence (see [6, Chapter I, §5, Lemma 4]), Λ has finite upper density. By Theorem 8.1 in [2], an invariant subspace in an unbounded convex domain admits spectral synthesis. Therefore, W coincides with the subspace $W(\Lambda, \Pi(a))$.

Let $K \subset \Pi(a)$ and $\beta \in (0,1)$ be fixed. Since K is a compact set in $\Pi(a)$, we have $\tau = a - H_K(1) > 0$. Let $\alpha \in (a - \tau, a)$ and $\varepsilon \in (0, (\alpha - a + \tau)/12) \cap (0, a - \alpha)$. We choose $\delta \in (0, \beta/5)$ such that $\max_{z \in K} \delta |z - \alpha| < \varepsilon$. Suppose that f is the function and $\{t_l\}_{l=1}^{\infty}$ the sequence whose existence was established in the proof of Theorem 2.2. Note that all points $\lambda_k \in \mathbb{C} \setminus B(0, t_1)$ lie in the angle $B(0, t_l)$, while by (2.2) and the choice of ε, the conjugate diagram of the function $f(\lambda)\exp(\alpha\lambda)$ lies in the half-plane $\Pi(a)$. Now, making use of residues, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_l} \frac{\omega_f(\lambda, \alpha, g)}{f(\lambda)} \exp(\lambda z)\, d\lambda = & \sum_{t_{l-1} < |\lambda_k| < t_l} \frac{1}{2\pi i} \int_{S_k} \frac{\omega_f(\lambda, \alpha, g)}{f(\lambda)} \exp(\lambda z)\, d\lambda \\ & + \sum_{t_{l-1} < |\lambda'_j| < t_l} \frac{1}{2\pi i} \int_{S'_j} \frac{\omega_f(\lambda, \alpha, g)}{f(\lambda)} \exp(\lambda z)\, d\lambda, \quad l \ge 1, \end{aligned}$$

where Γ_l is the boundary of the intersection of the annulus $B(0, t_{l+1}) \setminus B(0, t_l)$ with the angle $B(0, t_l)$, and the λ'_j are the zeros of f different from λ_k. In accordance with Property 2 of the function $\omega_f(\lambda, \alpha, g)$, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_l} \frac{\omega_f(\lambda, \alpha, g)}{f(\lambda)} \exp(\lambda z)\, d\lambda = \sum_{t_{l-1} < |\lambda_k| < t_l} \sum_{n=0}^{n_k - 1} d_{k,n} z^n \exp(\lambda_k z), \tag{3.4}$$

where $d_{k,n} = \mu_{k,n}(g)$, $\mu_{k,n} \in \Xi(\Lambda, \Pi(a))$, $k \ge 1$, $n = 0, \dots, n_k - 1$. Moreover, by Property 1 and inequality (2.1), we have (Ω is the boundary of a convex neighborhood of the conjugate diagram of the function $f(\lambda)\exp(\alpha\lambda)$)

$$\left| \frac{1}{2\pi i} \int_{\Gamma_l} \frac{\omega_f(\lambda, \alpha, g)}{f(\lambda)} \exp(\lambda z)\, d\lambda \right| \le t_{l+1} A(\varepsilon) \max_{z \in \Omega} |g(z)| \exp\Big(\max_{\lambda \in \Gamma_l} \big(3\varepsilon|\lambda| + \mathrm{Re}((z - \alpha)\lambda)\big)\Big).$$

Let $\lambda = x + iy$. Recalling the choice of ε and δ, we obtain

$$\begin{aligned} \max_{z \in K} \max_{\lambda \in \Gamma_l} \big(3\varepsilon|\lambda| + \mathrm{Re}((z - \alpha)\lambda)\big) & \le \max_{z \in K} \max_{\lambda \in \Gamma_l} (3\varepsilon t_{l+1} + x\,\mathrm{Re}(z - \alpha) - y\,\mathrm{Im}(z - \alpha)) \\ & \le \max_{z \in K} (3\varepsilon t_{l+1} + 2^{-1}(a - \tau - \alpha)t_l + \delta t_{l+1}|z - \alpha|) \\ & \le 4\varepsilon t_{l+1} - 6\varepsilon t_{l+1}(1 + \delta)^{-1} \le -\varepsilon t_{l+1}. \end{aligned}$$

Since Λ has finite upper density, the series $\sum_{t_l} \exp(-\varepsilon t_l)$ converges (summation is over all indices l for which there are points λ_k in the annulus $t_l < |\lambda| < t_{l+1}$). Thus, by (3.4), we obtain (3.3). Referring to Property 3 of the system $\Xi(\Lambda, D)$, we complete the proof. $\square$

Let $\Lambda = \{\lambda_k, n_k\}$. We introduce the following Banach spaces of numerical sequences:

$$\mathcal{B}_m(\Lambda) = \{d = \{d_{k,n}\} : \|d\|_m = \sup_{k,n} |d_{k,n}| m^n \exp((a - 1/m)|\lambda_k|) < \infty\}, \quad m \geq 1.$$

Denote by $\mathcal{B}(\Lambda, a)$ the projective limit of $\mathcal{B}_m(\Lambda)$, and by $\mathcal{L}(\Lambda, a)$ the operator acting from $\mathcal{B}(\Lambda, a)$ to $W(\Lambda, \Pi(a))$ by the following rule. To a sequence $d \in \mathcal{B}(\Lambda, a)$, we assign the sum of the series (1.1), provided that series converges uniformly on compact sets of the half-plane $\Pi(a)$. Let $K_m = \overline{B(0, m) \cap \Pi(a - 1/m)} \neq \varnothing$ if $m \geq m_0$.

Lemma 3.2. *Let $a \in \mathbb{R}$, and let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be an almost real sequence with finite upper density.*

1) *The operator $\mathcal{L}(\Lambda, a)$ is defined everywhere on the space $\mathcal{B}(\Lambda, a)$ and represents a linear, continuous, and injective operator . Moreover, for any $m \geq m_0$ there exists p and $C > 0$ (not depending on $d = \{d_{k,n}\} \in \mathcal{B}(\Lambda, a)$) such that*

$$\sum_{k=1,n=0}^{\infty, n_k - 1} \max_{z \in K_m} |d_{k,n} z^n \exp(\lambda_k z)| \leq C \|d\|_p, \quad d \in \mathcal{B}(\Lambda, a). \tag{3.5}$$

2) *If the operator $\mathcal{L}(\Lambda, a)$ is surjective, then it is an isomorphism of the linear topological spaces (Λ, a) and $W(\Lambda, \Pi(a))$. Furthermore, for any $m \geq m_0$ there exists p and $C > 0$ (not depending on $g \in W(\Lambda, \Pi(a))$) such that*

$$\sum_{k=1,n=0}^{\infty, n_k - 1} \max_{z \in K_m} |d_{k,n} z^n \exp(\lambda_k z)| \leq C \max_{z \in K_p} |g(z)|, \quad g = \mathcal{L}(\Lambda, a)(\{d_{k,n}\}). \tag{3.6}$$

Proof. 1) Since Λ has finite upper density, there exists (see [17, Chapter I, §1, Theorem 1.1.5]) an entire function f of exponential type with zeros at the points λ_k of multiplicities at least n_k. Let $K(\alpha)$ be a shifted conjugate diagram of f lying in the half-plane $\Pi(a)$. Then the function $f(\lambda)\exp(\alpha\lambda)$ has a conjugate diagram $K(\alpha) \subset \Pi(a)$.

As was mentioned above, in this case the representation (1.1) by a series uniformly convergent on the compact sets of the domain $\Pi(a)$ is unique. Consequently, the operator $\mathcal{L}(\Lambda, a)$ is injective.

Let $d = \{d_{k,n}\} \in \mathcal{B}(\Lambda, a)$. We fix an index $m \geq m_0$. Since Λ is an almost real sequence, we have

$$\begin{aligned}
\max_{z \in K_m} |\exp(\lambda_k z)| &\leq \exp(\operatorname{Re} \lambda_k (a - 1/m) + |\operatorname{Im} \lambda_k| \, |\operatorname{Im} z|) \\
&\leq \exp(\operatorname{Re} \lambda_k ((a - 1/m) + m |\operatorname{Im} \lambda_k| / \operatorname{Re} \lambda_k)) \\
&\leq \exp(\operatorname{Re} \lambda_k (a - 1/(m+1))) \\
&\leq \exp((a - 1/(m+2))|\lambda_k|), \quad k \geq k(m).
\end{aligned}$$

Thus, since $d \in \mathcal{B}_{m+3}(\Lambda)$, we get

$$\begin{aligned}
\sum_{k=k(m),n=0}^{\infty, n_k - 1} \max_{z \in K_m} |d_{k,n} z^n \exp(\lambda_k z)| &\leq \sum_{k=1,n=0}^{\infty, n_k - 1} |d_{k,n}| (m+2)^n \exp\left(\left(a - \frac{1}{m+2}\right)|\lambda_k|\right) \\
&\leq \|d\|_{m+3} \sum_{k=1,n=0}^{\infty, n_k - 1} \left(\frac{m+2}{m+3}\right)^n \exp\left(\left(\frac{1}{m+3} - \frac{1}{m+2}\right)|\lambda_k|\right) \leq C' \|d\|_{m+3}.
\end{aligned}$$

In the last inequality, we took into account the fact that the upper density of Λ is finite. Moreover, we have

$$\sum_{k=1,n=0}^{k(m)-1,n_k-1} \max_{z\in K_m} |d_{k,n} z^n \exp(\lambda_k z)| \le \|d\|_{m+3} \sum_{k=1,n=0}^{k(m)-1,n_k-1} \left(\frac{m}{m+3}\right)^n \exp\left(|\lambda_k| m - a + \frac{1}{m+3}\right).$$

This implies (3.5), from which it follows that $\mathcal{L}(\Lambda, a)$ is a linear operator defined on the entire space $\mathcal{B}(\Lambda, a)$.

2) Let $\mathcal{L}(\Lambda, a)$ be a surjective operator. Using item 1) and the Banach theorem on an inverse operator for Féchet spaces (which is the case for $\mathcal{B}(\Lambda, a)$ and $W(\Lambda, \Pi(a))$), we see that $\mathcal{L}(\Lambda, a)$ is an isomorphism of linear topological spaces. Now (3.6) follows from (3.5). The lemma is proved. □

Theorem 3.3. *Let $a \in \mathbb{R}$, and let W be a closed nontrivial invariant subspace of $H(\Pi(a))$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Suppose that $S_\Lambda = 0$. Then each function $g \in W$ is expanded in the series*

$$g(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n \exp(\lambda_k z), \quad z \in \Pi(a). \tag{3.7}$$

Moreover, for any $m \ge m_0$ there exists p and $C > 0$ (not depending on $g \in W$) such that

$$\sum_{k=1,n=0}^{\infty,n_k-1} \max_{z\in K_m} |d_{k,n} z^n \exp(\lambda_k z)| \le C \max_{z\in K_p} |g(z)|. \tag{3.8}$$

Proof. Applying Theorem 2.4 and arguments from the proof of Theorem 3.1, for any compact set $K \subset \Pi(a)$ we find a convex compact set $K'' \subset\subset \Pi(a)$ and $C > 0$ (not depending on $g \in W$) such that

$$\sum_{k=1}^{\infty} \max_{z\in K} \left| \sum_{n=0}^{n_k-1} d_{k,n} z^n \exp(\lambda_k z) \right| \le C \max_{z\in K''} |g(z)|, \tag{3.9}$$

where $d_{k,n} = \mu_{k,n}(g)$, $\mu_{k,n} \in \Xi(\Lambda, \Pi(a))$, $k \ge 1$, $n = 0, \dots, n_k - 1$.

Let us show that $d = \{d_{k,n}\} \in \mathcal{B}(\Lambda, a)$. First, we prove that

$$\limsup_{k\to\infty} \max_{\tau|\lambda_k|<n<n_k} |d_{k,n}| \exp(s|\lambda_k|) = 0 \tag{3.10}$$

for each $\tau > 0$ and $s \ge 1$. Suppose that this is not true. Then one can find $b, \tau > 0$, $s \ge 1$, and sequences of indices $\{k(l)\}$ and $\{n(l)\}$ such that $k(l) \to \infty$ as $l \to \infty$, $\tau|\lambda_{k(l)}| < n(l) < n_{k(l)}$, and

$$|d_{k(l),n(l)}| \ge b \exp(-s|\lambda_{k(l)}|), \quad l \ge 1. \tag{3.11}$$

Let

$$\sum_{n=0}^{n_k-1} d_{k,n} z^n = d_{k,n_k-1} \prod_{j=1}^{n_k-1} (z - \mu_j^k).$$

The coefficient $d_{k(l),n(l)}$ is the sum of products of zeros $\mu_j^{k(l)}$ and $d_{k(l),n_{k(l)}-1}$. Here, the number of all terms is at most $2^{n_{k(l)}}$. By (3.11), we may assume that, under a suitable enumeration of zeros,

$$\left| d_{k(l),n_{k(l)}-1} \prod_{j=1}^{n_{k(l)}-1-n(l)} \mu_{j^{k(l)}} \right| \ge 2^{-n_{k(l)}} b \exp(-s|\lambda_{k(l)}|), \quad l \ge 1. \tag{3.12}$$

Let $H > 0$. We choose an index m such that the length of the vertical part I_m of the boundary of the compact set K_m (symmetric about the real axis) is strictly greater than $4eH$ and does not exceed $4eH + 2$. Using an appropriate enumeration of zeros, we may assume that for some $0 \le p(l) \le n_{k(l)} - 1 - n(l)$, the following inequalities hold true (if $p(l) = n_{k(l)} - 1 - n(l)$, then the second group of inequalities is missing, while for $p(l) = 0$ the first group is missing; so in these cases the situation simplifies):

$$|\mu_j^{k(l)}| > 2 \max_{z \in I_m} |z| = 2A_m, \quad j = 1, \dots, p(l),$$
$$|\mu_j^{k(l)}| \le 2A_m, \quad j = p(l) + 1, \dots, n_{k(l)} - 1 - n(l).$$

Then (3.12) implies the estimates

$$\begin{aligned} &\left| d_{k(l), n_{k(l)}-1} \prod_{j=1}^{p(l)} \mu_j^{k(l)} \right| \ge (2A_m)^{-n_{k(l)}+1+n(l)+p(l)} 2^{-n_{k(l)}} b \exp(-s|\lambda_{k(l)}|), \quad l \ge 1, \\ &\left| \prod_{j=1}^{p(l)} \left(\frac{z}{\mu_j^{k(l)}} - 1 \right) \right| \ge 2^{-p(l)}, \quad z \in I_m, \quad l \ge 1. \end{aligned} \tag{3.13}$$

By Cartan's theorem on lower estimates for a polynomial (see [15, Chapter I, §4, Theorem 4.1]), we have

$$\left| \prod_{j=p(l)+1}^{n_{k(l)}-1} (z - \mu_j^{k(l)}) \right| \ge H^{n_{k(l)}-1-p(l)}, \quad l \ge 1,$$

outside of a collection of exceptional disks with the sum of radii $2eH$. In view of our choice of m,

$$\max_{z \in I_m} \left| \prod_{j=p(l)+1}^{n_{k(l)}-1} (z - \mu_j^{k(l)}) \right| \ge H^{n_{k(l)}-1-p(l)}, \quad l \ge 1.$$

Since Λ is almost real, we have $|\exp(\lambda_{k(l)} z)| \ge \exp(-(|a| + 1)|\lambda_{k(l)}|)$, $z \in I_m$, $l \ge l(m)$. Using (3.13), we see that

$$\max_{z \in I_m} \left| \sum_{n=0}^{n_{k(l)}-1} d_{k(l), n(l)} z^n \exp(\lambda_{k(l)} z) \right| \ge bT(m, l) \exp(-(|a| + 1 + s)|\lambda_{k(l)}|), \quad l \ge l(m),$$

where

$$\begin{aligned} T(m, l) &\ge 2^{-2n_{k(l)}} (2eH + 2 + |a|)^{-n_{k(l)}+1+n(l)+p(l)} H^{n_{k(l)}-1-p(l)} \\ &\ge 2^{-2n_{k(l)}} H^{n(l)} (2e + (2 + |a|)/H)^{-n_{k(l)}+1+n(l)+p(l)} \\ &\ge H^{n(l)} (8e + 4(2 + |a|)/H)^{-n_{k(l)}}. \end{aligned}$$

Since $n(l) > \tau|\lambda_{k(l)}|$ and for some $\sigma > 0$, thanks to the finiteness of $\bar{n}(\Lambda)$ (see Theorem 3.1), we have $n_k \le \sigma|\lambda_k|$, $k \ge 1$, it follows that for sufficiently large $H > 0$ the last estimates contradict (3.9). Therefore, (3.10) is true.

Let $\tau > 0$. We put $\widetilde{n}_k(\tau) = \min\{n_k, [\tau|\lambda_k|]\}$, where $[\tau|\lambda_k|]$ is the integral part of $\tau|\lambda_k|$. By (3.9), (3.10), and the inequality $n_k \le \sigma|\lambda_k|$, for any $m \ge m_0$ we can find $C_m(\tau) > 0$ such that

$$\max_{z \in K_m} \left| \sum_{n=0}^{\widetilde{n}_k(\tau)-1} d_{k,n} z^n \exp(\lambda_k z) \right| \le C_m(\tau), \quad k \ge 1. \tag{3.14}$$

Suppose that $d = \{d_{k,n}\} \notin \mathcal{B}(\Lambda, a)$. Then for some $s \ge 1$ we can find sequences of indices $\{k(l)\}$ and $\{n(l)\}$ such that $k(l) \to \infty$ as $l \to \infty$, and

$$|d_{k(l), n(l)}| s^{n(l)} \exp\left((a - 1/s)|\lambda_{k(l)}|\right) = r(l) \to \infty, \quad l \to \infty. \tag{3.15}$$

Passing to subsequences, we may assume that $n(l)/|\lambda_{k(l)}| \to \beta$ as $l \to \infty$. The condition $\beta > 0$ contradicts (3.10) because $n_k \le \sigma|\lambda_k|$, $k \ge 1$. Hence, $\beta = 0$. For each $\tau > 0$ we have $n(l) \le \widetilde{n}_k(\tau) - 1$, $l \ge l(\tau)$, and by (3.15) we get

$$|d_{k(l),n(l)}| \ge r(l) \exp\left((-a + 1/(s+1))|\lambda_{k(l)}|\right), \quad l \ge \widetilde{l}. \tag{3.16}$$

We choose an index $m \ge \max\{m_0, s+4\}$ such that the length of the vertical part I_m of the boundary of the compact set K_m is strictly greater than $4e$. Let

$$\widetilde{I}_m = I_m \cap \{z : |\operatorname{Im} z| \le 2e + 1\}$$

and write

$$\sum_{n=0}^{\widetilde{n}_k(\tau)-1} d_{k,n} z^n = d_{k,\widetilde{n}_k(\tau)-1} \prod_{j=1}^{\widetilde{n}_k(\tau)-1} (z - \widetilde{\mu}_j^k).$$

As above, we may assume that

$$|\widetilde{\mu}_j^{k(l)}| > 2 \max_{z \in \widetilde{I}_m} |z| = 2\widetilde{A}_m, \quad j = 1, \dots, p(l),$$
$$|\widetilde{\mu}_j^{k(l)}| \le 2\widetilde{A}_m, \quad j = p(l)+1, \dots, n_{k(l)} - 1 - n(l).$$

By the Cartan theorem,

$$\max_{z \in \widetilde{I}_m} \left| \prod_{j=p(l)+1}^{\widetilde{n}_{k(l)}(\tau)-1} (z - \widetilde{\mu}_j^{k(l)}) \right| \ge 1, \quad l \ge 1.$$

Moreover, since Λ is almost real, we have

$$\begin{aligned} |\exp(\lambda_k z)| &= \exp \operatorname{Re}(\lambda_k z) = \exp(\operatorname{Re}\lambda_k (a - 1/m) - \operatorname{Im}\lambda_k \operatorname{Im} z) \\ &\ge \exp(\operatorname{Re}\lambda_k((a - 1/m) - (2e+1)\operatorname{Im}\lambda_k/\operatorname{Re}\lambda_k)) \\ &\ge \exp(\operatorname{Re}\lambda_k(a - 1/(m-1))) \\ &\ge \exp((a - 1/(m-2))|\lambda_k|), \quad z \in \widetilde{I}_m, \quad k \ge \widetilde{k}. \end{aligned}$$

As above, recalling (3.16), we see that

$$\max_{z \in \widetilde{I}_m} \left| \sum_{n=0}^{\widetilde{n}_{k(l)}(\tau)-1} d_{k(l),n(l)} z^n \exp(\lambda_{k(l)} z) \right| \ge \widetilde{T}(m,l) r(l) \exp\left(\left(-\frac{1}{m-2} + \frac{1}{s+1}\right)|\lambda_{k(l)}|\right),$$

where $l \ge l(\tau)$, $k(l) \ge \widetilde{k}$, and

$$\widetilde{T}(m,l) \ge 2^{-2\widetilde{n}_{k(l)}(\tau)} (2e + 2 + |a|)^{-\widetilde{n}_{k(l)}(\tau)+1+n(l)+p(l)} \ge (8e + 8 + 4|a|)^{-\widetilde{n}_{k(l)}(\tau)}.$$

Since $\widetilde{n}_k(\tau) \le \tau|\lambda_k|$, choosing $\tau > 0$ to be arbitrarily small and taking our choice of m into account, we obtain

$$\max_{z \in K_m} \left| \sum_{n=0}^{\widetilde{n}_{k(l)}(\tau)-1} d_{k(l),n(l)} z^n \exp(\lambda_{k(l)} z) \right| \ge r(l) \to \infty, \quad l \to \infty.$$

This contradicts (3.14). Hence, $d = \{d_{k,n}\} \in \mathcal{B}(\Lambda, a)$.

As in the proof of Theorem 3.1, the subspace W coincides with $W(\Lambda, \Pi(a))$. Since $d \in \mathcal{B}(\Lambda, a)$, Property 3 of the system $\Xi(\Lambda, D)$ and statement 1) of Lemma 3.2 imply that formula (3.7) is true. As it has been proved, $\mathcal{L}(\Lambda, a)$ is a surjective operator. Consequently, in accordance with statement 2) of Lemma 3.2, we obtain (3.8). The theorem is proved. □

Now we show that the condition $S_\Lambda = 0$ in Theorem 3.3 is necessary. For this, we shall prove an analog of Theorem 3.1 in the paper [14]. In our situation, we cannot use this theorem itself, because it contains a restriction on the multiplicities of the points $\lambda_k\colon n_k/|\lambda_k| \to 0$.

Lemma 3.4. *Let D be a convex domain such that $H_D(1) = a < +\infty$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}$. Suppose that every function $g \in W$ can be expanded in a series of the form* (1.1) *uniformly convergent on all compact subsets of D. Then $S_\Lambda = 0$.*

Proof. By the same argument as at the beginning of the proof of Theorem 3.1, there exists an entire function of exponential type f that has zeros at the points λ_k of multiplicities at least n_k and whose conjugate diagram lies in D. Thus, on the space $H^*(D)$ there exists a system of functionals $\Xi(\Lambda, D)$ biorthogonal to $\mathcal{E}(\Lambda)$.

Suppose that $S_\Lambda \neq 0$. Then $S_\Lambda \leq -2\beta < 0$. By the definition of S_Λ, there is a sequence of positive numbers $\{\delta_p\}$ and a subsequence $\{\lambda_{k(p)}\}$ such that $\delta_p \to 0$ as $p \to \infty$, and

$$\ln |q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p)|/|\lambda_{k(p)}| \leq -\beta. \tag{3.17}$$

We may assume that

$$|\lambda_{k(p+1)}| \geq 2|\lambda_{k(p)}|, \quad \delta_p < 1/4, \quad p \geq 1. \tag{3.18}$$

Consider the functions

$$g_p(z) = \frac{1}{2\pi i} \int_{S(\lambda_{k(p)}, 5\delta_p|\lambda_{k(p)}|)} \frac{\exp(\lambda z)\, d\lambda}{(\lambda - \lambda_{k(p)}) q_\Lambda^{k(p)}(\lambda, \delta_p)}, \quad p \geq 1.$$

We estimate $|g_p|$ from above. We have

$$|q_\Lambda^{k(p)}(\lambda, \delta_p)| = \prod_{\substack{\lambda_k \in B(\lambda_{k(p)}, \delta_p|\lambda_{k(p)}|),\\ k \neq k(p)}} \left| \frac{\lambda - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq \left(\frac{4\delta_p|\lambda_{k(p)}|}{3\delta_p(1+\delta_p)|\lambda_{k(p)}|} \right)^{m(p)} \geq 1,$$
$$\lambda \in S(\lambda_{k(p)}, 5\delta_p|\lambda_{k(p)}|),$$

where $m(p)$ is the number of points λ_k, $k \neq k(p)$, with multiplicities taken into account, belonging to $B(\lambda_{k(p)}, \delta_p|\lambda_{k(p)}|)$. Let K be an arbitrary compact subset of D. Then $\operatorname{Re} z \leq H_D(1) - 2\tau = a - 2\tau$, $z \in K$, for some $\tau > 0$. Since Λ is almost real and $\delta_p \to 0$, we obtain

$$\begin{aligned}
|g_p(z)| &= \left| \frac{1}{2\pi i} \int_{S(\lambda_{k(p)}, 5\delta_p|\lambda_{k(p)}|)} \frac{\exp(\lambda z)\, d\lambda}{(\lambda - \lambda_{k(p)}) q_\Lambda^{k(p)}(\lambda, \delta_p)} \right| \\
&\leq 5\delta_p|\lambda_{k(p)}| \sup_{\lambda \in S(\lambda_{k(p)}, 5\delta_p|\lambda_{k(p)}|)} \left| \frac{\exp \lambda z}{(\lambda - \lambda_{k(p)})} \right| \\
&\leq \exp(\operatorname{Re}(\lambda_{k(p)} z) + 5\delta_p|\lambda_{k(p)}|\,|z|) \\
&\leq \exp(\operatorname{Re} \lambda_{k(p)} (\operatorname{Re} z + (|\operatorname{Im} z|\,|\operatorname{Im} \lambda_{k(p)}| + 5\delta_p|\lambda_{k(p)}|)/(\operatorname{Re} \lambda_{k(p)})) \\
&\leq \exp(\operatorname{Re} \lambda_{k(p)}(a - \tau)), \quad z \in K, \quad p \geq p(K).
\end{aligned} \tag{3.19}$$

Consider the function

$$g(z) = \sum_{p=1}^{\infty} c_p g_p(z), \tag{3.20}$$

where $c_p = \exp(-a \operatorname{Re} \lambda_{k(p)})$, $p \geq 1$. By (3.19),

$$\sum_{p=p(K)}^{\infty} |c_p g_p(z)| \leq \sum_{p=p(K)}^{\infty} \exp(-\tau \operatorname{Re} \lambda_{k(p)}) < \infty, \quad z \in K.$$

Hence, $g \in W$. Let us show that g admits no expansion in a series of the form (1.1) uniformly convergent on the compact subsets of D. By the definition of g_p, we have

$$g_p(z) = \widetilde{d}_{k(p),0} \exp(\lambda_{k(p)} z) + \sum_{\substack{\lambda_k \in B(\lambda_{k(p)}, \delta_p |\lambda_{k(p)}|) \\ k \neq k(p)}} \sum_{n=0}^{n_k - 1} \widetilde{d}_{k,n} z^n \exp(\lambda_k z),$$

where $\widetilde{d}_{k(p),0} = (q_\Lambda^{k(p)}(\lambda_{k(p)}, \delta_p))^{-1}$. By (3.18), the disks $B(\lambda_{k(p)}, \delta_p |\lambda_{k(p)}|)$, $p \geq 1$, are mutually disjoint. In particular, $\lambda_{k(p)} \in B(\lambda_{k(j)}, \delta_p |\lambda_{k(j)}|)$, if $j \neq p$. It follows that $\widetilde{d}_{k(p),0} c_p = \mu_{k(p),0}(g)$, $\mu_{k(p),0} \in \Xi(\Lambda, D)$. Suppose that g expands in the series (1.1), converging uniformly on the compact subsets of D. Then $d_{k(p),0} = \mu_{k(p),0}(g) = \widetilde{d}_{k(p),0} c_p$. On the other hand, by (3.17), for all $z \in D$ with $\operatorname{Re} z \geq a - \beta/2$ (such points z exist by the definition of H_D), we have

$$\begin{aligned} |d_{k(p),0} \exp(\lambda_{k(p)} z)| &= |\widetilde{d}_{k(p),0} c_p \exp(\lambda_{k(p)} z)| \\ &\geq \exp(\beta |\lambda_{k(p)}| - \operatorname{Re} \lambda_{k(p)} (\beta/2 - |\operatorname{Im} z|\, |\operatorname{Im} \lambda_{k(p)}| / \operatorname{Re} \lambda_{k(p)})) \geq 1, \quad p \geq p(z) \end{aligned}$$

(in the last inequality we have used the fact that Λ is almost real). This contradicts the convergence of the series (1.1) at the point $z \in D$. Hence, g cannot be expanded in a series like (1.1) uniformly convergent on the compact sets in D. This contradicts the assumptions of the lemma. Therefore, $S_\Lambda = 0$, and the lemma is proved. □

Now we are able to formulate and prove the fundamental principle criterion.

Theorem 3.5. *Let $a \in \mathbb{R}$, and let W be a closed nontrivial invariant subspace of $H(\Pi(a))$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. The following statements are equivalent.*

1) *Every function $g \in W$ expands in a series of the form* (1.1) *uniformly convergent on all compact subsets of $\Pi(a)$.*

2) $S_\Lambda = 0$.

3) *The operator $\mathcal{L}(\Lambda, a) \colon \mathcal{B}(\Lambda, a) \to W(\Lambda, \Pi(a))$ is an isomorphism of linear topological spaces.*

Proof. The implication 1) ⇒ 2) was proved in Lemma 3.4. By Theorem 3.3, 1) follows from 2). If 1) is true, the operator $\mathcal{L}(\Lambda, a)$ is a surjection because W is closed. According to item 2) of Lemma 3.2, statement 3) is valid. Since $W = W(\Lambda, \Pi(a))$ (see Theorem 3.1), the implication 3) ⇒ 1) follows from the definition of $\mathcal{L}(\Lambda, a)$. The theorem is proved. □

Therefore, the case of invariant subspaces with almost real spectrum on a half-plane with almost vertical boundary (with the half-plane to the left of the boundary) is analyzed completely. In what follows, we consider some other cases of unbounded domains. Let D be an unbounded convex domain. We set

$$\mathcal{J}(D) = \{\lambda \in \mathbb{C} : H_D(\lambda) = +\infty\}.$$

Since H_D is a convex positive homogeneous function, the set $\mathbb{C} \setminus \mathcal{J}(D)$ is a convex cone. Thus, only the following four cases are possible: $\mathbb{C} \setminus \mathcal{J}(D)$ is a point, a ray, a line, or an angle of size at most π. If $D = \mathbb{C}$, then $\mathcal{J}(D) = \mathbb{C} \setminus 0$. In the case where D is a half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z e^{i\varphi}) < a\}$, the set $\mathcal{J}(D)$ is the plane cut along the ray $\{\lambda = t e^{i\varphi} : t \geq 0\}$. However, if D is a strip $\{z \in \mathbb{C} : \operatorname{Re}(z e^{i\varphi}) < a,\ \operatorname{Re}(z e^{i(\varphi + \pi)}) < b\}$, then $\mathcal{J}(D)$ consists of two half-planes with the common boundary line $\{\lambda = t e^{i\varphi} : t \in \mathbb{R}\}$. In the other cases the domain D contains no lines. Nevertheless, D always contains some ray $\{z = z_0 + t e^{i\varphi}, t \geq 0\}$. Moreover, the set $\mathcal{J}(D)$ represents an angle of opening less than 2π and contains an open angle of opening π, i.e., the half-plane $\{\lambda = t e^{i\psi} : -\varphi - \pi/2 < \psi < -\varphi + \pi/2, t > 0\}$.

Since almost real sequences are dealt with, the direction of their accumulation $\lambda = 1 = (1,0)$ is exceptional. Depending on the domain D, one of the following four possibilities is realized.

1. H_D is bounded in a neighborhood of the point $\lambda = 1$, i.e., $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$ (int denotes the interior part of a set). For a bounded domain, this situation is the only possible.

2. H_D is unbounded in the neighborhood of the point $\lambda = 1$, i.e., $1 \in \operatorname{int} \mathcal{J}(D)$. This is the only possibility for $D = \mathbb{C}$.

3. $1 \in \partial \mathcal{J}(D)$ and $H_D(1) = +\infty$. This case is realized, for example, when D is a domain bounded by a parabola.

4. $1 \in \partial \mathcal{J}(D)$ and $H_D(1) < +\infty$. In this case either the boundary ∂D contains a vertical ray, or a vertical line is an asymptote for ∂D. The domain is located to the left of the corresponding ray or line.

The case treated in this paper corresponds to the fourth possibility. Let us show that, in fact, the fourth possibility reduces to that case. For this, we use a result on simultaneous analytic continuation of functions on invariant subspaces. Various criteria of such continuation were obtained in [18] and [19]. In [18], the one-dimensional case was investigated, while the multidimensional case was treated in [19].

Nevertheless, the most appropriate result for our situation is that from the paper [19] (Theorem 4.1). It gives the most general and simple (by their matter but, possibly, not by the form) sufficient conditions for analytic continuation. They are related to the principal invariant subspaces. As was mentioned in [19], in the one-dimensional case each invariant subspace is principal. Therefore, this result is applicable in our case.

We formulate a weaker version of [19, Theorem 4.1] only related to invariant subspaces with almost real spectrum.

Let D be a convex domain. Denote by $K(D)$ a sequence $\{K_m\}$ of compact convex subsets of D that strictly exhausts the domain D, i.e., $K_m \subset \operatorname{int} K_{m+1}$, $m \geq 1$, and $D = \bigcup_{m \geq 1} K_m$. We put $\Gamma^+(\delta) = \{t\lambda : \lambda \in B(1,\delta), t > 0\}$ and

$$F_\Lambda(\lambda) = \prod_{k=1}^{\infty} \left[\exp\Big(n_k \frac{\lambda}{\lambda_k}\Big) \Big(1 - \frac{\lambda}{\lambda_k}\Big)^{n_k} \right].$$

Let $\bar{n}(\Lambda) < +\infty$. This is the case if, for example, Λ is the spectrum of a closed nontrivial invariant subspace $W \subset H(D)$ (see the beginning of the proof of Theorem 3.1). Then the function F_Λ has growth of order one and possibly an infinite type (see [6, Chapter I, §4, the Borel theorem]). Following the terminology of [19], we note that the existence of the function F_Λ means that W is a principal invariant subspace. Therefore, the next statement is true ([19, Theorem 4.1]).

Lemma 3.6. *Let D be a convex domain, let $a = H_D(1)$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Assume that W admits spectral synthesis and that for any $\delta \in (0, 1/3)$ and $m \geq 1$ there exists a subharmonic function $\psi(\lambda)$ on C, an index p, and $R, R' > 0$ such that*

1) $\lambda_k \in U = \Gamma^+(\delta) \cup B(0, R')$, $k \geq 1$;

2) $\psi(\lambda) + \ln|F_\Lambda(\lambda)| \geq H_{K_m}(\lambda)$, $\lambda \in (U + B(0,1)) \setminus U$, $K_m \in K(D)$;

3) $\psi(\lambda) + \ln|F_\Lambda(\lambda)| \leq H_{K_p}(\lambda)$, $\lambda \in B(0,R)$, $K_p \in K(D)$.

Then every function on W admits analytic continuation to the domain $\Pi(a)$ (if $a = +\infty$, then $\Pi(a) = \mathbb{C}$), and is approximated on this domain (uniformly on the compact sets) by linear combinations of elements of the system $\mathcal{E}(\Lambda)$.

Remark. We explain how Lemma 3.6 is obtained from Theorem 4.1 in [19]. Following the notation of [19], note that in our case for any neighborhood V (the point $\lambda = 1$ is necessarily in V) one can find a disk $B(1,3\delta)$ lying in V. One can take $\overline{B(1,2\delta)}\cap S(0,1)$ to be the compact set X from that theorem. Then, under the assumptions of Lemma 3.6, all conditions of Theorem 4.1 are satisfied.

The next statement reduces possibility 4 to the situation considered above.

Lemma 3.7. *Let D be an unbounded convex domain such that $1 \in \partial\mathcal{J}(D)$, let $H_D(1) = a$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Then $W = W(\Lambda, \Pi(a))$.*

Proof. First, note that, since the domain D is unbounded, the subspace W admits spectral synthesis (see [2, Theorem 8.1]). Next, we shall show that all other assumptions of Lemma 3.6 are fulfilled.

Fix $\delta \in (0,1/3)$ and $m \geq 1$. Since $K(D)$ is a strict exhaustion of the domain D, we can find $\varepsilon > 0$ such that

$$H_{K_m}(\lambda) + \varepsilon|\lambda| \leq H_{K_{m+1}}(\lambda) \leq H_{K_{m+2}}(\lambda) - 2\varepsilon|\lambda|, \quad \lambda \in \mathbb{C}. \tag{3.21}$$

By assumption $1 \in \partial\mathcal{J}(D)$. Consequently, the set $\mathcal{J}(D)$ contains either the upper or the lower half-plane. Suppose that the first case is realized (in the second case the proof is similar).

By Theorem 2.2, for given $\varepsilon > 0$ and $\delta \in (0,1/3)$, there exist numbers $\gamma \in (0,1)$ and $\{t_i\}$ and an entire function f such that inequalities (2.1) and (2.2) are valid. Moreover, the function f has zeros at the points λ_k of multiplicities at least n_k. This implies that the function

$$\widetilde{\psi}(\lambda) = \pi \operatorname{Im} \lambda/\gamma + H_{K_{m+1}}(\lambda) + \ln|f(\lambda)| - \ln|F_\Lambda(\lambda)|$$

is subharmonic on $\mathbb{C}$. By (2.1) and (3.21), we have

$$\widetilde{\psi}(\lambda) + \ln|F_\Lambda(\lambda)| \geq H_{K_{m+1}}(\lambda) - \varepsilon|\lambda| \geq H_{K_m}(\lambda), \quad \lambda \in \Gamma^+(2\delta) \setminus (\Gamma^+(\delta) \cup B(0,t_1)).$$

Since Λ is almost real, we can find $R' > t_1$ such that condition 1) of Lemma 3.6 is satisfied, and the set $((U + B(0,1)) \setminus U) \setminus B(0,R'+1)$ lies in $\Gamma^+(2\delta) \setminus \Gamma^+(\delta)$. Hence,

$$\widetilde{\psi}(\lambda) + \ln|F_\Lambda(\lambda)| \geq H_{K_m}(\lambda), \quad \lambda \in ((U + B(0,1)) \setminus U) \setminus B(0,R'+1). \tag{3.22}$$

Denote by $b_1,\dots,b_n$ all zeros of the function $f(\lambda)$ (with multiplicities) that belong to the disk $B(0,R'+\tau)$ and differ from λ_k, $k \geq 1$. Here, $\tau \in (1,2)$ is chosen so that there are no zeros of $f(\lambda)$ on the circle $S(0,R'+\tau)$. We put

$$h(\lambda) = \prod_{j=1}^{n} \frac{(R'+\tau)(\lambda - b_j)}{(R'+\tau)^2 - \overline{b_j}\lambda}.$$

Then the function $\widetilde{\psi}(\lambda) - \ln|h(\lambda)|$ is subharmonic in the disk $B(0,R'+\tau)$ and lower bounded in the annulus $B(0,R+1) \setminus B(0,R')$. By the choice of the number R', the function $F_\Lambda(\lambda)$ has no zeros in the set $B(0,R'+1) \setminus (B(0,R') \cup \Gamma^+(\delta))$. It follows that $\ln|F_\Lambda(\lambda)|$ is also lower bounded on this set. Therefore, for some $\beta \in \mathbb{R}$

$$\widetilde{\psi}(\lambda) - \ln|h(\lambda)| + \ln|F_\Lambda(\lambda)| \geq \beta, \quad \lambda \in ((U + B(0,1)) \setminus U) \cap B(0,R'+1). \tag{3.23}$$

Since $|h(\lambda)| = 1$, $\lambda \in S(0,R'+2)$, we have $\widetilde{\psi}(\lambda) = \widetilde{\psi}(\lambda) - \ln|h(\lambda)|$, $\lambda \in S(0,R'+2)$. It follows that for any $C > 0$, the function defined by $\psi(\lambda) = \widetilde{\psi}(\lambda) + C$ if $\lambda \notin B(0,R'+\tau)$ and by $\psi(\lambda) = \max\{\widetilde{\psi}(\lambda), \widetilde{\psi}(\lambda) - \ln|h(\lambda)|\} + C$ if $\lambda \in B(0,R'+\tau)$, is subharmonic in the plane. We choose $C > 0$ such that

$$\beta + C \geq \max_{\lambda \in B(0,R'+1)} H_{K_m}(\lambda).$$

In accordance with to (3.22) and (3.23), this gives us condition 2) of Lemma 3.6.

It remains to prove condition 3). By (2.2), there exists $R \geq R' + \tau$ such that

$$\ln |f(\lambda)| \leq \pi |\operatorname{Im} \lambda|/\gamma + 2\varepsilon|\lambda|, \quad \lambda \notin B(0, R) \tag{3.24}$$

(see [6, Chapter I, §18, Theorem 28]). Using the definition of $\psi(\lambda)$ and (3.21), we have

$$\begin{aligned} \psi(\lambda) + \ln |F_\Lambda(\lambda)| &= \widetilde{\psi}(\lambda) + C + \ln |F_\Lambda(\lambda)| \\ &= \pi \operatorname{Im} \lambda/\gamma + H_{K_{m+1}}(\lambda) + \ln |f(\lambda)| + C \\ &\leq \pi \operatorname{Im} \lambda/\gamma + H_{K_{m+1}}(\lambda) + \pi |\operatorname{Im} \lambda|/\gamma + 2\varepsilon|\lambda| + C \\ &\leq H_{K_{m+2}+I}(\lambda) + C, \quad \lambda \notin B(0, R), \end{aligned} \tag{3.25}$$

where I is the segment with endpoints 0 and $-2\pi i/\gamma$. Let us show that $K_{m+2} + I$ is a compact set in the domain D. Since this is the case for K_{m+2}, we have

$$H_{K_{m+2}+I}(\lambda) \leq H_{K_{m+1}}(\lambda) < H_D(\lambda), \quad \operatorname{Im} \lambda \leq 0, \quad \lambda \neq 0.$$

Keeping in mind that $\mathcal{J}(D)$ contains the upper half-plane, we see that

$$H_{K_{m+2}+I}(\lambda) < H_D(\lambda), \quad \operatorname{Im} \lambda > 0.$$

Thus, $H_{K_{m+2}+I}(\lambda) < H_D(\lambda)$, $\lambda \neq 0$, i.e., $H_{K_{m+2}+I}$ is a compact subset of D. Since $K(D)$ is an exhaustion of the domain D, we can find an index p for which $H_{K_{m+2}+I} \subset K_{p-1}$. Then $H_{K_{m+2}+I}(\lambda) \leq H_{K_{p-1}}(\lambda)$, $\lambda \in \mathbb{C}$. Increasing $R > 0$ if necessary, we may assume (as in (3.21)) that $H_{K_{p-1}}(\lambda) + C \leq H_{K_p}(\lambda)$, $\lambda \notin B(0, R)$. Then by (3.25), we obtain condition 3) of Lemma 3.6.

Now we can apply this lemma. In accordance with it each function in W admits analytic continuation to the domain $\Pi(a)$ (for $a = +\infty$, $\Pi(a) = \mathbb{C}$) and is approximated in that domain by linear combinations of elements of the system $\mathcal{E}(\Lambda)$. In other words, $W \subset W(\Lambda, \Pi(a))$. The converse imbedding is obvious. Lemma 3.7 is proved. □

Lemma 3.7 allows us to obtain a fundamental principle criterion in the case corresponding to Possibility 4 (i.e., $1 \in \partial\mathcal{J}(D)$ and $H_D(1) = a < +\infty$).

Theorem 3.8. *Let D be an unbounded convex domain such that $1 \in \partial\mathcal{J}(D)$, let $H_D(1) = a < +\infty$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$.*

1) If $S_\Lambda = 0$, then each function $g \in W$ expands in a series of the form (1.1) *uniformly convergent on the compact subsets of $\Pi(a)$, and the operator $\mathcal{L}(\Lambda, a)\colon \mathcal{B}(\Lambda, a) \to W(\Lambda, \Pi(a))$ is an isomorphism of linear topological spaces.*

2) If each function $g \in W$ admits representation by the series (1.1) *uniformly convergent on the compact subsets of D, then $S_\Lambda = 0$.*

Proof. By Lemma 3.7, we have $W = W(\Lambda, \Pi(a))$. Theorem 3.5 shows that statement 1) is true. In view of Lemma 3.4, statement 2) is also valid. □

Let us consider the situations corresponding to Possibilities 2 and 3. The case where $1 \in \operatorname{int} \mathcal{J}(D)$ has been analyzed completely. In particular, in this case the fundamental principle criterion was obtained in Theorem 5.1 of the paper [4]. Moreover, in Proposition 2 of the paper [20] it was proved that in this case all functions in W are entire. Specifically, a criterion for a function belonging to an arbitrary closed nontrivial invariant subspace that admits spectral synthesis to be entire was obtained in [20, Proposition 2] provided that $\mathcal{J}(D) = \operatorname{int} \mathcal{J}(D)$. Lemma 3.7 (which is true for $a = +\infty$) allows us to eliminate this condition for invariant subspaces with almost real spectrum. Using Lemma 3.7 and Proposition 1 in [20], we obtain the following result.

Theorem 3.9. *Let D be a convex domain, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Each function in W is entire if and only if $1 \in \mathcal{J}(D)$.*

Therefore, the situations corresponding to Possibilities 2 and 3 reduce to the "entire" case ($D = \mathbb{C}$). Using Theorem 3.9 and [4, Theorem 5.1], we obtain the following result.

Theorem 3.10. *Let D be an unbounded convex domain such that $1 \in \mathcal{J}(D)$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$.*

1) *If $S_\Lambda > -\infty$, then each function $g \in W$ expands in a series of the form* (1.1) *uniformly convergent on all compact subsets of $\mathbb{C}$, and the operator $\mathcal{L}(\Lambda, a)\colon \mathcal{B}(\Lambda, a) \to W(\Lambda, \mathbb{C})$ is an isomorphism of linear topological spaces.*

2) *If each function $g \in W$ is expands in a series of the form* (1.1) *uniformly convergent on all compact subsets of $\mathbb{C}$, then $S_\Lambda > -\infty$.*

§4. Representation of functions in an invariant subspace on a convex domain

In this section we study the situation corresponding to Possibility 1 of the preceding section ($1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$).

One of the main results of the paper [5] (see Theorem 1 therein) states that the condition $m(\Lambda) = \limsup_{k\to\infty} n_k/|\lambda_k| = 0$ is necessary for the validity of the fundamental principle in the subspace $W \subset H(D)$ in the case where D is a bounded domain. Our nearest goal is to generalize the result related to the case of invariant subspaces with almost real spectrum to the case of arbitrary convex domains provided that $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$.

First, we introduce some notation. Let $t > 0$, and let D_t be a domain obtained from D by applying a homothetic transformation with coefficient t centered at the origin, i.e., $D_t = \{z' = tz : z \in D\}$. We put $\Lambda(t) = \{t^{-1}\lambda_k, n_k\}$. Obviously, for any $t > 0$, $m(\Lambda) = 0$ implies $m(\Lambda(t)) = 0$ and *vice versa.* In [5, Lemma 1] it was proved that each function in $W(\Lambda, D)$ expands in a series of the form (1.1) uniformly convergent on the compact subsets of D if and only if each function in $W(\Lambda(t), D_t)$ admits a similar expansion uniformly convergent on the compact subsets of D_t.

Theorem 4.1. *Let D be a convex domain such that $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Suppose that each function $g \in W$ expands in a series of the form* (1.1) *uniformly convergent on the compact subsets of D. Then $m(\Lambda) = 0$.*

Proof. First, note that, as in Theorem 3.1 and Lemma 3.4, the sequence Λ has finite upper density, and there exists a system of functionals $\Xi(\Lambda, D) \subset H^*(D)$ biorthorgonal to $\mathcal{E}(\Lambda)$.

Since $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$, we can find $\varphi \in (0, \pi/2)$ such that the function H_D is bounded in a neighborhood of the arc $\gamma = \{e^{i\alpha} : \alpha \in [-\varphi, \varphi]\}$. Putting $\Gamma = \{te^{i\psi} : \psi \in (\pi/2 + \varphi, 3\pi/2 - \varphi),\ t > 0\}$, we show that the set $D' = D \setminus \Gamma$ is bounded. Suppose the contrary. Then there exist points $t_l e^{i\psi(l)} \in D'$, $l \geq 1$, such that $t_l \to +\infty$ and $\psi(l) \to \psi(0) \in [-\varphi - \pi/2, \pi/2 + \varphi]$ as $l \to \infty$. Choose an arbitrary point $e^{i\psi}$ on the half-circle $S = \{e^{i\alpha} : \alpha \in (-\psi(0) - \pi/2, -\psi(0) + \pi/2)\}$. There is a number $b > 0$ and index $l(0)$ such that $\operatorname{Re}(e^{i\psi} e^{i\psi(l)}) \geq b$, $l \geq l(0)$. Hence,

$$H_D(e^{i\psi}) \geq \operatorname{Re}(e^{i\psi} t_l e^{i\psi(l)}) \geq t_l b \to +\infty, \quad l \to \infty,$$

i.e., we have $S \subset \mathcal{J}(D)$. This contradicts our choice of the arc γ, because S intersects any neighborhood of the arc. Therefore, D' is bounded.

Suppose that $H_D(1) \le 0$. By [5, Lemma 1], applying a homothetic transformation if necessary, we may assume that the set D' lies inside the disk $B(0,1/2)$ and

$$H_D(1) > -\sin\varphi/21. \tag{4.1}$$

Suppose that $m(\Lambda) \neq 0$. Then we can find $\tau > 0$ and a sequence of indices $\{k(j)\}$ such that $n_{k(j)} \ge \tau|\lambda_{k(j)}|$, $j \ge 1$. Let $\nu_0 = \min\{(\ln 21 - \ln\sin\varphi)^{-1}\sin\varphi/20, \tau/4\}$. We may assume that

$$n_{k(j)} \ge \tau|\lambda_{k(j)}| \ge 4\nu_0|\lambda_{k(j)}| \ge 4\nu(j) \ge 4\nu|\lambda_{k(j)}|, \tag{4.2}$$

where $\nu(j)$, $j \ge 1$, are natural numbers and $\nu > 0$. We set $D'' = D \setminus \Pi(-\sin\varphi/2)$. By construction (due to the inequality $H_D(1) \le 0$), the set D'' lies in the half-disk $B(0,1/2) \cap (\mathbb{C} \setminus \Pi(0))$, while its complement $D \setminus D''$ is contained in the truncated angle $\Gamma \cap \Pi(-\sin\varphi/2)$. Consider the series

$$\sum_{j=1}^{\infty} c_j \exp(\delta|\lambda_{k(j)}|)(zp(z))^{\nu(j)} \exp(\lambda_{k(j)} z), \tag{4.3}$$

where $p(z) = (z^2+1)(z+1)$ and

$$c_j = \exp\Big(-\sup_{z\in D''}\big(\nu(j)\ln|z| + \operatorname{Re}(\lambda_{k(j)} z)\big)\Big), \quad j \ge 1.$$

The series (4.3) resembles the series (11) in the paper [5]. Therefore, repeating word-for-word arguments at step 3) in the proof of Theorem 1 in that paper, we obtain the following. There exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ the series (4.3) is uniformly convergent on the set D''. There exists a set $\widetilde{D} \subset D''$ (depending only on δ) and a sequence of indices $j(p)$, $p \ge 1$, such that

$$\big|c_{j(p)} \exp\big(\delta|\lambda_{k(j(p))}|\big)(z')^{\nu(j(p))} \exp\big(\lambda_{k(j(p))} z'\big)\big| \ge 1, \quad p \ge 1, \quad z' \in \widetilde{D}. \tag{4.4}$$

Now we show that the series (4.3) converges uniformly on the set $\Gamma \cap \Pi(-\sin\varphi/2) \cap B(0,R)$. First, we estimate the coefficients c_j. By (4.1), we can find a point $\widetilde{z} \in D''$ such that $\operatorname{Re}\widetilde{z} \ge -\sin\varphi/21$. Since Λ is an almost real sequence, we have

$$\operatorname{Re}(\lambda_{k(j)}\widetilde{z}) = |\lambda_{k(j)}|\left(\operatorname{Re}\widetilde{z}\frac{\operatorname{Re}\lambda_{k(j)}}{|\lambda_{k(j)}|} - \frac{|\operatorname{Im}\widetilde{z}|\,|\operatorname{Im}\lambda_{k(j)}|}{|\lambda_{k(j)}|}\right) \ge -|\lambda_{k(j)}|\sin\varphi/20, \quad j \ge j_0.$$

Using (4.2), the definition of ν_0, and the inequality $|\widetilde{z}| < 1/2$, we get

$$\nu(j)\ln|\widetilde{z}| \ge \nu_0|\lambda_{k(j)}|\ln|\operatorname{Re}\widetilde{z}| \ge \frac{\sin\varphi|\lambda_{k(j)}|}{20(\ln 21 - \ln\sin\varphi)}\ln|\sin\varphi/21| \ge -|\lambda_{k(j)}|\frac{\sin\varphi}{20}.$$

Thus, $c_j \le \exp(\sin\varphi|\lambda_{k(j)}|/10)$, $j \ge j_0$. Let $\delta \in (0, \sin\varphi/10)$. Then

$$c_j \exp(\delta|\lambda_{k(j)}|) \le \exp(\sin\varphi|\lambda_{k(j)}|/5), \quad j \ge j_0. \tag{4.5}$$

Suppose $z \in \Gamma \cap \Pi(-\sin\varphi/2) \cap B(0,R)$. As above, we obtain

$$\operatorname{Re}(\lambda_{k(j)} z) \le |\lambda_{k(j)}|(\operatorname{Re} z + \sin\varphi/20), \quad j \ge j_1 \ge j_0.$$

By the definition of the angle Γ, we have $|z| \le |\operatorname{Re} z|/\sin\varphi$. Taking into account (4.2), (4.5), and the fact that $z \in \Pi(-\sin\varphi/2)$, we obtain

$$\begin{aligned}
&c_j \exp(\delta|\lambda_{k(j)}|)|zp(z)|^{\nu(j)}|\exp(\lambda_{k(j)}z)|\\
&\quad\le \exp\left(|\lambda_{k(j)}|(\sin\varphi/4 + \operatorname{Re} z)\right)|zp(z)|^{\nu(j)}\\
&\quad\le \exp\left(|\lambda_{k(j)}|\left(\frac{\sin\varphi}{4} - |\operatorname{Re} z| + \ln|zp(z)|^{\nu_0}\right)\right)\\
&\quad\le \exp\left(|\lambda_{k(j)}|\left(\frac{\sin\varphi}{4} - |\operatorname{Re} z| + 4\nu_0\ln(1+|z|)\right)\right)\\
&\quad\le \exp\left(|\lambda_{k(j)}|\left(\frac{\sin\varphi}{4} - |\operatorname{Re} z| + \frac{\sin\varphi|z|}{5}\right)\right)\\
&\quad\le \exp\left(|\lambda_{k(j)}|\left(\frac{\sin\varphi}{4} - \frac{4}{5}|\operatorname{Re} z|\right)\right)\\
&\quad\le \exp(-3|\lambda_{k(j)}|\sin\varphi/30), \quad j \ge j_1.
\end{aligned}$$

Since Λ has finite upper density, the last inequality means the uniform convergence of the series (4.3) on the set $\Gamma \cap \Pi(-\sin\varphi/2) \cap B(0,R)$.

Therefore, the series (4.3) converges uniformly on the compact sets of the domain D. Consequently, its sum $g(z)$ belongs to the subspace W, because it is closed and $n_{k(j)} \ge 4\nu(j)$ by (4.2). By assumptions, the function g expands in a series like (1.1) uniformly converging on the compact subsets of D. Let $\mu_{k(j),\nu(j)} \in \Xi(\Lambda, D)$. Then we have

$$|\mu_{k(j),\nu(j)}(g)| = |d_{k(j),\nu(j)}| = c_j \exp(\delta|\lambda_{k_j}|), \quad j \ge 1.$$

The last statement together with (4.4) contradicts the convergence of the series (1.1) at the points $z' \in \widetilde{D} \subset D$. Hence, our assumption is false, i.e., $m(\Lambda) = 0$ if $H_D(1) \le 0$.

Suppose now that $H_D(1) > 0$. As above, $D' = D \setminus \Gamma$ is bounded. We put

$$T = \{z : \operatorname{Re} z = H_D(1)\} \cap \partial D = \{z : \operatorname{Re} z = H_D(1)\} \cap \partial D'.$$

The set T is nonempty and bounded (it is a singleton or a segment). This implies that we can apply, if necessary, a homothetic transformation centered at the origin and assume that the disk $B(1,1)$ contains ?, and, moreover, the set $D'' = D \setminus \Pi(-\sin\varphi)$ lies in the disk $B(0,1)$, while its complement $D \setminus D''$ belongs to the truncated angle $\Gamma \cap \Pi(-\sin\varphi)$. Since the set T is compact, it is contained in $B(1,1)$ together with some its neighborhood. Hence, one can find $\varepsilon > 0$ such that the set

$$T(\varepsilon) = \{z : \operatorname{Re} z \ge H_D(1) - \varepsilon\} \cap D$$

is compactly included in $B(1,1)$. In other words, for some r_0 in the interval $(e^{-1}, 1)$, the set $T(\varepsilon)$ is contained in the disk $B(1, r_0)$. Suppose that $m(\Lambda) \ne 0$. Then we can find $\tau > 0$ and a sequence of indices $\{k(j)\}$ such that $n_{k(j)} \ge \tau|\lambda_{k(j)}|$, $j \ge 1$. We may assume that

$$\begin{aligned}
\gamma|\lambda_{k(j)}| &= \min\left\{\frac{\varepsilon}{4}, \frac{\tau}{2}, \frac{\sin\varphi}{4}\right\}|\lambda_{k(j)}| \le n(j)\\
&\le \min\left\{\frac{\varepsilon|\lambda_{k(j)}|}{2}, n_{k(j)}, \frac{|\lambda_{k(j)}|\sin\varphi}{2}\right\}, \quad j \ge 1,
\end{aligned} \tag{4.6}$$

where the $n(j)$ are natural numbers. Since Λ is an almost real sequence, and $r_0 < 1$, for any $R > 0$ and some $j(R)$ we have

$$\operatorname{Re}(\lambda_{k(j)}z) \le |\lambda_{k(j)}|(\operatorname{Re} z - \gamma\ln r_0/2), \quad z \in B(0,R), \quad j \ge j(R). \tag{4.7}$$

We put $c_j = \exp(-(H_D(1) + \gamma \ln r_0/4)|\lambda_{k(j)}|)$, $j \ge 1$, and consider the series

$$\sum_{j=1}^{\infty} c_j (z-1)^{n(j)} \exp(\lambda_{k(j)} z). \tag{4.8}$$

We shall show that this series converges uniformly on the compact subsets of the domain D.

By (4.6) and (4.7), taking into account the imbedding $T(\varepsilon) \subset B(1, r_0)$ and the definition of H_D, we obtain

$$\begin{aligned} &\big|c_j(z-1)^{n(j)} \exp(\lambda_{k(j)} z)\big| \\ &\qquad \le \exp\big(n(j)\ln r_0 - (H_D(1) - \operatorname{Re} z + 3\gamma \ln r_0/4)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big(\gamma|\lambda_{k(j)}| \ln r_0 - 3\gamma|\lambda_{k(j)}| \ln r_0/4\big) \\ &\qquad = \exp\big(\gamma|\lambda_{k(j)}| \ln r_0/4\big), \quad z \in T(\varepsilon), \quad j \ge j(1). \end{aligned} \tag{4.9}$$

Since $D'' \subset B(0,1)$ and $r_0 > e^{-1}$, by (4.6), (4.7) and the definition of $T(\varepsilon)$ we get

$$\begin{aligned} &\big|c_j(z-1)^{n(j)} \exp(\lambda_{k(j)} z)\big| \\ &\qquad \le \exp\big(n(j)\ln 2 - (H_D(1) - \operatorname{Re} z + 3\gamma \ln r_0/4)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big(-\varepsilon|\lambda_{k(j)}|/2 + 3\gamma|\lambda_{k(j)}|/4\big) \\ &\qquad \le \exp\big(-5\varepsilon|\lambda_{k(j)}|/16\big), \quad z \in D'' \setminus T(\varepsilon), \quad j \ge j(1). \end{aligned} \tag{4.10}$$

Since $D \setminus D'' \subset \Gamma \cap \Pi(-\sin\varphi)$, the inequality $H_D(1) > 0$ shows that

$$\begin{aligned} &\big|c_j(z-1)^{n(j)} \exp(\lambda_{k(j)} z)\big| \\ &\qquad \le \exp\big(n(j)\ln(1+|z|) - (H_D(1) - \operatorname{Re} z + 3\gamma \ln r_0/4)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big(2^{-1}|\lambda_{k(j)}| \sin\varphi \ln(1+|z|) + (\operatorname{Re} z + 3\sin\varphi/16)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big((|z|\sin\varphi/2 + \operatorname{Re} z + 3\sin\varphi/16)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big((|\operatorname{Re} z|/2 + \operatorname{Re} z + 3\sin\varphi/16)|\lambda_{k(j)}|\big) \\ &\qquad \le \exp\big((-5\sin\varphi/16)|\lambda_{k(j)}|\big), \quad z \in (D \setminus D'') \cap B(0,R), \quad j \ge j(R). \end{aligned}$$

As above, from (4.9) and (4.10) it follows that the series (4.8) uniformly converges on the compact subsets of the domain D and its sum g belongs to the subspace W.

By assumption, the function g expands in a series of the form (1.1) uniformly convergent on the compact subsets of D. Let $\mu_{k(j),0} \in \Xi(\Lambda, D)$. Then

$$|\mu_{k(j),0}(g)| = |d_{k(j),0}| = c_j, \quad j \ge 1.$$

Since $r_0 < 1$, the definition of the support function shows that we can find a point $z' \in D$ such that $\operatorname{Re} z' \ge H_D(1) + \gamma \ln r_0/8$. Like in (4.7), we obtain

$$\operatorname{Re}(\lambda_{k(j)} z') \ge |\lambda_{k(j)}|(\operatorname{Re} z' + \gamma \ln r_0/8), \quad j \ge j_0.$$

Hence, recalling the definition of c_j, we have

$$|d_{k(j),0}|\,|\exp(\lambda_{k(j)} z')| \ge 1, \quad j \ge j_0.$$

This contradicts the convergence of the series (1.1) at the point $z' \in D$. Therefore, our assumption is not true, i.e., $m(\Lambda) = 0$ if $H_D(1) > 0$. The theorem is proved. □

Theorem 4.1 allows us to obtain necessary conditions for the validity of the fundamental principle for invariant subspaces with arbitrary spectrum. These results were not obtained in the paper [4] and can be regarded as an improvement of its main result (Theorem 5.1).

First, we introduce some notation. Let $D(\varphi)$ be the domain obtained from the domain D by rotation by the angle φ, i.e., $D(\varphi) = \{z' = e^{i\varphi}z : z \in D\}$. We put $\Lambda_\varphi = \{e^{-i\varphi}\lambda_k, n_k\}$. Then $m(\Lambda) = m(\Lambda_\varphi)$. Using the same argument as in the proof of Lemma 1 in [5], we can prove the following result. Each function of class $W(\Lambda, D)$ expands in a series of the form (1.1) uniformly convergent on the compact subsets of D if and only if each function of class $W(\Lambda_\varphi, D(\varphi))$ expands in a series of the form (1.1) uniformly convergent on the compact subsets of $D(\varphi)$.

Theorem 4.2. *Let D be a convex domain, and let W be a closed nontrivial invariant subspace of $H(D)$ with spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. Suppose that each function $g \in W$ expands in a series of the form* (1.1) *uniformly convergent on the compact subsets of D. If the subsequence $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ converges to a point $\xi \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$, then* $\lim_{j\to\infty} n_{k(j)}/|\lambda_{k(j)}| = 0$.

Proof. Suppose that $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ converges to a point $\xi \in \operatorname{int}(\mathbb{C}\setminus\mathcal{J}(D))$. Consider the subsequence $\widetilde{\Lambda} = \{\lambda_{k(j)}, n_{k(j)}\}_{j=1)}^{\infty}$. We need to show that $m(\widetilde{\Lambda}) = 0$. By the hypothesis, since W is a closed subspace, each function $g \in \widetilde{W} = W(\widetilde{\Lambda}, D) \subset W$ expands in a series of the form (1.1) uniformly convergent on the compact subsets of D. Since W is nontrivial, there exists a system of functionals $\Xi(\Lambda, D) \subset H^*(D)$ biorthogonal to $\mathcal{E}(\Lambda)$. Then $d_{k,n} = \mu_{k,n}(g)$, $\mu_{k,n} \in \Xi(\Lambda, D)$. Since $g \in W(\widetilde{\Lambda}, D)$ is approximated only by elements of the system $\mathcal{E}(\widetilde{\Lambda})$, we have $d_{k,n} = 0$ for all $k \neq k(j)$, $j \geq 1$. Hence,

$$g(z) = \sum_{j=1,n=0}^{\infty,n_{k(j)}-1} d_{k(j),n} z^n \exp(\lambda_{k(j)}z), \quad z \in D, \quad g \in \widetilde{W},$$

and the series converges uniformly on the compact subsets of D. Using a rotation if necessary, we may assume that $\xi = 1$. Then

$$\frac{\lambda_{k(j)}}{|\lambda_{k(j)}|} = \frac{\operatorname{Re}\lambda_{k(j)}}{|\operatorname{Re}\lambda_{k(j)}|}\frac{1 + i(\operatorname{Im}\lambda_{k(j)}/\operatorname{Re}\lambda_{k(j)})}{\sqrt{1 + (\operatorname{Im}\lambda_{k(j)}/\operatorname{Re}\lambda_{k(j)})^2}} \to 1, \quad j \to 1.$$

Consequently, passing to subsequences, we may assume that the following is true:

$$\operatorname{Re}\lambda_{k(j)} > 0, \ j \geq 1, \ \text{ and } \ \operatorname{Im}\lambda_{k(j)}/\operatorname{Re}\lambda_{k(j)} \to 0 \ \text{ as } \ j \to \infty.$$

Therefore, $\widetilde{W}$ is a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\widetilde{\Lambda}$. By Theorem 4.1, we have $m(\widetilde{\Lambda}) = 0$. The theorem is proved. □

Remark. Theorem 4.2 contains Theorem 1 in the paper [5] as a particular case. Indeed, if D is a bounded domain, we have $m(\Lambda) = 0$ by Theorem 4.2.

Now we are able to prove a result that establishes the fundamental principle for arbitrary invariant subspaces and improves the result of [4, Theorem 5.1]. To formulate this result, we need additional definitions and notation.

Let D be a convex domain, and let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. In the paper [4], before Proposition 2.3, an analog $m(\Lambda)$ of the quantity $m_D(\Lambda)$ was introduced, which takes the geometry of the domain D into account. The definition of $m_D(\Lambda)$ immediately implies the following. We have $m_D(\Lambda) = 0$ if and only if $\lim_{j\to\infty} n_{k(j)}/|\lambda_{k(j)}| = 0$ for any subsequence $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ that converges to a point of the set $\mathbb{C} \setminus \mathcal{J}(D)$. A similar statement is also true for the quantity $S_\Lambda(D)$ (an analog of S_Λ, which involves the geometry of the domain D), introduced in [4] before Theorem 3.6. It vanishes if and only if $S_{\widetilde{\Lambda}} = 0$ for any subsequence $\widetilde{\Lambda} = \{\lambda_{k(j)}\}$ such that $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ converges to a point in $\mathbb{C}\setminus\mathcal{J}(D)$.

Let $K(D) = \{K_m\}$. We introduce the following Banach spaces of numerical sequences:

$$\mathcal{B}_m(\Lambda, D) = \big\{d = \{d_{k,n}\} : \|d\|_m = \sup_{k,n} |d_{k,n}| \exp(H_{K_m}(\lambda_k)) < \infty\big\}, \quad m \geq 1.$$

Denote by $\mathcal{B}(\Lambda, D)$ the projective limit of $\mathcal{B}_m(\Lambda, D)$ and by $\mathcal{L}(\Lambda, D)$ the operator acting from $\mathcal{B}(\Lambda, D)$ to $W(\Lambda, D)$ by the following rule: a sequence $d \in \mathcal{B}(\Lambda, D)$ is taken to the sum of the series (1.1), provided that it converges uniformly on the compact subsets of the domain D.

We say that the system $\mathcal{E}(\Lambda)$ is a Köthe basis in the subspace $W \subset H(D)$ if each function $g \in W$ is represented by a series of the form (1.1) in the domain D, and for any $m \geq 1$ there exists $p \geq 1$ and $C > 0$ (independent of g) such that

$$\sum_{k=1,n=0}^{\infty,n_k-1} \max_{z\in K_m} |d_{k,n} z^n \exp(\lambda_k z)| \leq C \max_{z\in K_p} |g(z)|.$$

Let $I_\Lambda(D)$ denote the set of all entire functions f of exponential type that have zeros at the points λ_k of multiplicities at least n_k and whose conjugate diagrams lie in D (i.e., $h_f(\lambda) < H_D(\lambda)$, $\lambda \neq 0$).

If Γ is an open angle with vertex at the origin, then $\Lambda(\Gamma)$ denotes the set of all points λ_k belonging to the angle Γ.

Theorem 4.3. *Let D be a convex domain, and let W be a closed nontrivial invariant subspace of $H(D)$ with the spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ that admits spectral synthesis. Suppose that $m(\widetilde{\Lambda}) = 0$ for any subsequence $\widetilde{\Lambda} = \{\lambda_{k(j)}\}$ such that $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ converges to a point $\xi \in \partial\mathcal{J}(D) \setminus \mathcal{J}(D)$. Then the following statements are are equivalent.*

1) *Every function $g \in W$ is represented by a series of the form* (1.1) *at any point $z \in D$, and for any subsequence $\{\lambda_{k(j)}/|\lambda_{k(j)}|\}$ that converges to a point in* $\operatorname{int}(\mathbb{C}\setminus\mathcal{J}(D))$ *we have*

$$\lim_{j\to\infty} n_{k(j)}/|\lambda_{k(j)}| = 0.$$

2) *Every function $g \in W$ is represented by a series of the form* (1.1) *that uniformly converges on the compact subsets of the domain D.*

3) *The system $\mathcal{E}(\Lambda)$ is a Köthe basis in W.*

4) *The operator $\mathcal{L}(\Lambda, D)\colon \mathcal{B}(\Lambda, D) \to W = W(\Lambda, D)$ is an isomorphism of linear topological spaces.*

5) *$S_\Lambda > -\infty$, $S_\Lambda(D) = 0$, and for every $m \geq 1$ and every compact set $F \subset S(0,1) \setminus \mathcal{J}(D)$ there exists $\varphi \in I_\Lambda(D)$ such that for any $\delta > 0$ we can find an open angle Γ and $T > 0$ such that $F \subset \Gamma$ and*

$$(\Lambda(\Gamma) \setminus B(0,T)) \subset \mathcal{R}^\delta, \quad \mathcal{R} = \{z : \ln|\varphi(z)| \geq H_{K_m}(z)\}.$$

Remark. 1. If $\mathbb{C} \setminus (\mathcal{J}(D) \cup \{0\})$ is an open set (in particular, if D is bounded) the condition $m(\widetilde{\Lambda}) = 0$ in Theorem 4.3 can be omitted (it is satisfied automatically). In this case, there is no subsequences $\widetilde{\Lambda}$ as in the theorem. Therefore, we obtain a criterion for the validity of the fundamental principle without any additional restrictions.

2. In the general case, the condition $m(\widetilde{\Lambda}) = 0$ cannot be lifted, otherwise the implication 2) $\Rightarrow$ 5) becomes wrong. Indeed, by Theorem 3.5, in the case of an almost real spectrum and $D = \Pi(a)$, statement 2) is equivalent to the fact that $S_\Lambda = 0$, i.e., 2) is valid even when $m(\Lambda) = \max_{\widetilde{\Lambda}} m(\widetilde{\Lambda}) \neq 0$. On the other hand, by the remark to Theorem 3.6 in [4], statement 5) implies the identity $m_D(\Lambda) = 0$ (in this case, $m_D(\Lambda) = m(\Lambda)$). Note that, by Theorems 2.2 and 2.4, the function $\varphi \in I_\Lambda(D)$, similar to that in 5), always exists. However, in contrast to 5), it depends on $\delta > 0$ (even with $S_\Lambda = 0$). Nevertheless, combined with the identity $S_\Lambda = 0$, this guaranties the validity of the fundamental principle for invariant subspace with almost real spectrum on the half-plane.

Proof. Suppose statement 1) is true. By the hypothesis of the theorem ($m(\widetilde{\Lambda}) = 0$), this provides the relation $m_D(\Lambda) = 0$. Theorem 5.1 in the paper [4] shows that the implications 1) $\Rightarrow$ 3) and 1) $\Rightarrow$ 5) are true. The implications 3) $\Rightarrow$ 2) and 4) $\Rightarrow$ 2) are obvious. By Theorem 4.2, the implication 2) $\Rightarrow$ 1) is also true. As was mentioned in Remark 2, statement 5) implies that $m_D(\Lambda) = 0$. Hence, by Theorem 5.1 in [4], the implication 5) $\Rightarrow$ 1) is true. It remains to prove that 2) $\Rightarrow$ 4). We have already seen that if 2) is true, then 1) is also true. Hence, by Proposition 2.5 in [4], the operator $\mathcal{L}(\Lambda, D)$ is a surjection. Then we obtain 4) as in Lemma 3.2. Theorem 4.3 is proved. □

At the end of the paper, we return to the (promised at the beginning of this section) case of invariant subspaces with almost real spectrum provided that $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(D))$. In this case, statement 5) of Theorem 4.3 can be replaced by simple geometric conditions on the sequence Λ and the domain D.

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be an almost real sequence with finite upper density and $m(\Lambda) = 0$. Then in accordance with the theorems of Abel and Cauchy–Hadamard (see [21, Theorems 3.1 and 4.1]), every series (1.1) converges either on the plane or on the half-plane $\Pi(a)$. Moreover, the abscissa of convergence is calculated by the formula

$$a = \liminf_{k\to\infty} \min_{0\le n<n_k} \frac{\ln|1/d_{k,n}|}{|\lambda_k|}.$$

Using this fact, it is not difficult to obtain analogs of Lemma 1 (for $m(\Lambda) = 0$) and Lemma 4 from the paper [13]. Using those analogs and Theorems 2.5, 4.1, 4.3, and repeating almost word-for-word the proof of Theorem 4 in [13], we obtain the following result.

Theorem 4.4. *Let D be a convex domain, let $1 \in \operatorname{int}(\mathbb{C} \setminus \mathcal{J}(\mathrm{D}))$, and let W be a closed nontrivial invariant subspace of $H(D)$ with almost real spectrum $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ that admits spectral synthesis. The following statements are equivalent.*

1) *Each function W is represented by a series of the form* (1.1) *on the half-plane $\Pi(H_D(1))$.*

2) *$S_\Lambda = 0$, $n_0(\Lambda) < +\infty$, and the intersection of the support line $\{z \in \mathbb{C} : \operatorname{Re} z = H_D(1)\}$ with the boundary of the domain D contains a segment of length $2\pi n_0(\Lambda)$.*

References

[1] I. F. Krasichkov-Ternovskiĭ, *Invariant subspaces of analytic functions.* I. *Spectral synthesis on convex domains*, Mat. Sb. **87** (1972), no. 4, 459–489; English transl., Math. USSR-Sb. **16** (1972), no. 4, 471–500. MR0422636 (54:10622)

[2] I. F. Krasichkov-Ternovskiĭ, *Invariant subspaces of analytic functions.* II. *Spectral synthesis on convex domains*, Mat. Sb. **88** (1972), no. 1, 1–30; English transl., Math. USSR-Sb. **17** (1972), no. 1, 1–29. MR0422636 (54:10623)

[3] A. A. Gol′dberg, B. Ya. Levin, and I. V. Ostrovskiĭ, *Entire and meromorphic functions*, Itogi Nauki i Techiki, Sovrem. Probl. Mat. Fund. Naprav., vol. 85, Akad. Nauk SSSR, VINITI, Moscow, 1991, pp. 5–185; English transl., Encyclopaedia Math. Sci., vol. 85, Springer, Berlin, 1997. MR1155417 (93a:30031)

[4] A. S. Krivosheev, *The fundamental principle for invariant subspaces in convex domains* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 2, 71–136; English transl., Izv. Math. **68** (2004), no. 2, 291–353. MR2058001

[5] O. A. Krivosheeva and A. S. Krivosheev, *Criterion for the fundamental principle to hold for invariant subspaces in bounded convex domains in the complex plane* (Russian, with Russian summary), Funktsional. Anal. i Prilozhen. **46** (2012), no. 4, 14–30; English transl., Funct. Anal. Appl. **46** (2012), no. 4, 249–261. MR3075093

[6] B. Ya. Levin, *Distribution of zeros of entire functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; English transl., Transl. Math. Monogr., vol. 5, Amer. Math. Soc., Providence, RI, 1964. MR0087740 (19:402c)

[7] A. F. Leont′ev, *Sequences of polynomials of exponentials*, Nauka, Moscow, 1980. (Russian) MR0577300 (81m:30002)

[8] A. S. Krivosheev, *An almost exponential basis*, Ufimsk. Mat. Zh. **2** (2010), no. 1, 87–96. (Russian)

[9] A. S. Krivosheyev, *Basis is broken by relatively small groups*, Ufimsk. Mat. Zh. **2** (2010), no. 2, 67–89. (Russian)

[10] A. S. Krivosheyev, *An almost exponential sequence of exponential polynomials*, Ufimsk. Mat. Zh. **4** (2012), no. 1, 88–106; English transl., Ufa Math. J. **4** (2012), no. 1, 82–99.

[11] A. S. Krivosheev and O. A. Krivosheeva, *A basis in an invariant subspace of analytic functions* (Russian, with Russian summary), Mat. Sb. **204** (2013), no. 12, 49–104; English transl., Sb. Math. **204** (2013), no. 11-12, 1745–1796. MR3185085

[12] A. S. Krivosheev and O. A. Krivosheeva, *A basis in an invariant subspace of entire functions* (Russian, with Russian summary), Algebra i Analiz **27** (2015), no. 2, 132–195; English transl., St. Petersburg Math. J. **27** (2016), no. 2, 273–316. MR3444464

[13] A. S. Krivosheyev and O. A. Krivosheyeva, *A closedness of set of Dirichlet series sums*, Ufa Math. J. **5** (2013), no. 3, 94–117. MR3430790

[14] O. A. Krivosheeva, *Singular points of the sum of series of exponential monomials on the boundary of the convergence domain* (Russian, with Russian summary), Algebra i Analiz **23** (2011), no. 2, 162–205; English transl., St. Petersburg Math. J. **23** (2012), no. 2, 321–350. MR2841675

[15] A. F. Leont′ev, *Entire functions. Series of exponentials*, Nauka, Moscow, 1983. (Russian) MR0753827 (86j:30005)

[16] I. F. Krasičkov, *Comparison of entire functions of finite order by means of the distribution of their roots* (Russian), Mat. Sb. (N.S.) **70 (112)** (1966), 198–230. MR0196087

[17] A. F. Leont′ev, *Exponential series*, Nauka, Moscow, 1976. (Russian) MR0584943 (58:2845)

[18] A. S. Krivosheev, *A criterion for the analytic continuation of functional belonging to invariant subspaces in convex domains of the complex plane* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 1, 43–78; English transl., Izv. Math. **68** (2004), no. 1, 43–76. MR2096937

[19] A. S. Krivosheev, *A criterion for the analytic continuation of functions from the principal invariant subspaces in convex domains in* $\mathbb{C}^n$ (Russian, with Russian summary), Algebra i Analiz **22** (2010), no. 4, 137–197; English transl., St. Petersburg Math. J. **22** (2011), no. 4, 615–655. MR2768963

[20] A. S. Krivosheev, *Analytic continuation of a function from invariant subspaces in convex domains of a complex space* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), no. 2, 75–102; English transl., Izv. Math. **62** (1998), no. 2, 287–312. MR1623826

[21] O. A. Krivosheeva, *The convergence domain for series of exponential monomials*, Ufimsk. Mat. Zh. **3** (2011), no. 2, 43–56; English transl., Ufa Math. J. **3** (2011), no. 2, 42–55. MR3428992

[22] L. Schwartz, *Approximation d'un fonction quelconque par des sommes d'exponentielles imaginaires* (French), Ann. Fac. Sci. Univ. Toulouse (4) **6** (1943), 111–176. MR0015553

[23] B. Ja. Levin, *Distribution of zeros of entire functions*, Revised edition, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980. Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR589888

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