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NISNEVICH SHEAFIFICATION OF A HOMOTOPY INVARIANT PRESHEAF WITH TRANSFERS

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Abstract. The definitions of the category of finite Witt-correspondences and of a presheaf with Witt-transfers are given. The injectivity on the affine line, the excision isomorphism on the affine line, and the excision isomorphism for an *étale* morphism of curves are proved. The homotopy invariance of the Nisnevich sheafification $\mathcal{F}_{\mathrm{nis}}$ of a homotopy invariant presheave with Witt-transfers $\mathcal{F}$ is proved, and the Nisnevich cohomologies $H^i_{\mathrm{nis}}(U, \mathcal{F}_{\mathrm{nis}})$ are shown to be trivial for any $U \subset \mathbb{A}^1$ and $i > 0$.

§1. Introduction

The triangulated category of motives $DM^-(k)$ over a perfect field k was constructed by Voevodsky in [4, 6] by the method that we shall call Voevodsky's method. The category of Voevodsky's motives $DM^-(k)$ equipped with the functor $Sm_k \to SM^-(k)$ is in a sense a universal object, and a certain class of cohomology theories on algebraic varieties can be passed through $DM^-(k)$. Thus, the higher Chow groups, the *étale* cohomologies with coefficients μ_n, and the motivic cohomologies $H^i(-, \mathbb{Z}(k))$ are well defined on the category $DM^-(k)$.

I. A. Panin posed the following problem: to construct the triangulated category of Witt-motives $DWM(k)$ over a perfect filed k, $\operatorname{char} k \neq 2$, by the Voevodsky method, using the category of the so-called finite Witt-correspondences as an initial object. This paper belongs to a series of publications where the category of Witt-motives $DWM(k)$ will be constructed, see [7]. It is expected that with rational coefficients the category $DWM(k)$ is equivalent to the minus part of $D^{\mathbb{A}^1}(k)_{\mathbb{Q}}$. Like in the category of Voevodsky's motives, many Hom-groups will be actually computable in the category $DWM(k)$.

Moreover, it is expected that the category $DWM(k)$, which will be constructed ultimately, will be equivalent to the category of Witt-motives constructed in [2] by Ananievsky, Levine, and Panin. The latter category with rational coefficients is equivalent to the minus part of $D^{\mathbb{A}^1}(k)_{\mathbb{Q}}$. The category in [2] was constructed as some $\mathbb{A}^1$-derived category of the category of Nisnevich sheaves over the Nisnevich sheaf of Witt rings $\underline{W}(-)$.

As an initial object, the Voevodsky method employs the preadditive category of correspondences Cor_k. The objects of Cor_k are smooth varieties, and the functor $\mathrm{Sm}_k \to \mathrm{Cor}_k$ is identity on objects. This method is based on the fundamental theorem proved by Voevodsky in [5] about homotopy invariant presheaves of Abelian groups on the category Cor_k, which are called the homotopy invariant presheaves with transfers. This theorem states in particular that the Nisnevich sheaf $\mathscr{F}_{\mathrm{nis}}$ associated with a homotopy invariant presheaf $\mathscr{F}$ with transfers is homotopy invariant, and that $\mathscr{F}_{\mathrm{nis}}$ is equipped with transfers in a canonical way. Our aim in the present paper is to define presheaves with

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Witt-transfers and to prove an analog the first part of Voevodsky's theorem mentioned above.

For this, we use the following definition of the category of Witt-correspondences WCor_k. The objects of WCor_k are smooth affine varieties over a perfect field k with $\mathrm{char}(k) \neq 2$. The morphism group $\mathrm{WCor}_k(X, Y)$ for smooth affine varieties X and Y is the Witt group of some category with involution related to X and Y. Often, a morphism from X to Y is determined by a quadratic space (P, q_P), where P is a $k[X \times Y]$-module that is finitely generated and projective over $k[X]$ and $q_P \colon P \to \mathrm{Hom}_{k[X]}(P, k[X])$ is a symmetric $k[X \times Y]$-linear isomorphism. Like Cor_k, the category WCor_k is additive, and it is equipped with a functor $\mathrm{Sm}_k \to \mathrm{WCor}_k$. A presheaf with Witt-transfers is simply an additive functor from WCor_k to the category of Abelian groups. A presheaf with Witt-transfers $\mathscr{F}$ is homotopy invariant if for any smooth affine X we have isomorphism $\mathscr{F}(X) \simeq \mathscr{F}(X \times \mathbb{A}^1)$. The main result of the paper is the following theorem (published without proof in [9]).

Theorem (main theorem)**.** *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers, then the Nisnevich sheafification $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

The proof is based on the following properties of presheaves with Witt-transfers proved in this paper: the injectivity on local schemes (proved in [12] by Chepurkin), injectivity on affine lines, Zariski excision on affine lines, and *étale* excision in dimension 1. Namely, the following statements hold true.

Theorem (injectivity on local schemes; see Chepurkin [12])**.** *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers, and U is a local scheme that is the localization of a smooth variety over k at some point. Then the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(\eta)$, where $\eta \in U$ is a generic point, is injective.*

Theorem (injectivity on the affine line)**.** *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers, and $U \subset V \subset \mathbb{A}^1{}_K$ is a pair of open subschemes on the affine line, over a filed $K = k(S)$ of fractions of some variety S over k.*

Then the restriction homomorphism $\pi^* \colon \mathscr{F}(V) \to \mathscr{F}(U)$ is injective.

Theorem (excision on the affine line)**.** *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers, and $U \subset V \subset \mathbb{A}^1{}_K$ is a pair of open subschemes on the affine line, over a filed $K = k(S)$ of fractions of some variety over k, and $z \in U$ is a closed point.*

Then the restriction homomorphism

$$\pi^* \colon \frac{\mathscr{F}(V - z)}{\mathscr{F}(V)} \to \frac{\mathscr{F}(U - z)}{\mathscr{F}(U)}$$

is an isomorphism (*the factor groups are well defined due to the preceding theorem*)*.*

Theorem (*étale* excision in dimension 1)**.** *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers, $\pi \colon X' \to X$ is an étale morphism of smooth curves over k, and $z \in X$, $z' \in X'$ are closed points such that π induces isomorphism $z' \simeq \pi(z') \simeq z$. Let $U = \operatorname{Spec} k[X]_z$ and $U' = \operatorname{Spec} k[X']_{z'}$.*

Then the inverse image homomorphism

$$\pi^* \colon \frac{\mathscr{F}(U - z)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(U' - z')}{\mathscr{F}(U')}$$

is an isomorphism (*the factor groups are well defined due to the Chepurkin theorem*)*.*

The last two theorems were published in [8] without proofs.

Proofs of the theorems listed above are based on constructions of some special morphisms in WCor_k, which are inverse to the regular maps described in the theorems.

In more detail, to prove injectivity on the affine line, which is the injectivity of the homomorphism $i^*\colon \mathscr{F}(V) \to \mathscr{F}(U)$ for an open embedding $i\colon U \hookrightarrow V$, it suffices to construct a morphism $\Phi \in \mathrm{WCor}(V, U)$ that is the left inverse to the morphism i up to an $\mathbb{A}^1$-homotopy. To prove the excision isomorphisms it suffices to construct some morphism in the category of pairs that is inverse to the morphism i up to an $\mathbb{A}^1$-homotopy.

Moreover, the injectivity and excision theorems stated above yield the following fact.

Theorem. *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers over $K = k(X)$ for some smooth variety X over k. Then for any open subscheme $U \subset \mathbb{A}^1{}_K$ we have*

$$\begin{cases} \mathscr{F}_{\mathrm{Nis}}(U) = \mathscr{F}_{\mathrm{Zar}}(U) = \mathscr{F}(U), \\ H^1_{\mathrm{Nis}}(U, \mathscr{F}_{\mathrm{Nis}}) = H^1_{\mathrm{Zar}}(U, \mathscr{F}_{\mathrm{Zar}}) = 0. \end{cases}$$

The author thanks I. Panin for formulation of the problem and help with its solution.

§2. The category of Witt-correspondences

Let k be a field, $\operatorname{char} k \neq 2$. Denote by SmAff_k the category of smooth varieties over k.

For any pair of affine varieties X and Y, let $\mathrm{Proj}(X, Y)$ denote the full subcategory in the category of $k[X \times Y]$-modules spanned by all P that are finitely generated and projective over $k[X]$. The functor $P \mapsto D(P) = \mathrm{Hom}_{k[X]}(P, k[X])$ determines a duality on $\mathrm{Proj}(X, Y)$, where the structure of a $k[X \times Y]$-module on $\mathrm{Hom}_{k[X]}(P, k[X])$ is induced by the structure of $k[X \times Y]$-module on P. So, for any pair of smooth affine varieties we get an exact category with duality $(\mathrm{Proj}(X, Y), D)$.

Definition 1 (WCor_k)**.**

- ◊ $\mathrm{Ob}\,\mathrm{WCor}_k = \mathrm{Ob}\,\mathrm{SmAff}_k$;
- ◊ $\mathrm{WCor}_k(X, Y) = W(\mathrm{Proj}(X, Y), D)$ (see [3] for the definition of the Witt group of an exact category with duality).

  A typical example of a morphism from X to Y is determined by a quadratic space $({}_{k[Y]}P_{k[X]}, q_P)$, where ${}_{k[Y]}P_{k[X]}$ is a $k[Y \times X]$-module that is finitely generated and projective as a $k[X]$-module and $q_P\colon P \to \mathrm{Hom}_{k[X]}(P, k[X])$ is a $k[Y \times X]$-linear isomorphism.
- The composition of

$$\Phi \in \mathrm{WCor}_k(X, Y) \quad \text{and} \quad \Psi \in \mathrm{WCor}_k(Y, Z)$$

  is defined in terms of the tensor product of quadratic spaces.
- The identity morphism is determined by the diagonal. Precisely, for a smooth variety X, $\mathrm{id}_X \in \mathrm{WCor}(X, X)$ is defined as the class of the quadratic space $({}_{k[X]}k[X]_{k[X]}, (1))$, where

$$(1)_X = ({}_{k[X]}k[X]_{k[X]}, 1\colon k[X]) \simeq \mathrm{Hom}_{k[X]}(k[X], k[X])$$

  denotes the quadratic form on ${}_{k[X]}k[X]_{k[X]}$ given by the canonical isomorphism

$$k[X] \simeq \mathrm{Hom}_{k[X]}(k[X], k[X]).$$

There is a useful functor

$$\mathrm{SmAff}_k \to \mathrm{WCor}_k,$$

which takes a regular map $f\colon X \to Y$ to the morphism determined by the bi-module ${}_{k[Y]}k[X]_{k[X]}$ and the canonical isomorphism

$${}_{k[Y]}k[X] \simeq \mathrm{Hom}_{k[X]}\big({}_{k[Y]}k[X]\,, (k[X]\big).$$

This functor gives the restriction of a presheaf defined on the category WCor_k to the category SmAff_k, and, in this paper, talking about such a restriction we always mean this functor.

Example 1. In this example we describe a certain class of morphisms on the category WCor_k. Most of morphisms constructed and used in this paper belong to this class.

(1) Let X, S be k-smooth affine schemes, let $X \leftarrow Y : \pi$ be a finite flat morphism of affine k-schemes, and let $l\colon k[Y] \to k[X]$ be a $k[X]$-linear homomorphism such that the homomorphism

$$q_l\colon k[Y] \to \mathrm{Hom}_{k[X]}(k[Y], k[X])$$

defined by the rule $b \mapsto q_b$, where $q_b(b') = l(bb')$, is an isomorphism. The homomorphism q is $k[Y]$-linear, because for any $\psi \in \mathrm{Hom}_{k[X]}(k[Y], k[X])$ by definition we have $(b' \cdot \psi)(b) = \psi(b'b)$. Moreover, q determines a symmetric quadratic form due to the commutativity of $k[Y]$. Let $f\colon Y \to S$ be a morphism of k-schemes. Consider the homomorphism of k-algebras

$$(\mathrm{id}_X \times f)^*\colon k[X \times S] \to k[X \times Y].$$

We view $k[Y]$ and $\mathrm{Hom}_{k[X]}(k[Y], k[X])$ as $k[X \times S]$-modules via the homomorphism $(\mathrm{id}_X \times f)^*$. Then the morphism q_l is a $k[X \times S]$-linear symmetric isomorphism. Thus, we have the class $[k[Y], q_l]$ of the quadratic space $(k[Y], q_l)$ in the Witt group $\mathrm{WCor}(X, S)$. In other words, we get a morphism $[k[Y], q_l]$ in $\mathrm{WCor}(X, S)$. We denote this morphism by $(\pi, l, f)\colon X \to S$.

(2) If $g\colon S \to S'$ is a morphism of k-smooth affine schemes, then $g \circ (\pi, l, f) = (\pi, l, g \circ f)$ in $\mathrm{WCor}(X, S')$. Suppose $j\colon X' \to X$ is a morphism of k-smooth affine schemes; then $(\pi, l, f) \circ j = (\pi', l', f \circ j')$ in $\mathrm{WCor}(X', S)$, where $\pi'\colon Y' \to X'$ is the base change of a morphism π along j, and $j'\colon Y' \to Y$ is the base change of j along π and $l' = k[X'] \otimes_{k[X]} l$.

(3) Let $i\colon S' \hookrightarrow S$ be a smooth embedding of k-smooth affine schemes, and let $j\colon X' \to X$ be a morphism of k-smooth affine schemes. As above,

$$(\pi, l, f) \circ j = (\pi', l', f \circ j') \in \mathrm{WCor}(X', S).$$

Suppose that $(f \circ j')(Y')$ is contained in S', and let $f'\colon Y' \to S'$ be a unique morphism of schemes such that $i \circ f' = f \circ j'\colon Y' \to S$. Then a morphism $(\pi', l', f')\colon X' \to S'$ in $\mathrm{WCor}(X', S')$ arises, and

$$(\pi, l, f) \circ j = i \circ (\pi', l', f') \in \mathrm{WCor}(X', S).$$

Definition 2 (Presheaves and sheaves with Witt-transfers)**.** A presheaf with Witt-transfers is a presheaf $F\colon\ \mathrm{WCor}_k \to \mathrm{Ab}$ such that

$$\mathscr{F}\bigl(X_1 \coprod X_2\bigr) = \mathscr{F}(X_1) \oplus \mathscr{F}(X_2)$$

for any X_1 and X_2. A sheaf with Witt-transfers is a presheaf with Witt-transfers that is a sheaf as a functor on SmAff_k. A homotopy invariant presheaf with Witt-transfers is a presheaf with Witt-transfers that is homotopy invariant as a presheaf on SmAff_k.

Now we define a certain subcategory in the category of arrows. It will be used in the proofs for excision isomorphisms.

Definition 3 (The category $\mathrm{WCor}^{\cdot\hookrightarrow\cdot}$)**.** The objects of the category $\mathrm{WCor}_k^{\cdot\hookrightarrow\cdot}$ are pairs (X_1, X_2), where X_1 is a smooth variety and X_2 is an open subscheme. A morphism $\Phi \in \mathrm{WCor}_k^{\cdot\hookrightarrow\cdot}((X_1, X_2), (Y_1, Y_2))$ is a pair of morphisms $\Phi_i \in \mathrm{WCor}(X_i, Y_i)$, $i = 1, 2$, such that $\Phi_1 \circ i_X = i_Y \circ \Phi_2$, where $i_X\colon X_2 \hookrightarrow X_1$, $i_Y\colon Y_2 \hookrightarrow Y_1$ are inclusions.

Example 2. Let $(\pi, l, f)\colon X \to S$ be the morphism in $\mathrm{WCor}(X, S)$ occurring in item (1) of Example 1. Let $i\colon S' \hookrightarrow S$ be a morphism of k-smooth affine schemes, and let $j\colon X' \to X$ be a morphism of k-smooth affine schemes as in item (3) of Example 1. Let $(\pi', l', f')\colon X' \to S'$ be the morphism in $\mathrm{WCor}(X', S')$ occurring in item (3) of Example 1. Suppose j is an open embedding. Then the pair of morphisms

$$\Phi = (\pi, l, f)\colon X \to S \quad \text{and} \quad \Phi' = (\pi', l', f')\colon X' \to S'$$

is a morphism

$$(\Phi, \Phi')\colon (X, X') \to (S, S')$$

in the category $\mathrm{WCor}^{\cdot\hookrightarrow\cdot}$.

Definition 4 (The category of pairs $\mathrm{WCor}^{\mathrm{pair}}$)**.** The additive category $\mathrm{WCor}^{\mathrm{pair}}_k$ is the factorcategory of the additive category $\mathrm{WCor}^{\cdot\hookrightarrow\cdot}_k$ relative to the ideal generated by the identity morphism of objects (X, X) for all varieties X.

Remark 1. More explicitly, the Hom-groups in $\mathrm{WCor}^{\mathrm{pair}}_k$ are defined as

$$\begin{aligned}&\mathrm{WCor}^{\mathrm{pair}}_k((X_1, X_2), (Y_1, Y_2)) \\ &\quad \stackrel{\mathrm{def}}{=} H(\mathrm{WCor}_k(X_1, Y_2) \xrightarrow{i_Y\circ -, -\circ i_X} \mathrm{WCor}_k(X_1, Y_1) \oplus \mathrm{WCor}_k(X_2, Y_2) \\ &\qquad\qquad \xrightarrow{-\circ i_X, i_Y\circ -} \mathrm{WCor}_k(X_2, Y_1)),\end{aligned}$$

where H denotes the homology group in the middle term of the complex of length 3.

Thus, any morphism $\Phi\colon (X_1, X_2) \to (Y_1, Y_2)$ in the category of pairs is determined by a pair $\Phi_i \in \mathrm{WCor}(X_i, Y_i)$, $i = 1, 2$, such that the left diagram below is commutative.

A pair (Φ_1, Φ_2) gives rise to the zero morphism whenever there is a morphism $\Omega \in \mathrm{WCor}_k(X_1, Y_2)$ such that the right diagram is commutative.

$$\begin{array}{ccc} X_2 & \xhookrightarrow{i_X} & X_1 \\ \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_1}, \\ Y_2 & \xhookrightarrow{i_Y} & Y_1 \end{array} \qquad \begin{array}{ccc} X_2 & \xhookrightarrow{i_X} & X_1 \\ {\scriptstyle \Phi_2}\downarrow & \swarrow{\scriptstyle \Omega} & \downarrow{\scriptstyle \Phi_2}. \\ Y_2 & \xhookrightarrow{i_Y} & Y_1 \end{array}$$

Remark 2. To define a morphism

$$\Phi\colon (X_1, X_2) \to (Y_1, Y_2) \in \mathrm{WCor}^{\mathrm{pair}}_k((X_1, X_2), (Y_1, Y_2))$$

it suffices to construct a quadratic space

$$(P_1, q_{P_1}),\ q_{P_1}\colon P_1 \simeq \mathrm{Hom}_{k[X_1]}(P_1, k[X_1]),$$

such that the homomorphism

$$\alpha\colon P_1 \otimes_{k[X_1]} k[X_2] \to k[Y_2] \otimes_{k[Y_1]} P_1 \otimes_{k[X_1]} k[X_2] =: P_2$$

of the form $p \otimes f \mapsto 1 \otimes p \otimes f$ is an isomorphism.

The module P_2 has a canonical structure of a module over $k[Y_2] \otimes k[X_2]$, which is finitely generated and projective as a $k[X_2]$-module. Let $\beta = \alpha^{-1}$. The symmetric $k[Y_1 \times X_2]$-linear isomorphism

$$q_{P_1} \otimes_{k[X_1]} k[X_2]\colon P_1 \otimes_{k[X_1]} k[X_2] \to \mathrm{Hom}_{k[X_2]}(P_1 \otimes_{k[X_1]} k[X_2], k[X_2])$$

determines a symmetric $k[Y_2 \times X_2]$-linear isomorphism

$$q_{P_2} = \beta^\vee \circ (q_{P_1} \otimes_{k[X_1]} k[X_2]) \circ \beta\colon P_2 \to \mathrm{Hom}_{k[X_2]}(P_2, k[X_2]).$$

Then $\Phi_1 \circ i_X = i_Y \circ \Phi_2$, where

$$\Phi_1 = (P_1, q_{P_1})\colon X_1 \to Y_1, \quad \Phi_2 = (P_2, q_{P_2})\colon X_2 \to Y_2.$$

Therefore, the pair (Φ_1, Φ_2) is a morphism of pairs $(X_1, X_2) \to (Y_1, Y_2)$. We define

$$\Phi_{(P_1, q_{P_1})} = (\Phi_1, \Phi_2)\colon (X_1, X_2) \to (Y_1, Y_2) \in \mathrm{WCor}^{\mathrm{pair}}((X_1, X_2), (Y_1, Y_2)).$$

Observe that not every morphism of the category of pairs $\mathrm{WCor}^{\mathrm{pair}}$ can be defined in this way. *However, we shall consider and employ only such morphisms.*

Remark 3. For a presheaf with Witt-transfers $\mathscr{F}$, we introduce a presheaf $\mathscr{F}^{\mathrm{pair}}$ on the category $\mathrm{WCor}^{\mathrm{pair}}$ such that

$$\mathscr{F}^{\mathrm{pair}}(X_1, X_2) = \frac{\mathscr{F}(X_2)}{i_X^*(\mathscr{F}(X_1))}.$$

If

$$(\Phi_1, \Phi_2)\colon (X_1, X_2) \to (Y_1, Y_2),$$

then

$$(\Phi_1, \Phi_2)^*\colon \mathscr{F}^{\mathrm{pair}}(Y_1, Y_2) \to \mathscr{F}^{\mathrm{pair}}(X_1, X_2)$$

is defined as a unique homomorphism induced by the homomorphism

$$\Phi_1^*\colon \mathscr{F}(Y_1) \to \mathscr{F}(X_1).$$

Now we are going to discuss homotopy invariant presheaves and define the category $\overline{\mathrm{WCor}}_k$.

Definition 5 (The category $\overline{\mathrm{WCor}_k}$)**.** The objects of the category $\overline{\mathrm{WCor}_k}$ are the same as in WCor_k, and the morphisms are defined by the rule

$$\overline{\mathrm{WCor}_k}(X, Y) = \operatorname{coker}(\mathrm{WCor}_k(\mathbb{A}^1 \times X, Y) \xrightarrow{(-\circ i_0) - (-\circ i_1)} \mathrm{WCor}_k(X, Y)),$$

where $i_0, i_1\colon X \hookrightarrow \mathbb{A}^1 \times X$ denote the zero and unit section of the projection $\mathbb{A}^1 \times X \to X$.

Define the category $\overline{\mathrm{WCor}_k^{\mathrm{pair}}}$. Its objects are the same as the objects of $\mathrm{WCor}_k^{\mathrm{pair}}$, and the morphisms are defined by the rule

$$\begin{aligned}&\overline{\mathrm{WCor}_k^{\mathrm{pair}}}((X_1, X_2), (Y_1, Y_2)) \\ &\quad = \operatorname{coker}[\mathrm{WCor}_k^{\mathrm{pair}}((\mathbb{A}^1 \times X_1, \mathbb{A}^1 \times X_2), (Y_1, Y_2))) \\ &\qquad \xrightarrow{(-\circ i_0) - (-\circ i_1)} \mathrm{WCor}_k^{\mathrm{pair}}((X_1, X_2), (Y_1, Y_2)]).\end{aligned}$$

Remark 4 (About homotopy invariance). The homotopy invariant presheaves with Witt-transfers are precisely those on the category $\overline{\mathrm{WCor}_k}$ (i.e., the presheaves on WCor_k that can be passed through the functor $\mathrm{WCor}_k \to \overline{\mathrm{WCor}_k}$). If a presheaf $\mathscr{F}$ with Witt-transfers is homotopy invariant, then the presheaf $\mathscr{F}^{\mathrm{pair}}$ on the category $\mathrm{WCor}^{\mathrm{pair}}$ can be passed through the category $\overline{\mathrm{WCor}^{\mathrm{pair}}}$.

§3. Injectivity on the affine line

Theorem 1. *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers and $U \subset V \subset \mathbb{A}^1{}_K$ is a pair of Zariski open subschemes of the affine line over a filed K that is the field of rational functions of a smooth variety S. Let $i\colon U \to V$ denote the injection. Then the homomorphism*

$$i^*\colon \mathscr{F}(V) \to \mathscr{F}(U)$$

is an injection.

Lemma 1. *Suppose $U \subset V \subset \mathbb{A}^1{}_K$ is an injection in a subscheme $\mathbb{A}^1{}_K$, and i denotes the injection of U into V; then there is a morphism $\Phi \in \mathrm{WCor}(V, U)$ such that*

$$[i \circ \Phi] = [\mathrm{id}_V]$$

in $\overline{\mathrm{WCor}}(V, V)$.

Proof of the theorem. Let $a \in \mathscr{F}(V)$ be a section such that $i^*(a) = 0$. By item 2 of Remark 4, the presheaf $\mathscr{F}$ viewed as a functor form WCor to Ab can be passed through $\overline{\mathrm{WCor}}$. Since $[i \circ \Phi] = [\mathrm{id}_V]$, we have $a = \Phi^*(i^*(a) = 0$. Thus, the injectivity of i^* will be proved if we prove Lemma 1.

Proof of Lemma 1. To simplify the notation, we assume that $K = k$. (Actually, we can use base change along the extension of the base filed K/k and the fact that a presheaf defined on smooth schemes over k can be defined naturally on schemes over K, because it is the residue filed of a generic point of a smooth scheme.) If suffices to construct Witt-correspondences

$$\Phi \in \mathrm{WCor}(V, U) \quad \text{and} \quad H \in \mathrm{WCor}(V \times \mathbb{A}^1{}_k, V)$$

such that $H_0 = i \circ \Phi$ and $H_1 = \mathrm{id}_V$.

Let $T = \mathbb{A}^1 \setminus V$, $D = V \setminus U$. We view $V \times U$ as a subset in $V \times \mathbb{A}^1$. Let X be a coordinate on $\mathbb{A}^1$ and Y a coordinate on V.

For some sufficiently large *odd* integer n there is a polynomial of degree n whose leading coefficient in X is 1 and such that

$$f \in k[V \times \mathbb{A}^1] = k[V][X] \,:\, \deg_X(f) = n, \quad f\big|_{V\times T} = (X - Y)^n|_{V\times T}, \quad f|_{V\times D} = 1.$$

Let t be a coordinate on the left factor $\mathbb{A}^1$ in the product $\mathbb{A}^1 \times V \times \mathbb{A}^1$. Consider the polynomial

$$h = f \cdot (1 - t) + (X - Y)^n \cdot t \in k[\mathbb{A}^1 \times V \times \mathbb{A}^1].$$

Then $h\big|_{\mathbb{A}^1\times V\times T} = (X - Y)^n$ is invertible. Consider the map

$$\mathbb{A}^1 \times V \times \mathbb{A}^1 \leftarrow \mathbb{A}^1 \times V \times \mathbb{A}^1 \,:\, (\mathrm{pr}_{\mathbb{A}^1\times V}, h) = \Pi.$$

Denote by $(\mathbb{A}^1 \times V \times \mathbb{A}^1)_l$ the left copy of $\mathbb{A}^1 \times V \times \mathbb{A}^1$ and by $(\mathbb{A}^1 \times V \times \mathbb{A}^1)_r$ the right one $\mathbb{A}^1 \times V \times \mathbb{A}^1$. Put $A = k[(\mathbb{A}^1 \times V \times \mathbb{A}^1)_l]$, $B = k[(\mathbb{A}^1 \times V \times \mathbb{A}^1)_r]$. Let $\Pi^* \colon A \to B$ be the homomorphism of k-algebras induced by Π. Since h has the leading coefficient 1 with respect to X, it follows that B is a free A-module of rank n. By Proposition 2.1 in [10], we have isomorphism of B-modules

$$\omega_{B/k} \otimes_A \omega_{A/k}^{-1} \simeq \mathrm{Hom}_A(B, A) = .$$

The B-module $\omega_{B/k}$ and the A-module $\omega_{A/k}$ are free and have rank one. Choosing some trivializations of these modules, we get an isomorphism of B-modules $Q \colon B \to \mathrm{Hom}_A(B, A)$. Consider the morphism of A-modules $L = Q(1) \colon B \to A$. It is easy to check that, in the notation of item (1) of Example 1, we have $Q = q_L$. Now the triple (Π, L, pr_3) is a morphism $(\mathbb{A}^1\times V\times\mathbb{A}^1)_l \to \mathbb{A}^1$ in $\mathrm{WCor}(k)$. Denote by $Y_t \subset (\mathbb{A}^1\times V\times\mathbb{A}^1)_r$ the scheme preimage of the scheme $\mathbb{A}^1 \times V \times 0$ along Π. Since $h\big|_{\mathbb{A}^1\times V\times T} = (X - Y)^n$ is invertible, it follows that $Y_t \subset \mathbb{A}^1 \times V \times V$ and $\mathrm{pr}_3(Y_t) \subset V$. Hence, by items (2) and (3) of Example 1 there is a morphism

$$\widetilde{H}_t = (\pi_t, l_t, g_t) \colon \mathbb{A}^1 \times V = \mathbb{A}^1 \times V \times 0 \to V$$

in $\mathrm{WCor}(k)$, where $\pi_t \colon Y_t \to \mathbb{A}^1\times V\times 0$ is the restriction of Π to Y_t, $l_t = k[\mathbb{A}^1\times V\times 0]\otimes_A L$, $\mathrm{pr}_3^V \colon \mathbb{A}^1 \times V \times V \to V$ is the projection to the first factor, and $g_t = \mathrm{pr}_3^V|_{Y_t} \colon Y_t \to V$.

Consider the morphism $\widetilde{H}_0 = \widetilde{H} \circ j_0 \colon 0 \times V \to V$, where $j_0 \colon 0 \times V \to \mathbb{A}^1 \times V$ is a closed embedding. Since $f|_{V\times D} = 1$, the scheme preimage $Y_0 := \pi^{-1}(0 \times V)$ is contained in $0 \times V \times U$. By item (3) of Example 1, we have $\widetilde{H}_0' = i \circ (\pi_0, l_0, g_0')$, where $g_0' \colon Y_0 \to U$ is a unique morphism of schemes such that $i \circ g_0' = g_0 \colon Y_0 \to V$, and where $\pi_0 \colon Y_0 \to 0 \times V$ is the restriction of π_t to Y_0 and $l_0 = k[0 \times V] \otimes_{k[\mathbb{A}^1\times V]} l_t$.

Consider the morphism $\widetilde{H}_1 = \widetilde{H} \circ j_1 \colon 1 \times V \to V$, where $j_1 \colon 1 \times V \to \mathbb{A}^1 \times V$ is a closed embedding. By item (3) of Example 1, we have $\widetilde{H}_1 = (\pi_1, l_1, g_1)$, where $g_1 =$

$g_t \circ \mathrm{in}_1 \colon Y_1 \to V$, $Y_1 = \pi^{-1}(1 \times V)$, $\mathrm{in}_1 \colon Y_1 \hookrightarrow Y_t$, $\pi_1 \colon Y_1 \to 1 \times V$ is the restriction of π_t to Y_1, and $l_1 = k[1 \times V] \otimes_{k[\mathbb{A}^1 \times V]} l_t$. Observe the isomorphism $k[Y_1] = k[V \times V]/(X-Y)^n$ of $k[V \times V]$-modules.

Sublemma 1. *For some* $\lambda \in K[V]^\times$ *we have*

$$\widetilde{H}_1 = \lambda \cdot \mathrm{id}_V$$

in $\mathrm{WCor}(V, V)$.

Proof. As was mentioned above, the morphism $\widetilde{H}_1$ is the triple (π_1, l_1, g_1), i.e., it is equal to the class of the quadratic space $(k[Y_1], q_{l_1})$ in $\mathrm{WCor}(V, V)$, where

$$q_{1_1} \colon k[Y_1] \to \mathrm{Hom}_{k[V]}(k[Y_1], k[V])$$

is the $k[V \times V]$-linear isomorphism defined by the rule $b \mapsto \psi_b$, where $\psi_b(b') = l_1(bb')$. Let $n = 2m + 1$. Consider the ideal

$$J = ((X - Y)^{m+1}) \subset K[Y_1].$$

It is clear that J is a sublagrangian subspace in $(k[Y_1], q)$. Hence, by Theorem 32 in [3], the class of $(k[Y_1], q)$ in $\mathrm{WCor}(V, V)$ is equal to the class $(J^\perp/J, q_{\mathrm{new}})$ for some $k[V \times V]$-linear isomorphism

$$q_{\mathrm{new}} \colon J^\perp/J \to \mathrm{Hom}_{k[V]}(J^\perp/J, k[V]).$$

Since $J^\perp$ coincides with the ideal

$$I = ((X - Y)^m) \subset K[Y_1],$$

it follows that $J^\perp/J$ is a free module of rank 1 over $K[V]$. Since q_{new} is a $k[V \times V]$-linear isomorphism, q_{new} is simply the homomorphism of multiplication by an element $\lambda \in K[V]^\times$. □

Now we put

$$H_t = \lambda^{-1} \cdot \widetilde{H}_t \colon \mathbb{A}^1 \times V \to V \quad \text{and} \quad \Phi = (\pi_0, \lambda^{-1} \cdot l_0, g_0') \colon V \to U.$$

Then in $\mathrm{WCor}(V, V)$ we have $H_0 = H_t \circ \langle j_0 \rangle = i \circ \Phi$, and $H_1 = H_t \circ \langle j_1 \rangle = \mathrm{id}_V$, as required. □

□

§4. Excision on the affine line

In this section we prove the Zariski excision of $\mathbb{A}^1{}_K$ over the filed of functions $K = k(X)$ of a smooth variety X.

Theorem 2. *Suppose* $\mathscr{F}$ *is a homotopy invariant presheaf with Witt-transfers, and let* $z \in U \subset V \subset \mathbb{A}^1{}_K$ *be a closed point and a pair of Zariski open subschemes of the affine line over the filed* K *that is the filed of rational functions of a smooth variety* S. *Then the restriction homomorphism*

$$i^* \colon \frac{\mathscr{F}(V - z)}{\mathscr{F}(V)} \to \frac{\mathscr{F}(U - z)}{\mathscr{F}(U)},$$

where $i \colon U \to V$ *denotes injection, is an isomorphism.*

Remark 5. The notation of the factor groups in the theorem is consistent due to injectivity on local schemes.

Proof. Let $i\colon U \to V$, $i'\colon U - Z \to V - Z$ be embeddings as in Theorem 2. In terms of the presheaf $\mathscr{F}^{\mathrm{pair}}$ on $\mathrm{WCor}_k^{\mathrm{pair}}$, the claim of the theorem means that

$$(i,i')^*\colon \mathscr{F}^{\mathrm{pair}}(V-z,V) \to \mathscr{F}^{\mathrm{pair}}(U-z,U)$$

is an isomorphism. Hence, it suffices to prove the following lemma.

Lemma 2. *Let $\langle i\rangle$ be the class of a pair (i,i') in $\mathrm{WCor}_k^{\mathrm{pair}}((U,U-z),\,(V,V-z))$. Let $[i]$ be the class of $\langle i\rangle$ in $\overline{\mathrm{WCor}_k^{\mathrm{pair}}}((U,U-z),\,(V,V-z))$. Then $[i]$ is an isomorphism in $\overline{WCor_k}$.*

Proof of Lemma 2. To simplify the notation, we assume that $K=k$. First, we construct morphisms

$$\Phi \in \mathrm{WCor}_k^{\mathrm{pair}}((V,V-z),(U,U-z)),$$
$$\Theta \in \mathrm{WCor}_k^{\mathrm{pair}}((V\times\mathbb{A}^1,(V-z)\times\mathbb{A}^1),(V,V-z))$$

such that

$$\Theta\circ\langle j_0\rangle = \langle i\rangle\circ\Phi,\ \ \Theta\circ\langle j_1\rangle = \langle \mathrm{id}_{(V,V-z)}\rangle,$$

where the $j_s\colon (V,V-z)\hookrightarrow (V,V-z)\times\mathbb{A}^1$, $s=0,1$, are the embeddings determined by the points 0 and 1 on $\mathbb{A}^1$. This will imply that

$$[i\circ\Phi] = [\mathrm{id}_{(V,V-z)}] \in \overline{\mathrm{Wor}_k}((V,V-z),(V,V-z)),$$

or, in other words, that $[i]$ is left invertible in $\overline{\mathrm{Wor}_k}$, *proving the first part of the lemma.*

Let $T=\mathbb{A}^1\setminus V$, $D=V\setminus U$. We view $V\times U$ as a subset in $V\times\mathbb{A}^1$. Let X be a coordinate on $\mathbb{A}^1$ and Y a coordinate on V. For some sufficiently large *odd* integer n, there is a polynomial of degree n whose leading coefficient in X is equal to 1 and such that

$$f\in K[V\times\mathbb{A}^1] = K[V][X]\colon$$
$$f\big|_{V\times T} = (X-Y)^n|_{V\times T}, f|_{V\times D}=1, f|_{V\times z} = (X-Y)^n|_{V\times z}.$$

Let t be a coordinate on the left factor $\mathbb{A}^1$ in the product $\mathbb{A}^1\times V\times\mathbb{A}^1$. Consider the polinomial

$$h = f\cdot(1-t) + (X-Y)^n\cdot t \in K[\mathbb{A}^1\times V\times\mathbb{A}^1].$$

Now, let

$$\widetilde{H}_t = (\pi_t,l_t,g_t)\colon \mathbb{A}^1\times V = \mathbb{A}^1\times V\times 0 \to V,$$
$$\widetilde{H}_0 = \widetilde{H}\circ j_0\colon 0\times V\to V$$

be the morphisms in the category WCor_k occurring in the proof of Theorem 1, and let $\widetilde{H}_0 = i\circ(\pi_0,l_0,g_0')$, where $g_0'\colon Y_0\to U$ is a unique morphism of schemes such that $i\circ g_0' = g_0\colon Y_0\to V$, and where $\pi_0\colon Y_0\to 0\times V$ is the restriction of π_t on Y_0, and $l_0 = k[0\times V]\otimes_{k[\mathbb{A}^1\times V]} l_t$.

Let $V' = V-z$. The function

$$h|_{\mathbb{A}^1\times(V-Z)\times Z} = (X-Y)^n|_{\mathbb{A}^1\times V'\times z}$$

is invertible. Therefore, Example 2 shows that the morphism $\widetilde{H}_t = (\pi_t,l_t,g_t)$ gives rise to a morphism

$$\widetilde{H}_t' = (\pi_t',l_t',g_t')\colon \mathbb{A}^1\times V'\to V'$$

such that the pair $(\widetilde{H}_t,\widetilde{H}_t')$ is a morphism in the category

$$\mathrm{WCor}_k^{\mathrm{pair}}(\widetilde{H}_t,\widetilde{H}_t')\colon (\mathbb{A}^1\times V,\mathbb{A}^1\times V')\to(V,V').$$

Put $(\widetilde{H}_0,\widetilde{H}_0') := (\widetilde{H}_t,\widetilde{H}_t')\circ\langle j_0\rangle\colon (0\times V,0\times V')\to(V,V')$.

Let $U' = U - z$ and $h_0 = h|_{t=0}$. Observe that

$$h_0|_{(V-Z)\times(Z\sqcup D)}$$

is invertible. Hence, Example 2 shows that the morphism

$$\widetilde{H}_{0.V,U} := (\pi_0, l_0, g_0')\colon V \to U$$

determines some morphism

$$\widetilde{H}_{0.V',U'} := (\pi_0', l_0', (g_0')')\colon V' \to U'$$

such that the pair $(\widetilde{H}_{0.V,U}, \widetilde{H}_{0.V',U'})$ is a morphism in the category

$$\mathrm{WCor}^{\mathrm{pair}}((0\times V, 0\times V'), (U, U')).$$

In the proof of Theorem 1 it was checked that $i\circ\widetilde{H}_{0.V,U} = \widetilde{H}_0$ in $\mathrm{WCor}\,k(V,V)$. Similarly, $i'\circ\widetilde{H}_{0.V',U'} = \widetilde{H}_0'$ in $\mathrm{WCor}\,k(V',V')$, where $i'\colon U' \hookrightarrow V'$ is the natural embedding. Therefore,

$$\begin{aligned}\langle i\rangle\circ(\widetilde{H}_{0.V,U}, \widetilde{H}_{0.V',U'}) &= (i,i')\circ(\widetilde{H}_{0.V,U}, \widetilde{H}_{0.V',U'})\\ &= (\widetilde{H}_0, \widetilde{H}_0') = (\widetilde{H}_t, \widetilde{H}_t')\circ\langle j_0\rangle \in \mathrm{WCor}_k^{\mathrm{pair}}((V,V'),(V,V')).\end{aligned}$$

Put $\widetilde{\Phi} := (\widetilde{H}_{0.V,U}, \widetilde{H}_{0.V',U'})$ and $\widetilde{\Theta} := (\widetilde{H}_t, \widetilde{H}_t')$. Then

$$\langle i\rangle\circ\widetilde{\Phi} = \widetilde{\Theta}\circ\langle j_0\rangle \in \mathrm{WCor}_k^{\mathrm{pair}}((V,V'),(V,V')).$$

Consider the morphism

$$(\widetilde{H}_1, \widetilde{H}_1') = \widetilde{\Theta}\circ\langle j_1\rangle\colon 1\times(V,V') \to (V,V').$$

In the proof of Theorem 1 it was shown that the morphism $\widetilde{H}_1'$ is equal to the class of the quadratic space $(k[Y_1], q_{l_1})$ in the Witt group, where

$$Y_1 = \operatorname{Spec} k[1\times V\times V]/(X-Y)^n.$$

In the proof of Lemma 1 it was shown that the class of the quadratic space $(k[Y_1], q_{l_1})$ is equal to the class of the space $(k[\Delta], \lambda)$, where Δ is the diagonal in $1\times V\times V$ and λ is an invertible function on V. Thus, $\widetilde{H}_1 = \lambda\cdot\langle \mathrm{id}_V\rangle$. Also, this implies that the morphism $\widetilde{H}_1'$ determined by the space $(k[Y_1', q_{l_1}')$ is equal to $\lambda\cdot\langle\mathrm{id}_{V'}\rangle$. Hence, $\widetilde{\Theta}\circ\langle j_1\rangle = \lambda\cdot\langle\mathrm{id}_{(V,V')}\rangle$. Consequently, putting

$$\Theta = \lambda^{-1}\cdot\widetilde{\Theta}, \Phi = \lambda^{-1}\cdot\widetilde{\Phi},$$

we get $\Theta\circ\langle j_0\rangle = \Phi$, $\Theta\circ\langle j_1\rangle = \mathrm{id}_{(V,V')}$. This completes the proof of the first part of the lemma.

Now we construct *the right inverse* to the morphism $[i]$ in $\overline{\mathrm{WCor}_k^{\mathrm{pair}}}$. For this, we construct morphisms

$$\Psi \in \mathrm{WCor}^{\mathrm{pair}}((V,V'),(U,U')), \quad \Xi \in \mathrm{WCor}^{\mathrm{pair}}(\mathbb{A}^1\times(U,U'),(U,U'))$$

such that

$$\Xi\circ\langle j_0\rangle = \Psi\circ\langle i\rangle, \quad \Xi\circ\langle j_1\rangle = \langle\mathrm{id}_(V, V-z)\rangle.$$

This will imply that $[\Psi\circ i] = [\mathrm{id}_V] \in \overline{\mathrm{Wor}_k}((V,V'),(V,V'))$. *This will be the second part of the proof of the lemma.*

We view $V\times U$ and $\mathbb{A}^1\times U\times U$ as subsets in $V\times\mathbb{A}^1$ and $\mathbb{A}^1\times U\times\mathbb{A}^1$. Let X be a coordinate on the first factor $\mathbb{A}^1$, Y a coordinate on the factors V and U, and t a coordinate on the left factor $\mathbb{A}^1$. Let Δ denote the graph of the embedding $U\hookrightarrow\mathbb{A}^1$, i.e., $\Delta = \operatorname{Spec}\ K[\mathbb{A}^1\times U]/(X-Y)$. For some sufficiently large n, using interpolation theorem, we can find $f \in K[V \times \mathbb{A}^1]$ and $g \in K[U \times \mathbb{A}^1]$ with degrees n and $n-1$ in X (respectively) and with leading term 1 such that

$$f\big|_{V\times(D\amalg T)} = 1, \qquad f\big|_{V\times z} = X - Y,$$
$$g\big|_{U\times(D\amalg T)} = \big((X-Y)\big|_{(D\amalg T)\times U}\big)^{-1}, \quad g\big|_{U\times z} = 1, \qquad g\big|_{\Delta} = 1.$$

These conditions can be satisfied because $(D \amalg T) \cap U = \varnothing$ and $X - Y$ is invertible on $(D \amalg T) \times U$.

Consider the polynomial

$$f = f \cdot (1-t) + g \cdot (X-Y) \cdot t \in K[X][U][t] = K[\mathbb{A}^1 \times U \times \mathbb{A}^1].$$

To f and h, we apply simultaneously the construction that was applied to h in the first part of the proof of the lemma. Specifically, we consider regular maps

$$V \times \mathbb{A}^1 \leftarrow V \times \mathbb{A}^1 \,:\, (\mathrm{pr}_V, f) = \Pi^f, \mathbb{A}^1 \times U \times \mathbb{A}^1 \leftarrow \mathbb{A}^1 \times U \times \mathbb{A}^1 \,:\, (\mathrm{pr}_{\mathbb{A}^1\times U}, h) = \Pi^h,$$

and let ${\Pi_f}^*\colon A_f \to B_f$ and ${\Pi_h}^*\colon A_h \to B_h$ be homomorphisms of k-algebras induced by the morphisms Π_f and Π_h. Denote $C_f = k[V]$ and $C_h = k[\mathbb{A}^1 \times U]$. Then $B_f = C_f[X]$, $A_f = C_f[T]$, $B_h = C_h[X]$, $A_h = C_h[T]$, and since Π_f and Π_h are morphisms of the relative affine lines over V and $\mathbb{A}^1 \times U$ (respectively), we see that Π_f^* and Π_h^* are homomorphisms of C_f and C_h-algebras. Since the leading coefficients of f and h in X are 1, it follows that B_f and B_h are free modules of rank n over A_f and A_h, and Proposition 2.1 in [10] shows that there are isomorphisms

$$\widetilde{Q_f}\colon \omega_{B_f/C_f} \otimes_{A_f} \omega^{-1}_{A_f/C_f} \simeq \mathrm{Hom}_{A_f}(B_f, A_f),$$
$$\widetilde{Q_h}\colon \omega_{B_h/C_h} \otimes_{A_h} \omega^{-1}_{A_h/C_h} \simeq \mathrm{Hom}_{A_h}(B_h, A_h)\colon$$
$$k[0 \times U \times \mathbb{A}^1] \otimes_{A_h} \widetilde{Q_h} = k[U] \otimes_{k[V]} \widetilde{Q_f}$$

that agree upon base changes along i and j_0, because

$$h\big|_{0\times U\times\mathbb{A}^1} = f\big|_{U\times\mathbb{A}^1}.$$

Now, consider the affine lines determined by the coordinates

$$\omega_{B_f/C_f} = (dX) \cdot B_f, \quad \omega_{A_f/C_f} = (dT) \cdot A_f,$$
$$\omega_{B_h/C_h} = (dX) \cdot B_h, \quad \omega_{A_h/C_h} = (dT) \cdot A_h;$$

using trivializations of the canonical classes of these lines, we get homomorphisms

$$Q_f\colon B_f \simeq \mathrm{Hom}_{A_f}(B_f, A_f),\ Q_h\colon B_h \simeq \mathrm{Hom}_{A_h}(B_h, A_h)\colon$$
$$k[0 \times U \times \mathbb{A}^1] \otimes_{A_h} Q_h = k[U] \otimes_{k[V]} Q_f,$$

which agree upon base changes. Item (1) of Example 1 yields morphisms

$$(\Pi^f, L_f, \mathrm{pr}_2)\colon V \times \mathbb{A}^1 \to \mathbb{A}^1 (\Pi^h, L_h, \mathrm{pr}_3)\colon \mathbb{A}^1 \times U \times \mathbb{A}^1 \to \mathbb{A}^1\colon$$
$$(\Pi^h, L_h, \mathrm{pr}_3) \circ \langle j_0 \times \mathbb{A}^1 \rangle = (\Pi^f, L_f, \mathrm{pr}_2) \circ \langle i \times \mathbb{A}^1 \rangle,$$

where $\mathrm{pr}_2\colon V \times \mathbb{A}^1 \to \mathbb{A}^1$ and $\mathrm{pr}_3\colon \mathbb{A}^1 \times U \times \mathbb{A}^1 \to \mathbb{A}^1$ are the projections to the first factors $\mathbb{A}^1$, and $L_f = Q_f(1)$, $L_h = Q_h(1)$.

Put $Y = \Pi_f^{-1}(V \times 0)$ and $Y_t = \Pi_h^{-1}(\mathbb{A}^1 \times U \times 0)$ (these are scheme preimages). Then, since $f\big|_{V\times(T\amalg D)} = 1$ and $h\big|_{\mathbb{A}^1\times U\times(T\amalg D)} = 1$, we have

$$Y \subset V \times U, \quad Y_t = \subset \mathbb{A}^1 \times U \times U,$$

and in accordance with items (2) and (3) of Example 1 we get morphisms

$$\widetilde{P} = (\pi, l, g)\colon V \to U,\ \widetilde{H}_t = (\pi_t, l_t, g_t)\colon \mathbb{A}^1 \times U = \mathbb{A}^1 \times U \times 0 \to U\colon \widetilde{H}_t \circ j_0 = \widetilde{P},$$

in WCor_k, where $\pi\colon Y \to V\times 0$ is the restriction of Π to the scheme Y,

$$l_t = k[V\times 0]\otimes_A Q_f(1)\colon k[Y]\to k[V],$$

$\mathrm{pr}_2^U\colon V\times U\to U$ is the first projection, $g=\mathrm{pr}_2^U|_Y\colon Y\to U$, and $\pi_t\colon Y_t\to \mathbb{A}^1\times U\times 0$ is the restriction of Π to the scheme Y_t,

$$l_t = k[\mathbb{A}^1\times U\times 0]\otimes_A Q_h(1)\colon k[Y_t]\to k[\mathbb{A}^1\times U],$$

$\mathrm{pr}_3^U\colon \mathbb{A}^1\times U\times U\to U$ is the projection to the first factor, and $g_t=\mathrm{pr}_3^U|_{Y_t}\colon Y_t\to U$.

Since $f\big|_{V'\times z}$ and $h\big|_{\mathbb{A}^1\times U'\times z}$ are invertible, Example 2 shows that the polynomials $\widetilde{P}$ and $\widetilde{H}$ can be completed to the pairs

$$(1)\quad \begin{aligned}&\widetilde{\Psi}=(\widetilde{P},\widetilde{P}')\in \mathrm{WCor}^{\mathrm{pair}}((V,V'),(U,U')),\\ &\widetilde{\Xi}=(\widetilde{H}_t,\widetilde{H}_t')=(\pi_t,l_t,g_t)\in \mathrm{WCor}^{\mathrm{pair}}(\mathbb{A}^1\times(U,U'),(U,U'))\colon \widetilde{\Xi}\circ\langle j_0\rangle=\widetilde{\Psi}\circ\langle i\rangle.\end{aligned}$$

Consider the morphism $\Xi\circ j_1 = (\widetilde{H}_1,\widetilde{H}_1')$. By the definition of $\widetilde{H}_t$, we see that $\widetilde{H}_1=(\pi_1,l_1,g_1)$, where

$$\begin{aligned}&\pi_1\colon Y_1\to U,\\ &Y_1=\Pi_t^{-1}(1\times U\times 0)=\mathrm{Spec}\,k[U\times\mathbb{A}^1]/(g\cdot(X-Y)),\\ &l_1=k[1\times U]\otimes_{k[\mathbb{A}^1\times U]},\\ &g_1\colon Y_1\to U.\end{aligned}$$

Since $g\big|_\Delta=1$, we have $Y_1=\Delta\amalg R$, where $R=\mathrm{Spec}\,k[U\times\mathbb{A}^1]/(g)$. Hence, $\widetilde{H}_1$ splits in the sum of morphisms determined by the restrictions of the morphisms π_1, l_1, and g_1 to the components Δ and R. Since $g\big|_{U\times z}=1$ and $R\subset U\times U'$, it follows that $\widetilde{H}_1=\lambda\cdot\mathrm{id}_U+\widetilde{G}$, where $\widetilde{G}\in\mathrm{WCor}(U,U')$. Therefore, $\widetilde{H}_1'=\lambda\cdot\mathrm{id}_{U'}+\widetilde{G}'$, where $\widetilde{G}'\in\mathrm{WCor}(U',U')$. Thus,

$$(2)\qquad (\widetilde{H}_1,\widetilde{H}_1')=\lambda\cdot\mathrm{id}_{(U,U')}\,.$$

To finish the proof, we put

$$\Psi=\langle\lambda^{-1}\rangle\circ\widetilde{\Psi},\ \Xi=\langle\lambda^{-1}\rangle\circ\widetilde{\Xi},$$

where $\langle\lambda^{-1}\rangle=\langle(k[\Delta],\lambda^{-1})\rangle\in\mathrm{WCor}(U,U)$. (Note that the above compositions with the morphism $\langle\lambda^{-1}\rangle\in\mathrm{WCor}((U,U-z),(U,U-z))$ lead to multiplication of the quadratic spaces by λ viewed as a function on the second factors in $\mathbb{A}^1\times V\times U$ and $U\times U$.) Then (1) and (2) show that $\Psi\circ\langle i\rangle=\Xi\circ\langle j_0\rangle$, $\Xi\circ\langle j_1\rangle=\mathrm{id}_{(U,U')}$. □

□

§5. Étale excision in dimension 1

In this section we prove *étale* excision for smooth curves over the filed of functions of some smooth variety $K=k(X)$.

Theorem 3. *Suppose $\mathscr{F}$ is a homotopy invariant presheaf with Witt-transfers and $\pi\colon X'\to X$ is an étale morphism of smooth varieties of dimension one over a field K that is the field of rational functions of a smooth variety over k. Suppose $z\in X$ and $z'\in X'$ are closed points such that π induces an isomorphism $\pi\colon z'\simeq z$, and $U=\mathrm{Spec}(\mathcal{O}_{X,z})$, $U'=\mathrm{Spec}(\mathcal{O}_{X',z'})$ are local schemes at the points z and z'. Then the homomorphism of inverse image along π induces an isomorphism*

$$\pi^*\colon \frac{\mathscr{F}(U-z)}{\mathscr{F}(U)}\xrightarrow{\sim}\frac{\mathscr{F}(U'-z')}{\mathscr{F}(U')}.$$

Remark 6. The notation of the factor groups in the theorem is consistent due to the injectivity on the local schemes.

Proof. Like in the preceding section, it suffices to prove the corresponding property in the category $\mathrm{WCor}^{\mathrm{pair}}$. In this case we formulate this property as the existence of certain morphisms that are week versions of the left and right morphisms to the morphism $\pi\colon \mathrm{WCor}^{\mathrm{pair}}((U', U'-z'), (U, U-z))$.

Lemma 3. *Let*

$$\langle i\rangle \in \mathrm{WCor}^{\mathrm{pair}}_K((U, U-z), (X, X-z)),$$
$$\langle i'\rangle \in \mathrm{WCor}^{\mathrm{pair}}_K((U', U'-z'), (X', X'-z')),$$

and

$$\langle \pi\rangle \in \mathrm{WCor}^{\mathrm{pair}}_K((X', X'-z'), (X, X-z))$$

be the classes determined by the morphisms $i\colon U \hookrightarrow X$, $i'\colon U' \hookrightarrow X'$, and π. Suppose that $[i]\rangle \in \overline{\mathrm{WCor}}^{\mathrm{pair}}_K((U, U-z), (X, X-z))$, $[i']\rangle \in \overline{\mathrm{WCor}}^{\mathrm{pair}}_K((U', U'-z'), (X', X'-z'))$, and $[\pi] \in \overline{\mathrm{WCor}}^{\mathrm{pair}}_K((X', X'-z'), (X, X-z))$. Then

a) *there exists $\Phi \in \mathrm{WCor}^{\mathrm{pair}}_K((U, U-z), (X', X'-z'))$ such that $[\pi\circ\Phi] = [i]$ in the category $\overline{\mathrm{WCor}}^{\mathrm{pair}}_K((U, U-z), (X, X-z))$;*

b) *there exists $\Psi \in \mathrm{WCor}^{\mathrm{pair}}_K((U, U-z), (X', X'-z'))$ such that $[\Psi\circ\pi] = [i']$ in the category $\overline{\mathrm{WCor}}^{\mathrm{pair}}_K((U', U'-z'), (X', X'-z'))$.*

Proof of the theorem. We show that item a) in Lemma 3 implies the injectivity of π^*. Let $a \in \mathscr{F}'(U-z, U)$, and let $\pi^*(a) = 0$. Since $\mathscr{F}'(U-z, U) = \varinjlim_{z'\in V'\subset X'} \mathscr{F}'(V-z, V)$, shrinking X and X' shows that $a = j^*(a_X)$, $a_X \in \mathscr{F}'(X-z, X)$, where $j\colon U\to X$, and moreover, the canonical classes of X and X' are trivial. Then by Lemma 3a) applied to the new X and X', we have $j^*(a_X) = \Phi^*(\pi^*(a_X)) = 0$. Consequently, the kernel of π^* is equal to 0.

Now we show that item b) in Lemma 3 implies the surjectivity of π^*. Suppose that $a\in\mathscr{F}'(U'-z, U')$. Shrinking X and X', we see that $a = i'^*(a'_X)$ with $a'_X \in \mathscr{F}'(X'-z, X')$. Then, by Lemma 3b) applied to X and X', we have $i'^*(a'_X) = \pi^*(\Phi^*(a'_X))$. Thus, π^* is surjective.

Proof of Lemma 3a). To simplify the notation, we assume that $K = k$. By the definition of the category $\overline{\mathrm{WCor}}^{\mathrm{pair}}$, to prove item a) it suffices to construct morphisms

$$\Phi \in \mathrm{WCor}^{\mathrm{pair}}_k((U, U-z), (X', X'-z')),$$
$$\Theta \in \mathrm{WCor}^{\mathrm{pair}}_k((U\times\mathbb{A}^1, (U-z)\times\mathbb{A}^1), (X, X-z))$$

such that

$$\Theta\circ\langle j_0\rangle = \langle\pi\rangle\circ\Phi,\ \Theta\circ\langle j_1\rangle = \langle i\rangle,$$

where $j_0, j_1 \in \mathrm{WCor}^{\mathrm{pair}}((U, U-z), (U\times\mathbb{A}^1, (U-z)\times\mathbb{A}^1))$ are the unit and zero sections, and π and i are viewed as morphisms in $\mathrm{WCor}^{\mathrm{pair}}_k$, actually,

$$\pi \in \mathrm{WCor}^{\mathrm{pair}}((X', X'-z'), (X, X-z)) \ \text{ and } i \in \mathrm{WCor}^{\mathrm{pair}}((U, U-z), (X, X-z)).$$

In the proof of the injectivity of the excision homomorphism on the affine line, we constructed Witt-correspondences by using regular functions on $\mathbb{A}^1{}_k$; now we construct the required Witt-correspondences by using sections of linear bundles on relative curves $\bar{X}_U = \bar{X}\times U$ and $\overline{X'}_U = \overline{X'}\times U$, where $\bar{X}$ and $\overline{X'}$ are smooth projective curves with open dense immersions $j\colon X\hookrightarrow \bar{X}$, $j'\colon X'\hookrightarrow \overline{X'}$.

We introduce the following notations-definitions.

Definition 6 (The symbols $\mathcal{I}(s)$, $Z(s)$, and $S(D)$)**.** Let $s \in \Gamma(X, \mathscr{L})$ be a regular section of the invertible sheaf on X. We denote by $\mathcal{I} \subset \mathscr{O}(X)$ the ideal determined by s ($\mathcal{I}$ is equal to the image of the homomorphism $\mathscr{L}^{-1} \stackrel{s}{\hookrightarrow} \mathscr{O}(X) = (\mathscr{O}(X) \stackrel{s}{\hookrightarrow} \mathscr{L}) \otimes \mathscr{L}^{-1}$). Let $Z(s) = Z(\mathcal{I}) \subset X$ be the closed subscheme determined by the ideal $\mathcal{I}$.

Finally, for any divisor D in X, we denote by $S(D)$ the closed subscheme in X determined by the sheaf of ideals in $\mathcal{I} \subset \mathscr{O}(X)$, $\mathcal{I}(U) = \{f \in bO(U) : \operatorname{div} f \geq D\}$.

Let $\bar{\pi}\colon \overline{X'} \to \bar{X}$ be a morphism of smooth projective curves such that $\bar{\pi} \circ j' = j \circ \pi$, where $j\colon X \hookrightarrow \bar{X}$, $j'\colon X' \hookrightarrow \overline{X'}$ are the open immersions defined above. Since j' is dense, $\bar{\pi}$ is quasifinite. Then π is finite, because it is projective, and it is quasi-finite because j' is dense. Let r be a rational function on $\bar{X}$ such that $r(z) = 0$, $r\big|_{\bar{X} \setminus X} = 1$, and $r\big|_{\bar{X}' \setminus X'} = 1$ (it can be defined as a regular function on an affine neighborhood of $z \cup (\bar{X} \setminus X) \cup \bar{\pi}(\bar{X}' \setminus X'))$. A nonconstant rational function gives rise to a finite morphism to the projective line and an ample invertible sheaf that is the inverse image of $\mathscr{O}(1)$. Denote $D = r^{-1}(1)$, $D' = r \circ \pi$. Then, by definition, $\mathscr{O}(D)$ and $\mathscr{O}(D')$ are ample, and $\bar{X} - D \subset X$, $\bar{X}' - D' \subset X'$. Since the claim of the lemma for the curves $\bar{X} - D$ and $\bar{X}' - D'$ implies the same claim for X and X', and since $\omega(\bar{X} - D)$ and $\omega(\bar{X}' - D')$ are trivial, without loss of generality we may assume that $\bar{X} - D = X$ and $\bar{X}' - D' = X'$. Let $\mu\colon \omega(X) \simeq \mathscr{O}(X)$ denote any trivialization of the canonical class of X, and let $\Delta \subset X \times U$ be the graph of the embedding $i\colon U \hookrightarrow X$.

Put $\deg \bar{\pi} = l$. Let $d \in \Gamma(\bar{X}, \mathscr{L}(lD))$ be such that $\operatorname{div} d = lD$, and denote by the same symbol the inverse images in $\Gamma(\bar{X} \times U \times \mathbb{A}^1, \mathscr{L}(nlD \times U \times \mathbb{A}^1)$ that trivializes the linear bundle on $X \times U \times \mathbb{A}^1$.

Sublemma 2. *For some sufficiently large n, there are sections*

$$s' \in \Gamma(\overline{X'} \times U), \mathscr{L}(nD' \times U)), \quad s_0 \in \Gamma(\bar{X} \times U, \mathscr{L}(nlD \times U)),$$
$$s_1 \in \Gamma(\bar{X} \times U, \mathscr{L}(nlD \times U)), \quad s \in \Gamma(\bar{X} \times U \times \mathbb{A}^1, \mathscr{L}(nlD \times U \times \mathbb{A}^1))\colon$$

$$\begin{aligned} &Z(s') \cap (U \times \bar{\pi}^{-1}(z')) = \mathbf{z}', && Z(s') \cap (D' \times U) = \varnothing, \\ &Z(s_0) \cap (U \times z) = \mathbf{z}\,, && Z(s_0) \cap (D \times U) = \varnothing, \\ &Z(s_1) \cap (U \times z) = \mathbf{z}\,, && Z(s_1) \cap (D \times U) = \varnothing, \\ &Z(s) \cap (\mathbb{A}^1 \times U \times z) = \mathbb{A}^1 \times \mathbf{z}\,, && Z(s) \cap (D \times U \times \mathbb{A}^1) = \varnothing, \end{aligned} \tag{3}$$

$$\begin{gathered} s\big|_{\bar{X} \times U \times 0} = s_0, \quad s\big|_{\bar{X} \times U \times 1} = s_1, \quad s_1\big|_{(z \cup D) \times U} = s_0\big|_{(z \cup D) \times U}, \\ \pi_U\colon Z(s') \simeq Z(s), \quad s_1\big|_{\Delta} = 0, \end{gathered} \tag{4}$$

where $\mathbf{z}$ denotes the diagonal in $z \times z$, $\mathbf{z}'$ is the graph of the map $\pi\colon z' \to z$, which in fact is isomorphic to the diagonal, because π gives an isomorphism of z and z'.

Before we describe the construction of the sections in the sublemma above, we show how to construct the morphisms Φ and Θ using this sections.

We construct the required quadratic spaces with the help of the same construction as in Lemmas 1 and 2 applied now to the regular function $h = \frac{s}{d^n} \in k[X \times U]$.

Consider the regular map

$$\Pi = (\mathrm{pr}_{\mathbb{A}^1 \times U}, h) = (\mathrm{pr}_{\mathbb{A}^1 \times U}, \frac{s}{d^n})\colon \mathbb{A}^1 \times U \times X \to \mathbb{A}^1 \times U \times \mathbb{A}^1$$

and denote by B_A the algebra corresponding to the map Π, so that $B = K[\mathbb{A}^1]$, $A = K[\mathbb{A}^1 \times U \times \mathbb{A}^1]$. We denote by $\mu_{\mathbb{A}^1 \times U}\colon \omega_{\mathbb{A}^1 \times U}(\mathbb{A}^1 \times U \times X) \simeq \mathscr{O}(\mathbb{A}^1 \times U \times X)$ the inverse image of μ along the projection $\mathbb{A}^1 \times U \times X \to X$, and by dT the trivialization of the canonical class $\omega_{\mathbb{A}^1 \times U}(\mathbb{A}^1 \times U \times \mathbb{A}^1)$ determined by the coordinate T on the second factor $\mathbb{A}^1$.

To continue our construction and apply Proposition 2.1 from [10] to the map Π, we need to show that B is finitely generated and projective over A. Note that Π is obtained by the base change from the projective morphism

$$\bar{\Pi} = (\mathrm{id}_{U\times\mathbb{A}^1}, [s : d^l])\colon \mathbb{A}^1 \times U \times \bar{X} \to \mathbb{A}^1 \times U \times \mathbb{P}^1,$$

along the map $\mathbb{A}^1 \times U \times \mathbb{A}^1 \to \mathbb{A}^1 \times U \times \mathbb{P}^1$. The morphism $\bar{\Pi}$ is a morphism from the relative projective curve $\mathbb{A}^1 \times U \times \bar{X}$ to the relative projective line $\mathbb{A}^1 \times U \times \mathbb{P}^1$ determined by a pair of noncollinear sections of a line bundle, because s is invertible on $D_{\mathbb{A}^1\times U}$. Hence, by the following sublemma, the morphism $\bar{\Pi}$ is finite, surjective, and flat. Therefore, Π is also finite, surjective, and flat. Thus, since any finitely generated flat module is projective, B is a projective $K[\mathbb{A}^1 \times U \times \mathbb{A}^1]$-module.

Sublemma 3. *Let a morphism $F\colon X_T \to \mathbb{P}^1_T$ of a relative projective curve X_T to the relative projective line $\mathbb{P}^1_T$ over an essential smooth scheme T be determined by linearly independent sections s, d of some linear bundle X_T. Then the morphism F is finite, surjective, and flat.*

Proof. The preimage $F^{-1}(t) \subset X_T$ of a point $t \in \mathbb{P}^1_T$ is isomorphic to $Z(s\cdot t_1 - d\cdot t_2) \subset \bar{X}_t$, where t_1, t_2 are noncollinear sections of $\mathscr{O}(1)$ on $\mathbb{P}^1_T$. Since s is not collinear to d, it follows that $s \cdot t_1 - d \cdot t_2 \not\equiv 0$, and so $Z(s \cdot t_1 - d \cdot t_2)$ is a nonempty proper closed subset of X_t. Hence, $\dim F^{-1}(t) = 0$ for any point t. Thus, F is surjective and quasifinite.

Now, since a quasifinite projective morphism is finite, we see that F is finite. Now observe that X_T and $\mathbb{P}^1_T$ are essentially smooth and $\dim X_T = \dim \mathbb{P}^1_T$. Hence, F is flat (see [1, Corollary V.3.9. and Theorem II.4.7]). □

Now we apply [10, Proposition 2.1] to the morphism Π, obtaining an isomorphism $q^\omega\colon \omega_\Pi \simeq \mathrm{Hom}_A(B, A)$. Using trivializations of the canonical classes of X and $\mathbb{A}^1$, we define a B-linear isomorphism

$$q_B = (\Pi^*(dT)^{-1} \otimes \mu) \circ q^\omega\colon B \simeq \mathrm{Hom}_A(B, A).$$

By item (1) of Example 1, q_B gives rise to a morphism

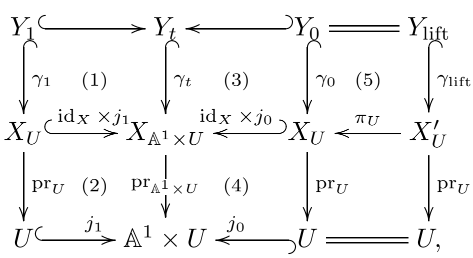
$$\Upsilon = (\Pi, l_B, \mathrm{pr}_X) \in \mathrm{WCor}(\mathbb{A}^1 \times U \times \mathbb{A}^1, X),$$

where $l_B = q_B(1)\colon \mathrm{Hom}_A(B, A)$.

Denote

$$Y_t = Z(s) \subset \mathbb{A}^1 \times U \times \bar{X}, \quad Y_1 = Z(s_1) \subset U \times \bar{X},$$
$$Y_0 = Z(s_0 \subset U \times \bar{X}, \quad Y_{\mathrm{lift}} = Z(s') \subset U \times \overline{X'}.$$

Now (3) implies that Y_t , Y_0, and Y_{lift} are closed subsets in $\mathbb{A}^1 \times U \times X$, $U \times X$, and $U' \times X$, and relations (4) yield the commutative diagram



where γ_t, γ_0, γ_1, and γ_{lift} are the corresponding closed embeddings. The morphism $\pi_U\colon U \times X' \to U \times X$ induces isomorphism of Y_{lift} and Y_0; we denote this morphism by π^Y and the inverse morphism by $lift\colon Y_0 \to Y_{\mathrm{lift}}$. Let p_* $(* = t, 0, 1, \mathrm{lift})$ denote the projections of Y_* to $\mathbb{A}^1 \times U$ $(* = t)$ or U $(* = 0, 1, \mathrm{lift})$, and let q_* $(* = t, 0, 1, \mathrm{lift})$ be the projections to X ($*=t,0,1$) or X' ($*=\mathrm{lift}$), respectively. Finally, we denote by Y'_t, Y'_0, Y'_1, and Y'_{lift} the fibered products of Y_t, Y_0, Y_1, and Y_{lift} with the scheme $U-z$ over U.

Sine $h=\frac{s}{d^n}$, we have $Y_t=\Pi^{-1}(\mathbb{A}^1\times U\times 0)$. Using item (2) of Example 1, we get

$$\Upsilon\circ\langle\rho\rangle=(p_t,l_t,g_t)=:\widetilde{H}_t\in\mathrm{WCor}_k(\mathbb{A}^1\times U,X),$$

where $\rho\colon\mathbb{A}^1\times U\times 0\hookrightarrow\mathbb{A}^1\times U\times\mathbb{A}^1$), and $l_t\colon k[Y_t]\to k[\mathbb{A}^1\times U]$ is a $k[\mathbb{A}^1\times U]$-linear homomorphism that is the base change of l_B. The first row in (3)) implies that $Y'_t\subset g_t^{-1}(X-z)$. Hence, by Example 2, there is a morphism in the category of pairs

$$\widetilde{\Theta}=(\widetilde{H}_t,\widetilde{H}'_t)\in\mathrm{WCor}^{\mathrm{pair}}((\mathbb{A}^1\times U,\mathbb{A}^1\times(U-z)),(X,X-z)).$$

By item (2) of Example 1, the morphism

$$\widetilde{\Theta}\circ\langle j_0\rangle=(\widetilde{H}_0,\widetilde{H}_0)$$

is given by the pair of triples

$$\big(p_0,l_0\colon k[Y_0]\to k[U],g_0\big),\quad(p'_0,l'_0,g'_0).$$

Since lift: $Y_0\simeq Y_{\mathrm{lift}}$, we can define linear homomorphisms $l_{\mathrm{lift}}=l_0\circ(\mathrm{lift}^*)$ and $l'_{\mathrm{lift}}=l'_0\circ(\mathrm{lift}^{*\prime})$ (where $\mathrm{lift}'=\mathrm{lift}\times_U(U-z)$). So we get a morphism of pairs

$$\begin{aligned}\widetilde{\Phi}:=(\widetilde{P},\widetilde{P}'):=((p_{\mathrm{lift}},l_{\mathrm{lift}},g_{\mathrm{lift}}),(p'_{\mathrm{lift}},l'_{\mathrm{lift}},g'_{\mathrm{lift}}))\\ \in\mathrm{WCor}^{\mathrm{pair}}((U,U-z),(X',X'-z'))\colon\quad\widetilde{\Phi}=\widetilde{\Theta}\circ\langle j_0\rangle.\end{aligned}$$

Now consider the morphism

$$\widetilde{\Theta}\circ j_1=(\widetilde{H}_1,\widetilde{H}'_1)=((p_1,l_1,g_1),(p'_1,l'_1,g'_1))\in\mathrm{WCor}^{\mathrm{pair}}((U,U-z),(X,X-z)).$$

We show that $Y_1=\Delta\amalg R$ for some closed subscheme

$$R\subset(X-z)\times U.$$

Since $s_1\big|_\Delta=0$, we get

$$s_1=\delta\cdot r,\ \delta\in\Gamma(U\times\bar{X},\mathscr{L}(\Delta)),\ Z(\delta)=\Delta,\ r\in\Gamma(U\times\bar{X},\mathcal{R}).$$

Since $Z(s_1\big|_{z\times U}=\mathbf{z}=Z(\delta\big|_{z\times U})$, we see that $r\big|_{z\times U}$ is invertible. Hence, r is zero at $\mathbf{z}$. Since $\mathbf{z}$ is a unique closed point of Δ, it follows tht $r\big|_\Delta$ is invertible. Thus,

$$Y_1=Z(s_1)=Z(\delta)\amalg Z(r)=\Delta\amalg R.$$

Then any $k[Y_1]$-linear quadratic $k[U]$-form on $k[Y_1]$ splits into a sum of forms with supports Δ and R. Consequently,

$$\widetilde{H}_1=\big(p_1\big|_\Delta,l_1\big|_\Delta,g_1\big|_\Delta\big)+\big(p_1\big|_R,l_1\big|_R,g_1\big|_R\big).$$

Since $R\subset(X-z)\times U$, item (3) of Example 1 shows that there is a morphism

$$G\in\mathrm{WCor}(U,X-z)\colon\big(p_1\big|_R,l_1\big|_R,g_1\big|_R\big)=\langle X-z\hookrightarrow X\rangle\circ G.$$

Since $\Delta\simeq U$, and the homomorphism $l_1\big|_\Delta\colon k[\Delta]\to k[U]$ is determined by an invertible function $\lambda\in k[U]^*$, it follows that $\big(p_1\big|_\Delta,l_1\big|_\Delta,g_1\big|_\Delta\big)=\lambda\cdot\langle i\rangle\in\mathrm{WCor}(U,X)$.

Thus, $\widetilde{H}_1=\lambda\cdot\langle i\rangle+G$. Now, by Example 2, $\widetilde{H}_1$ determines a morphism $\widetilde{H}'_1$, whence

$$(\widetilde{H}_1,\widetilde{H}'_1)=\lambda\cdot\langle i\rangle+(G,G').$$

On the other hand, since $G\in\mathrm{WCor}(U,X-z)$, the definition of $\mathrm{WCor}^{\mathrm{pair}}$ shows that the morphism (G,G') is equal to zero. Therefore, $(\widetilde{H}_1,\widetilde{H}'_1=\lambda\cdot\langle i\rangle$ Finally, putting

$$\Phi=\lambda^{-1}\cdot\widetilde{\Phi},\Theta=\lambda^{-1}\cdot\widetilde{\Theta},$$

we get the required morphisms in $\mathrm{WCor}^{\mathrm{pair}}$.

So, all that we need to complete the proof of item a) is to construct the sections s', s_0, s_1, and s. Using the following lemma, which is a consequence of the Serre theorem (see [11, Theorem 5.2, Chapter 3]), we construct the required sections consecutively on some closed subsets.

Sublemma 4. *Suppose X is a projective scheme over a Noetherian ring, Z is a closed subscheme, $\mathscr{F}$ is a coherent sheaf, and $\mathscr{L}$ is a very ample invertible sheaf on X. Then for some k, the restriction $\Gamma(\mathscr{F}\otimes\mathscr{L}^{\otimes n})\to\Gamma((\mathscr{F}\otimes\mathscr{L}^{\otimes n})\big|_Z)$ is surjective for all $n>k$.*

Let W denote the local scheme of $\mathbf{z}$ in $z\times U$, and let W' be the local scheme of $\mathbf{z}$ in $U\times z'$, δ' the local parameter in $K[W']$, and $N'=\operatorname{Spec}K[W']/I(\delta')^2$ a closed subscheme in W'.

First, we construct the section s' on $\overline{X'}_z$. For this, we prove the following statement.

Sublemma 5. *Suppose $\pi\colon X'\to X$ is a finite morphism of curves over an infinite field, z is a closed point in X', Y is a closed subscheme of X', $Y\not\ni z$, and $\mathscr{L}$ is a very ample locally free sheaf of rank 1 on X'. Then there exists n_0 such that for all $n>n_0$, there is a global section s of $\mathscr{L}^{\otimes n}$ with the property that s is zero at z, s is invertible on Y, and the restriction of π to $Z(s)$ is a closed embedding.* (*In detail, we mean the restriction of π to the closed subscheme in X' determined by the sheave of ideals $\mathcal{I}\subset\mathscr{O}_{X'}$, $\mathcal{I}(U)=\{f\in\mathscr{O}(U)|\operatorname{div}f\geq\operatorname{div}s\}$*).

Proof. The morphism $\pi^s\colon Z(s)\to X$ is a closed embedding if and only if the homomorphism $\varepsilon_\pi^s\colon\mathscr{O}(X))\to\pi_*(\mathscr{O}(\operatorname{div}s))$ induced by π^s is surjective. Denote by Γ the affine space formed by the global sections of $\mathscr{L}^{\otimes n}$ that are zero at z. By Sublemma 4, Γ is not empty whenever n is sufficiently large.

First, we show that there is an open subscheme $U\subset\Gamma$ such that ε_π^s is surjective provided $s\in U$. Consider the map

$$\mu=\pi_\Gamma\colon X'\times\Gamma\to X\times\Gamma$$

and the universal section

$$s_\Gamma\in\Gamma\big(\operatorname{pr}_{X'}^*(\mathscr{L}^{\otimes n})\big),$$

where $\operatorname{pr}\colon X'\times\Gamma\to X'$ is the projection along Γ. Let $Z_{\rm in}\subset X'\times\Gamma$ be the support of the cokernel $\varepsilon_\mu\colon\mathscr{O}(X\times\Gamma)\to\mu_*\left(\mathscr{O}(\operatorname{div}s_\Gamma)\right)$, and let $Z\subset\Gamma$ be the union of the subspaces $\Gamma_y\subset\Gamma$, $\Gamma_y=\{s\in\Gamma|s(y)=0\}$, $y\in Y$. Then ε_π^s is surjective if and only if s is a rational point in Γ such that $s\notin Z$, and $s\notin\operatorname{pr}_\Gamma(Z_{\rm in})$, where pr_Γ is the projection along X'.

Since Γ is an affine space, $\Gamma-(\operatorname{pr}_\Gamma(Z_i)\cup Z_n)$ has a rational point whenever $\Gamma\neq\operatorname{pr}_\Gamma(Z_i)\cup Z_n$, (as schemes over the ground field).

Since $Y\not\ni z$, Sublemma 4 shows that for all sufficiently large n there is a section $s\in\Gamma$ such that $s(z)=0$, and s is invertible on Y. Hence $\Gamma\neq Z$. Thus, since Γ is irreducible, it suffices to prove that $\Gamma\neq\operatorname{pr}_\Gamma(Z_{\rm in})$.

A base change argument allows us to assume that $k=F$ is algebraically closed. If $\pi\colon Z(s)\to X$ is not an embedding, then $\operatorname{div}s\geq p_1+p_2$ for some $p_1,p_2\in X'$ with $\pi(p_1)=\pi(p_2)$ (p_1 and p_2 may coincide). We compute the codimension of $Z_{\rm in}$ in Γ. Observe that for all n and any pair of points $p_1,p_2\in X'$, the restriction homomorphism

$$r_{p_1,p_2,n}\colon\Gamma(\mathscr{L}^{\otimes n})\to\Gamma(\mathscr{L}^{\otimes n})\big|_{S(p_1+p_2+z)})=F^2$$

is surjective. Indeed, for any fixed n the surjectivity of $r_{p_1,p_2,n}$ is an open condition for a pair (p_1,p_2); on the other hand, for any pair p_1,p_2 for all sufficiently large n $r_{p_1,p_2,n}$ is surjective by Sublemma 4. Hence, the codimension of the subspace in Γ_0 spanned by the sections div $s\geq p_1+p_2$ is equal to the codimension of the subspace of regular functions

$$\big\{f\in F[S(p_1+p_2+z)]\,:\,\operatorname{div}f\geq p_1+p_2,\,\operatorname{div}f\geq z\big\}$$

in the space of functions that vanish at z. So, this codimension is 2 when p_1, $p_2 \neq z$, and it is 1 otherwise.

For any $p \in X$ there is a finite set of pairs $p_1, p_2 \in X$ with $\pi(p_1) = \pi(p_2) = p$. Since for $p \neq \pi(z)$ and for any such pair, the condition $\operatorname{div} s \geq p_1 + p_2$ determines a subspace in Γ of codimension 2, we have $\dim(Z \cap (p \times \Gamma)) \leq \dim \Gamma - 2$. If $p = \pi(z)$, then these conditions have codimension at least 1, whence $\dim(Z \cap (\pi(z) \times \Gamma)) \leq \dim \Gamma - 1$. Thus, $\dim Z \leq \dim \Gamma - 1$, so that $\Gamma \neq \mathrm{pr}_\Gamma(Z_{\mathrm{in}})$. □

Proof of Sublemma 2. Applying Sublemma 5 to $\pi_z \colon \bar{X}' \to \bar{X}$ and the sheaf $\mathscr{L}(z \times D')$, for all integers n larger than some $\bar{k}$, we find a section $\bar{s} \in L(n \cdot z \times D')$ on $\overline{X'}_z$ such that the restriction of $\bar{\pi}_z$ to $Z(\bar{s})$ is a closed embedding, $\bar{s}$ is zero at $\mathbf{z}'$, and $\bar{s}$ is invertible on $z \times (\bar{\pi}^{-1}(z) - \mathbf{z}') \cup U \times D'$. Since the inverse image functor takes ample bundles to ample ones, Sublemma 4 applied to $(U \times X', \mathscr{O}(U \times X'), \mathscr{L}(U \times D'))$ and $(U \times X, \mathscr{O}(U \times X), \mathscr{L}(U \times D))$ says that for all n larger than some k, the restriction homomorphisms

$$\Gamma(U \times X', \mathscr{L}(nU \times D'')) \to \Gamma\big(\mathscr{L}(nU \times D'')\big|_{U \times z' \cup U \times D' \cup z \times \overline{X}'}\big),$$
$$\Gamma(U \times X, \mathscr{L}(nlU \times D)) \to \Gamma\big(\mathscr{L}(nlU \times D)\big|_{U \times z \cup U \times D \cup \Delta}\big)$$

are surjective. Choose any n larger than $\bar{k}$ and k, and choose a section $\bar{s}$ satisfying the conditions above.

Now we can find $s' \in \Gamma(\bar{X}' \times U, \mathscr{L}(nU \times D')$ such that $s'\big|_{z \times X'} = \bar{s}$, s' is invertible on $U \times D'$, and $s'\big|_{N'} = \delta$ (here we use some trivialization of $\mathscr{L}(nU \times D')\big|'_N$). It follows that

$$Z(s') \cap (U \times \bar{\pi}^{-1}(z')) = \mathbf{z}', \quad Z(\operatorname{div} s') \cap (U \times D') = 0. \tag{5}$$

Indeed, we check the first identity. The closed points of the semilocal scheme $U \times \bar{\pi}^{-1}(z')$ are $\bar{\pi}^{-1}(z) \times z$. The section s' is invertible on $(U \times \bar{\pi}^{-1}(z')) - \mathbf{z}'$ because $\bar{s}$ is invertible on $s'\big|_{N'} = \delta'$. Hence, $U \times Z(s') \cap \bar{\pi}^{-1}(z')$ is contained in a neighborhood of $\mathbf{z}'$. At the same time, $Z(s') \cap W' = \mathbf{z}'$ because $Z(s') \cap N' = \mathbf{z}'$. The second identity in (5) is a reformulation of the fact that s' is invertible on $U \times D'$.

Let s_0 be a global section of $\mathscr{L}(nlD_U)$ such that $Z(s_0) = \bar{\pi}_{U_*}(Z(s'))$. Then (5) implies

$$Z(s_0) \cap (U \times z') = \mathbf{z}, \quad Z(s_0) \cap (U \times D) = 0. \tag{6}$$

Now we choose a section s_1 of $\mathscr{L}(U \times nlD)$ such that

$$s_1\big|_\Delta = 0,\ s_1\big|_{U \times (z \cup D)} = s_0\big|_{U \times (z \cup D)}$$

(these conditions agree on the intersection because $\Delta \cap (U \times z) = \mathbf{z}$ and s_0 is zero at $\mathbf{z}$). Let $s = s_0 \cdot (1 - t) + s_1 \cdot t$ be a section of $\mathscr{L}(nlD \times U)$. Then by (6) we get

$$Z(s) \cap (\mathbb{A}^1 \times U \times z) = \mathbb{A}^1 \times \mathbf{z}, \quad Z(s) \cap (\mathbb{A}^1 \times U \times D) = 0,$$

because $s\big|_{\mathbb{A}^1 \times U \times (z \cup D)} = s_0\big|_{U \times (z \cup D)}$. □

This proves item a) in Lemma 3. □

Proof of Lemma 3b). To prove the claim it suffices to find morphisms

$$\Phi \in \mathrm{WCor}_k^{\mathrm{pair}}((U, U - z), (X', X' - z')),$$
$$\Xi \in \mathrm{WCor}_k^{\mathrm{pair}}((U' \times \mathbb{A}^1, (U' - z') \times \mathbb{A}^1), (X', X' - z')),$$

such that

$$\Xi \circ \langle j_0 \rangle = \Phi \circ \langle \pi \rangle \in \mathrm{WCor}_k^{\mathrm{pair}}((U', U' - z'), (X', X' - z')),$$
$$[\Xi \circ \langle j_1 \rangle] = [\langle i' \rangle] \in \overline{\mathrm{WCor}_k^{\mathrm{pair}}}((U', U' - z'), (X', X' - z')),$$

where $j_0, j_1 \in \mathrm{WCor}^{\mathrm{pair}}((U', U'-z'), (U'\times\mathbb{A}^1, (U'-z')\times\mathbb{A}^1))$ are the zero and unit sections, π and i' are viewed as morphisms in the category WCor',

$$\pi \in \mathrm{WCor}^{\mathrm{pair}}((U', U'-z'), (U, U-z)), \text{ and } i' \in \mathrm{WCor}^{\mathrm{pair}}((U', U'-z'), (X', X'-z')).$$

Like in the proof of statement a), we construct quadratic spaces via some sections of line bundles

$$\begin{aligned} \mathscr{L}(nD'_U) &\text{ on } \bar{X}'_U, \\ \mathscr{L}(nD'_{U'\times\mathbb{A}^1}) &\text{ on } \bar{X}'_{U'\times\mathbb{A}^1} \end{aligned}$$

and

$$\mathscr{L}(nD'_{U'}) \text{ on } \bar{X}'_{U'}$$

for some sufficiently large n.

Let $\mathbf{z}''$ denote the diagonal in $z'\times z'$, let W'' be the local scheme of $z'\times U'$ at $\mathbf{z}''$, and let $N'' = \operatorname{Spec} K[W'']/I(\delta'')^2$. Choose any trivialization of the canonical class $\mu' \colon \omega(X') \simeq \mathscr{O}(X')$.

By Sublemma 4, for all n larger than some k, there is a global section s' of $\mathscr{L}(nD'_U)$ on $U\times\overline{X'}$ such that s' is invertible on $U\times D$, s' is invertible on $z'\times U-\mathbf{z}'$, and $s'\big|_{N'} = \delta'$ (here we use some trivialization of $\mathscr{L}(nD'_U)$ on N'). Let $s_0 = (\pi\times\mathrm{id}_{\overline{X}'})^*(s')$ be a section of $\mathscr{L}(nD'_{U'})$ that is the inverse image of s' along $\pi\times\mathrm{id}_{\overline{X}'}$. Then s_0 is invertible on $U'\times D'$ and on $z'\times U'-\mathbf{z}''$, and $s_0\big|_{N''} = \delta''$, where δ'' is the inverse image of δ along $\mathrm{id}_{\pi\times\overline{X}'}$.

Now we choose a section s_1 of the sheaf $\mathscr{L}(nU'\times D')$ on $U'\times\bar{X}'$ such that

$$s_1\big|_{U'\times(z'\cup D')} = s_0\big|_{U'\times(z'\cup D')},\ s_1\big|_{\Delta'} = 0;$$

the conditions agree because s_0 is zero on

$$\Delta'\cap(U\times(z'\cup D')) = \mathbf{z}''.$$

Let $s = s_0\cdot(1-t)+s_1\cdot t$ be a section of $\mathscr{L}(n\mathbb{A}^1\times U'\times D')$ on $\mathbb{A}^1\times U'\times\bar{X}'$; then s is invertible on $\mathbb{A}^1\times U'\times D'$ and $\mathbb{A}^1\times(U'-\mathbf{z}')\times z'$, and $s\big|_{\mathbb{A}^1\times W'} = \delta'$ (here we use a trivialization $\mathscr{L}(lnD_{\mathbb{A}^1\times U'})\big|_{\mathbb{A}^1\times W'} = \mathscr{O}(\mathbb{A}^1\times W')$), $s\big|_{0\times U'\times W'\times} = s_0$, $s\big|_{1\times U'\times W'\times} = s_1$).

Thus, we have proved the following

Sublemma 6. *There are sections*

$$s' \in \Gamma(\overline{X'}_U, \mathscr{L}(nD')), \quad s_0, s_1 \in \Gamma(\overline{X'}_{U'}, \mathscr{L}(nD')), \quad s \in \Gamma(\overline{X'}_{U'\times\mathbb{A}^1}, \mathscr{L}(nD'))$$

$$\begin{aligned} Z(s'\big|_{U\times z'}) &= \mathbf{z}', & Z(s'\big|_{U\times D'}) &= \varnothing, \\ Z(s\big|_{\mathbb{A}^1\times U'\times z'}) &= \mathbb{A}^{\mathbf{1}}\times\mathbf{z}'', & Z(s\big|_{\mathbb{A}^1\times U\times D'}) &= \varnothing, \end{aligned} \tag{7}$$

$$s_0 = (\pi\times\mathrm{id}_X)^*(s'), \quad s\big|_{0\times U'\times\overline{X'}} = s_0, \quad s\big|_{1\times U'\times\overline{X'}} = s_1, \quad s_1\big|_{\Delta'} = 0. \tag{8}$$

Also, we choose a section $d' \in \Gamma(\bar{X}, \mathscr{O}(D')$ with $Z(d') = D'$.

Now we apply the same construction as in a) to three functions simultaneously:

$$h = \frac{s}{d'^n} \in k[\mathbb{A}^1\times U'\times X'], \quad h_0 = \frac{s_0}{d'^n} \in k[U'\times X'], \quad f = \frac{s'}{d'^n} \in k[U\times X'.$$

So, we consider morphisms of relative affine curves

$$\begin{array}{llcll} \Pi & = (\mathrm{pr}_{\mathbb{A}^1\times U'}, h) & : & \mathbb{A}^1\times U'\times X' & \to \mathbb{A}^1\times U'\times\mathbb{A}^1, \\ \Pi_0 & = (\mathrm{pr}_{U'}, h_0) & : & U'\times X' & \to U'\times\mathbb{A}^1, \\ \Pi' & = (\mathrm{pr}_U, f) & : & U\times X' & \to U\times\mathbb{A}^1, \end{array}$$

$$(\mathbb{A}^1\times j_0)^*(\Pi) = \dot{\pi}^*(\Pi')$$

(here $\dot{\pi}$ denotes morphisms of bases of the relative curves $\dot{\pi}\colon U' \to U$, the last identity follows form (8)). Like in item a), Π, Π_0, and Π' can be represented as base changes of some morphisms of relative projective curves, and Sublemma 3 shows that Π, Π_0, and Π' are surjective, finite, flat morphisms of smooth curves. Let B_A, B_{0A_0}, and $B'_{A'}$ denote the algebras corresponding to the morphisms Π, Π_0, and Π'. Then we apply [10, Proposition 2.1] to Π, Π_0, and Π', obtaining isomorphisms q^ω, q_0^ω, and $q^{\omega\prime}$. Now we consider the trivializations $\mu^\Pi = \mu' \otimes \Pi^*(dT)^{-1}$, $\mu^{\Pi_0} = \mu' \otimes \Pi_0^*(dT)^{-1}$, and $\mu^{\Pi'} = \mu' \otimes \Pi'^*(dT)^{-1}$, and define symmetric isomorphisms

$$q_B\colon B_B \simeq \mathrm{Hom}_A(B_B, A), \quad q_{B_0}\colon B_{0B_0} \simeq \mathrm{Hom}_{A_0}(B_{0B_0}, A_0),$$
$$q_{B'}\colon B'_{B'} \simeq \mathrm{Hom}_{A'}(B'_{B'}, A')\colon j_0^*(q_B) = q_{B_0} = \bar{\pi}_U(q_{B'}),$$

as compositions of q^ω, q_0^ω, and $q^{\omega\prime}$ with μ^Π, μ^{Π_0}, and $\mu^{\Pi'}$. By items (1) and (2) of Example 1, this gives us morphisms

$$\Upsilon = (\Pi, L, \mathrm{pr}_{X'}) \in \mathrm{WCor}(\mathbb{A}^1 \times U' \times \mathbb{A}^1, X'), \qquad \widetilde{H}_t = \Upsilon \circ \rho \in \mathrm{WCor}(\mathbb{A}^1 \times U, X'),$$
$$\Upsilon_0 = (\Pi_0, L_0, \mathrm{pr}_{X'}) \in \mathrm{WCor}(U' \times \mathbb{A}^1, X'), \qquad \widetilde{H}_0 = \Upsilon_0 \circ \rho_0 \in \mathrm{WCor}(U', X'),$$
$$\Upsilon' = (\Pi', L', \mathrm{pr}_{X'}) \in \mathrm{WCor}(U \times \mathbb{A}^1, X'), \qquad \widetilde{P} = \Upsilon' \circ \rho' \in \mathrm{WCor}(U, X')\colon$$
$$\Upsilon \circ (\mathbb{A}^1 \times j_0) = \Upsilon_0, \quad \Upsilon_0 = \Upsilon' \circ \dot{\pi} \qquad \widetilde{H}_t \circ j_0 = \widetilde{H}_0, \quad \widetilde{P} \circ \pi = \widetilde{H}_0,$$

where

$$\rho\colon \mathbb{A}^1 \times \times U'0 \hookrightarrow \mathbb{A}^1 \times U' \times \mathbb{A}^1, \quad \rho_0\colon U' \times 0 \hookrightarrow U' \times \mathbb{A}^1, \text{ and } \rho'\colon 0 \times U \hookrightarrow U \times \mathbb{A}^1$$

are the zero section embeddings with respect to the first factor.

Moreover, we get morphisms of pairs

$$\widetilde{\Xi} = (\widetilde{H}_t, \widetilde{H}'_t) \in \mathrm{WCor}^{\mathrm{pair}}(\mathbb{A}^1 \times (U', U' - z'), (X', X' - z')),$$
$$\widetilde{\Phi} = (\widetilde{P}, \widetilde{P}') \in \mathrm{WCor}^{\mathrm{pair}}((U, U - z), (X', X' - z')).$$

Indeed, by item (2) of Example 1, H_t and P are determined by triples $\widetilde{H}_t = (p_t, l_t, g_t)$ and $\widetilde{P} = (p', l', g')$, where $p_t\colon Y_t \to \mathbb{A}^1 \times U'$, $g_t\colon Y_t \to X'$, $Y_t = \Pi^{-1}(\mathbb{A}^1 \times U' \times 0) = h^{-1}(0) \subset X' \times \mathbb{A}^1 \times U'$, and $p'\colon Y' \to U$, $g'\colon Y' \to X'$, $Y' = \Pi'^{-1}(0 \times U) = f^{-1}(0) \subset U \times X'$.

By definition, we have $h = s \cdot d'^{-n}$, whence $Y_t = Z(h) = Z(s)$. However, by (7),

$$Z(s\big|_{\mathbb{A}^1 \times U' \times z'}) = \mathbb{A}^1 \times \mathbf{z}'',$$

so that $Y_t \times_{U'} (U' - z') \subset g_t^{-1}(X' - z')$. Then, by Example 2, there is a morphism

$$\widetilde{H}'_t \in \mathrm{WCor}(U' - z', X' - z')$$

such that $(X' - z' \hookrightarrow X') \circ \widetilde{H}'_t = \widetilde{H}_t \circ (\mathbb{A}^1 \times (U' - z') \hookrightarrow \mathbb{A}^1 \times U')$. In other words, there is a morphism of pairs $(\widetilde{H}_t, \widetilde{H}'_t) \in \mathrm{WCor}^{\mathrm{pair}}(\mathbb{A}^1 \times (U', U' - z'), (X', X' - z'))$. Similarly, since $Y' = Z(f) = Z(s')$ and $s'\big|_{z' \times (U - z)}$ is invertible, we get $p'^{-1}(Y') \subset g'^{-1}(X - z)$, and the construction in Example 2 yields a morphism of pairs $(\widetilde{P}, \widetilde{P}')$.

Thus, we get morphisms $\widetilde{\Xi}$ and $\widetilde{\Phi}$ such that

$$\widetilde{\Xi} \circ j_0 = \widetilde{\Phi} \circ \pi.$$

Consider the morphism $\widetilde{\Xi} \circ \langle j_1 \rangle$. It is determined by the pair of triples $(\widetilde{H}_1, \widetilde{H}'_1)$ with $\widetilde{H}_1 = (p_1, l_1, g_1)$, $\widetilde{H}'_1 = (p_1{}', l_1{}', g_1{}')$, $p_1\colon Y_1 \to U'$, $g_1\colon Y_1 \to X'$, $Y_1 = p_t^{-1}(U' \times 1) \subset U' \times X'$ (again, see Examples 1 and 2). Since $s_1\big|_{\Delta'} = 0$ and $s_1\big|_{N''} = \delta''$, and since s_1 is invertible on $(z' \times U) - \mathbf{z}''$, and δ'' is a local parameter on W'', it follows that $Z(s_1) \cap (z' \times U') = \mathbf{z}'' = \Delta \cap (U' \times z' \times)$. $K[Y_1] = K[\Delta'] \times K[R']$ for some $R' \subset U' \times (X' - z)$, and $\widetilde{H}_1 = (p_1\big|_{\Delta'}, l_1\big|_{\Delta'}, g_1\big|_{\Delta'}) + (p_1\big|_{R'}, l_1\big|_{R'}, g_1\big|_{R'})$. The restriction of the linear homomorphism $l_1\colon k[Y_1] \to k[U']$ to $k[\Delta']$ is defined by an invertible function $\lambda \in K[U']^*$, On the other hand, the triple $(p_1\big|_{R'}, l_1\big|_{R'}, g_1\big|_{R'})$ gives rise to the zero morphism in the category of pairs. Hence,

$$(\widetilde{H}_1, \widetilde{H}_1') = \lambda \cdot \langle i' \rangle.$$

Choose an invertible function $\lambda' \in K[U]^*$ such that $\lambda'(z) = \lambda(z)^{-1}$ and put

$$\Xi = \lambda' \cdot \Xi', \Phi = \lambda' \cdot \Phi'.$$

The following Sublemma 7 implies that

$$\lambda' \cdot \lambda \cdot [i'] = [i'] \ \text{ in } \ \overline{\mathrm{WCor}}((U', U' - z'), (X', X' - z')),$$

thus giving us the required identity $[\Xi \circ \langle j_1 \rangle] = [i]$.

Sublemma 7. *Suppose X is a smooth scheme, U is a local scheme at a point z in X, and i is the embedding $U \hookrightarrow X$. Let the morphism $\varepsilon \in \mathrm{WCor}((U, U-z), (X, X-z))$ be determined by a $K[X \times U]$-module $K[\Delta]$, where Δ is the graph of i, and q is the quadratic form determined by the function $e \in K[U]^\times$ such that $e(z) = 1$; then*

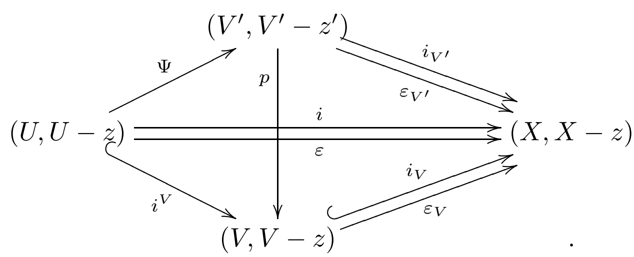
$$[\varepsilon] = [i] \in \overline{\mathrm{WCor}}((U, U-z), (X, X-z)).$$

Proof. Let a Zariski neighborhood V of z in X be such that there is a morphism $\varepsilon_V \in \mathrm{WCor}((V, V-z), (X, X-z))$ satisfying $\varepsilon_V \circ i^V = \varepsilon$, where i^V denotes the embedding $U \hookrightarrow V$.

Consider the covering $p\colon V' = \operatorname{Spec} K[V][b]/(b^2 = e) \to V$. The morphism p is *étale* over z because $e(z) = 1$; hence, it is an *étale* covering of U. Let z' be the preimage of z such that $b(z') = 1$. Shrinking V and V', we see that p is *étale* and $p^{-1}(z) = z'$.

Denote by i_V the embedding $V \hookrightarrow X$ and denote by the same symbol the corresponding element in $\mathrm{WCor}((V, V-z), (X, X-z))$. Since $p^{-1}(z) = z'$, p gives rise to a morphism in $\mathrm{WCor}((V', V'-z'), (V, V-z))$.

Consider the morphisms $i_{V'} = i_V \circ p$, $\varepsilon_{V'} = \varepsilon_V \circ p \in \mathrm{WCor}((V', V'-z'), (X, X-z))$. The morphism $i_{V'}$ is determined by the module $K[\Delta']$ and the unit function (where Δ' the graph of the embedding $p \circ i_V\colon V' \to X$). The morphism $\varepsilon \circ p$ is determined by the same module $K[\Delta']$ and by the function $p^*(e)$. Since $p^*(e) = b^2 \in K[V']$, i.e., $p^*(e)$ is a square, it follows that the quadratic form determined by e gives rise to the same morphism in $\varepsilon \circ p = i'_V \in \mathrm{WCor}((V', V'-z'), (X, X-z))$ as the unit form.



.

Now we note that item a) in Lemma 3 applied to the morphism $p\colon V' \to V$ implies that there exists

$$\Psi \in \mathrm{WCor}((U, U-z), (V', V'-z')) \ \text{ with } \ [\Psi \circ p] = [i^V] \in \overline{\mathrm{WCor}}((U, U-z), (V, V-z)).$$

Hence,

$$[i^V \circ \varepsilon] = [\Psi] \circ [p \circ \varepsilon] = [\Psi] \circ [p \circ i_V] = [i]. \qquad \square$$

$\square$

$\square$

§6. Homotopy invariance of the associated sheaf

Theorem 4. *For a homotopy invariant presheaf $\mathscr{F}$ with Witt-transfers, the associated Zariski sheaf $\mathscr{F}_{Zar}$ is homotopy invariant.*

Proof. The theorem follows form the next lemma by a standard argument.

Lemma 4. *Let $\mathscr{F}$ be a homotopy invariant sheaf with Witt-transfers. Then the canonical embedding $\mathscr{F}(U) \to \mathscr{F}_{\mathrm{Zar}}(U)$ is surjective for any Zariski open subset $U \subset \mathbb{A}_K$, where K is the filed of functions of a smooth variety over k.*

Let X be a k-smooth irreducible variety, and let K be its field of functions. It suffices to prove that the homomorphism $\mathscr{F}_{\mathrm{Zar}}(\mathbb{A}_X) \to \mathscr{F}_{\mathrm{Zar}}(X)$ induced by the embedding $i_{0,X}\colon X \to \mathbb{A}_X$ is injective.

Consider the commutative square

$$\begin{array}{ccc} \mathscr{F}_{\mathrm{Zar}}(\mathbb{A}_X) & \xhookrightarrow{J^*} & \mathscr{F}_{\mathrm{Zar}}(\mathbb{A}_{k(X)}) \\ \downarrow{\scriptstyle i^*_{0,X}} & & \downarrow{\scriptstyle i^*_{0,k(X)}} \\ \mathscr{F}_{\mathrm{Zar}}(X) & \xhookrightarrow{j^*} & \mathscr{F}_{\mathrm{Zar}}(k(X)). \end{array}$$

By the injectivity theorem for homotopy invariant presheaves with Witt-transfers, for any irreducible variety Y the homomorphism $\mathscr{F}_{Zar}(Y) \to \mathscr{F}_{Zar}(k(Y))$ is injective. Hence, J^* is a monomorphism. By Lemma 4, the homomorphism $\mathscr{F}(\mathbb{A}_X) \to \mathscr{F}_{\mathrm{Zar}}(\mathbb{A}_X)$ is an epimorphism. At the same time, it is a monomorphism, because the presheaf $\mathscr{F}$ is homotopy invariant. Hence, $\mathscr{F}(\mathbb{A}_X) \to \mathscr{F}_{\mathrm{Zar}}(\mathbb{A}_X)$ is an isomorphism. Since $\mathscr{F}(k(X)) = \mathscr{F}_{\mathrm{Zar}}(k(X))$, it follows that $i^*_{0,k(X)}$ is an isomorphism. Now the injectivity of $J*$ implies that of $i^*_{0,X}$. Since at the same time $i^*_{0,X}$ is an epimorphism, it is an isomorphism. Thus, it suffices to prove the lemma.

Proof of Lemma 4. Let $s \in F_{\mathrm{Zar}}(U)$. Let $c\colon \mathfrak{U} \to U$ be a Zariski covering such that there is $s_{\mathfrak{U}}$ with $c^*(s) = \varepsilon(s_{\mathfrak{U}})$, where ε is the natural homomorphism $\mathscr{F} \to \mathscr{F}_{\mathrm{Zar}}$. Denote by V any open subset U in $\mathfrak{U}$, and for any point $z \in U \setminus V$ we denote by U_z any subset in $s_{\mathfrak{U}}$ containing z.

Choose a point $z \in U \setminus V$. Let V_1 be the smallest open subset in $\mathbb{A}_K$ containing V and z. Consider the element $s_V \in \mathscr{F}(V)$ that is the restriction of $s_{\mathfrak{U}}$ to V, and consider its image $r_z \in \frac{\mathscr{F}(V)}{\mathscr{F}(V_1)}$. By Theorem 2,

$$\frac{\mathscr{F}(V)}{\mathscr{F}(V_1)} \simeq \lim_{z \in W \subset \mathbb{A}_K} \frac{\mathscr{F}(W - z)}{\mathscr{F}(W)}.$$

The restriction of $s_{\mathfrak{U}}$ to V_z gives us an element $s_z \in \mathscr{F}(V_z)$ compatible with s_V on $V_z - z$; hence, $r_z = 0$, and there exists $s_{V_1} \in \mathscr{F}(V_1)$ such that s_{V_1} coincides with s_V under the restriction to V. Then we add points of $U \setminus V$ inductively, finding an element $s_U \in \mathscr{F}(U)$ such that the germs of s_U and s_V coincide at the generic point. By injectivity for presheaves with Witt-transfers, the germs of s_U and s_V coincide at all points. □

□

Theorem 5. *Let $\mathscr{F}$ be a homotopy invariant presheave with Witt-transfers, and let $K = k(X)$ for a smooth variety X over k.*

Then

$$\begin{cases} \mathscr{F}_{\mathrm{Nis}}\big|_{\mathbb{A}^1{}_K} \simeq \mathscr{F}\big|_{\mathbb{A}^1{}_K}, \\ h^1(\mathscr{F}_{\mathrm{Nis}})\big|_{\mathbb{A}^1{}_K} \simeq 0. \end{cases}$$

Proof. The first statement is equivalent to saying that $\mathscr{F}(U) \to \mathscr{F}_{\mathrm{Zar}}(U)$ is an isomorphism for any $U \subset \mathbb{A}^1{}_K$. Indeed, since $\mathscr{F}(U) \to \mathscr{F}(W)$ is injective for any pair of open subschemes $W \subset U \subset \mathbb{A}^1{}_K$, we see that the restriction to the generic point $\mathscr{F}(U) \to \mathscr{F}(\eta)$ is injective for any open $U \subset \mathbb{A}^1$, and finally, $\mathscr{F}(U) \to \mathscr{F}_{\mathrm{Zar}}(U)$ is injective for any open U.

The second relation is equivalent to saying that $H^1_{\mathrm{Nis}}(U) = 0$ for any $U \subset \mathbb{A}^1{}_K$, because all higher cohomologies are trivial because of dimension.

Consider the sequence

$$0 \to \mathscr{F}(U) \xrightarrow{i} \mathscr{F}(\eta) \xrightarrow{d^1} \sum_{z \in \mathrm{MaxSp}\,(U)} \frac{\mathscr{F}(U_z - z)}{\mathscr{F}(U_z)} \to 0, \tag{9}$$

where z runs over all closed points of U, and U_z denotes a local neighborhood of z. This is a short exact sequence. Indeed, the arrow i is injective by Theorem 1 on the injectivity on $\mathbb{A}^1$. The exactness at the middle term follows form the injectivity of the excision homomorphisms. The surjectivity of the second arrow follows from the surjectivity of the excision homomorphism.

This sequence is a sequence of sections of the presheet $\mathscr{F}\big|_{\mathbb{A}^1{}_K}$ and sections of the following flasque resolvent of it in the Nisnevich topology:

$$\mathscr{F} \to \eta_*(\mathscr{F}(\eta) \xrightarrow{d} \sum_{z \in \mathrm{MaxSp}\,(\mathbb{A}^1)} z_*(\frac{\mathscr{F}(U^h_z - z)}{\mathscr{F}(U^h_z)}),$$

where η is the generic point, η_* is the inverse image homomorphism along $\eta \to \mathbb{A}^1$, z in the second term runs over all closed points on $\mathbb{A}^1$, and z_* is the inverse image along $z \to \mathbb{A}^1$. The injectivity of a homotopy invariant presheaf with Witt-transfers on local schemes and the excision isomorphism imply that this is an exact sequence of sheaves. Thus, it is a resolvent of length 1, whence $H^0_{\mathrm{Nis}}(U) = \ker(d(U))$, $H^1_{\mathrm{Nis}}(U) = \mathrm{coker}(d(U))$, and the higher cohomologies are trivial. Now the exactness of the sequence (9) implies that $H^0_{\mathrm{Nis}}(U) = \mathscr{F}(U)$ (this proves again that $\mathscr{F}$ is a sheaf) and $H^1_{\mathrm{Nis}}(U) = 0$. □

Proof of the main theorem. The proof is similar to the proof of Theorem 4 if we apply the first statement of Proposition 5 instead of Lemma 4. □

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