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# HOMOTOPY THEORY OF NORMED SETS I. BASIC CONSTRUCTIONS

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Abstract. We would like to present an extension of the theory of $\mathbb{R}_{\geq 0}$-graded (or "$\mathbb{R}_{\geq 0}$-normed") sets and monads over them as defined in recent paper by Frederic Paugam.

The theory of graded sets is extended in three directions. First of all, it is shown that $\mathbb{R}_{\geq 0}$ can be replaced with more or less arbitrary (partially) ordered commutative monoid $\Delta$, yielding a symmetric monoidal category $\mathcal{N}_\Delta$ of $\Delta$-*normed sets.* However, this category fails to be closed under some important categorical constructions. This problem is dealt with by embedding $\mathcal{N}_\Delta$ into a larger *category* $\mathit{Sets}^\Delta$ *of* $\Delta$-*graded sets.*

Next, it is shown show that most constructions make sense with $\Delta$ replaced by a small symmetric monoidal category $\mathcal{I}$. In particular, we have a symmetric monoidal category $\mathit{Sets}^{\mathcal{I}}$ of $\mathcal{I}$-graded sets.

These foundations are used for two further developments: a *homotopy theory for normed and graded sets*, essentially consisting of a well-behaved combinatorial model structure on simplicial $\mathcal{I}$-graded sets and a *theory of* $\Delta$-*graded monads.* This material will be exposed elsewhere.

## §1. Normed sets

**1.0.** We start with the definition of $\Delta$-*normed sets*, which corresponds to the definition of [Pa] when $\Delta = \mathbb{R}_{\geq 0}$.

**Definition 1.1.** Let $\Delta$ be a partially ordered set (in most cases, $\Delta$ will be a partially ordered commutative monoid, e.g. $\Delta = \mathbb{R}_{\geq 0}$). A **$\Delta$-normed set** is a pair $X = (X, |\cdot|_X)$, where $X$ is a set, and $|\cdot|_X\colon X \to \Delta$ is an arbitrary map of sets, called **a ($\Delta$-valued) norm on $X$.**

If $x$ is an element of $X$, we say that $|x|_X := |\cdot|_X(x) \in \Delta$ is the *norm of* $x$.

A **morphism** $f$ of the $\Delta$-normed set $(X, |\cdot|_X)$ to $(Y, |\cdot|_Y)$ is a **norm-shrinking map** $f\colon X \to Y$, i.e., a map of sets such that $|f(x)|_Y \leq |x|_X$ for all $x \in X$. This condition can also be written as $|\cdot|_Y \circ f \leq |\cdot|_X$.

The category of all $\Delta$-normed sets with norm-shrinking maps as morphisms will be denoted by $\mathcal{N}_\Delta$.

**1.1. Other variants of maps between normed sets.** In his work [Pa], apart from the norm-shrinking maps, Paugam considered two other classes of maps between $\Delta$-normed sets (for $\Delta = \mathbb{R}_{\geq 0}$): the *norm-preserving maps*, having the property that $|\cdot|_Y \circ f = |\cdot|_X$, which thus determine the slice category $\mathit{Sets}_{/\Delta}$, and the *Lipschitz maps*, having that property that for some $C = C_f \in \Delta$, inequality $|f(x)|_Y \leq C \cdot |x|_X$ holds for all $x \in X$ (here $\Delta$ has to be an ordered monoid, which is true for $\Delta = \mathbb{R}_{\geq 0}$ considered in *loc.cit.*).

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We think that norm-preserving maps are "uninteresting" in the sense that they lead to the well-known slice category $Sets_{/\Delta}$, equivalent to the category product $Sets^{|\Delta|}$, where $|\Delta|$ is the underlying set of $\Delta$.

As to the Lipschitz maps from $X$ to $Y$, they are quite important; however, we are going to interpret them as elements of the underlying set of the *inner Hom* $X \multimap Y = \mathbf{Hom}(X, Y)$ from $X$ to $Y$ with respect to a natural monoidal structure on $\mathcal{N}_\Delta$. We think this is a more natural way of dealing with Lipschitz maps than treating them as morphisms of a category.

### 1.2. Limits of $\Delta$-normed sets.

**Proposition 1.** *Let $\Delta$ be a poset. Then*:

a) *If the binary sup's* ($\vee$) *exist in $\Delta$, then the binary direct products $X \times Y$ exist in $\mathcal{N}_\Delta$.*
b) *If $\Delta$ admits a minimal element* $0$*, the final object $e_{\mathcal{N}_\Delta}$ exists in $\mathcal{N}_\Delta$.*
c) *If the finite sup's exist in $\Delta$, the finite limits exist in $\mathcal{N}_\Delta$.*
d) *If any subset $A \subset \Delta$ that is bounded from above admits a supremum, arbitrary* (*small*) *limits exist in $\mathcal{N}_\Delta$.*

1.2.1. *Construction of limits in $\mathcal{N}_\Delta$.* All of the above statements follow from an explicit construction of limits in $\mathcal{N}_\Delta$ that we are going to describe now.

Let $X \colon \mathcal{J} \to \mathcal{N}_\Delta$ be a diagram in $\mathcal{N}_\Delta$. Denote by $S \colon \mathcal{N}_\Delta \to Sets$ the underlying set functor: $S \colon (Z, |\cdot|_Z) \rightsquigarrow Z$. Put $Y := \lim_{\mathcal{J}} S \circ X$, and denote by $\pi_\alpha \colon Y \to X(\alpha)$ the canonical projection of sets for all $\alpha : \mathcal{J}$. Now let $Y^\circ$ be the subset of $Y$ consisting of all $y \in Y$ such that the set of norms of projections $\{|\pi_\alpha(y)|_{X(\alpha)}\}_{\alpha:\mathcal{J}}$ is bounded from above. This is automatically true in cases a)–c), so in these cases $Y^\circ = Y$.

Next, put

$$|y|_{Y^\circ} := \sup_{\alpha:\mathcal{J}} |\pi_\alpha(y)|_{X(\alpha)}. \tag{1.2.1.1}$$

It is easy to see that $(Y^\circ, |\cdot|_{Y^\circ})$, together with the restrictions of $\pi_\alpha$ to $Y^\circ$, is the limit in $\mathcal{N}_\Delta$ of the original diagram $X$.

1.2.2. *Special case: binary direct products in $\mathcal{N}_\Delta$.* In particular, we see that the binary direct product $X \times Y$ of two $\Delta$-normed sets $X = (X, |\cdot|_X)$ and $Y = (Y, |\cdot|_Y)$ is given by the Cartesian product $X \times Y$ of underlying sets together with the *sup-norm*

$$\big(x, y\big)_{X \times Y} = |x|_X \vee |y|_Y. \tag{1.2.2.1}$$

1.2.3. *Special case: final object of $\mathcal{N}_\Delta$.* The final object of $\mathcal{N}_\Delta$ is given by an the singleton $\mathbf{1} = \{1\}$ with the norm of its unique element equal to the smallest element $0$ of $\Delta$.

1.2.4. *Special case: infinite products in $\mathcal{N}_\Delta$.* An infinite product $\prod_\alpha X_\alpha$ is slightly more complicated to compute: it consists of all families $\mathbf{x} = (x_\alpha)_\alpha$, $x_\alpha \in X_\alpha$, such that $\{|x_\alpha|_{X_\alpha}\}$ is bounded from above, with $|\mathbf{x}|$ equal to $\sup_\alpha |x_\alpha|_{X_\alpha}$.

1.2.5. *Special case: fibered products in $\mathcal{N}_\Delta$.* Given two norm-shrinking maps $f \colon X \to S$, $g \colon Y \to S$ of $\Delta$-normed sets, we can construct their fibered product in $\mathcal{N}_\Delta$ as follows. We take their set-theoretical fibered product $X \times_S Y = \big\{(x, y) \in X \times Y \,:\, f(x) = g(y)\big\}$, and endow it with the restriction of the sup-norm on $X \times Y \supset X \times_S Y$ given by (1.2.2.1).

1.2.6. *Special case: equalizers in $\mathcal{N}_\Delta$.* Given two parallel morphisms $f, g \colon X \rightrightarrows Y$ in $\mathcal{N}_\Delta$, we can always construct their equalizer $\mathrm{Eq}(f, g)$ in $\mathcal{N}_\Delta$, without any additional assumptions on $\Delta$, as follows. Take the set-theoretical equalizer $\mathrm{Eq}(f, g) := \{x \in X \,:\, f(x) = g(x)\}$, and endow it with the restriction of the $\Delta$-norm $|\cdot|_X$ of $X$.

1.2.7. *Monomorphisms in $\mathcal{N}_\Delta$.* Since $f\colon X \to Y$ is a monomorphism if and only if the relative diagonal $\Delta_f\colon X \to X \times_Y X$ is an isomorphism, we see that $f$ is a monomorphism in $\mathcal{N}_\Delta$ if and only if it is set-theoretically injective, i.e., if and only if $S(f)$ is a monomorphism in *Sets*. In other words, $S\colon \mathcal{N}_\Delta \to$ *Sets reflects monomorphisms.*

1.2.8. *Strict monomorphisms in $\mathcal{N}_\Delta$.* Recall that *strict monomorphisms* are those monomorphisms that appear as common equalizers of some family of pairs of parallel morphisms with common source. Our explicit description of equalizers shows that $i\colon Y \to X$ is a strict monomorphism in $\mathcal{N}_\Delta$ if and only if $i$ is set-theoretically injective and norm-preserving, i.e., $|i(y)|_X = |y|_Y$ for all $y \in Y$. If we identify $Y$ with a subset of $X$, this means that $|\cdot|_Y$ is a restriction of the norm of $X$. In this way the set of strict subobjects of a $\Delta$-normed set $X = (X, |\cdot|_X)$ is in one-to-one correspondence with the power-set $\mathfrak{P}(X)$ of $X$.

### 1.3. Colimits of normed sets.

**Proposition 2.** *Let $\Delta$ be a poset. Then:*

a) *Arbitrary (small) coproducts exist in $\mathcal{N}_\Delta$. They can be computed with the aid of disjoint unions of underlying sets.*
b) *If any nonempty subset of $\Delta$ admits an infimum, arbitrary (small) colimits exist in $\mathcal{N}_\Delta$.*

**Corollary 1.3.1.** *Let $\Delta$ be a poset such that any nonempty subset of $\Delta$ admits an infimum. Then the category $\mathcal{N}_\Delta$ is* bicomplete*, i.e., admits arbitrary (small) limits and colimits.*

*Proof.* We know that this condition is sufficient for the existence of arbitrary colimits. On the other hand, the existence of infima of nonempty subsets of $\Delta$ is *equivalent* to the existence of suprema of subsets bounded from above, because of well-known relations between infima (of nonempty subsets) and suprema (of subsets bounded from above):

$$\sup A = \inf\{b \,:\, A \le b\}; \tag{1.3.1.1}$$

$$\inf B = \sup\{a \,:\, a \le B\}. \tag{1.3.1.2}$$

We already know from **1.2** that the existence of suprema of subsets bounded from above implies the existence of arbitrary limits. Another way of expressing the same thing is this: once we know that arbitrary colimits exist in $\mathcal{N}_\Delta$, and we know that $\mathcal{N}_\Delta$ admits a small family of generators (namely, all one-element $\Delta$-normed sets with underlying set equal to $\mathbf{1} = \{1\}$), we can deduce the existence of arbitrary (small) limits. □

In order to prove the proposition, we have to consider an explicit construction of colimits in $\mathcal{N}_\Delta$.

1.3.2. *Construction of coproducts in $\mathcal{N}_\Delta$.* Suppose that $X_\alpha = (X_\alpha, |\cdot|_{X_\alpha})$ be a family of $\Delta$-normed sets. Their coproduct in $\mathcal{N}_\Delta$ is simply the disjoint union $X := \coprod_\alpha X_\alpha$, with the norm $|\cdot|_X$ restricting to $|\cdot|_{X_\alpha}$ on each component $X_\alpha \subset X$.

1.3.3. *Quotient norms.* Now suppose that $p\colon (X, |\cdot|_X) \to (Y, |\cdot|_Y)$ is a surjective map of $\Delta$-normed sets. We say that the norm on $Y$ is the *quotient norm* of that of $X$ *with respect to* $p$ if for any $y \in Y$ we have

$$|y|_Y = \inf_{x \in p^{-1}(y)} |x|_X. \tag{1.3.3.1}$$

Notice that the quotient norm on $Y$ is uniquely determined by means of this formula by the map $p$ and the norm $|\cdot|_X$ on $X$. In other words, given a $\Delta$-normed set $(X, |\cdot|_X)$

and a surjective map $p\colon X \to Y$, there is at most one quotient norm $|\cdot|_Y$ on $Y$. If the infima of nonempty subsets exist in $\Delta$, a quotient norm always exists.

1.3.4. *Coequalizers in* $\mathcal{N}_\Delta$. We are ready to compute the coequalizers $\mathrm{Coeq}(f, g\colon X \rightrightarrows Y)$ in $\mathcal{N}_\Delta$, provided the infima of nonempty subsets of $\Delta$ exist. Let $Z$ be the corresponding coequalizer computed in *Sets*, and let $p\colon Y \to Z$ be the projection. Endow $Z$ by the quotient norm of $|\cdot|_Y$ with respect to $p$. It is easy to see that $Z$ together with this quotient norm is a coequalizer of $f$ and $g$.

1.3.5. *Arbitrary small colimits in* $\mathcal{N}_\Delta$. Since any colimit can be expressed with the aid of coproducts and coequalizers, we see that the small colimits exist in $\mathcal{N}_\Delta$, provided the infima of nonempty subsets exist in $\Delta$. We can describe these colimits more explicitly by combining the above results. Given a diagram $X\colon \mathcal{J} \to \mathcal{N}_\Delta$, let $Y := \operatorname{colim} S \circ X$ be the corresponding colimit of underlying sets, and let $p_\alpha\colon X(\alpha) \to Y$ be the projection of a component into the colimit. Now $Y$ becomes a colimit of the diagram $X$ in $\mathcal{N}_\Delta$, provided we introduce a norm on the elements of $Y$ by

$$|y|_Y = \inf_{\alpha, p_\alpha(x)=y} |x|_{X_\alpha}. \tag{1.3.5.1}$$

1.3.6. *Strict epimorphisms in* $\mathcal{N}_\Delta$. Recall that strict epimorphisms are those morphisms that appear as the common coequalizers of some family of pairs of parallel morphisms with common target. This implies that if $p\colon Y \to Z$ is a strict epimorphism in $\mathcal{N}_\Delta$, then $p$ is a surjective map of sets, and $|\cdot|_Z$ is the quotient norm of $|\cdot|_Y$ with respect to $p$. The converse is true provided binary $\vee$'s exist in $\Delta$, which is needed for the existence of a kernel pair $Y \times_Z Y \rightrightarrows Y$. In this case $p\colon Y \to Z$ is even an effective epimorphism, i.e., the coequalizer of its kernel pair.

1.3.7. *Epimorphisms in* $\mathcal{N}_\Delta$. Recall that $p\colon Y \to Z$ is an epimorphism if and only if the codiagonal $Z \to Z \coprod_Y Z$ is an isomorphism. We deduce from the explicit construction of colimits given above that $p\colon Y \to Z$ *is an epimorphism of normed sets if and only if it is surjective as a map of sets.* This statement can easily be shown directly, without any reference to colimits or additional requirements on $\Delta$.

1.3.8. *Case* $\Delta = \mathbb{R}_{\geq 0}$. Of course, $\mathbb{R}_{\geq 0}$ admits the infima of arbitrary nonempty subsets, hence $\mathcal{N}_{\mathbb{R}_{\geq 0}}$ is bicomplete, as already shown in [Pa].

**1.4. Adjoining $\infty$ to $\Delta$.** We saw in **1.3.1** that $\mathcal{N}_\Delta$ *is bicomplete if and only if arbitrary nonempty infima exist in* $\Delta$. This condition is a bit cumbersome: it would be more convenient to work with $\Delta$ where arbitrary infima (hence also suprema) exist, i.e., with $\Delta$ a complete lattice.

This problem can be circumvented by adjoining a new largest element to $\Delta$.

**Notation 1.4.1.** We denote by $\Delta_\infty$ the partially ordered set $\Delta \sqcup \{\infty\}$, obtained from $\Delta$ by adjoining a new element $\infty$, larger than any element of $\Delta$.

1.4.2. *Bicompleteness of* $\mathcal{N}_\Delta$ *and completeness of* $\Delta_\infty$. Clearly, arbitrary nonempty infima exist in $\Delta$ if and only if arbitrary infima (and suprema) exist in $\Delta_\infty$, i.e., if and only if $\Delta_\infty$ is a complete lattice. Combining this with **1.3.1**, we see that $\mathcal{N}_\Delta$ *is bicomplete if and only if* $\Delta_\infty$ *is a complete lattice.*

1.4.3. *Adjoint functors between $\mathcal{N}_\Delta$ and $\mathcal{N}_{\Delta_\infty}$.* We have an obvious "embedding functor" $e_!\colon \mathcal{N}_\Delta \to \mathcal{N}_{\Delta_\infty}$, transforming a $\Delta$-normed set $X = (X, |\cdot|_X\colon X \to \Delta)$ into the $\Delta_\infty$-normed set $(X, e \circ |\cdot|_X)$, where $e\colon \Delta \to \Delta_\infty$ is the natural embedding. We usually denote $e_! X$ by the same letter $X$, and its norm by the same symbol $|\cdot|_X$, assuming $\Delta$ to be canonically embedded into $\Delta_\infty$ as a subset "of finite elements".

The functor $e_!$ is fully faithful; it enables us to identify the $\Delta$-normed sets with those $\Delta_\infty$-normed sets "whose norm takes only finite values".

The functor $e_!$ admits a right adjoint $e^*\colon \mathcal{N}_{\Delta_\infty} \to \mathcal{N}_\Delta$ transforming $(X, |\cdot|_X\colon X \to \Delta_\infty)$ into $X_{<\infty} := \{x \in X : |x|_X < \infty\}$ with the norm given by restriction of $|\cdot|_X$. One can also write $X_{<\infty} = X \times_{\Delta_\infty} \Delta$. The adjointness relation

$$\mathrm{Hom}_{\mathcal{N}_{\Delta_\infty}}(e_! X, Y) \cong \mathrm{Hom}_{\mathcal{N}_\Delta}(X, e^* Y) \tag{1.4.3.1}$$

is immediate.

1.4.4. *Consequence for limits and colimits.* Once we have adjoint functors $e_! \dashv e^*$ with fully faithful $e_!$ (hence $e^*$ a colocalization functor), we can relate limits and colimits in the two categories under consideration:

$$e_!(\operatorname{colim} F) \cong \operatorname{colim} e_! \circ F; \tag{1.4.4.1}$$

$$\operatorname{colim} F \cong e^*(\operatorname{colim} e_! \circ F); \tag{1.4.4.2}$$

$$\lim F \cong e^*(\lim e_! \circ F). \tag{1.4.4.3}$$

In particular, we can compute a limit in $\mathcal{N}_\Delta$, say, an infinite product $\prod_\alpha X_\alpha$, by computing it first in $\mathcal{N}_{\Delta_\infty}$, and then applying $e^*$ to it, i.e., considering only elements of finite norm in the product thus obtained.

That said, limits in $\mathcal{N}_{\Delta_\infty}$ are easier to compute and to understand, since the *forgetful functor $\mathcal{N}_{\Delta_\infty} \to$ Sets preserves limits.* This is due to the following fact.

**Proposition 1.4.5.** *If $\Delta$ has a largest element, then the forgetful functor $S\colon \mathcal{N}_\Delta \to$ Sets admits a left adjoint, hence it preserves all limits.*

*Proof.* Denote the largest element of $\Delta$ by $\omega$. It is easy to see that the functor transforming any set $A$ into $A^\omega$, the same set endowed with the $\Delta$-norm taking the value $\omega$ on all elements of $A$, is a left adjoint to $S$:

$$\mathrm{Hom}_{Sets}(A, S(X)) \cong \mathrm{Hom}_{\mathcal{N}_\Delta}(A^\omega, X). \tag{1.4.5.1}$$

Indeed, any map $f\colon A \to X$ is norm-shrinking in this case. □

### 1.5. Generators of $\mathcal{N}_\Delta$.

1.5.1. *Notation for one-element normed ets.* We denote by $\mathbf{1}^\alpha$ the standard singleton $\mathbf{1} = \{1\}$ with the norm of its unique element set to $\alpha \in \Delta$. The full subcategory of $\mathcal{N}_\Delta$ consisting of one-element normed sets is equivalent to its full subcategory $\mathbf{1}_\Delta$ consisting of all $\mathbf{1}^\alpha$, $\alpha \in \Delta$, which in its turn is isomorphic to the category $\Delta^{\mathrm{op}}$ (we view partially ordered set $\Delta$ as a category with at most one morphism between any two objects in a standard fashion), since a (necessarily unique) morphism from $\mathbf{1}^\alpha$ to $\mathbf{1}^\beta$ exists if and only if $\alpha \geq \beta$.

1.5.2. $\mathbf{1}_\Delta \cong \Delta^{\mathrm{op}}$ *generates* $\mathcal{N}_\Delta$. Notice that the standard one-element normed sets generate $\mathcal{N}_\Delta$ under strict epimorphisms. In fact, any object $X = (X, |\cdot|_X)$ is not merely the target of a strict epimorphism from a coproduct of some family of objects from $\mathbf{1}_\Delta$; it is even isomorphic to such a coproduct:

$$(X, |\cdot|_X) \cong \coprod_{x \in X} \mathbf{1}^{|x|_X}. \tag{1.5.2.1}$$

1.5.3. *Embedding of $\mathcal{N}_\Delta$ into presheaves on $\Delta^{\mathrm{op}}$.* Since $\mathbf{1}_\Delta \cong \Delta^{\mathrm{op}}$ generates $\mathcal{N}_\Delta$, composition of Yoneda embedding $h\colon \mathcal{N}_\Delta \to \widehat{\mathcal{N}_\Delta}$ with the restriction of presheaves from $\mathcal{N}_\Delta$ to $\mathbf{1}_\Delta$ induces a *fully faithful* limit-preserving embedding $I$ of $\mathcal{N}_\Delta$ into $\widehat{\Delta^{\mathrm{op}}} \cong$ *Funct*$(\Delta, \mathit{Sets}) = \mathit{Sets}^\Delta$.

This embedding transforms $X = (X, |\cdot|_X)$ into the functor

$$I(X)\colon \alpha \rightsquigarrow \mathrm{Hom}_{\mathcal{N}_\Delta}(\mathbf{1}^\alpha, X) \cong X_{\leq\alpha} := \{x \in X \,:\, |x|_X \leq \alpha\}. \tag{1.5.3.1}$$

We are going to study this embedding in more detail later.

1.5.4. *Finite normed sets and compact objects of $\mathcal{N}_\Delta$.* Recall that an object $X$ of some category $\mathcal{C}$ is called *compact* (or sometimes *finitely presented*) if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves filtered colimits.

One might expect finite normed sets to be compact objects of $\mathcal{N}_\Delta$, and the category $\mathcal{N}_\Delta$ to be compactly generated. In fact, this is usually not true. We claim, however, that *any compact object of $\mathcal{N}_\Delta$ is necessarily finite.*

Indeed, suppose that $X = (X, |\cdot|_X)$ is compact. Write $X$ as the filtered colimit of all its finite subsets $Y \subset X$ with norms given by restriction of $|\cdot|_X$. By compactness, we must have

$$\mathrm{id}_X \in \mathrm{Hom}(X, X) \cong \mathrm{colim}_{\text{finite } Y \subset X} \mathrm{Hom}(Y, X). \tag{1.5.4.1}$$

This shows that $\mathrm{id}_X\colon X \to X$ factorizes through some finite $Y \subset X$, which is possible only if $X = Y$ is finite itself.

Now we might want to study when finite $\Delta$-normed sets are compact. Since any finite normed set is a finite coproduct of generators $\mathbf{1}^\alpha$, and in any cocomplete category $S \coprod T$ is compact if and only if both $S$ and $T$ are compact, it suffices to understand when $\mathbf{1}^\alpha$ is compact, i.e., the functor $X \rightsquigarrow \mathrm{Hom}(\mathbf{1}^\alpha, X) = X_{\leq\alpha} = \{x \in X \,:\, |x|_X \leq \alpha\}$ commutes with filtered inductive limits. Here is a partial result in this direction.

**Proposition 1.5.5.** *Suppose $\Delta$ is* well ordered*, i.e., its order is linear, and any nonempty subset of $\Delta$ has the smallest element. Then* $\mathrm{Hom}(\mathbf{1}^\alpha, -)$ *commutes with filtered colimits, for any $\alpha \in \Delta$; in particular, all finite $\Delta$-normed sets are compact, and $\mathcal{N}_\Delta$ is compactly generated, since any $\Delta$-normed set is the filtered colimit of its finite subsets with induced norms.*

Since we don't need this result later, we leave it as an exercise for the reader.

1.5.6. *Notation for standard finite normed sets.* Any finite normed set is isomorphic to a *standard finite normed set*, with the underlying set isomorphic to $\mathbf{n} = \{1, 2, \ldots, n\}$ for some $n$. We introduce the notation

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle := (\mathbf{n}, \alpha\colon \mathbf{n} \to \Delta) \tag{1.5.6.1}$$

for a standard finite set $\mathbf{n}$ with the norm $\alpha\colon \mathbf{n} \to \Delta$ taking the values $\alpha_i = \alpha(i)$ at $1 \leq i \leq n$.

### 1.6. Monoidal structure on normed sets.

1.6.1. *Bifunctor on $\mathcal{N}_\Delta$ defined by a multiplication.* Suppose we are given a poset map $\mu\colon \Delta \times \Delta \to \Delta$. It enables us to define a bifunctor $\otimes_\mu\colon \mathcal{N}_\Delta \times \mathcal{N}_\Delta \to \mathcal{N}_\Delta$, characterized uniquely up to isomorphism by the following properties:

- $\otimes_\mu$ preserves arbitrary coproducts in each argument;
- we have $\mathbf{1}^\alpha \otimes_\mu \mathbf{1}^\beta \cong \mathbf{1}^{\mu(\alpha,\beta)}$ on the generators $\mathbf{1}^\alpha$ of $\mathcal{N}$.

Conversely, any such bifunctor $\otimes$ having the property that $\mathbf{1}_\Delta$ is stable under $\otimes$ can be obtained with the aid of some $\mu$.

1.6.2. *Properties of $\otimes_\mu$ in terms of $\mu$.* Notice that $\otimes_\mu$ is associative, commutative and with unit if and only if the "multiplication" $\mu\colon \Delta \times \Delta \to \Delta$ has the corresponding properties.

1.6.3. *Canonical tensor product $\otimes$ on $\mathcal{N}_\Delta$.* In particular, if $\Delta$ is a (partially) ordered commutative monoid, its multiplication defines a "canonical" symmetric monoidal structure $\otimes$ on $\mathcal{N}_\Delta$, characterized by the property that $\otimes$ preserve arbitrary coproducts and

$$\mathbf{1}^\alpha \otimes \mathbf{1}^\beta \cong \mathbf{1}^{\alpha\beta}. \tag{1.6.3.1}$$

1.6.4. *Direct description of $X \otimes Y$.* Of course, we can describe $X \otimes_\mu Y$ explicitly by using the decomposition (1.5.2.1): its underlying set is $X \times Y$, with the norm given by $\big|(x,y)\big|_{X\otimes_\mu Y} = \mu\big(|x|_X, |y|_Y\big)$. In particular, if $\mu$ is the multiplication of a partially ordered commutative monoid $\Delta$, we obtain

$$\big|(x,y)\big|_{X\otimes Y} = |x|_X \cdot |y|_Y. \tag{1.6.4.1}$$

In this situation, we denote $(x,y)$, viewed as an element of $X \otimes Y$, by $x \otimes y$, so that the following nicer-looking formula holds:

$$\big|x \otimes y\big|_{X\otimes Y} = |x|_X \cdot |y|_Y. \tag{1.6.4.2}$$

1.6.5. *The special case, where $\Delta = \mathbb{R}_{\geq 0}$.* In [Pa], several tensor products were considered on $\mathbb{R}_{\geq 0}$-normed sets. All of these tensor products are special cases of the above construction for the following choices of $\mu$:

- $\mu(x,y) = \max(x,y)$ yields the Cartesian monoidal structure, with $X \otimes_\mu Y = X \times Y$. In fact, taking binary sup $\vee$ for $\mu$ yields a Cartesian monoidal structure for an arbitrary poset $\Delta$ with binary suprema.
- The functions $\mu_p(x,y) = (x^p + y^p)^{1/p}$, for $0 < p \leq \infty$, with $\mu_\infty = \max$, define a family of monoidal structures $\otimes_p$ on $\mathcal{N}_{\mathbb{R}_{\geq 0}}$.
- The function $\mu_m(x,y) = xy$ defines the "multiplicative" monoidal structure in the terminology of [Pa], which corresponds to the "canonical" monoidal structure on $\mathcal{N}_{\mathbb{R}_{\geq 0}}$ in our terminology.

1.6.6. *Extension to $\Delta_\infty$.* Given a map $\mu\colon \Delta \times \Delta \to \Delta$ as above, we can extend it to $\mu_\infty\colon \Delta_\infty \times \Delta_\infty \to \Delta_\infty$ by putting

$$\mu_\infty(x,y) = \begin{cases} \mu(x,y) & \text{if } x, y \neq \infty; \\ \infty & \text{if } x = \infty \text{ or } y = \infty. \end{cases} \tag{1.6.6.1}$$

In this way we obtain a bifunctor $\otimes_{\mu_\infty}$ on $\mathcal{N}_{\Delta_\infty}$, having the property that $e_!(X \otimes_\mu Y) \cong e_!X \otimes_{\mu_\infty} e_!Y$ and $e^*(\bar{X} \otimes_{\mu_\infty} \bar{Y}) \cong e^*\bar{X} \otimes_\mu e^*\bar{Y}$.

If $\mu$ is commutative, associative, or with unity, $\mu_\infty$ has the same property. In particular, when $\Delta$ is a commutative ordered monoid, the same is true about $\Delta_\infty$, and the canonical monoidal structure of $\mathcal{N}_\Delta$ extends to $\mathcal{N}_{\Delta_\infty}$.

**1.7. Closedness of the monoidal structure on $\mathcal{N}_\Delta$.** We would like to give some conditions sufficient for $\mathcal{N}_\Delta$, with $\Delta$ an ordered commutative monoid, to be a *closed* monoidal category. Since $\mathcal{N}_\Delta$ contains the $\otimes$-subcategory $\mathbf{1}_\Delta$, equivalent to $\Delta^{\mathrm{op}}$ as a monoidal category, it is reasonable to expect $\Delta^{\mathrm{op}}$ to be closed as well.

1.7.1. *Notation for the inner Hom.* Given two objects $Y$ and $Z$ of a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes)$, we denote by $\mathbf{Hom}(Y,Z)$, or by $Y \multimap_{\mathcal{C}} Z$ or $Y \multimap Z$ the object of $\mathcal{C}$ representing the functor $X \rightsquigarrow \mathrm{Hom}(X \otimes Y, Z)$. In other words, we expect to have a canonical isomorphism

$$\mathrm{Hom}(X, Y \multimap Z) \cong \mathrm{Hom}(X \otimes Y, Z) \qquad \text{for all } X \in \mathcal{C}. \tag{1.7.1.1}$$

1.7.2. *Closedness for $\mathcal{N}_\Delta$ and $\mathcal{N}_{\Delta_\infty}$.* We would like to study the relationship between $\multimap$ for $\Delta$-normed sets and $\Delta_\infty$-normed sets, where $\Delta_\infty = \Delta \cup \{\infty\}$ is obtained from $\Delta$ by adjoining a new largest element as described in **1.4** and **1.6.6**.

Namely, if $Y \in \mathcal{N}_\Delta$ and $\bar{Z} \in \mathcal{N}_{\Delta_\infty}$, then

$$e^*(e_!Y \multimap_{\Delta_\infty} \bar{Z}) \cong (Y \multimap_\Delta e^*\bar{Z}). \tag{1.7.2.1}$$

In particular, if $e_!Y \multimap \bar{Z}$ is representable in $\mathcal{N}_{\Delta_\infty}$, then $Y \multimap e^*\bar{Z}$ is representable in $\mathcal{N}_\Delta$.

Putting here $\bar{Z} = e_!Z$ for some $Z \in \mathcal{N}_\Delta$ and using $e^*e_!Z \cong Z$, we obtain

$$(Y \multimap_\Delta Z) \cong e^*(e_!Y \multimap_{\Delta_\infty} e_!Z) \tag{1.7.2.2}$$

1.7.3. In particular, *if the monoidal category $\mathcal{N}_{\Delta_\infty}$ is closed, the same is true for $\mathcal{N}_\Delta$.*

1.7.4. *Proof of* (1.7.2.1). We are going to show that the two sides of (1.7.2.1) represent canonically isomorphic covariant functors on $\mathcal{N}_\Delta$. Indeed, for any $X \in \mathcal{N}_\Delta$ we have

$$\mathrm{Hom}_\Delta\big(X, e^*(e_!Y \multimap_{\Delta_\infty} \bar{Z})\big) \cong \mathrm{Hom}_{\Delta_\infty}(e_!X, e_!Y \multimap_{\Delta_\infty} \bar{Z}) \tag{1.7.4.1}$$

$$\cong \mathrm{Hom}_{\Delta_\infty}(e_!X \otimes e_!Y, \bar{Z}) \tag{1.7.4.2}$$

$$\cong \mathrm{Hom}_{\Delta_\infty}\big(e_!(X \otimes Y), \bar{Z}\big) \tag{1.7.4.3}$$

$$\cong \mathrm{Hom}_\Delta(X \otimes Y, e^*\bar{Z}) \tag{1.7.4.4}$$

$$\cong \mathrm{Hom}_\Delta(X, Y \multimap_\Delta e^*\bar{Z}). \tag{1.7.4.5}$$

**Proposition 1.7.5.** (*Properties of $\mathbf{1}^\beta \multimap \mathbf{1}^\gamma$.*) *Let $\beta$, $\gamma \in \Delta$, and suppose that $\Delta$ is an ordered monoid with largest element $\omega$, which is a zero for the multiplication of $\Delta$. Then $\mathbf{1}^\beta \multimap \mathbf{1}^\gamma$, if representable in $\mathcal{N}^\Delta$ at all, consists of exactly one element, i.e., is isomorphic to some $\mathbf{1}^\alpha$.*

*Proof.* Suppose $\mathbf{1}^\beta \to \mathbf{1}^\gamma$ is representable by some $X = (X, |\cdot|_X)$. Since $\omega$ is the largest element of $\Delta$, we have $X = X_{\leq\omega} = \{x \in X : |x|_X \leq \omega\} = \mathrm{Hom}(\mathbf{1}^\omega, X) \cong \mathrm{Hom}(\mathbf{1}^\omega \otimes \mathbf{1}^\beta, \mathbf{1}^\gamma) = \mathrm{Hom}(\mathbf{1}^{\omega\beta}, \mathbf{1}^\gamma) = \mathrm{Hom}(\mathbf{1}^\omega, \mathbf{1}^\gamma) = \{*\}$ since $\gamma \leq \omega$. This shows that the underlying set of $X$ is a singleton, hence $X \cong \mathbf{1}^\alpha$ for some uniquely determined $\alpha \in \Delta$. $\square$

**Corollary 1.7.6.** *Under the previous assumptions on $\Delta$, all $\mathbf{1}^\beta \multimap \mathbf{1}^\gamma$ are representable in $\mathcal{N}_\Delta$ if and only if monoidal category $\Delta^{\mathrm{op}} \cong \mathbf{1}_\Delta \subset \mathcal{N}_\Delta$ is closed.*

1.7.7. *Closedness of $\Delta^{\mathrm{op}}$.* Notice that, for any commutative ordered monoid $\Delta$, the closedness of $\Delta^{\mathrm{op}}$ is equivalent to the following condition.

- For any $\beta$, $\gamma \in \Delta$, there is an element $\lceil\gamma/\beta\rceil \in \Delta$ such that

$$\alpha \geq \lceil\gamma/\beta\rceil \Leftrightarrow \alpha\beta \geq \gamma \tag{1.7.7.1}$$

Indeed, $\lceil\gamma/\beta\rceil$ is merely another notation for $\beta \multimap_{\Delta^{\mathrm{op}}} \gamma$.

1.7.8. *Existence of $\multimap$ in $\Delta$ and $\Delta_\infty$.* Notice that, if $\lceil\gamma/\beta\rceil$ exists in $\Delta$ for some $\beta$ and $\gamma \in \Delta$, the same is true in $\Delta_\infty$.

1.7.9. *Case of invertible $\beta$.* If $\beta$ is invertible, $\lceil\gamma/\beta\rceil$ always exists and equals $\gamma\beta^{-1}$. In particular, $\lceil\gamma/\beta\rceil$ exists in $\mathbb{R}_{\geq 0}$, hence also in $\bar{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, if $0 \leq \gamma < \infty$, $0 < \beta < \infty$, because all such $\beta$ are invertible in $\mathbb{R}_{\geq 0}$.

1.7.10. *Example:* $\Delta = \bar{\mathbb{R}}_{\geq 0}$. Previous remark enables us to show that $\Delta^{\mathrm{op}}$ is a closed monoidal category for $\Delta = \bar{\mathbb{R}}_{\geq 0}$. Indeed, the existence of $\lceil\gamma/\beta\rceil$ for $\beta \neq 0, \infty$ has already been shown. We must only complement this with the following observations:

$$\lceil 0/0 \rceil = 0; \tag{1.7.10.1}$$

$$\lceil \gamma/0 \rceil = \infty \quad \text{if } \gamma \neq 0; \tag{1.7.10.2}$$

$$\lceil \gamma/\infty \rceil = 0. \tag{1.7.10.3}$$

Now we would like to prove a partial converse to **1.7.6**.

**Proposition 1.7.11.** *Suppose that a commutative ordered monoid $\Delta$ is a complete lattice, and the monoidal category $\Delta^{\mathrm{op}}$ is closed. Then the monoidal category $\mathcal{N}_\Delta$ is also closed. The inner Hom $X \multimap Y$ of two $\Delta$-normed sets $X = (X, |\cdot|_X)$ and $Y = (Y, |\cdot|_Y)$ can be computed as follows: the underlying set of $X \multimap Y$ is $\mathrm{Hom}_{\mathit{Sets}}(X, Y)$, and the norm of a map $\phi\colon X \to Y$ is given by the classical formula for the norm of a linear operator*:

$$|\phi|_{X \multimap Y} = \inf\{\alpha \in \Delta \,:\, \forall x \in X,\ |\phi(x)|_Y \leq \alpha \cdot |x|_X\} \tag{1.7.11.1}$$

$$= \sup_{x \in X} \lceil |\phi(x)|_Y / |x|_X \rceil. \tag{1.7.11.2}$$

*Proof.* First of all, observe that both expressions on the right-hand side of (1.7.11.1) and (1.7.11.2) are well defined and equal to each other. Indeed, for any $\alpha \in \Delta$ we have

$$\alpha \geq \sup_{x \in X} \lceil |\phi(x)|_Y / |x|_X \rceil$$

if and only if for all $x \in X$, $\alpha \geq \lceil |\phi(x)|_Y / |x|_X \rceil$ if and only if for all $x \in X$, $\alpha \cdot |x|_X \geq |\phi(x)|_Y$ if and only if $\alpha$ belongs to the set on the right-hand side of (1.7.11.1). In other words, the infimum in (1.7.11.1) is taken along the set of all upper bounds for the family of (1.7.11.2), so the two expressions have to be equal by (1.3.1.1). □

Now we are going to prove the statement for some special cases.

1.7.12. $X = \mathbf{1}^\beta$, $Y = \mathbf{1}^\gamma$. In this case $X \multimap Y = \mathbf{1}^\beta \multimap \mathbf{1}^\gamma = \mathbf{1}^{\lceil\gamma/\beta\rceil}$ by **1.7.5**. This is also what is prescribed by (1.7.11.2).

1.7.13. $X = \mathbf{1}^\beta$, $Y$ *arbitrary.* Notice that for $X = \mathbf{1}^\beta$ and any coproduct (i.e., disjoint union) decomposition $Y = \coprod_\iota Y_\iota$, the canonical homomorphism

$$\theta\colon \coprod_\iota (X \multimap Y_\iota) \to \big(X \multimap \coprod_\iota Y_\iota\big) \tag{1.7.13.1}$$

is an isomorphism. Indeed, since $\mathbf{1}_\Delta$ generates $\mathcal{N}_\Delta$, it suffices to check that $\theta$ becomes an isomorphism after applying any functor $\mathrm{Hom}(\mathbf{1}^\alpha, -)$, i.e., that

$$\mathrm{Hom}(\mathbf{1}^\alpha, \theta) \,:\, \mathrm{Hom}\big(\mathbf{1}^\alpha, \coprod_\iota (\mathbf{1}^\beta \multimap Y_\iota)\big) \to \mathrm{Hom}(\mathbf{1}^{\alpha\beta}, \coprod_\iota Y_\iota) \tag{1.7.13.2}$$

is an isomorphism, which is immediate, both sides being canonically isomorphic to $\coprod_\iota \mathrm{Hom}(\mathbf{1}^{\alpha\beta}, Y_\iota)$ since $\mathrm{Hom}(\mathbf{1}^\alpha, -)$ preserves arbitrary coproducts.

Furthermore, this argument shows that both sides of (1.7.13.1) corepresent the same functor on $\mathcal{N}_\Delta$, so we can conclude that the right-hand side of (1.7.13.1) exists whenever the left-hand side exists, and then they are isomorphic.

Representing an arbitrary $\Delta$-normed set $Y$ as $\coprod_{y \in Y} \mathbf{1}^{|y|}$, we see that

$$(\mathbf{1}^\beta \multimap Y) \cong \coprod_{y \in Y} \mathbf{1}^{\lceil |y|/\beta \rceil}, \tag{1.7.13.3}$$

which is what is required by (1.7.11.2) for this case.

1.7.14. *$X$ and $Y$ arbitrary.* Notice that $- \multimap Y$ transforms arbitrary coproducts into products; writing $X = \coprod_{x\in X} \mathbf{1}^{|x|}$, we see that

$$(X \multimap Y) \cong \prod_{x\in X} (\mathbf{1}^{|x|} \multimap Y). \tag{1.7.14.1}$$

Recalling the construction of products in $\mathcal{N}_\Delta$ with the aid of sup-norms discussed in **1.2.4**, we obtain (1.7.11.2) for arbitrary $X$ and $Y$.

1.7.15. *A priori properties of $X \multimap Y$.* The above proof may seem more complicated than it is necessary. In fact, it is easy to see that $X \multimap Y$, if exists at all, must satisfy

$$\begin{aligned}(X \multimap Y)_{\leq\alpha} &\cong \mathrm{Hom}(\mathbf{1}^\alpha, X \multimap Y) \cong \mathrm{Hom}(\mathbf{1}^\alpha \otimes X, Y)\\ &\cong \{\phi\colon X \to Y \,:\, |\phi(x)|_Y \leq \alpha\cdot|x|_X \quad \text{for all } x\in X\}.\end{aligned} \tag{1.7.15.1}$$

In other words, the set of elements of $X \multimap Y$ with norm $\leq \alpha$ exactly coincides with the set of "Lipschitz maps $X \to Y$ with Lipschitz constant $\alpha$". This implies that $X \multimap Y$ must be equal to the set of all Lipschitz maps $X \to Y$, and that the norm of a Lipschitz map $\phi\colon X \to Y$ must be equal to the infimum of all Lipschitz constants acceptable for $\phi$.

The complicated point is to show that this infimum is a Lipschitz constant for $\phi$ as well, and that the set of all Lipschitz maps $X \to Y$ with the norm given by this infimum satisfies the universal property required from $X \multimap Y$. This can be shown directly provided $\inf \beta E = \beta\cdot\inf E$ for any $\beta\in\Delta$ and any $E \subset \Delta$. This property, in turn, follows from closedness of the monoidal structure on $\Delta^{\mathrm{op}}$.

**Corollary 1.7.16.** *Suppose that a commutative ordered monoid $\Delta$ admits the infima of nonempty subsets, and that $\lceil\gamma/\beta\rceil$ exists in $\Delta$ for all $\beta$, $\gamma\in\Delta$ such that $\alpha\beta\geq\gamma$ for at least one $\alpha\in\Delta$. Then the monoidal structure of $\mathcal{N}_\Delta$ is closed, with $X \multimap Y$ given by the set of all Lipschitz maps $X \to Y$ (or, equivalently, by the maps $X \to Y$ with finite $\Delta_\infty$-norm).*

*Proof.* Using (1.7.2.2) and **1.7.11**, we see that it would suffice to show that $\Delta_\infty$ is a complete lattice, which is true because $\Delta$ admits arbitrary nonempty infima, and that $\Delta_\infty^{\mathrm{op}}$ is closed as a monoidal category, i.e., $\lceil\gamma/\beta\rceil$ exists for all $\beta$, $\gamma\in\Delta_\infty$. This is also clear:

$$\lceil\gamma/\beta\rceil_{\Delta_\infty} = \begin{cases}\lceil\gamma/\beta\rceil_\Delta & \text{if } \beta,\, \gamma<\infty \text{ and } \alpha\beta\geq\gamma \text{ for some } \alpha\in\Delta;\\ \infty & \text{if } \beta,\, \gamma<\infty, \text{ but } \alpha\beta\ngeq\gamma \text{ for all } \alpha\in\Delta;\\ \infty & \text{if } \beta<\infty,\ \gamma=\infty;\\ \inf\Delta & \text{if } \beta=\infty.\end{cases} \tag{1.7.16.1}$$

□

**Corollary 1.7.17.** *Suppose that a partially ordered set $\Delta$ is such that its opposite $\Delta^{\mathrm{op}}$ is a complete Heyting algebra. Then $\mathcal{N}_\Delta$ is Cartesian closed.*

*Proof.* Since $\Delta$ is a complete lattice, we know that $\mathcal{N}_\Delta$ is a bicomplete category. In particular, it is Cartesian. We introduce a product on $\Delta$ by taking binary sup $\vee$, or, equivalently, consider binary inf $\wedge$ as a product on $\Delta^{\mathrm{op}}$. On one hand, we know that the bifunctor $\otimes_\vee$ is isomorphic to the binary direct product bifunctor $\times$, i.e., the chosen monoid structure on $\Delta$ determines Cartesian monoidal structure on $\mathcal{N}_\Delta$. On the other hand, according to **1.7.11**, for this monoidal structure to be closed, i.e., for $\mathcal{N}_\Delta$ to be Cartesian closed, it suffices that $\Delta^{\mathrm{op}}$ be closed for $\wedge$, i.e., that $\Delta^{\mathrm{op}}$ be a Heyting algebra. □

**1.8. Fuzzy sets.** We have already remarked on several occasions that working with $\Delta^{\mathrm{op}}$ instead of $\Delta$ has its advantages. This is due to the fact that $\mathcal{N}_\Delta$ contains a full (monoidal, if $\Delta$ is an ordered monoid) subcategory equivalent to $\Delta^{\mathrm{op}}$, not to $\Delta$. One might want to rewrite the theory of $\Delta$-normed sets using "inverse norms", i.e., $\Delta^{\mathrm{op}}$-valued norms. A map is then required to be "inverse norm-extending" instead of "norm-shrinking". Such "inverse norms" turn out to have properties similar to those of *membership functions* of fuzzy sets, and some fuzzy set and fuzzy logic constructions happen to be special cases of those considered above for normed sets, when recast in terms of "inverse norms" or "membership functions".

1.8.1. *Notation for elements of the opposite poset* $\Delta^{\mathrm{op}}$. We need sometimes a notation for elements of $\Delta^{\mathrm{op}}$ suitable for distinguishing them from "the same" elements of $\Delta$. We suggest denoting by $\alpha \mapsto \alpha^\circ$ the natural bijection $\Delta \xrightarrow{\sim} \Delta^{\mathrm{op}}$, as well as its inverse. Set-theoretically, this is simply $\mathrm{id}_\Delta$, since the partially ordered sets $\Delta$ and $\Delta^{\mathrm{op}}$ have same underlying sets. In particular, $(\alpha^\circ)^\circ = \alpha$ and $\alpha^\circ \leq \beta^\circ$ if and only if $\alpha \geq \beta$.

These properties suggest another notation $\alpha \to \alpha^{-1}$ for this same bijection. However, it may be misleading when $\Delta$ is a multiplicatively-written commutative monoid or even group, so we shall not use it.

1.8.2. *Inverse norms as membership functions. Fuzzy sets.* A $\Delta$-valued *inverse norm* or *membership function* on a set $X$, where $\Delta$ is any poset, is by definition the same thing as a $\Delta^{\mathrm{op}}$-valued norm, i.e., a map $|\cdot|_X^\circ$ or $m_X\colon X \to \Delta$. A $\Delta$*-fuzzy set* is a pair $(X, m_X\colon X \to \Delta)$ consisting of a set $X$ and a membership function on $X$. A *morphism of* $\Delta$*-fuzzy sets* $f\colon (X, m_X) \to (Y, m_Y)$ is a map of sets $f\colon X \to Y$ such that $m_Y(f(x)) \geq m_X(x)$ for any $x \in X$. The category of all $\Delta$-fuzzy sets and their morphisms will be denoted by $\mathcal{F}_\Delta$. By definition, $\mathcal{F}_\Delta$ is isomorphic to $\mathcal{N}_{\Delta^{\mathrm{op}}}$.

1.8.3. *Interpretation of membership function for* $\Delta = [0,1]$. The most classical version of fuzzy sets involves membership functions with values in $\Delta = [0,1]$. In this case, a fuzzy set $\mathcal{X} = (X, m_X\colon X \to [0,1])$ is to be thought of as a "fuzzy subset" of a "crisp" (i.e., usual) set $X$ such that an element $x \in X$ belongs to $\mathcal{X}$ with "certainty", "probability" or "grade" $m_X(x)$. When $m_X$ takes only values 0 and 1, i.e., is a characteristic function $\chi_A$ of a "crisp" subset $A \subset X$, we may say that $\mathcal{X}$ "is" the "crisp" subset $A$ of $X$.

This shows that membership functions with values in $\Delta_0 := \{0,1\}$ define an important subclass of fuzzy sets, namely, those fuzzy sets that are actually crisp.

1.8.4. *Restricting* $\Delta$ *to* $(0,1]$. It sometimes makes sense to forbid the membership function to take the value 0, thus setting $\Delta := (0,1]$ and $\Delta_0 = \{1\}$. In this case the "underlying set" or "support" $X$ of a fuzzy set $\mathcal{X} = (X, m_X)$ will not have "superfluous elements", i.e., the elements that belong to $\mathcal{X}$ with certainty 0.

1.8.5. $\Delta^{\mathrm{op}}$ *isomorphic to* $\mathbb{R}_{\geq 0}$. Notice that $(0,1]^{\mathrm{op}}$ and $[0,1]^{\mathrm{op}}$ are isomorphic to $\mathbb{R}_{\geq 0}$ and $\bar{\mathbb{R}}_{\geq 0}$, respectively, via the map $p \mapsto 1/p - 1$. Another way of establishing such an isomorphism is given by $p \mapsto -\log p$. This means that the study of $(0,1]$ or $[0,1]$-fuzzy sets is essentially equivalent to the study of $\mathbb{R}_{\geq 0}$ or $\bar{\mathbb{R}}_{\geq 0}$-normed sets.

There are other ways to transform norms into membership functions and conversely. For example, one can convert a normed set $(X, |\cdot|_X)$ into the fuzzy set $(X, e^{-|\cdot|_X})$ using the "Laplace distribution" associated with the original norm. As long as we are not interested in additional operations on $(0,1]$ or $\mathbb{R}_{\geq 0}$, such as multiplication, any isomorphism $(0,1]^{\mathrm{op}} \cong \mathbb{R}_{\geq 0}$ would do.

1.8.6. *Operations with fuzzy sets.* An important part of the theory of fuzzy sets is given by the so-called *t-norm fuzzy logics* and associated operations with fuzzy sets. Essentially, one has to define on $[0,1]$ an associative commutative operation $\mu$ (called a *t-norm*), making $[0,1]$ a commutative monoid, and then a corresponding operation $\otimes_\mu$ is defined on fuzzy sets, essentially in the same way as in **1.6.1**. Some choices of the t-norm $\mu$ have been considered historically:

- The Łukasiewich t-norm $\mu(x,y) := \max(x+y-1,0)$.
- The minimum t-norm $\mu(x,y) := \min(x,y)$, defining the direct product of fuzzy sets.
- The product t-norm $\mu(x,y) := xy$.

## §2. Graded sets and presheaves

**2.0. Embedding normed sets into presheaves.** We already saw in **1.5.3** that there is a natural way to embed the category $\mathcal{N}_\Delta$ of $\Delta$-normed sets into the category $\mathit{Sets}^\Delta$ of functors $\Delta \to \mathit{Sets}$, or, equivalently, into the category $\widehat{\Delta^{\mathrm{op}}}$ of presheaves on $\Delta^{\mathrm{op}}$. This embedding $I\colon \mathcal{N}_\Delta \to \mathit{Sets}^\Delta$ is given by (1.5.3.1):

$$I(X) = I_\Delta(X)\colon \alpha \rightsquigarrow X_{\leq\alpha} := \{x \in X \,:\, |x|_X \leq \alpha\} \\ \cong \mathrm{Hom}_{\mathcal{N}_\Delta}(\mathbf{1}^\alpha, X). \tag{2.0.0.1}$$

We would like to study this fully faithful functor in more detail, and extend most constructions of the previous section from normed sets to presheaves. In particular, we are going to show that the essential image of the embedding $I$ consists exactly of *projective objects* of $\mathit{Sets}^\Delta$, so that $\mathcal{N}_\Delta \cong \operatorname{Proj} \mathit{Sets}^\Delta$, and extend the closed monoidal structure from $\mathcal{N}_\Delta$ to $\mathit{Sets}^\Delta$.

2.0.1. *Replacing $\Delta$ by a small category $\mathcal{I}$.* This leads to a further generalization: while considering $\mathit{Sets}^\Delta$, it is natural to replace a poset or an ordered monoid $\Delta$ by small (symmetric monoidal) category $\mathcal{I}$. We say that the objects of $\mathit{Sets}^{\mathcal{I}}$ are *$\mathcal{I}$-graded sets.* It might be more convenient to replace $\mathcal{I}$ with the opposite category $\mathcal{I}^{\mathrm{op}}$, studying the category of presheaves $\widehat{\mathcal{I}}$ on a small (symmetric monoidal) category $\mathcal{I}$. This leads to better-looking statements, cf. **1.8**. For example, if $\mathcal{I}$ is a small closed symmetric monoidal category, the same is true for $\widehat{\mathcal{I}}$.

**2.1. Normed sets as projective graded sets.** Our next goal is to show that the essential image of the fully faithful functor $I\colon \mathcal{N}_\Delta \to \mathit{Sets}^\Delta$ consists exactly of projective objects of $\mathit{Sets}^\Delta$. This suggests that $\operatorname{Proj}(\mathit{Sets}^{\mathcal{I}})$ is a nice candidate for the role of the "category of $\mathcal{I}$-normed sets" for any small category $\mathcal{I}$.

**Theorem 2.1.1.** (*Embedding normed sets into graded.*) *Let $\Delta$ be a partially ordered set. Denote by $I = I_\Delta\colon \mathcal{N}_\Delta \to \mathit{Sets}^\Delta$ the functor given by* (2.0.0.1)*. Then $I$ is fully faithful and limit-preserving, and its essential image consists of* **projective objects** *of $\mathit{Sets}^\Delta$, i.e., the objects $P$ such that* $\mathrm{Hom}(P,-)$ *transforms* (*strict*) *epimorphisms in $\mathit{Sets}^\Delta$ into surjective maps of sets. Therefore, $I$ establishes an equivalence between the category of $\Delta$-normed sets $\mathcal{N}_\Delta$ and the full subcategory* $\operatorname{Proj} \mathit{Sets}^\Delta$ *of projective objects of $\mathit{Sets}^\Delta$.*

The proof of this theorem will be given as a series of observations.

2.1.2. *$\mathit{Sets}^\Delta$ is a topos; all epimorphisms are strict and effective.* Notice that $\mathit{Sets}^\Delta$ is a topos, actually the topos of presheaves of sets on $\Delta^{\mathrm{op}}$. This implies that all epimorphisms in $\mathit{Sets}^\Delta$ are strict and effective, so we do not have to think about such distinctions while discussing projective objects of $\mathit{Sets}^\Delta$, and that $\xi\colon X \to Y$ is an epimorphism in $\mathit{Sets}^\Delta$ if and only if $\xi_\alpha\colon X(\alpha) \to Y(\alpha)$ is surjective for all $\alpha \in \Delta$.

2.1.3. *Underlying set of $X$ can be recovered as* $\operatorname{colim} I_\Delta(X)$. Let

$$X = (X, |\cdot|_X : X \to \Delta)$$

be a $\Delta$-normed set. Consider the corresponding functor

$$IX : \alpha \rightsquigarrow X_{\leq\alpha} = \{x \in X \,:\, |x|_X \leq \alpha\}$$

and its colimit $\operatorname{colim} IX$. By definition, this colimit can be computed as the quotient of the disjoint union of all $(IX)(\alpha) = X_{\leq\alpha}$ modulo the smallest equivalence relation identifying $x$ as an element of $X_{\leq\alpha}$ with $x$ as an element of $X_{\leq\beta}$ whenever $x \in X$ and both $\alpha$ and $\beta$ are greater than or equal to $|x|$. This description immediately shows that $\operatorname{colim} IX$ is canonically isomorphic to the underlying set of the $\Delta$-normed set $X$.

2.1.4. *Underlying set of a $\Delta$-graded set.* We may want to extend our terminology from $\Delta$-normed to $\Delta$-graded sets by saying that $|\mathcal{X}| := \operatorname{colim} \mathcal{X}$ is the *underlying set* of a $\Delta$-graded set $\mathcal{X}$, even if it does not come from a $\Delta$-normed set.

2.1.5. *$I_\Delta$ is faithful.* Since $X = \operatorname{colim} IX$, we see that any morphism, i.e., norm-shrinking map $f : X \to Y$ can be recovered from $If$ simply by applying the colimit functor, which gives $f$ on underlying sets. This implies the faithfulness of $I$.

2.1.6. *$I_\Delta$ is fully faithful.* Suppose we are given a natural transformation $\xi : IX \to IY$, i.e., a compatible collection of maps $\xi_\alpha : X_{\leq\alpha} \to Y_{\leq\alpha}$. Put $f := \operatorname{colim} \xi : X \to Y$. By the definition of colimit, $\xi_\alpha$ coincides with the restriction of the map of sets $f$ to $X_{\leq\alpha}$; in other words, $f : X \to Y$ is a map of sets mapping $X_{\leq\alpha}$ to $Y_{\leq\alpha}$, for all $\alpha \in \Delta$, i.e., a norm-shrinking map, or a morphism $X \to Y$ in $\mathcal{N}_\Delta$. Obviously, $\xi = I(f)$, hence $I = I_\Delta$ is fully faithful.

2.1.7. *$I_\Delta$ preserves arbitrary limits.* Notice that $I_\Delta$ preserves all limits that exist in $\mathcal{N}_\Delta$. This follows from the fact that the limits in $\mathit{Sets}^\Delta$ are computed componentwise, and the fact that each component $X \rightsquigarrow (IX)(\alpha)$ is given by $\operatorname{Hom}_{\mathcal{N}_\Delta}(\mathbf{1}^\alpha, -)$, which commutes with any limits by definition.

2.1.8. *$I_\Delta$ preserves arbitrary coproducts.* This follows again from the fact that the colimits in $\mathit{Sets}^\Delta$ are computed componentwise, and that each component functor

$$\operatorname{Hom}_{\mathcal{N}_\Delta}(\mathbf{1}^\alpha, -) : X \rightsquigarrow X_{\leq\alpha}$$

preserves arbitrary coproducts by the explicit construction of **1.3.2**.

2.1.9. *$I_\Delta$ transforms any normed singleton $\mathbf{1}^\alpha$ into the corepresentable functor $h^\alpha$.* Notice that $(I\mathbf{1}^\alpha)(\beta) = \operatorname{Hom}_{\mathcal{N}_\Delta}(\mathbf{1}^\beta, \mathbf{1}^\alpha) \cong \operatorname{Hom}_\Delta(\alpha, \beta) = h^\alpha(\beta)$, where $h^\alpha$ denotes the covariant functor $\Delta \to \mathit{Sets}$ corepresentable by $\alpha$. This means that $I_\Delta \mathbf{1}^\alpha \cong h^\alpha$, i.e., $I_\Delta$ *transforms the standard normed singletons into the corresponding corepresentable functors.* Since $I_\Delta$ is fully faithful, we might want to identify $\mathbf{1}^\alpha$ with $h^\alpha$ and denote the corepresentable functor $h^\alpha$ by $\mathbf{1}^\alpha$. This is consistent with the notation $[S]^\alpha$ introduced below in **2.4.4**.

2.1.10. *Any $I_\Delta X$ is projective.* Since $I_\Delta$ preserves arbitrary coproducts by **2.1.8** and transforms $\mathbf{1}^\alpha$ into $h^\alpha$, we obtain by (1.5.2.1)

$$I_\Delta X = \coprod_{x \in X} h^{|x|_X}. \tag{2.1.10.1}$$

Since all $h^\alpha$ are projective in the presheaf category $\mathit{Sets}^\Delta$, the functor

$$\operatorname{Hom}(h^\alpha, -) : Y \rightsquigarrow Y(\alpha)$$

transforming presheaf epimorphisms into surjective maps of sets, and any coproduct of projective objects is again projective, we see that $I_\Delta X$ *is a projective object of* $Sets^\Delta$, *for any normed set* $X = (X, |\cdot|_X)$.

2.1.11. *Any projective object* $P$ *of* $Sets^\Delta$ *is a retract of some* $I_\Delta X$. Suppose that $P$ is a projective $\Delta$-graded set. Since the $h^\alpha$ generate $Sets^\Delta$, we can find an epimorphism $p : F := \coprod_{x\in X} h^{\phi(x)} \twoheadrightarrow P$ from a "free" object $F$ into $P$. One can take for example $X := \coprod_{\alpha\in\Delta} P(\alpha)$ with $\phi\colon X \to \Delta$ equal to $\alpha$ on $P(\alpha)$. Clearly, $F \cong I_\Delta(X)$, where $X = (X, \phi\colon X \to \Delta)$ is a $\Delta$-normed set.

Now $p$ is an effective epimorphism and $P$ is projective, so $p_*\colon \mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$ has to be surjective, hence $p$ admits a section $\sigma\colon P \to F$, so $p$ is a split epimorphism, $e := \sigma \circ p$ is an idempotent on $F \cong I_\Delta(X)$, and $P$ is a retract of $F$ with respect to the idempotent $e$: $P \cong \mathrm{Eq}(e, \mathrm{id}_F) \cong \mathrm{Coeq}(e, \mathrm{id}_F)$.

2.1.12. $\mathcal{N}_\Delta$ *is idempotent-complete.* Let $X = (X, |\cdot|_X)$ be a $\Delta$-normed set, and let $e = e^2\colon X \to X$ be an idempotent on $X$. We claim that the *corresponding retract* $Y := \mathrm{Eq}(e, \mathrm{id}_X)$ *exists in* $\mathcal{N}_\Delta$, hence $\mathcal{N}_\Delta$ is idempotent-complete. Indeed, by **1.2.6** $Y = \mathrm{Eq}(e, \mathrm{id}_X)$ can be computed as the subset $Y = \{x \in X : e(x) = x\}$ with norm given by restriction of $|\cdot|_X$.

2.1.13. *Any projective* $\Delta$*-graded set* $P$ *is free, i.e., isomorphic to some* $I_\Delta Y$. We have seen that any projective $P$ is a retract of some $I_\Delta X$ with respect to an idempotent $e_P \in \mathrm{End}(I_\Delta X)$. We have already shown that $I_\Delta$ is fully faithful, so $e_P$ comes from an idempotent $e \in \mathrm{End}_{\mathcal{N}_\Delta}(X)$. Now $\mathcal{N}_\Delta$ is idempotent complete, so $e$ defines a retract $X \leftrightarrows Y$. Any functor preserves retracts of idempotents whenever they exist; in particular, $I_\Delta Y$ is the retract of $I_\Delta X$ corresponding to $I_\Delta(e) = e_P$, hence is isomorphic to $P$.

This completes the proof of **2.1.1**.

**2.2. Free and projective objects of $\boldsymbol{Sets^{\mathcal{I}}}$.** We would like to extend some of the previous considerations to the more general case of $Sets^{\mathcal{I}}$, where $\mathcal{I}$ is any small category. In particular, we would like to study free and projective objects of this category, and discuss whether they admit a description as "$\mathcal{I}$-normed sets".

2.2.1. *Set of connected components of* $\mathcal{I}$*-graded set.* Let $\mathcal{X}\colon \mathcal{I} \to Sets$ be an $\mathcal{I}$-graded set (cf. **2.0.1**). Denote by $\underline{S}$ the constant functor $\mathcal{I} \to Sets$ with value $S$, for any set $S$. By the definition of colimit,

$$\mathrm{Hom}_{Sets^{\mathcal{I}}}(\mathcal{X}, \underline{S}) \cong \mathrm{Hom}_{Sets}(\operatorname{colim}\mathcal{X}, S). \tag{2.2.1.1}$$

On the other hand, recall that for any object $X$ of a (Grothendieck) topos $\mathscr{E}$, one can define the *proset* $\pi_0 X\colon \mathrm{Pro}(Sets)$ *of connected components of* $X$ by

$$\mathrm{Hom}_{\mathscr{E}}(X, \underline{S}_{\mathscr{E}}) \cong \mathrm{Hom}_{\mathrm{Pro}(Sets)}(\pi_0 X, S). \tag{2.2.1.2}$$

Comparing these two statements, we see that *the colimit* $\operatorname{colim}\mathcal{X}$ *is the set of connected components* $\pi_0\mathcal{X}$ *of* $\mathcal{X}$. This is true in any presheaf topos.

2.2.2. *Terminology: the underlying set of an* $\mathcal{I}$*-graded set.* We sometimes say that the colimit $\operatorname{colim}\mathcal{X}$ is the *underlying set* of an $\mathcal{I}$-graded set $\mathcal{X}$. This terminology is compatible with that of **2.1.4**.

2.2.3. *Decomposition into a disjoint union of connected components.* Since $\pi_0\mathcal{X}$ is a set, we obtain a canonical decomposition of $\pi_0\mathcal{X}$ into a disjoint union of connected objects, indexed by $\pi_0\mathcal{X}$. It can be obtained as follows. For any set $S$ and any element $s \in S$, denote by $i_s\colon \mathbf{1} \to S$ the only map from a singleton into $S$ with image $s$. By adjunction, we have a morphism $\mathcal{X} \to \underline{\pi_0\mathcal{X}}$. For any $c \in \pi_0\mathcal{X}$, we construct the pullback

(2.2.3.1)
$$\begin{tikzcd} \mathcal{X}_c \arrow[r] \arrow[d] & \underline{\mathbf{1}} \arrow[d, "\underline{i_c}"] \\ \mathcal{X} \arrow[r] & \underline{\pi_0\mathcal{X}}. \end{tikzcd}$$

Since $\underline{\pi_0\mathcal{X}} = \coprod_{c\in\pi_0\mathcal{X}} \underline{\mathbf{1}}$ is a coproduct decomposition in the topos $Sets^{\mathcal{I}}$, and the coproducts in topoi are universal and disjoint, we see that

(2.2.3.2)
$$\mathcal{X} \cong \coprod_{c\in\pi_0\mathcal{X}} \mathcal{X}_c.$$

Each $\pi_0\mathcal{X}_c$ is a singleton, identified with the subset $\{c\} \subset \pi_0\mathcal{X}$; this means that each $\mathcal{X}_c$ is a connected object of the topos $Sets^{\mathcal{I}}$ (in fact, $\mathcal{Y}$ is connected if and only if $\pi_0\mathcal{Y}$ is a singleton), and (2.2.3.2) is a canonical decomposition of the object $\mathcal{X}$ into a coproduct (or "disjoint union") of its "connected components" (i.e., some connected objects), indexed by $\pi_0\mathcal{X}$.

One can actually construct such decompositions of an object $\mathcal{X}$ in any topos, provided $\pi_0\mathcal{X}$ is a set, and not merely a proset.

2.2.4. *$P$ is projective in a topos if and only if any epimorphism $X \twoheadrightarrow P$ splits.* Since (effective) epimorphisms are stable under pullbacks in a topos, *$P$ is a projective object of a topos $\mathscr{E}$ if and only if any epimorphism $\pi\colon X \twoheadrightarrow P$ splits, i.e., admits a section.* Indeed, necessity is obvious: since all epimorphisms in a topos are effective, $\pi_*\colon \mathrm{Hom}_{\mathscr{E}}(P,X) \to \mathrm{Hom}_{\mathscr{E}}(P,P)$ has to be surjective; in particular, one can find a preimage $\sigma$ of $\mathrm{id}_P$. Conversely, suppose that all epimorphisms with target $P$ split, and consider an epimorphism $p\colon X \to Y$ and a morphism $f\colon P \to Y$. We would like to show that $p_*\colon \mathrm{Hom}_{\mathscr{E}}(P,X) \to \mathrm{Hom}_{\mathscr{E}}(P,Y)$ is surjective, i.e., that $f$ factorizes through $p$. This is equivalent to the pullback $p'\colon X \times_Y P \to P$ admitting a section $\sigma$, because $P \xrightarrow{\sigma} X \times_Y P \xrightarrow{\mathrm{pr}_1} X$ will then provide the required factorization for $f$; but epimorphisms are stable under pullback in a topos, so $p'$ is an epimorphism as well, and it splits by assumption.

2.2.5. *$P = P' \sqcup P''$ is projective if and only if both $P'$ and $P''$ are.* Recall that any coproduct of projective objects is again projective, because any product of surjective maps of sets is surjective. We want to show that, conversely, *if a coproduct $P = P' \sqcup P''$ is a projective object of a topos $\mathscr{E}$, the same holds for $P'$ and $P''$.* By symmetry, one has to show only the projectivity of $P'$, i.e., that any epimorphism $p'\colon X' \twoheadrightarrow P'$ splits. Consider the pushout diagram

(2.2.5.1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X' \sqcup P'' \arrow[d, "p"] \\ P' \arrow[r, "i_1"] & P' \sqcup P''. \end{tikzcd}$$

Since $p$ is a pushout of $p'$, it is an epimorphism. By assumption, $P = P' \sqcup P''$ is projective, so $p$ admits a section $\sigma$. Now the above diagram is bi-Cartesian in any topos $\mathscr{E}$, because all coproducts are universally disjoint in a topos. This means that $p'$ is a pullback of $p$, so the corresponding pullback $\sigma'\colon P' \to X'$ of $\sigma$ is a section of $p'$, i.e., any epimorphism $X' \to P'$ splits.

2.2.6. *$P$ is projective if and only if all its components are.* Now suppose that $P$ is an object of a topos $\mathscr{E}$ such that $\pi_0 P$ is a set. Consider its decomposition into connected components:

$$P = \coprod_{c\in\pi_0 P} P_c. \tag{2.2.6.1}$$

The above results show that *$P$ is projective if and only if all its connected components $P_c$ are.*

2.2.7. *Application to projective presheaves.* We saw in **2.2.1** that, for any object $\mathcal{P}$ of a presheaf topos $\mathit{Sets}^{\mathcal{I}}$, the proset of its connected components $\pi_0\mathcal{P} = \operatorname{colim}\mathcal{P}$ is actually a set. This means that *any projective object $\mathcal{P}$ of $\mathit{Sets}^{\mathcal{I}}$ admits a canonical decomposition into a "disjoint union" (i.e., coproduct)* of connected projective objects, indexed by $\pi_0\mathcal{P} = \operatorname{colim}\mathcal{P}$.

2.2.8. *Connected projective presheaves are retracts of representable functors.* We claim that the *connected projective objects of $\mathit{Sets}^{\mathcal{I}}$ are precisely the retracts of corepresentable functors $h^a$.* Indeed, any corepresentable functor $h^a$ is projective, since

$$\operatorname{Hom}_{\mathit{Sets}^{\mathcal{I}}}(h^a, -)\colon \mathcal{X} \rightsquigarrow \mathcal{X}(a)$$

transforms epimorphisms into surjective maps of sets, and connected, since

$$\operatorname{Hom}_{\mathit{Sets}}(\operatorname{colim} h^a, S) \cong \operatorname{Hom}_{\mathit{Sets}^{\mathcal{I}}}(h^a, \underline{S}) \cong \underline{S}(a) = S \cong \operatorname{Hom}_{\mathit{Sets}}(\mathbf{1}, S)$$

for any set $S$, so $\pi_0 h^a = \operatorname{colim} h^a \cong \mathbf{1}$. Any retract of the projective object $h^a$ is still projective; connectedness is also stable under retracts, because $\pi_0$ of such a retract is a retract of singleton, hence also is a singleton.

Conversely, suppose that $\mathcal{P}$ is a connected projective object of $\mathit{Sets}^{\mathcal{I}}$. Since $\mathit{Sets}^{\mathcal{I}}$ is generated by corepresentable functors, we can find an epimorphism $p : F := \coprod_{\iota\in J} h^{a_\iota} \twoheadrightarrow \mathcal{P}$ for some family $(a_\iota)_{\iota\in J}$ of objects of $\mathcal{I}$. The projectivity of $\mathcal{P}$ implies that $p$ admits a section $\sigma\colon \mathcal{P} \to \coprod_\iota h^{a_\iota}$. The connectedness of $\mathcal{P}$ implies that the image $\sigma(\mathcal{P})$ lies inside one of $h^{a_\iota}$, i.e., that $\sigma$ factorizes through one of $i_\iota\colon h^{a_\iota} \to F = \coprod_\iota h^{a_\iota}$: otherwise the pullback along $\sigma$ of this coproduct decomposition of $F$ would provide a nontrivial decomposition of the connected $\mathcal{P}$ into a coproduct, coproducts in a topos being universally disjoint. Now if $\sigma = i_\iota\sigma'$ for some $\sigma'\colon \mathcal{P} \to h^{a_\iota}$, then $\sigma'$ is a section of $pi_\iota$, and therefore $\mathcal{P}$ is a retract of the corepresentable functor $h^{a_\iota}$ as claimed.

2.2.9. *Replacing $\mathcal{I}$ with an idempotent-complete rigid category does not change $\mathit{Sets}^{\mathcal{I}}$.* Notice that replacing $\mathcal{I}$ with its idempotent completion replaces the presheaf category $\mathit{Sets}^{\mathcal{I}}$ with an equivalent one. Furthermore, replacing $\mathcal{I}$ with its "rigidification" (the full subcategory containing exactly one representative of each isomorphism class of objects) again replaces $\mathit{Sets}^{\mathcal{I}}$ with an equivalent category. This means that *it suffices to study $\mathit{Sets}^{\mathcal{I}}$ assuming $\mathcal{I}$ to be idempotent-complete and rigid.*

2.2.10. *Projective objects of $\mathit{Sets}^{\mathcal{I}}$ as $\mathcal{I}$-graded sets.* Let $\mathcal{I}$ be a rigid idempotent-complete small category. Consider a projective object $\mathcal{P}$ of $\mathit{Sets}^{\mathcal{I}}$. We have already established that the connected component decomposition $\mathcal{P} = \coprod_{c\in\pi_0\mathcal{P}} \mathcal{P}_c$ of $\mathcal{P}$ exists and is naturally indexed by $\pi_0\mathcal{P} = \operatorname{colim}\mathcal{P}$. Any its component $\mathcal{P}_c$ is a retract of a corepresentable functor $h^a$, hence representable itself, $\mathcal{I}$ being assumed idempotent-complete. In this way $\mathcal{P}_c$ is isomorphic to $h^{a_c}$ for an object $a_c : \mathcal{I}$, usually determined up to isomorphism. But $\mathcal{I}$ has been assumed rigid, so isomorphic objects of $\mathcal{I}$ have to be equal. This means that each $a_c \in \operatorname{Ob}\mathcal{I}$ is uniquely determined, and we obtain a map $a\colon \pi_0\mathcal{P} \to \operatorname{Ob}\mathcal{I}$ such that

$$\mathcal{P} \cong \coprod_{c\in\pi_0\mathcal{P}} h^{a(c)}. \tag{2.2.10.1}$$

2.2.11. *Definition of an $\mathcal{I}$-normed set.* Let $\mathcal{I}$ be any small category. We *define* an *$\mathcal{I}$-normed set* to be a couple $(X, |\cdot|_X)$ consisting of a set $X$ and a map $|\cdot|_X\colon X \to \operatorname{Ob}\mathcal{I}$. Any $\mathcal{I}$-graded set $X = (X, |\cdot|_X)$ defines a projective object $IX$ of $\mathit{Sets}^{\mathcal{I}}$ by

$$IX := \coprod_{x\in X} h^{|x|_X}. \tag{2.2.11.1}$$

All projective objects of $\mathit{Sets}^{\mathcal{I}}$ arise in this way, at least if $\mathcal{I}$ is idempotent complete. If we define morphisms between $\mathcal{I}$-graded sets $X$ and $Y$ as morphisms between $IX$ and $IY$ in $\mathit{Sets}^{\mathcal{I}}$, we obtain a *category $\mathcal{N}_{\mathcal{I}}$ of $\mathcal{I}$-normed sets* such that $\operatorname{Proj}\mathit{Sets}^{\mathcal{I}}$ is equivalent to $\mathcal{N}_{\mathcal{I}}$ if $\mathcal{I}$ is idempotent complete. In general, $\mathcal{N}_{\mathcal{I}}$ is equivalent only to $\operatorname{Free}\mathit{Sets}^{\mathcal{I}}$, the category of "free" objects of $\mathit{Sets}^{\mathcal{I}}$, i.e., coproducts of corepresentable functors.

2.2.12. *Explicit description of morphisms between $\mathcal{I}$-normed sets.* Let

$$X = (X, |\cdot|_X\colon X \to \operatorname{Ob}\mathcal{I}) \text{ and } Y = (Y, |\cdot|_Y\colon Y \to \operatorname{Ob}\mathcal{I})$$

be two $\mathcal{I}$-normed sets. We would like to compute $\operatorname{Hom}_{\mathcal{N}_{\mathcal{I}}}(X,Y) = \operatorname{Hom}_{\mathit{Sets}^{\mathcal{I}}}(IX, IY)$ explicitly. By definition,

$$IX = \coprod_{x\in X} h^{|x|_X}, \tag{2.2.12.1}$$

$$IY = \coprod_{y\in Y} h^{|y|_Y}. \tag{2.2.12.2}$$

Any morphism $\phi\colon IX \to IY$ induces a map $\bar\phi := \pi_0(\phi)\colon X \cong \pi_0(IX) \to Y \cong \pi_0(IY)$ between the underlying sets of $X$ and $Y$, i.e., sets of connected components of $IX$ and $IY$. This means that a connected component $h^{|x|_X}$ of $IX$ is mapped into the connected component $h^{|\bar\phi(x)|_Y}$ of $IY$. By Yoneda,

$$\operatorname{Hom}_{\mathit{Sets}^{\mathcal{I}}}\big(h^{|x|_X}, h^{|\bar\phi(x)|_Y}\big) \cong \operatorname{Hom}_{\mathcal{I}}\big(|\bar\phi(x)|_Y, |x|_X\big),$$

so $\phi$ is completely determined by the map of sets $\bar\phi\colon X \to Y$ and by $\xi\colon X \to \operatorname{Ar}\mathcal{I}$ such that $\xi_x \in \operatorname{Hom}_{\mathcal{I}}\big(|\bar\phi(x)|_Y, |x|_X\big)$ for any $x \in X$.

We arrive to the following definition.

**Definition 2.2.13.** (Elementary definition of $\mathcal{I}$-normed sets.) Let $\mathcal{I}$ be a small category. An **$\mathcal{I}$-normed set** $X$ is a couple $(X, |\cdot|_X)$, consisting of a set $X$ (called the **underlying set** of $X$) and a map $|\cdot|_X\colon X \to \operatorname{Ob}\mathcal{I}$, called an **$\mathcal{I}$-valued norm on** $X$. A morphism $(X, |\cdot|_X) \to (Y, |\cdot|_Y)$ is a couple $(f, \xi)$ consisting of a map of underlying sets $f\colon X \to Y$ and a map $\xi\colon X \to \operatorname{Ar}\mathcal{I}$ such that $\xi_x$ is a morphism from $|f(x)|_Y$ to $|x|_X$ for any $x \in X$. Normed sets together with the morphisms thus defined constitute a category $\mathcal{N}_{\mathcal{I}}$, called the **category of $\mathcal{I}$-normed sets.**

We have also proved the following theorem, generalizing **2.1.1**.

**Theorem 2.2.14.** (*Projective objects of $\mathit{Sets}^{\mathcal{I}}$.*)

a) *Let $\mathcal{I}$ be a small category, $\mathcal{I}'$ its idempotent completion (sometimes called "Karoubi closure"), $\mathcal{I}''$ the rigidification of $\mathcal{I}'$. Then restriction along natural functors $\mathcal{I} \to \mathcal{I}' \leftarrow \mathcal{I}''$ induces equivalences between categories $\mathit{Sets}^{\mathcal{I}} \leftarrow \mathit{Sets}^{\mathcal{I}'} \to \mathit{Sets}^{\mathcal{I}''}$, as well as between the corresponding categories of projective objects.*

b) *Let $\mathcal{I}$ be an idempotent-complete small category. Denote by $I\colon \mathcal{N}_{\mathcal{I}} \to Sets^{\mathcal{I}}$ the natural functor from the category of $\mathcal{I}$-normed sets into $Sets^{\mathcal{I}}$ given by* (2.2.11.1). *Then $I$ is a fully faithful functor with essential image equal to the full subcategory* $\operatorname{Proj} Sets^{\mathcal{I}}$ *of projective objects of $Sets^{\mathcal{I}}$.*

c) *Let $\mathcal{I}$ be an arbitrary small category. Then $I\colon \mathcal{N}_{\mathcal{I}} \to Sets^{\mathcal{I}}$ is a fully faithful functor with essential image equal to the full subcategory* $\operatorname{Free} Sets^{\mathcal{I}}$ *of $Sets^{\mathcal{I}}$ consisting of "free" objects, i.e., coproducts of corepresentable functors.*

2.2.15. *Elements of* $\operatorname{Hom}_{\mathcal{I}}(|f(x)|, |x|)$ *as witnesses for* $|f(x)| \leq |x|$. Comparison with the case of $\Delta$-graded and $\Delta$-normed sets, for $\Delta$ a partially ordered set (cf. **2.1.1**), shows us some distinctions with the more general case. For example, a morphism $(f, \xi)\colon (X, |\cdot|_X) \to (Y, |\cdot|_Y)$ is no longer determined by the map of underlying sets $f\colon X \to Y$, because now there can be more than one morphism $\xi_x : |f(x)|_Y \to |x|_X$ in $\mathcal{I}$. One might think of $\xi_x$ as a "witness for $|f(x)|_Y \leq |x|_X$", with potentially more than one "witness" for such an inequality. This way of thinking sometimes helps generalize statements about $\Delta$-normed sets to $\mathcal{I}$-normed sets.

2.2.16. *Components $IX(\alpha)$ of $IX$.* Let $X = (X, |\cdot|_X\colon X \to \operatorname{Ob}\mathcal{I})$ be an $\mathcal{I}$-normed set, and $IX = \coprod_{x\in X} h^{|x|}$ its image in $Sets^{\mathcal{I}}$. When $\mathcal{I} = \Delta$, we could write

$$IX(\alpha) \cong X_{\leq\alpha} = \bigl\{x \in X \,:\, |x|_X \leq \alpha\bigr\}. \tag{2.2.16.1}$$

In the general case, we have

$$IX(\alpha) = \Bigl(\coprod_{x\in X} h^{|x|}\Bigr)(\alpha) = \coprod_{x\in X} h^{|x|}(\alpha) = \bigsqcup_{x\in X} \operatorname{Hom}_{\mathcal{I}}\bigl(|x|_X, \alpha\bigr). \tag{2.2.16.2}$$

Again, instead of subset of $X$ characterized by the property $|x|_X \leq \alpha$, we consider the set of pairs $(x, \xi)$, where $x$ is an element of $X$, and $\xi\colon |x|_X \to \alpha$ is a "witness for $|x| \leq \alpha$".

However, $IX(\alpha)$ can no longer be identified with a subset of $X$: we only have a canonical map $IX(\alpha) \to X$, compatible with all morphisms $(IX)(\phi)$, $\phi\colon \alpha \to \beta$, such that the collection of all these maps induces an *iso*morphism $\operatorname{colim} IX \to X$.

2.2.17. *Limits and colimits in $\mathcal{N}_{\mathcal{I}}$ and $Sets^{\mathcal{I}}$.* Arbitrary (small) limits and colimits obviously exist in the functor category $Sets^{\mathcal{I}}$; they can be computed componentwise. As to the category $\mathcal{N}_{\mathcal{I}}$, equivalent to full subcategory $\operatorname{Proj} Sets^{\mathcal{I}}$ of $Sets^{\mathcal{I}}$ (for idempotent-complete $\mathcal{I}$), we can say that *whenever certain limits or colimits preserve projectivity of objects of $Sets^{\mathcal{I}}$, they exist in $\mathcal{N}_{\mathcal{I}}$ as well, and are preserved by the functor $I$.* Furthermore, this condition is *necessary* for the existence of limits: *$I$ preserves all limits that exist in $\mathcal{N}_{\mathcal{I}}$*, because all component functors $X \rightsquigarrow (IX)(\alpha) \cong \operatorname{Hom}_{\mathcal{N}_{\mathcal{I}}}(\mathbf{1}^\alpha, X)$ do, hence the *limit* $\lim_{\mathcal{D}} \mathcal{X}$ *of a diagram $\mathcal{X}\colon \mathcal{D} \to \mathcal{N}_{\mathcal{I}}$ exists if and only if* $\lim_{\mathcal{D}} I \circ \mathcal{X}$ *is a projective object of $Sets^{\mathcal{I}}$.*

2.2.18. *Coproducts in $\mathcal{N}_{\mathcal{I}}$.* For example, any coproduct of projective objects is still projective (and any coproduct of free objects is still free), so arbitrary coproducts exist in $\mathcal{N}_{\mathcal{I}}$ and are preserved by $I$. The explicit formula (2.2.11.1) shows that coproducts in $\mathcal{N}_{\mathcal{I}}$ can be computed with the aid of disjoint unions as in **1.3.2**.

2.2.19. *Binary products in $\mathcal{N}_{\mathcal{I}}$.* Notice that for any $X = (X, |\cdot|_X)$, $Y = (Y, |\cdot|_Y) : \mathcal{N}_{\mathcal{I}}$,

$$IX \times IY = \coprod_{(x,y)\in X\times Y} h^{|x|} \times h^{|y|}. \tag{2.2.19.1}$$

Suppose that the binary coproducts exist in $\mathcal{I}$; let us denote them by $\vee$. Then $h^\alpha \times h^\beta \cong h^{\alpha\vee\beta}$, so the binary products exists in $\mathcal{N}_\mathcal{I}$ in this case, and can be computed by introducing the $\vee$-norm on $X \times Y$:

$$\big|(x,y)\big|_{X\times Y} := |x|_X \vee |y|_Y. \tag{2.2.19.2}$$

This corresponds of course to the "sup-norm" in the case of $\mathcal{I} = \Delta$.

2.2.20. *Arbitrary products in $\mathcal{N}_\mathcal{I}$.* Let $J$ be a set, let $\{\mathcal{X}_\iota\}_{\iota\in J}$ be an $J$-indexed family of $\mathcal{I}$-normed sets, and let $\mathcal{X}_\iota = (X_\iota, |\cdot|_{\mathcal{X}_\iota})$. Suppose that $J$-indexed coproducts exist in $\mathcal{I}$. Then the direct product $\mathcal{X} := \prod_{\iota\in J}\mathcal{X}_\iota$ exists in $\mathcal{N}_\mathcal{I}$, and can be computed as follows. The underlying set of $\mathcal{X}$ is $X := \prod_{\iota\in J} X_\iota$, and the $\mathcal{I}$-valued norm $|\cdot|_\mathcal{X}\colon X \to \operatorname{Ob}\mathcal{I}$ is given by

$$\big|(x_\iota)_\iota\big|_\mathcal{X} := \bigvee_{\iota\in J} |x_\iota|_{\mathcal{X}_\iota}. \tag{2.2.20.1}$$

Here $\bigvee_{\iota\in J}$ is a notation for $J$-indexed coproducts in $\mathcal{I}$.

This statement can be proved for example by checking that $\mathcal{X} = (X, |\cdot|_\mathcal{X})$ thus constructed satisfies the universal property expected from the direct product $\prod_\iota \mathcal{X}_\iota$. Since $I\colon \mathcal{N}_\mathcal{I} \to \mathit{Sets}^\mathcal{I}$ preserves all limits that exist in $\mathcal{N}_\mathcal{I}$, we see that $I\mathcal{X}$ remains a product of $I\mathcal{X}_\iota$ in $\mathit{Sets}^\mathcal{I}$. In particular, the functor $\pi_0 = \operatorname{colim}\colon \operatorname{Proj}(\mathit{Sets}^\mathcal{I}) \to \mathit{Sets}$ commutes with $J$-indexed products in this case.

2.2.21. *Equalizers in $\mathcal{N}_\mathcal{I}$.* Suppose that the coequalizers exist in $\mathcal{I}$. Then the equalizers exists in $\mathcal{N}_\mathcal{I}$ (and are preserved by $I\colon \mathcal{N}_\mathcal{I} \to \mathit{Sets}^\mathcal{I}$, together with all other limits); compare with **1.2.6**. Indeed, given two parallel morphisms $f = (f,\xi)$ and $g = (g,\eta)\colon \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} = (X, |\cdot|_\mathcal{X})$ and $\mathcal{Y} = (Y, |\cdot|_\mathcal{Y})$, we construct their equalizer $\mathcal{Z} = (Z, |\cdot|_\mathcal{Z})$ and $j = (j,\kappa)\colon \mathcal{Z} \to \mathcal{X}$ as follows. Put $Z := \operatorname{Eq}(f,g)$ in the category of sets; thus $Z = \{x \in X : f(x) = g(x)\}$. For any $x \in Z$, we have two parallel morphisms $\xi_x$, $\eta_x\colon |f(x)|_\mathcal{Y} = |g(x)|_\mathcal{Y} \rightrightarrows |x|_\mathcal{X}$ in $\mathcal{I}$. Denote their coequalizer in $\mathcal{I}$ by $|x|_\mathcal{Z}$, together with the corresponding map $\kappa_x\colon |x|_\mathcal{X} \to |x|_\mathcal{Z}$. Then $\mathcal{Z} := (Z, |\cdot|_\mathcal{Z})$ is an object of $\mathcal{N}_\mathcal{I}$, and $j = (j,\kappa)$, with $j\colon Z \hookrightarrow X$ the natural embedding, is an equalizer of $f$ and $g$ in $\mathcal{N}_\mathcal{I}$.

2.2.22. *Arbitrary limits in $\mathcal{N}_\mathcal{I}$.* Combining **2.2.20** with **2.2.21**, we see that *whenever arbitrary (small) colimits exist in $\mathcal{I}$, arbitrary (small) limits exist in $\mathcal{N}_\mathcal{I}$ as well, and are preserved by $I\colon \mathcal{N}_\mathcal{I} \to \mathit{Sets}^\mathcal{I}$. In other words,* $\operatorname{Proj}(\mathit{Sets}^\mathcal{I}) = \operatorname{Free}(\mathit{Sets}^\mathcal{I})$ *is stable under (small) limits in $\mathit{Sets}^\mathcal{I}$.* In fact, one can prove a refined version of this result: if for some small category $\mathcal{D}$ all $\mathcal{D}^{\mathrm{op}}$-indexed colimits exist in $\mathcal{I}$, then all $\mathcal{D}$-indexed limits exist in $\mathcal{N}_\mathcal{I}$ and commute with the underlying set functor $\pi_0\colon \mathcal{N}_\mathcal{I} \to \mathit{Sets}$, $\mathcal{X} = (X, |\cdot|_\mathcal{X}) \rightsquigarrow X$.

We see that, when $\mathcal{I}$ is cocomplete (i.e., arbitrary small colimits exist in $\mathcal{I}$), then $\mathcal{N}_\mathcal{I}$ is complete, and $I\colon \mathcal{N}_\mathcal{I} \to \mathit{Sets}^\mathcal{I}$ commutes with arbitrary (small) limits, as if it had a left adjoint. We are going to prove a (dual) result in this direction.

2.2.23. *Connected categories, limits and colimits.* Recall that a category $\mathcal{D}$ is said to be *connected* if it is nonempty, and if it cannot be written as the coproduct (i.e., disjoint union) of two nonempty categories: $\mathcal{D} \neq \varnothing$, $\mathcal{D} \neq \mathcal{D}' \sqcup \mathcal{D}''$ for $\mathcal{D}'$, $\mathcal{D}'' \neq \varnothing$. We say that a limit or colimit is *connected* if it is taken along a (usually small) connected category.

It is worthwhile to remark that, given a presheaf $\mathcal{X} : \widehat{\mathcal{I}}$, the category $\mathcal{I}_{/\mathcal{X}}$ is connected if and only if $\mathcal{X}$ is connected as an object of the topos $\widehat{\mathcal{I}}$. Furthermore, if $\mathcal{X} = \operatorname{inj\,lim}_{\iota:\mathcal{D}} h_{\alpha(\iota)}$ is a colimit of representable presheaves, then $\mathcal{X}$ is connected if and only if the index category $\mathcal{D}$ is connected, i.e., if and only if the colimit expressing $\mathcal{X}$ is connected.

**Theorem 2.2.24.** ($\mathcal{I}$*-normed sets as a localization of* $Sets^{\mathcal{I}}$*.*) *Suppose that the* (*small*) *connected limits exist in the small category* $\mathcal{I}$ (*for example,* $\mathcal{I}$ *is complete*); *in particular,* $\mathcal{I}$ *is idempotent complete. Then the fully faithful functor* $I\colon \mathcal{N}_{\mathcal{I}} \to Sets^{\mathcal{I}}$ *admits a left adjoint* $Q\colon Sets^{\mathcal{I}} \to \mathcal{N}_{\mathcal{I}}$*, exhibiting a full subcategory* $\mathrm{Proj}(Sets^{\mathcal{I}}) = \mathrm{Free}(Sets^{\mathcal{I}})$ *of* $Sets^{\mathcal{I}}$*, equal to the essential image of* $I$*, as a localization* (*or "reflection" in another terminology*) *of the category* $Sets^{\mathcal{I}}$*.*

*Proof.* We want to give an explicit construction of a left adjoint $Q\colon Sets^{\mathcal{I}} \to \mathcal{N}_{\mathcal{I}}$. Given any $\mathcal{X} : Sets^{\mathcal{I}}$, consider the set $\pi_0\mathcal{X} = \mathrm{colim}_{\mathcal{I}}\,\mathcal{X}$ and the connected component decomposition

$$\mathcal{X} = \coprod_{c\in\pi_0\mathcal{X}} \mathcal{X}_c \tag{2.2.24.1}$$

of (2.2.3.2). Each $\mathcal{X}_c$ can be written as a connected colimit of corepresentable functors, for example, in the canonical way:

$$\mathcal{X}_c \cong \operatorname*{inj\,lim}_{\alpha\,:\,\mathcal{I}^{\mathrm{op}}_{/\mathcal{X}_c}} h^{\alpha}. \tag{2.2.24.2}$$

Here we use the fact that the category $\mathcal{I}^{\mathrm{op}}_{/\mathcal{X}}$ is connected if and only if presheaf $\mathcal{X} : \widehat{\mathcal{I}^{\mathrm{op}}}$ is connected.

Define an object $|c|$ of $\mathcal{I}$ to be "the same" limit computed in $\mathcal{I}$ (recall that the Yoneda embedding $\mathcal{I}^{\mathrm{op}} \to Sets^{\mathcal{I}}$ is contravariant in this case):

$$|c| := \operatorname*{proj\,lim}_{\alpha\,:\,(\mathcal{I}^{\mathrm{op}}_{/\mathcal{X}_c})^{\mathrm{op}}} \alpha. \tag{2.2.24.3}$$

By construction, we have an arrow $\mathcal{X}_c = \mathrm{inj\,lim}_{\alpha:\mathcal{I}^{\mathrm{op}}_{/\mathcal{X}_c}} h^{\alpha} \to h^{|c|}$, which is universal with respect to all morphisms from $\mathcal{X}_c$ into corepresentable functors. Since $\mathcal{X}_c$ is connected, this arrow is also universal with respect to all morphisms from $\mathcal{X}_c$ into objects of $\mathrm{Free}(Sets^{\mathcal{I}})$, i.e., $Q(\mathcal{X}_c)$ is singleton $\mathbf{1}^{|c|}$ with the norm of its only point put equal to $|c|$. Now, $Q$ has to respect coproducts; applying this (partially defined) functor to (2.2.24.1), we see that $Q(\mathcal{X})$ has to be the set of connected components $\pi_0\mathcal{X}$ with the norm of each $c \in \pi_0\mathcal{X}$ given by (2.2.24.3). It is easy to see that, indeed, $(\pi_0\mathcal{X}, |\cdot|)$ has the universal property required from $Q(\mathcal{X})$. □

**Corollary 2.2.25.** *If* $\mathcal{I}$ *admits the* (*small*) *connected limits, for example, it is complete, then* $\mathcal{N}_{\mathcal{I}}$ *is cocomplete, i.e., arbitrary* (*small*) *colimits exist in* $\mathcal{N}_{\mathcal{I}}$*. They can be computed by applying the localization functor* $Q$ *to the corresponding colimit computed in* $Sets^{\mathcal{I}}$*. The underlying set functor* $\pi_0\colon \mathcal{N}_{\mathcal{I}} \to Sets$ *is cocontinuous, i.e., commutes with all colimits.*

*Proof.* Only the last statement requires a proof. It follows from the fact that $\pi_0 = \mathrm{colim}_{\mathcal{I}}$ commutes with all colimits in $Sets^{\mathcal{I}}$, being a colimit itself, and the fact that $Q$ preserves $\pi_0$; more precisely, the "localization" (or "reflection") morphism $\mathcal{X} \to IQ(\mathcal{X})$ becomes an isomorphism after applying $\pi_0$. This is a consequence of the explicit construction of $Q(\mathcal{X})$ given during the proof of **2.2.24**. □

2.2.26. *Explicit construction of* $Q$ *for* $\mathcal{I} = \Delta$. Suppose that $\mathcal{I}$ is a partially ordered set $\Delta$ with infima of nonempty subsets, for example $\mathbb{R}_{\geq 0}$ or $\bar{\mathbb{R}}_{\geq 0}$. Then the functor $Q\colon Sets^{\Delta} \to \mathcal{N}_{\Delta}$ can be described as follows. The underlying set of $Q(\mathcal{X})$ for some $\mathcal{X} : Sets^{\Delta}$ has to be equal to $\pi_0\mathcal{X} = \mathrm{colim}_{\Delta}\,\mathcal{X}$. Denote by $\phi_\alpha$ the natural map $\mathcal{X}(\alpha) \to \pi_0\mathcal{X}$ for each $\alpha \in \Delta$; then the norm $|\cdot|_{Q(\mathcal{X})}\colon \pi_0\mathcal{X} \to \Delta$ is defined by

$$|x|_{Q(\mathcal{X})} = \inf\{\alpha \in \Delta : \phi_\alpha^{-1}(x) \neq \varnothing\}. \tag{2.2.26.1}$$

**Proposition 2.2.27.** *Suppose $\mathcal{I}$ admits all connected limits and finite coproducts, and that $\mathcal{I}^{\mathrm{op}}$ is Cartesian closed. Then the left adjoint $Q\colon Sets^{\mathcal{I}} \to \mathcal{N}_{\mathcal{I}}$ to $I$ preserves the finite products.*

*Proof.* Our assumptions imply the existence of $Q$ by **2.2.24** and the existence of finite products in $\mathcal{N}_{\mathcal{I}}$ by **2.2.20**, preserved by the fully faithful functor $I\colon \mathcal{N}_{\mathcal{I}} \to Sets^{\mathcal{I}}$. We want to show that $\gamma_{\mathcal{X},\mathcal{Y}}\colon Q(\mathcal{X}\times\mathcal{Y}) \to Q(\mathcal{X})\times Q(\mathcal{Y})$ is an isomorphism for any $\mathcal{X}$, $\mathcal{Y}$ : $Sets^{\mathcal{I}}$, since the preservation by $Q$ of the final object $h^{o}$, where $o$ is the initial object of $\mathcal{I}$, is trivial. Now the binary products are distributive with respect to arbitrary coproducts both in $Sets^{\mathcal{I}}$ (which is a topos) and $\mathcal{N}_{\mathcal{I}}$ (where distributivity follows from explicit constructions), and $Q$ preserves the coproducts, being a left adjoint, so we can express both $\mathcal{X}$ and $\mathcal{Y}$ as the coproducts ("disjoint unions") of their connected components. Therefore, we may assume both $\mathcal{X}$ and $\mathcal{Y}$ connected.

Now write $\mathcal{X}$ and $\mathcal{Y}$ as (connected) colimits of the corepresentable functors: $\mathcal{X} = \operatorname{inj\,lim}_{\iota:\mathcal{D}} h^{\alpha(\iota)}$, $\mathcal{Y} = \operatorname{inj\,lim}_{\kappa:\mathcal{D}'} h^{\beta(\kappa)}$. Since $Sets^{\mathcal{I}}$ is Cartesian closed, binary products in $Sets^{\mathcal{I}}$ commute with colimits, and we obtain

$$\mathcal{X}\times\mathcal{Y} \cong \operatorname*{inj\,lim}_{(\iota,\kappa):\mathcal{D}\times\mathcal{D}'} h^{\alpha(\iota)}\times h^{\beta(\kappa)} \cong \operatorname*{inj\,lim}_{\iota,\kappa} h^{\alpha(\iota)\vee\beta(\kappa)}. \tag{2.2.27.1}$$

The explicit construction of $Q$ given during the proof of **2.2.24** tells us that $Q(\mathcal{X}\times\mathcal{Y})$ is the normed singleton $\mathbf{1}^{\gamma}$ with $\gamma$ equal to "the same" colimit computed in $\mathcal{I}^{\mathrm{op}}$ (or the dual limit computed in $\mathcal{I}$):

$$Q(\mathcal{X}\times\mathcal{Y}) = \mathbf{1}^{\gamma} \quad \text{where } \gamma = \operatorname*{inj\,lim}_{\iota,\kappa}\bigl(\alpha(\iota)\times_{\mathcal{I}^{\mathrm{op}}}\beta(\kappa)\bigr) \quad \text{in } \mathcal{I}^{\mathrm{op}}. \tag{2.2.27.2}$$

Now, since $\mathcal{I}^{\mathrm{op}}$ has been assumed to be Cartesian closed, the binary products commute with arbitrary colimits in this category as well, so we have

$$\gamma \cong \gamma'\times_{\mathcal{I}^{\mathrm{op}}}\gamma'' \quad \text{where } \gamma' = \operatorname*{inj\,lim}_{\iota}\alpha(\iota), \quad \gamma'' = \operatorname*{inj\,lim}_{\kappa}\beta(\kappa). \tag{2.2.27.3}$$

It remains to observe that $Q(\mathcal{X}) = \mathbf{1}^{\gamma'}$ and $Q(\mathcal{Y}) = \mathbf{1}^{\gamma''}$, so that $Q(\mathcal{X})\times Q(\mathcal{Y}) = \mathbf{1}^{\gamma'\vee\gamma''} \cong \mathbf{1}^{\gamma} = Q(\mathcal{X}\times\mathcal{Y})$ as claimed. □

2.2.28. *The case of $\mathcal{I} = \Delta$ with infima of nonempty subsets and finite suprema.* Proposition **2.2.27** is applicable in particular to $\mathcal{I} = \Delta$, where $\Delta$ is a partially ordered set with infima of nonempty subsets and finite suprema, distributive with respect to each other:

$$\inf_{\iota\in I}(\beta\vee\alpha_{\iota}) = \beta\vee\inf_{\iota\in I}\alpha_{\iota}, \quad I\neq\varnothing. \tag{2.2.28.1}$$

For example, $\Delta = \mathbb{R}_{\geq 0}$ or $\bar{\mathbb{R}}_{\geq 0}$ have these properties.

Notice that the case of $\mathcal{I} = \Delta$ with infima of nonempty subsets already appeared in several statements, such as **1.3.1**. Is **2.2.25** indeed a generalization of this statement?

All our examples of categories $\mathcal{I}$ with arbitrary (small) connected limits and finite coproducts come from partially ordered sets, so we might wonder whether there are any other examples. The answer turns out to be negative.

2.2.29. *Small $\mathcal{I}$ with connected limits and binary coproducts is a poset.* We claim that *any small category $\mathcal{I}$ admitting arbitrary* (*small*) *connected limits and binary coproducts is equivalent to a partially ordered set $\Delta$ with infima of nonempty subsets and joins.* Let $\mathcal{I}$ be such a small category; we have to show that $\mathcal{I}$ is a preorder, i.e., that $\operatorname{card}\operatorname{Hom}_{\mathcal{I}}(X,Y)\leq 1$ for any $X$, $Y$ : $\mathcal{I}$. Suppose that this is not the case; then we can choose two distinct parallel morphisms $u\neq v\colon X \rightrightarrows Y$ between two objects of $\mathcal{I}$.

We have assumed the existence of binary coproducts in $\mathcal{I}$; in particular, we have $X\sqcup X$ with two embeddings $i, j\colon X\rightrightarrows X\sqcup X$ and the codiagonal map $\delta = \langle\mathrm{id}_X,\mathrm{id}_X\rangle$, characterized by $\delta\circ i = \delta\circ j = \mathrm{id}_X$. By the universal property of $X\sqcup X$, the morphisms

$u$ and $v$ induce $w := \langle u, v\rangle \colon X \sqcup X \to Y$ such that $w \circ i = u$, $w \circ j = v$. Since $u \neq v$, we must also have $i \neq j$. We see that *without loss of generality, we may assume that* $Y = X \sqcup X$, *and* $u$ *and* $v$ *are standard embeddings* $i$, $j \colon X \rightrightarrows X \sqcup X$.

Now we want to use the existence of connected limits in $\mathcal{I}$. For any small set $S$, denote by $P_S \xrightarrow{\delta_S} X$ the fibered product of $S$ copies of $\delta \colon X \sqcup X \to X$. The set of sections of $\delta_S$ is

$$\mathrm{Hom}_{\mathcal{I}_{/X}}(X, P_S) \cong \mathrm{Hom}_{\mathcal{I}_{/X}}(X, X \sqcup X)^S \supset \{i, j\}^S. \tag{2.2.29.1}$$

We see that

$$\mathrm{Hom}_{\mathcal{I}}(X, P_S) \supset \mathrm{Hom}_{\mathcal{I}_{/X}}(X, P_S) \supset \{i, j\}^S. \tag{2.2.29.2}$$

Since $i \neq j$, we obtain

$$\operatorname{card} \operatorname{Ar} \mathcal{I} \geq \operatorname{card} \mathrm{Hom}_{\mathcal{I}}(X, P_S) \geq 2^{\operatorname{card} S} \tag{2.2.29.3}$$

for any small set $S$. Since $\mathcal{I}$ was assumed to be a small category, we might take $S := \operatorname{Ar} \mathcal{I}$; then (2.2.29.3) implies $\operatorname{card} S \geq 2^{\operatorname{card} S}$, which is absurd.

2.2.30. *$\mathcal{I}$-fuzzy sets and presheaves.* One can dualize all of the above statements and constructions, replacing $\mathcal{I}$ with $\mathcal{I}^{\mathrm{op}}$, as discussed in **1.8**. The category $\mathcal{N}_{\mathcal{I}}$ of $\mathcal{I}$-normed sets becomes the category $\mathcal{F}_{\mathcal{I}}$ of "$\mathcal{I}$-fuzzy sets", consisting of pairs $\mathcal{X} = (X, m_{\mathcal{X}})$, where $X$ is any set, and $m_{\mathcal{X}} \colon X \to \operatorname{Ob} \mathcal{I}$ is an "$\mathcal{I}$-valued membership function". Constructions with $\mathcal{I}$-fuzzy sets turn out to be related to "$\mathcal{I}$-valued logics" in a more natural way than for $\mathcal{I}$-normed sets. For example, the direct product of two $\mathcal{I}$-fuzzy sets $\mathcal{X} = (X, m_{\mathcal{X}})$ and $\mathcal{Y} = (Y, m_{\mathcal{Y}})$ is given by $X \times Y$ with membership function $m_{\mathcal{X}\times\mathcal{Y}}(x, y) := m_{\mathcal{X}}(x) \wedge m_{\mathcal{Y}}(y)$, where $\wedge$ is the direct product of $\mathcal{I}$ corresponding to (additive) logical conjunction under the Curry–Howard correspondence. This might be understood as "$(x, y) \in \mathcal{X} \times \mathcal{Y}$ if and only if $x \in \mathcal{X}$ and $y \in \mathcal{Y}$", interpreted in the $\mathcal{I}$-valued logic.

**2.3. Monoidal structures on $\mathcal{I}$-graded sets.** We would like to introduce monoidal structures on $\mathcal{I}$-graded and $\mathcal{I}$-normed sets induced by a monoidal structure on $\mathcal{I}$, much as in **1.6**. Such considerations are more convenient to make in the context of contravariant functors, i.e., presheaves, and $\mathcal{I}$-fuzzy sets; statements for $\mathcal{I}$-graded sets, i.e., objects of $\mathit{Sets}^{\mathcal{I}}$, and $\mathcal{I}$-normed sets, can be obtained afterwards by replacing $\mathcal{I}$ with the dual category $\mathcal{I}^{\mathrm{op}}$.

2.3.1. *Colimit-preserving extension of a bifunctor from $\mathcal{I}$ to $\widehat{\mathcal{I}}$.* We claim that *any bifunctor $\otimes : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ can be uniquely* (*up to isomorphism*) *extended to a bifunctor $\widehat{\otimes} : \widehat{\mathcal{I}} \times \widehat{\mathcal{I}} \to \widehat{\mathcal{I}}$, which commutes with colimits in each argument and restricts to* (*a functor isomorphic to*) $\otimes$ *via Yoneda embedding*:

$$h_S \widehat{\otimes} h_T \cong h_{S\otimes T} \quad \text{for any } S,\, T \in \mathcal{I}. \tag{2.3.1.1}$$

This can be deduced from the fact that any presheaf can be written as a colimit of representable presheaves:

$$X \cong \operatorname{colim}_{S:\mathcal{I}_{/X}} h_S. \tag{2.3.1.2}$$

This implies that necessarily

$$X \widehat{\otimes} Y \cong \operatorname{colim}_{(S,T)\,:\,\mathcal{I}_{/X}\times\mathcal{I}_{/Y}} h_{S\otimes T}. \tag{2.3.1.3}$$

2.3.2. *Alternative construction via left Kan extension.* Notice that

$$\mathcal{I}_{/X} \times \mathcal{I}_{/Y} \cong (\mathcal{I} \times \mathcal{I})_{/X \boxtimes Y},$$

where $X \boxtimes Y : \widehat{\mathcal{I} \times \mathcal{I}}$ is given by

$$(X \boxtimes Y)(S,T) := X(S) \times Y(T). \tag{2.3.2.1}$$

Then formula (2.3.1.3) turns out to be the formula for the image of $X \boxtimes Y$ under left Kan extension $\otimes_! \colon \widehat{\mathcal{I} \times \mathcal{I}} \to \widehat{\mathcal{I}}$ of $\otimes \colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$. Therefore,

$$X \widehat{\otimes} Y \cong \otimes_!(X \boxtimes Y). \tag{2.3.2.2}$$

2.3.3. *Alternative description via "bilinear maps".* Consider the trifunctor

$$\mathrm{Bilin}_{\widehat{\mathcal{I}}} \colon \widehat{\mathcal{I}}^{\mathrm{op}} \times \widehat{\mathcal{I}}^{\mathrm{op}} \times \widehat{\mathcal{I}} \to \mathit{Sets},$$

given by

$$\mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; Z) := \mathrm{Hom}_{\widehat{\mathcal{I}}}(X \widehat{\otimes} Y, Z). \tag{2.3.3.1}$$

Using (2.3.2.2) together with the universal property of the left Kan extensions, we obtain

$$\mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; Z) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}\big(\otimes_!(X \boxtimes Y), Z\big) \cong \mathrm{Hom}_{\widehat{\mathcal{I} \times \mathcal{I}}}(X \boxtimes Y, Z \circ \otimes). \tag{2.3.3.2}$$

In other words, the "bilinear maps" $\Phi \in \mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; Z)$ can be described as collections of maps of sets

$$\Phi_{S,T} \colon X(S) \times Y(T) \to Z(S \otimes T), \quad \text{where } S,\ T : \mathcal{I}, \tag{2.3.3.3}$$

functorial in $S$ and $T$ in a natural fashion: for any $f \colon S' \to S$, $g \colon T' \to T$, we have

$$\begin{tikzcd} X(S) \times Y(T) \arrow[r, "X(f) \times Y(g)"] \arrow[d, "\Phi_{S,T}"] & X(S') \times Y(T') \arrow[d, "\Phi_{S',T'}"] \\ Z(S \otimes T) \arrow[r, "Z(f \otimes g)"] & Z(S' \otimes T'). \end{tikzcd} \tag{2.3.3.4}$$

This direct description enables us to define first the trifunctor $\mathrm{Bilin}_{\widehat{\mathcal{I}}}$, and then $X \widehat{\otimes} Y$ as the object of $\widehat{\mathcal{I}}$ corepresenting $\mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; -)$.

2.3.4. *Generalization to polyfunctors.* In fact, **2.3.1** can be generalized to polyfunctors: *any functor $M \colon \mathcal{I}^n \to \mathcal{I}$ can be uniquely (up to isomorphism) extended to a functor $\widehat{M} \colon \widehat{\mathcal{I}}^n \to \widehat{\mathcal{I}}$ commuting with arbitrary colimits separately in each argument.* The proof is essentially the same. For example, $\widehat{M}(X_1, \dots, X_n)$ can be constructed as $M_!(X_1 \boxtimes X_2 \boxtimes \cdots \boxtimes X_n)$, where $M_! \colon \widehat{\mathcal{I}^n} \to \widehat{\mathcal{I}}$ is the left Kan extension of $M$.

2.3.5. *Consequence for associativity and commutativity constraints.* This generalization is useful when a bifunctor $\otimes \colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ admits an associativity constraint $\alpha$,

$$\alpha_{S,T,U} \colon (S \otimes T) \otimes U \xrightarrow{\sim} S \otimes (T \otimes U).$$

In this case, applying existence and uniqueness statements for extensions of trifunctors $(S,T,U) \rightsquigarrow (S \otimes T) \otimes U \xrightarrow{\sim} S \otimes (T \otimes U)$, we obtain the existence of a functorial isomorphism $\widehat{\alpha}$ between $(X \widehat{\otimes} Y) \widehat{\otimes} Z$ and $X \widehat{\otimes} (Y \widehat{\otimes} Z)$. Next, applying uniqueness for $n = 4$, we see that $\widehat{\alpha}$ satisfies the pentagon axiom, so $\widehat{\alpha}$ is an associativity constraint for $\widehat{\otimes}$. Similar arguments apply to commutativity and unit constraints and their relationship with the associativity constraint.

2.3.6. *Monoidal structure on $\widehat{\mathcal{I}}$.* We see that *whenever $\mathcal{I} = (\mathcal{I}, \otimes)$ is a (symmetric) monoidal category, the same is true about $\widehat{\mathcal{I}}$, with (symmetric) monoidal structure given by the extension $\widehat{\otimes}$ of $\otimes$ constructed above. This monoidal structure is closed, and the Yoneda embedding $h\colon \mathcal{I} \to \widehat{\mathcal{I}}$ has a natural structure of a monoidal functor. These properties characterize the monoidal structure on $\widehat{\mathcal{I}}$ uniquely.*

2.3.7. *Closedness of the monoidal structure of $\widehat{\mathcal{I}}$.* We would like to show that the functor $H_{Y,Z}\colon X \rightsquigarrow \mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; Z)$ is representable for any fixed $Y$, $Z : \widehat{\mathcal{I}}$ and any bifunctor $\otimes\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$. This would imply in particular the closedness of the monoidal structure of $\widehat{\mathcal{I}}$, when $\otimes$ is part of the monoidal structure on $\mathcal{I}$.

Indeed, $H_{Y,Z}\colon X \rightsquigarrow \mathrm{Bilin}_{\widehat{\mathcal{I}}}(X, Y; Z) = \mathrm{Hom}_{\widehat{\mathcal{I}}}(X \boxtimes Y, Z \circ \otimes)$ transforms arbitrary colimits into limits, because $X \rightsquigarrow X \boxtimes Y$ preserves arbitrary colimits, and $\mathrm{Hom}_{\widehat{\mathcal{I}}}(-, Z \circ \otimes)$ transforms colimits into limits. This is sufficient for representability of a functor $\widehat{\mathcal{I}}^{\mathrm{op}} \to$ *Sets*.

Presheaf $Y \multimap Z$ representing $H_{Y,Z}$ can easily be recovered by Yoneda:

$$(Y \multimap Z)(S) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_S, Y \multimap Z) \cong H_{Y,Z}(h_S) = \mathrm{Bilin}_{\widehat{\mathcal{I}}}(h_S, Y; Z).$$

Writing any $X : \widehat{\mathcal{I}}$ as a colimit of representable sheaves, we see that $H_{Y,Z}(X) \to \mathrm{Hom}(X, Y \multimap Z)$ is an isomorphism for all $X$, not only for representable.

2.3.8. *Formulas for the inner Hom $Y \multimap Z$.* We have established the following formula for the inner Hom $Y \multimap Z$:

$$(Y \multimap Z)(S) \cong \mathrm{Bilin}_{\widehat{\mathcal{I}}}(h_S, Y; Z) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_S \widehat{\otimes} Y, Z). \tag{2.3.8.1}$$

In particular, for any $T : \mathcal{I}$ we have

$$(h_T \multimap Z)(S) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_S \widehat{\otimes} h_T, Z) \cong Z(S \otimes T). \tag{2.3.8.2}$$

Denote by $r_T\colon S \rightsquigarrow S \otimes T$ the "right multiplication by $T$" endofunctor on $\mathcal{I}$. Then

$$(h_T \multimap Z) \cong Z \circ r_T = r_T^* Z. \tag{2.3.8.3}$$

By the universal property of left Kan extensions, this implies that

$$X \widehat{\otimes} h_T \cong r_{T,!} X \tag{2.3.8.4}$$

and by interchanging the arguments

$$h_S \widehat{\otimes} Y \cong l_{S,!} Y, \quad \text{where } l_S\colon T \rightsquigarrow S \otimes T. \tag{2.3.8.5}$$

Finally, this implies

$$\begin{aligned} (Y \multimap Z)(S) &\cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_S \widehat{\otimes} Y, Z) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(l_{S,!} Y, Z) \\ &\cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(Y, l_S^* Z) = \mathrm{Hom}_{\widehat{\mathcal{I}}}(Y, Z \circ l_S). \end{aligned} \tag{2.3.8.6}$$

In other words, $(Y \multimap Z)(S)$ consists of collections of maps of sets $\phi = (\phi_T)_{T:\mathcal{I}}$, $\phi_T\colon Y(T) \to Z(S \otimes T)$ compatible with all morphisms $g\colon T' \to T$ in $\mathcal{I}$.

2.3.9. *Restriction of $\widehat{\otimes}$ to $\mathcal{I}$-fuzzy sets.* Notice that $\widehat{\otimes}$ preserves coproducts in each arguments, and transforms pairs of representable presheaves into representable presheaves. This implies that $\mathrm{Free}(\widehat{\mathcal{I}})$, the full subcategory of "free" presheaves (i.e., presheaves isomorphic to a coproduct of representable presheaves), is stable under $\widehat{\otimes}$. In particular, when $\otimes$ is a (symmetric) monoidal structure on $\mathcal{I}$, we see that $\mathrm{Free}(\widehat{\mathcal{I}})$ is a monoidal subcategory of $\widehat{\mathcal{I}}$.

On the other hand, the functor $I\colon \mathcal{F}_{\mathcal{I}} \to \widehat{\mathcal{I}}$ transforming $\mathcal{I}$-fuzzy set $\mathcal{X} = (X, m_{\mathcal{X}})$ into the presheaf

$$I\mathcal{X} := \coprod_{x\in X} h_{m_{\mathcal{X}}(x)} \tag{2.3.9.1}$$

establishes equivalence between $\mathcal{F}_{\mathcal{I}}$ and $\mathrm{Free}(\widehat{\mathcal{I}})$. This means that we can transport the restriction of $\widehat{\otimes}$ to $\mathrm{Free}(\widehat{\mathcal{I}})$ along the quasi-inverse of this equivalence and obtain a bifunctor $\otimes = \otimes_{\mathcal{F}}$ on $\mathcal{F}_{\mathcal{I}}$. When $\otimes$ is a (symmetric) monoidal structure on $\mathcal{I}$, this $\otimes_{\mathcal{F}}$ determines a symmetric monoidal structure on $\mathcal{F}_{\mathcal{I}}$, and $I$ becomes a monoidal functor.

2.3.10. *Explicit formula for the tensor product of $\mathcal{I}$-fuzzy sets.* More explicitly, let $\mathcal{X} = (X, m_{\mathcal{X}})$ and $\mathcal{Y} = (Y, m_{\mathcal{Y}})$ be two $\mathcal{I}$-fuzzy sets. Then, since $\widehat{\mathcal{I}}$ preserves coproducts in each argument and restricts to the original $\otimes$ on representable presheaves, we have

$$I\mathcal{X}\widehat{\otimes}I\mathcal{Y} \cong \coprod_{x\in X, y\in Y} h_{m_{\mathcal{X}}(x)\otimes m_{\mathcal{Y}}(y)}. \tag{2.3.10.1}$$

This is canonically isomorphic to $I(\mathcal{X}\otimes\mathcal{Y})$ if we put

$$\mathcal{X}\otimes\mathcal{Y} := (X\times Y, m_{\mathcal{X}\otimes\mathcal{Y}}), \quad \text{where} \tag{2.3.10.2}$$

$$m_{\mathcal{X}\otimes\mathcal{Y}}(x,y) := m_{\mathcal{X}}(x)\otimes m_{\mathcal{Y}}(y). \tag{2.3.10.3}$$

Our next goal is to provide a formula for the inner Hom of $\mathcal{I}$-fuzzy sets. The following statement is a generalization of **1.7.11**.

**Theorem 2.3.11.** (*Inner Homs for $\mathcal{I}$-fuzzy sets.*) *Suppose that $\mathcal{I}$ is a closed symmetric monoidal category such that arbitrary* (*small*) *products exist in $\mathcal{I}$. Then the category of $\mathcal{I}$-fuzzy sets $\mathcal{F}_{\mathcal{I}}$ is also a closed symmetric monoidal category, and the functor $I\colon \mathcal{F}_{\mathcal{I}} \to \widehat{\mathcal{I}}$ preserves the inner Hom. Furthermore, given two $\mathcal{I}$-fuzzy sets $\mathcal{X} = (X, m_{\mathcal{X}})$ and $\mathcal{Y} = (Y, m_{\mathcal{Y}})$, the inner Hom $\mathcal{X} \multimap \mathcal{Y}$ can be constructed as follows. Its underlying set is the set* $\mathrm{Hom}(X,Y) = \mathrm{Hom}_{\mathit{Sets}}(X,Y)$ *of all maps of sets $\phi\colon X \to Y$, and the membership function is given by*

$$m_{\mathcal{X}\multimap\mathcal{Y}}(\phi) = \prod_{x\in X}\left(m_{\mathcal{X}}(x) \multimap m_{\mathcal{Y}}\big(\phi(y)\big)\right). \tag{2.3.11.1}$$

*Proof.* We have to show that $I\mathcal{X} \multimap I\mathcal{Y}$, computed in $\widehat{\mathcal{I}}$, is isomorphic to $I\mathcal{Z}$, where $\mathcal{Z} = (\mathrm{Hom}(X,Y), m_{\mathcal{Z}})$, with $m_{\mathcal{Z}}$ given by formula (2.3.11.1). This would imply $\mathcal{Z} \cong (\mathcal{X} \multimap \mathcal{Y})$ in $\mathcal{F}_{\mathcal{I}}$, since $I\colon \mathcal{F}_{\mathcal{I}} \to \widehat{\mathcal{I}}$ is a fully faithful monoidal functor. Computation of $I\mathcal{X} \multimap I\mathcal{Y}$ will be done in several steps: □

2.3.12. *$\multimap$ in $\widehat{\mathcal{I}}$ preserves the limits.* Observe that $\multimap$ in $\widehat{\mathcal{I}}$ transforms the colimits in first argument into limits, since $\widehat{\otimes}$ preserves colimits separately in each argument. This is applicable in particular to the coproduct defining $I\mathcal{X} = \coprod_{x\in X} h_{m_{\mathcal{X}}(x)}$:

$$(I\mathcal{X} \multimap I\mathcal{Y}) \cong \prod_{x\in X}(h_{m_{\mathcal{X}}(x)} \multimap I\mathcal{Y}). \tag{2.3.12.1}$$

2.3.13. *$h_S \multimap -$ preserves the colimits.* Our next claim is that $W \rightsquigarrow (h_S \multimap W)$ preserves arbitrary colimits in $\widehat{\mathcal{I}}$. Indeed, by (2.3.8.3), $h_S \multimap W$ is isomorphic to the composition $r_S^*(W) = W \circ r_S$, where $r_S\colon T \rightsquigarrow T\otimes S$ is the right multiplication by $S$, and $r_S^*$ preserves arbitrary colimits, for example because it admits a right adjoint, the right Kan extension $r_{S,*}$ of $r_S$.

2.3.14. *Application to $h_S \multimap I\mathcal{Y}$.* We can apply the previous remark to the coproduct defining $I\mathcal{Y}$. We obtain

$$(h_S \multimap I\mathcal{Y}) \cong \coprod_{y\in Y} (h_S \multimap h_{m_{\mathcal{Y}}(y)}). \tag{2.3.14.1}$$

2.3.15. *Computation of $h_S \multimap h_T$.* Finally, we have to compute $h_S \multimap h_T$, assuming the monoidal structure of $\mathcal{I}$ closed. For any $U : \mathcal{I}$, we have

$$\begin{aligned}(h_S \multimap h_T)(U) &\cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_U, h_S \multimap h_T) \cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_U \widehat{\otimes} h_S, h_T)\\ &\cong \mathrm{Hom}_{\widehat{\mathcal{I}}}(h_{U\otimes S}, h_T) \cong \mathrm{Hom}_{\mathcal{I}}(U\otimes S, T)\\ &\cong \mathrm{Hom}_{\mathcal{I}}(U, S \multimap_{\mathcal{I}} T) \cong h_{S\multimap T}(U).\end{aligned} \tag{2.3.15.1}$$

This proves that

$$(h_S \multimap h_T) \cong h_{S\multimap T} \tag{2.3.15.2}$$

whenever the monoidal structure of $\mathcal{I}$ is closed.

2.3.16. *Computation of $I\mathcal{X} \multimap I\mathcal{Y}$.* Combining together formulas (2.3.12.1), (2.3.14.1), and (2.3.15.2), we get

$$(I\mathcal{X} \multimap I\mathcal{Y}) \cong \prod_{x\in X} \coprod_{y\in Y} h_{m_{\mathcal{X}}(x)\multimap m_{\mathcal{Y}}(y)} \tag{2.3.16.1}$$

in the case when the monoidal structure of $\mathcal{I}$ is closed. Assuming the existence of arbitrary products in $\mathcal{I}$, we can apply **2.2.20** to compute the product on the right-hand side of (2.3.16.1), which is the $X$-indexed product of $I\mathcal{Y}_x$, where $\mathcal{Y}_x$ is the $\mathcal{I}$-fuzzy set $\mathcal{Y}_x = (Y, m_{\mathcal{Y}_x})$, $m_{\mathcal{Y}_x}(y) = (m_{\mathcal{X}}(x) \multimap m_{\mathcal{Y}}(y))$. We see that this product is isomorphic to $I\mathcal{Z}$, where the underlying set of $\mathcal{Z}$ is $\prod_{x\in X} Y \cong \mathrm{Hom}_{Sets}(X,Y)$, and the membership function $m_{\mathcal{Z}}\colon \mathrm{Hom}(X,Y) \to \mathrm{Ob}\,\mathcal{I}$ is given by

$$m_{\mathcal{Z}}(\phi) = \prod_{x\in X} \bigl(m_{\mathcal{X}}(x) \multimap m_{\mathcal{Y}}(\phi(x))\bigr)\,. \tag{2.3.16.2}$$

This is exactly formula (2.3.11.1), and we have completed the proof of **2.3.11**.

2.3.17. *Interpretation in the $\mathcal{I}$-valued linear logic.* Formula (2.3.16.2) can be understood as an interpretation of the statement "$\phi$: $X \to Y$ is a map $\mathcal{X} \to \mathcal{Y}$ if and only if for all $x \in X$, condition $x \in \mathcal{X}$ linearly implies $\phi(x) \in \mathcal{Y}$" in the $\mathcal{I}$-valued linear logic. Formula (2.3.10.3) admits a similar interpretation: "$(x, y) \in \mathcal{X} \otimes \mathcal{Y}$ if and only if the linear conjunction of $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ holds".

2.3.18. *Dual case: the monoidal structure on $\mathcal{I}$-normed and $\mathcal{I}$-graded sets.* Replacing $\mathcal{I}$ with $\mathcal{I}^{\mathrm{op}}$, we obtain statements about the category $\mathcal{N}_{\mathcal{I}} \cong \mathcal{F}_{\mathcal{I}^{\mathrm{op}}}$ of $\mathcal{I}$-normed sets and the category $Sets^{\mathcal{I}} \cong \widehat{\mathcal{I}^{\mathrm{op}}}$ of $\mathcal{I}$-graded sets. Instead of restating all previous results in the dual situation, we are going to refer to them directly, leaving the task of dualizing statements to the reader.

We are also going to discuss the more general case of a monoidal structure on $\mathcal{C}^{\mathcal{I}}$, where $\mathcal{I}$ is a small monoidal category as before, but $\mathcal{C}$ is any monoidal category. When $\mathcal{C}$ is *Sets* with the Cartesian monoidal structure, we recover counterparts of our previous constructions for the covariant functor case.

**2.4. Monoidal structure on $\mathcal{C}^{\mathcal{I}}$.** The reader may have noticed that one can replace $(Sets, \times)$ with any other monoidal category $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}})$ in the definition (2.3.3.3) of a "bilinear natural transformation" of presheaves. This will be our starting point for defining a monoidal structure on the functor category $\mathcal{C}^{\mathcal{I}}$ for any small monoidal category $\mathcal{I}$ and any (cocomplete closed) monoidal category $\mathcal{C}$. This monoidal structure is by no means new; it is known as the "Day convolution product", cf. [Day].

We shall be particularly interested in the cases where $\mathcal{C} = Ab$ and $\mathcal{C} = K\text{-}Mod$, $K$ a commutative ring, apart from the already considered case of $\mathcal{C} = Sets$.

2.4.0. *Conditions on $\mathcal{I}$ and $\mathcal{C}$.* In what follows, $\mathcal{I}$ denotes a small monoidal category, and $\mathcal{C}$ a cocomplete (i.e., closed under colimits) *closed* monoidal category. We want to construct a monoidal structure on $\mathcal{C}^{\mathcal{I}} = Funct(\mathcal{I}, \mathcal{C})$ under these assumptions, symmetric if both original monoidal structures are symmetric. Our first steps require $\mathcal{I}$ to be merely a small category.

2.4.1. *Terminology: $\mathcal{I}$-graded Abelian groups, $K$-modules, objects of $\mathcal{C}$.* We say that Abelian objects of $\mathcal{C}^{\mathcal{I}}$, i.e., covariant functors $\mathcal{I} \to \mathcal{C}$, are *$\mathcal{I}$-graded objects of $\mathcal{C}$.* When $\mathcal{C} = Sets$, $\mathcal{C} = Ab$, $K\text{-}Mod$ for a commutative ring $K$ etc., we speak about *$\mathcal{I}$-graded sets* (cf. **2.0.1**), *$\mathcal{I}$-graded Abelian groups*, *$\mathcal{I}$-graded $K$-modules* and so on.

2.4.2. *The case of discrete $\mathcal{I} = \Delta$.* Consider the case of discrete $\mathcal{I}$, when $\mathcal{I} = \Delta$, with $\Delta$ a monoid. Then $\mathcal{C}^{\Delta}$ consists of families $\mathbf{X} = (X_\alpha)_{\alpha\in\Delta}$ of objects of $\mathcal{C}$, with morphisms $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ given by families $\mathbf{f} = (f_\alpha)_{\alpha\in\Delta}$ of morphisms $f_\alpha\colon X_\alpha \to Y_\alpha$. In other words, we recover the classical category of $\Delta$-graded objects in $\mathcal{C}$.

2.4.3. *Underlying object of an $\mathcal{I}$-graded object.* Let $\mathcal{X}\colon \mathcal{I} \to \mathcal{C}$ be an $\mathcal{I}$-graded object of $\mathcal{C}$. We say that the colimit $\operatorname{colim}_{\mathcal{I}} \mathcal{X}$ is the *underlying object of $\mathcal{X}$*, and sometimes denote it by $|\mathcal{X}|$. When the natural morphism $\mathcal{X}(a) \to |\mathcal{X}|$ is a monomorphism, we may want to identify $\mathcal{X}(a)$ with a subobject of $|\mathcal{X}|$. This terminology is compatible with **2.1.4** and **2.2.2**.

On the other hand, when $\mathcal{C}$ is additive and $\mathcal{I} = \Delta$ is discrete, we have $|\mathcal{X}| = \bigoplus_{\alpha\in\Delta} X_\alpha$ for $\mathcal{X} = (X_\alpha)_{\alpha\in\Delta}$, and each component $X_\alpha$ is canonically identified with the corresponding subobject in the direct sum $|\mathcal{X}|$, so we recover the "elementary" definition of graded Abelian groups or $K$-modules in this case.

2.4.4. *Special objects $[M]^\alpha$ of $\mathcal{C}^{\mathcal{I}}$.* Given objects $M : \mathcal{C}$ and $\alpha : \mathcal{I}$, we can define a *special object* $[M]^\alpha$ of $\mathcal{C}^{\mathcal{I}}$ by requiring the existence of isomorphism

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}\big([M]^\alpha, \mathcal{X}\big) \cong \operatorname{Hom}_{\mathcal{C}}\big(M, \mathcal{X}(\alpha)\big) \tag{2.4.4.1}$$

functorial in $\mathcal{X}$ from $\mathcal{C}^{\mathcal{I}}$. Sometimes we write $M^\alpha$ instead of $[M]^\alpha$ when no confusion may arise.

Clearly, $[M]^\alpha$ is determined uniquely (up to unique isomorphism) by the above property. It always exists since $\mathcal{C}$ was assumed cocomplete, and can be computed for example with the aid of the left Kan extension $I_{\alpha,!}M$ of the embedding $I_\alpha\colon \mathbf{1} \to \mathcal{I}$ that maps the only object of the unit category into $\alpha : \mathcal{I}$. This description yields a formula for $[M]^\alpha$:

$$[M]^\alpha\colon \beta \rightsquigarrow M^{(\operatorname{Hom}_{\mathcal{I}}(\alpha,\beta))}. \tag{2.4.4.2}$$

Here $M^{(S)}$ denotes the $S$-indexed coproduct of copies of the object $M$, characterized by property $\operatorname{Hom}_{\mathcal{C}}(M^{(S)}, N) \cong \operatorname{Hom}_{\mathcal{C}}(M, N)^S$. One might also write $M \otimes S$ or $M \oslash S$ instead of $M^{(S)}$, thinking of $\mathcal{C}$ as naturally tensored (or enriched) over *Sets*.

One can also check directly that the right-hand side of (2.4.4.2) satisfies the universal property of (2.4.4.1).

Notice that $(\alpha, M) \rightsquigarrow [M]^\alpha$ defines a bifunctor $\mathcal{I}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, commuting with arbitrary colimits in the second variable. It is worthwhile to remark that (2.4.4.2) together with (2.4.4.1) imply

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}\bigl([M]^\alpha, [N]^\beta\bigr) &\cong \mathrm{Hom}_{\mathcal{C}}\bigl(M, [N]^\beta(\alpha)\bigr)\\ &\cong \mathrm{Hom}_{\mathcal{C}}\bigl(M, N^{(\mathrm{Hom}_{\mathcal{I}}(\beta,\alpha))}\bigr).\end{aligned} \tag{2.4.4.3}$$

2.4.5. *Special objects generate $\mathcal{C}^{\mathcal{I}}$ under colimits.* The special objects $[M]^\alpha$, where $\alpha$ runs through all objects of $\mathcal{I}$, and $M$ through all objects, or merely a set of generators for $\mathcal{C}$, constitute a dense system of generators for $\mathcal{C}^{\mathcal{I}}$. In other words, the functor mapping $\mathcal{X} : \mathcal{C}^{\mathcal{I}}$ into presheaf

$$(\alpha, M) \rightsquigarrow \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}([M]^\alpha, \mathcal{X}) \cong \mathrm{Hom}_{\mathcal{C}}(M, \mathcal{X}(\alpha))$$

on $\mathcal{I}^{\mathrm{op}} \times \mathcal{C}$ or $\mathcal{I}^{\mathrm{op}} \times \mathcal{G}$, where $\mathcal{G}$ is a full subcategory generating $\mathcal{C}$, is fully faithful.

2.4.6. *Coends.* We have seen that any object $\mathcal{X} : \mathcal{C}^{\mathcal{I}}$ can be represented as a colimit of special objects. In fact, there is a canonical way of expressing $\mathcal{X}$ in this way, best described in terms of *coends*.

Recall that a *coend* $\int^{\mathcal{I}} F$ of a functor $F\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a small category and $\mathcal{C}$ is any category, is a special kind of colimit along the "morphism decomposition" category $\mathrm{Ar}^{\to}\mathcal{I}$ of all morphisms $\phi\colon \alpha \to \beta$ in $\mathcal{I}$, with morphisms from $\phi\colon \alpha \to \beta$ to $\phi'\colon \alpha' \to \beta'$ given by couples $(\chi_0, \chi_1)$, where $\chi_0\colon \alpha \to \alpha'$ and $\chi_1\colon \beta' \to \beta$ are morphisms in $\mathcal{I}$, providing a decomposition $\phi = \chi_1 \circ \phi' \circ \chi_0$, of the functor $\widetilde{F}\colon (\phi\colon \alpha \to \beta) \rightsquigarrow F(\beta, \alpha)$, with $\widetilde{F}((\chi_0, \chi_1))$ given by $F(\chi_1, \chi_0)$:

$$\int^{\mathcal{I}} F = \mathrm{colim}_{\mathrm{Ar}^{\to}\mathcal{I}}\, \widetilde{F}. \tag{2.4.6.1}$$

Another expression for coends in terms of iterated colimits is given by the following coequalizer of coproducts expression:

$$\coprod_{\phi\colon \alpha\to\beta} F(\beta, \alpha) \rightrightarrows \coprod_{\alpha : \mathcal{I}} F(\alpha, \alpha) \dashrightarrow \int^{\mathcal{I}} F. \tag{2.4.6.2}$$

In other words, $\mathrm{Hom}_{\mathcal{C}}(\int^{\mathcal{I}} F, X)$ is canonically isomorphic to the set of all "extranatural transformations" $\theta\colon F \to X$, i.e., families of morphisms

$$\theta = (\theta_\alpha)_{\alpha : \mathcal{I}},\ \theta_\alpha\colon F(\alpha, \alpha) \to X$$

such that $\theta_\alpha \circ F(\phi, \mathrm{id}_\alpha) = \theta_\beta \circ F(\mathrm{id}_\beta, \phi)$ for any morphism $\phi\colon \alpha \to \beta$ in $\mathcal{I}$:

$$\begin{tikzcd} F(\beta,\alpha) \arrow[r, "F(\mathrm{id}_\beta,\phi)"] \arrow[d, "F(\phi,\mathrm{id}_\alpha)"] & F(\beta,\beta) \arrow[d, "\theta_\beta"] \\ F(\alpha,\alpha) \arrow[r, "\theta_\alpha"] & X. \end{tikzcd} \tag{2.4.6.3}$$

We sometimes write $\int^{\iota : \mathcal{I}} \mathcal{E}$ or $\int^{\iota} \mathcal{E}$, where $\mathcal{E} = \mathcal{E}\{\iota, \iota\}$ is an expression where $\iota$ occurs exactly twice, once covariantly and once contravariantly, to denote the coend $\int^{\mathcal{I}} \bigl((\iota, \iota') \rightsquigarrow \mathcal{E}\{\iota, \iota'\}\bigr)$.

2.4.7. *Coend expression for an $\mathcal{I}$-graded object.* Now let $\mathcal{X}\colon \mathcal{I} \to \mathcal{C}$ be any $\mathcal{I}$-graded object of $\mathcal{C}$. We claim that

$$\mathcal{X} \cong \int^{\alpha : \mathcal{I}} [\mathcal{X}(\alpha)]^\alpha. \tag{2.4.7.1}$$

Indeed, by definition $\mathrm{Hom}_{\mathcal{C}^\mathcal{I}}\left(\int^\alpha [\mathcal{X}(\alpha)]^\alpha, \mathcal{Y}\right)$ consists of families of morphisms $\theta = (\theta_\alpha)_{\alpha : \mathcal{I}}$, $\theta_\alpha\colon [\mathcal{X}(\alpha)]^\alpha \to \mathcal{Y}$, "extranatural" in the sense that

$$\theta_\beta \circ [\mathcal{X}(\phi)]^\beta = \theta_\alpha \circ [\mathcal{X}(\alpha)]^\phi \quad \text{for any } \phi\colon \alpha \to \beta \text{ in } \mathcal{I}. \tag{2.4.7.2}$$

By the definition of the special object $[\mathcal{X}(\beta)]^\alpha$, this is the same thing as a collection $\theta^\flat_\alpha\colon \mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)$ such that

$$\left(\theta_\beta \circ [\mathcal{X}(\phi)]^\beta\right)^\flat = \left(\theta_\alpha \circ [\mathcal{X}(\alpha)]^\phi\right)^\flat, \tag{2.4.7.3}$$

i.e.,

$$\theta^\flat_\beta \circ \mathcal{X}(\phi) = \mathcal{Y}(\phi) \circ \theta^\flat_\alpha. \tag{2.4.7.4}$$

This is precisely the condition for $\left(\theta^\flat_\alpha\colon \mathcal{X}(\alpha) \to \mathcal{Y}(\alpha)\right)_{\alpha : \mathcal{I}}$ to constitute a natural transformation $\mathcal{X} \to \mathcal{Y}$, hence $\mathrm{Hom}_{\mathcal{C}^\mathcal{I}}\left(\int^{\alpha : \mathcal{I}} [\mathcal{X}(\alpha)]^\alpha, \mathcal{Y}\right) \cong \mathrm{Hom}_{\mathcal{C}^\mathcal{I}}(\mathcal{X}, \mathcal{Y})$ for any $\mathcal{Y} : \mathcal{C}^\mathcal{I}$, so $\mathcal{X} \cong \int^{\alpha : \mathcal{I}} [\mathcal{X}(\alpha)^\alpha]$ as claimed.

2.4.8. *Special objects in $Sets^\mathcal{I}$.* When $\mathcal{C} = Sets$, $[\mathbf{1}]^\alpha$ coincides with the corepresentable functor $h^\alpha$, and $[S]^\alpha$ is isomorphic to the coproduct of $S$ copies of $[\mathbf{1}]^\alpha = h^\alpha$. Since $\mathbf{1}$ generates *Sets*, we see that $[\mathbf{1}]^\alpha = h^\alpha$ generate $Sets^\mathcal{I}$. The brief notation $\mathbf{1}^\alpha$ for $[\mathbf{1}]^\alpha$ turns out to be compatible with **1.5.1** in view of **2.1.9**.

2.4.9. *Special generators of $Ab^\mathcal{I}$ and $K$-$Mod^\mathcal{I}$.* The category $\mathcal{C} = K$-*Mod* is generated by one object $K = K_s$ (the ring $K$, viewed as a left module over itself). This implies that $[K]^\alpha$ generate $K$-$Mod^\mathcal{I}$, and $[\mathbb{Z}]^\alpha$ generate $Ab^\mathcal{I}$. Furthermore, there generators are projective:

2.4.10. $[P]^\alpha$ *is projective when $P$ is.* Notice that $[P]^\alpha$ is a projective object of $\mathcal{C}^\mathcal{I}$ whenever $P$ is projective in $\mathcal{C}$, because effective epimorphisms of $\mathcal{C}^\mathcal{I}$ are detected componentwise. Therefore, if $\mathcal{G} \subset \mathcal{C}$ is a set (full subcategory) of projective generators for $\mathcal{C}$, then $[P]^\alpha$, $P : \mathcal{G}$, $\alpha : \mathcal{I}$, constitute a family of projective generators for $\mathcal{C}^\mathcal{I}$, and any projective object of $\mathcal{C}^\mathcal{I}$ has to be a retract of a coproduct of some objects of this form. For example, any projective $\mathcal{I}$-graded Abelian group has to be a retract of the direct sum of copies of $[\mathbb{Z}]^\alpha$.

2.4.11. *Underlying object of* $[M]^\alpha$. The underlying object $\left|[M]^\alpha\right| = \mathrm{colim}_\mathcal{I}[M]^\alpha$ is canonically isomorphic to $M$. This follows for example from the fact that the underlying object functor $\mathrm{colim}_\mathcal{I}\colon \mathcal{C}^\mathcal{I} \to \mathcal{C}$ is the left Kan extension $P_!$ along the canonical projection $P\colon \mathcal{I} \to \mathbf{1}$, and $[M]^\alpha$ is the value of a left Kan extension $I_{\alpha,!}(M)$ as well, hence $\mathrm{colim}_\mathcal{I}[M]^\alpha \cong P_! I_{\alpha,!}(M) \cong (P \circ I_\alpha)_!(M) \cong M$.

2.4.12. *Bilinear natural transformations in $\mathcal{C}^\mathcal{I}$.* Now we start assuming both $\mathcal{I}$ and $\mathcal{C}$ monoidal. In this case, we can give a definition similar to (2.3.3.3).

Given three $\mathcal{I}$-graded objects $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{C}$, we define a *bilinear natural transformation* $\Phi\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ to be a collection $\Phi = (\Phi_{\alpha,\beta})_{\alpha,\beta : \mathcal{I}}$ of morphisms in $\mathcal{C}$, where

$$\Phi_{\alpha,\beta}\colon \mathcal{X}(\alpha) \otimes_\mathcal{C} \mathcal{Y}(\beta) \to \mathcal{Z}(\alpha \otimes_\mathcal{I} \beta). \tag{2.4.12.1}$$

This collection must be natural in $\alpha$ and $\beta$, meaning that for any morphisms $f\colon \alpha \to \alpha'$ and $g\colon \beta \to \beta'$ in $\mathcal{I}$ the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta) \arrow[r, "\mathcal{X}(f)\otimes_{\mathcal{C}}\mathcal{Y}(g)"] \arrow[d, "\Phi_{\alpha,\beta}"] & \mathcal{X}(\alpha') \otimes_{\mathcal{C}} \mathcal{Y}(\beta') \arrow[d, "\Phi_{\alpha',\beta'}"] \\
\mathcal{Z}(\alpha \otimes_{\mathcal{I}} \beta) \arrow[r, "\mathcal{Z}(f\otimes_{\mathcal{I}} g)"] & \mathcal{Z}(\alpha' \otimes_{\mathcal{I}} \beta').
\end{tikzcd} \tag{2.4.12.2}$$

We denote by $\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X},\mathcal{Y};\mathcal{Z})$ the set of all natural bilinear transformations

$$\Phi\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}.$$

Another equivalent way of defining $\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X},\mathcal{Y};\mathcal{Z})$ is given by the formula

$$\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X},\mathcal{Y};\mathcal{Z}) = \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}\times\mathcal{I}}}\big(\otimes_{\mathcal{C}} \circ (\mathcal{X}\times\mathcal{Y}), \mathcal{Z}\circ \otimes_{\mathcal{I}}\big). \tag{2.4.12.3}$$

2.4.13. *Polylinear natural transformations.* The above definition can be immediately generalized to the case of *$n$-polylinear natural transformations*, for any integer $n \geq 0$. Given $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}$ in $\mathrm{Ob}\,\mathcal{C}^{\mathcal{I}}$, we define

$$\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Y}) := \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}^n}}\big(\otimes_{\mathcal{C}}^{(n)} \circ (\mathcal{X}_1\times\cdots\times\mathcal{X}_n), \mathcal{Y}\circ\otimes_{\mathcal{I}}^{(n)}\big), \tag{2.4.13.1}$$

where $\otimes_{\mathcal{C}}^{(n)}\colon \mathcal{C}^n \to \mathcal{C}$ denotes the iterated tensor product of $\mathcal{C}$, and similarly for $\mathcal{I}$.

2.4.14. *Polycategory structure on $\mathcal{C}^{\mathcal{I}}$.* The sets $\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Y})$ introduce on $\mathcal{C}^{\mathcal{I}}$ a polycategory structure (i.e., a generalization of the notion of a category where one has "morphisms" from finite sequences of objects into one object), compatible with the original category structure since 1-polylinear natural transformations coincide with the usual natural transformations:

$$\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X};\mathcal{Y}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X},\mathcal{Y}). \tag{2.4.14.1}$$

Actually,a polycategory is a special version of a colored operad, with set of colors equal to the set of objects of the polycategory in question. This means that the structure of a polycategory must also include "operadic composition" maps

$$\begin{split}\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{Y}_1,\ldots,\mathcal{Y}_r;\mathcal{Z}) \times \prod_{i=1}^{r} \mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_{\alpha_{i-1}+1},\ldots,\mathcal{X}_{\alpha_i};\mathcal{Y}_i)\\ \to \mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n;\mathcal{Z})\end{split} \tag{2.4.14.2}$$

for any $n$, $r \geq 0$, $0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r = n$. Such maps are quite easy to define; we do not wish to write out explicit expressions for them at this point.

2.4.15. *Construction of tensor product on $\mathcal{C}^{\mathcal{I}}$.* We denote by $\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$ or $\mathcal{X}_1 \otimes_{\mathcal{C}^{\mathcal{I}}} \cdots \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{X}_n$ the object of $\mathcal{C}^{\mathcal{I}}$ representing $\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n;-)$. Such an object always exists: indeed, according to (2.4.13.1), it can be defined with the aid of the left Kan extension $\otimes_{\mathcal{I},!}^{(n)}$ of the functor $\otimes_{\mathcal{I}}^{(n)}\colon \mathcal{I}^n \to \mathcal{I}$:

$$\mathcal{X}_1 \otimes_{\mathcal{C}^{\mathcal{I}}} \cdots \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{X}_n \cong \otimes_{\mathcal{I},!}^{(n)}\big(\otimes_{\mathcal{C}}^{(n)} \circ (\mathcal{X}_1\times\cdots\times\mathcal{X}_n)\big). \tag{2.4.15.1}$$

Such left Kan extensions always exist in our situation, category $\mathcal{C}$ being assumed cocomplete.

In particular, we have a "tensor product" bifunctor $\otimes = \otimes_{\mathcal{C}^{\mathcal{I}}}\colon \mathcal{C}^{\mathcal{I}} \times \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$ such that $\mathcal{X}\otimes\mathcal{Y}$ represents $\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X},\mathcal{Y};-)$.

2.4.16. *Iterated tensor products and associativity.* We would like to show the existence of canonical isomorphisms

$$\gamma_{n,m}\colon \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_{m+n} \xrightarrow{\sim} (\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n) \otimes (\mathcal{X}_{n+1} \otimes \cdots \otimes \mathcal{X}_{n+m}). \tag{2.4.16.1}$$

This would imply that $\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$ can be constructed as an iterated binary tensor the product $\otimes_{\mathcal{C}^{\mathcal{I}}}$, with any choice of the order of operations. In turn, this would mean the existence of associativity and unit constraints for $\otimes_{\mathcal{C}^{\mathcal{I}}}$, i.e., a monoidal structure on $\mathcal{C}^{\mathcal{I}}$.

Notice that the existence of a natural transformation $\gamma_{n,m}$ follows from the existence of the polycategoric composition maps of (2.4.14.2); the problem is to prove that $\gamma_{n,m}$ is an isomorphism.

2.4.17. *Iterated tensor products in $\mathcal{C}^{\mathcal{I}}$ commute with colimits whenever this holds in $\mathcal{C}$.* We claim that the (*iterated*) *tensor products* $\otimes_{\mathcal{C}^{\mathcal{I}}}\colon \mathcal{C}^{\mathcal{I}} \times \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$ *and* $\otimes^{(n)}_{\mathcal{C}^{\mathcal{I}}}\colon (\mathcal{C}^{\mathcal{I}})^n \to \mathcal{C}^{\mathcal{I}}$ *commute with colimits in each variable.* Indeed, this is equivalent to saying that

$$\mathrm{Polylin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_1, \ldots, \mathcal{X}_n; \mathcal{Y})$$

transforms the colimits in each argument $\mathcal{X}_i$ into the corresponding limits of sets; let us show this for the first argument $\mathcal{X}$ of $\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}, \mathcal{Y}; \mathcal{Z})$, the general case being similar. Suppose that $\mathcal{X} = \operatorname{inj\,lim}_{\iota\colon\mathcal{D}} \mathcal{X}_\iota$. We are writing here a colimit as a "generalized inductive limit along a small category $\mathcal{D}$"; in other words, $\operatorname{inj\,lim}_\iota \mathcal{X}_\iota$ is simply a shorthand for $\mathrm{colim}(\lambda\iota.\mathcal{X}_\iota)$ or $\mathrm{colim}(\iota \rightsquigarrow \mathcal{X}_\iota)$, so we are actually dealing with the case of an arbitrary (small) colimit. In this case, $\mathrm{Bilin}(\mathcal{X}, \mathcal{Y}; \mathcal{Z})$ consists of collections $\Phi = (\Phi_{\alpha,\beta})$ of morphisms

$$\Phi_{\alpha,\beta}\colon \mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta) \to \mathcal{Z}(\alpha \otimes \beta) \tag{2.4.17.1}$$

natural in $\alpha$ and $\beta : \mathcal{I}$. Write $\mathcal{X}(\alpha)$ as $\operatorname{inj\,lim}_\iota \mathcal{X}_\iota(\alpha)$ and use the fact that $\otimes_{\mathcal{C}}$ commutes with colimits in the first argument, the monoidal structure of $\mathcal{C}$ being assumed to be closed (we use the closedness of $\otimes_{\mathcal{C}}$ for the first time here!). We see that $\Phi$ is given by a collection

$$\Phi^\iota_{\alpha,\beta}\colon \mathcal{X}_\iota(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta) \to \mathcal{Z}(\alpha \otimes \beta) \tag{2.4.17.2}$$

natural in $\alpha$, $\beta : \mathcal{I}$ and $\iota : \mathcal{D}$. Fixing $\iota$ and letting $\alpha$ and $\beta$ vary, we obtain a collection of elements $\Phi^\iota \in \mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_\iota, \mathcal{Y}; \mathcal{Z})$, natural with respect to $\iota$. This is the same thing as an element of $\operatorname{proj\,lim}_\iota \mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}_\iota, \mathcal{Y}; \mathcal{Z})$ as desired.

2.4.18. *Reduction to the case of special objects.* Recall that $\mathcal{C}^{\mathcal{I}}$ is generated under colimits by the special objects $[M]^\alpha$ of **2.4.4**; one can even write arbitrary $\mathcal{X}$ as a colimit of $[\mathcal{X}(\alpha)]^\beta$ for all morphisms $\phi\colon \alpha \to \beta$ in $\mathcal{I}$, i.e., a *coend* $\int^\alpha [\mathcal{X}(\alpha)]^\alpha$, cf. (2.4.7.1) and (2.4.6.1). Together with the previous result **2.4.17**, this implies that it suffices to check that $\gamma_{n,m}$ of (2.4.16.1) is an isomorphism whenever all $\mathcal{X}_i$ are special: $\mathcal{X}_i = [M_i]^{\alpha_i}$. This would follow from the explicit formula (2.4.19.1) for tensor products of special objects given below.

2.4.19. *Tensor product of special objects of $\mathcal{C}^{\mathcal{I}}$.* We claim that

$$[M_1]^{\alpha_1} \otimes_{\mathcal{C}^{\mathcal{I}}} \otimes \cdots \otimes_{\mathcal{C}^{\mathcal{I}}} [M_n]^{\alpha_n} \cong [M_1 \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} M_n]^{\alpha_1 \otimes \cdots \otimes \alpha_n} \tag{2.4.19.1}$$

for any $n \geq 0$, $M_i : \mathcal{C}$, $\alpha_i : \mathcal{I}$. In particular, for $n = 2$ and $n = 0$ we obtain

$$[M]^\alpha \otimes [N]^\beta \cong [M \otimes N]^{\alpha \otimes \beta} \tag{2.4.19.2}$$

and

$$\mathbf{1}_{\mathcal{C}^{\mathcal{I}}} \cong [\mathbf{1}_{\mathcal{C}}]^{\mathbf{1}_{\mathcal{I}}} \tag{2.4.19.3}$$

Let us prove (2.4.19.1). Put $\mathcal{X}_i := [M_i]^{\alpha_i}$, $\mathcal{X} := \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$. According to (2.4.15.1), $\mathcal{X}$ can be computed as $\otimes^{(n)}_{\mathcal{I},!}(\bar{\mathcal{X}})$, where $\bar{\mathcal{X}} = \otimes^{(n)}_{\mathcal{C}} \circ (\mathcal{X}_1 \times \cdots \times \mathcal{X}_n) \colon \mathcal{I}^n \to \mathcal{C}$. Explicitly,

$$\bar{\mathcal{X}} \colon (\beta_1, \ldots, \beta_n) \rightsquigarrow \mathcal{X}_1(\beta_1) \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \mathcal{X}_n(\beta_n). \tag{2.4.19.4}$$

Recall that $\mathcal{X}_i = [M_i]^{\alpha_i}$, so that $\mathcal{X}_i(\beta_i) \cong M_i^{(\mathrm{Hom}(\alpha_i,\beta_i))}$ by (2.4.4.2). Since $\otimes_{\mathcal{C}}$ is closed, it commutes with arbitrary coproducts, and we arrive at

$$\bar{\mathcal{X}} \colon (\beta_1, \ldots, \beta_n) \rightsquigarrow (M_1 \otimes \cdots \otimes M_n)^{(\mathrm{Hom}(\alpha_1,\beta_1)\times\cdots\times\mathrm{Hom}(\alpha_n,\beta_n))}. \tag{2.4.19.5}$$

Again by (2.4.4.2), this can be identified with a special object of $\mathcal{C}^{\mathcal{I}^n}$:

$$\bar{\mathcal{X}} \cong [M_1 \otimes \cdots \otimes M_n]^{(\alpha_1,\ldots,\alpha_n)}. \tag{2.4.19.6}$$

Recall that special objects can be written as images of left Kan extensions. In particular, $\bar{\mathcal{X}} = I_{\alpha,!}(M)$, where $M := M_1 \otimes \cdots \otimes M_n$, $I_\alpha \colon \mathbf{1} \to \mathcal{I}^n$ is the the functor from the one-point category into $\mathcal{I}$ with image $\alpha := (\alpha_1, \ldots, \alpha_n)$.

Next recall that $\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n$ is given by the left Kan extension $\otimes^{(n)}_{\mathcal{I},!}(\bar{\mathcal{X}})$. Using the transitivity of the left Kan extensions, we obtain

$$\begin{aligned} \mathcal{X} = \otimes^{(n)}_{\mathcal{I},!}(\bar{\mathcal{X}}) &= (\otimes^{(n)}_{\mathcal{I},!} \circ I_{\alpha,!})(M) = (\otimes^{(n)}_{\mathcal{I}} \circ I_\alpha)_!(M) \\ &= I_{\otimes^{(n)}(\alpha),!}(M) = I_{\alpha_1\otimes\cdots\otimes\alpha_n,!}(M) = [M]^{\alpha_1\otimes\cdots\otimes\alpha_n}. \end{aligned} \tag{2.4.19.7}$$

This is formula (2.4.19.1) as required. According to **2.4.16** and **2.4.18**, the existence of this formula implies the existence of associativity and unit constraints for $\otimes_{\mathcal{C}^{\mathcal{I}}}$, i.e., $\mathcal{C}^{\mathcal{I}}$ with $\otimes_{\mathcal{C}^{\mathcal{I}}}$ is a monoidal category.

2.4.20. *Formula for iterated tensor products in terms of coends.* Combining together formula (2.4.19.1) for iterated tensor products of special objects, the fact that $\otimes_{\mathcal{C}^{\mathcal{I}}}$ commutes with colimits in each argument (cf. **2.4.17**), and the explicit expression (2.4.7.1) of any object of $\mathcal{C}^{\mathcal{I}}$ as a coend, i.e., a special type of colimit of special objects, we obtain an expression for the (iterated) tensor product of arbitrary objects of $\mathcal{C}^{\mathcal{I}}$ in terms of iterated coends:

$$\mathcal{X}_1 \otimes_{\mathcal{C}^{\mathcal{I}}} \cdots \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{X}_n \cong \int^{\alpha_1 : \mathcal{I}, \ldots, \alpha_n : \mathcal{I}} \bigl[\mathcal{X}_1(\alpha_1) \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \mathcal{X}_n(\alpha_n)\bigr]^{\alpha_1 \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \alpha_n}. \tag{2.4.20.1}$$

2.4.21. *Components of a tensor product of functors.* Combining (2.4.20.1) with (2.4.4.2), and observing that the coends are computed componentwise in a functor category, we obtain an explicit formula for the components of $\mathcal{X} \otimes \mathcal{Y}$:

$$\mathcal{X} \otimes \mathcal{Y} \colon \gamma \rightsquigarrow \int^{\alpha : \mathcal{I}, \beta : \mathcal{I}} \bigl(\mathcal{X}(\alpha) \otimes \mathcal{Y}(\beta)\bigr)^{(\mathrm{Hom}(\alpha\otimes\beta,\gamma))}. \tag{2.4.21.1}$$

2.4.22. *The case of discrete $\mathcal{I}$.* When the category $\mathcal{I}$ is discrete, i.e., $\mathcal{I} = \Delta$ for some monoid $\Delta$, the above constructions and results reduce to their classical counterparts, especially when $\mathcal{C}$ is additive. For example, the coend $\int^{\iota : \Delta} F(\iota, \iota)$ of a functor

$$F \colon \Delta^{\mathrm{op}} \times \Delta \to \mathcal{C}$$

is isomorphic to the coproduct $\coprod_{\iota\in\Delta} F(\iota, \iota)$ along the diagonal, better written as a direct sum $\oplus_\iota F(\iota, \iota)$ when $\mathcal{C}$ is additive. In particular, (2.4.21.1) becomes

$$\mathcal{X} \otimes_{\mathcal{C}^\Delta} \mathcal{Y} \colon \gamma \rightsquigarrow \bigoplus_{\alpha+\beta=\gamma} \mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta) \tag{2.4.22.1}$$

if we write the monoid operation of $\Delta$ additively.

2.4.23. *$\mathcal{C}^{\mathcal{I}}$ is symmetric whenever both $\mathcal{C}$ and $\mathcal{I}$ are.* Notice that the monoidal category $\mathcal{C}^{\mathcal{I}}$ is symmetric (i.e., admits a commutativity constraint) whenever both $\mathcal{C}$ and $\mathcal{I}$ are. This follows for example from formula (2.4.20.1).

2.4.24. *Tensor product with special object.* When one or several of $\mathcal{X}_i$ in (2.4.20.1) are already special, this formula can be simplified. For example, we have a coend formula for the tensor product of a special object with arbitrary object:

$$[M]^\alpha \otimes \mathcal{Y} \cong \int^{\beta\,:\,\mathcal{I}} \big[M \otimes \mathcal{Y}(\beta)\big]^{\alpha\otimes\beta}. \tag{2.4.24.1}$$

2.4.25. *Monoidal structure on $\mathcal{C}^{\mathcal{I}}$ is closed.* Since $\otimes_{\mathcal{C}^{\mathcal{I}}}$ commutes with colimits in each argument by **2.4.17**, and $\mathcal{C}^{\mathcal{I}}$ admits a reasonably small system of generators consisting of special objects $[M]^\alpha$, by **2.4.5**, it is reasonable to expect the monoidal structure of $\mathcal{C}^{\mathcal{I}}$ to be closed, especially in view of the assumption of closedness for the monoidal structure of $\mathcal{C}$. This is indeed the case: $(\mathcal{C}^{\mathcal{I}}, \otimes_{\mathcal{C}^{\mathcal{I}}})$ *is closed, assuming $\mathcal{C}$ to be closed monoidal, complete, and cocomplete.*

We show this by providing a formula for the linear Hom $\multimap_{\mathcal{C}^{\mathcal{I}}}$ in two steps.

2.4.26. *Formula for $[N]^\beta \multimap \mathcal{Z}$.* Fix $N : \mathcal{C}$, $\beta : \mathcal{I}$, and $\mathcal{Z} : \mathcal{C}^{\mathcal{I}}$. Then for any $M : \mathcal{C}$ and $\alpha : \mathcal{I}$ we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}\big([M]^\alpha, [N]^\beta \multimap \mathcal{Z}\big) &\cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}\big([M \otimes_{\mathcal{C}} N]^{\alpha\otimes\beta}, \mathcal{Z}\big)\\ &\cong \operatorname{Hom}_{\mathcal{C}}\big(M \otimes_{\mathcal{C}} N, \mathcal{Z}(\alpha\otimes\beta)\big)\\ &\cong \operatorname{Hom}_{\mathcal{C}}\big(M, N \multimap_{\mathcal{C}} \mathcal{Z}(\alpha\otimes\beta)\big)\\ &\cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{I}}}([M]^\alpha, \mathcal{W}),\end{aligned} \tag{2.4.26.1}$$

where the functor $\mathcal{W}\colon \mathcal{I} \to \mathcal{C}$ is given by

$$\mathcal{W}\colon \alpha \rightsquigarrow \big(N \multimap_{\mathcal{C}} \mathcal{Z}(\alpha\otimes\beta)\big). \tag{2.4.26.2}$$

Since the $[M]^\alpha$ generate $\mathcal{C}^{\mathcal{I}}$ under colimits, this proves that $\mathcal{W}$ is $([N]^\beta \multimap \mathcal{Z})$, i.e., $([N]^\beta \multimap \mathcal{Z})$ *exists for any $N : \mathcal{C}$, $\beta : \mathcal{I}$, $\mathcal{Z} : \mathcal{C}^{\mathcal{I}}$, and is given by*

$$\big([N]^\beta \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z}\big)\colon \alpha \rightsquigarrow \big(N \multimap_{\mathcal{C}} \mathcal{Z}(\alpha\otimes\beta)\big). \tag{2.4.26.3}$$

2.4.27. *General case: the end formula for $\mathcal{Y} \multimap \mathcal{Z}$.* Notice that $\otimes_{\mathcal{C}^{\mathcal{I}}}$ commutes with colimits in the second argument by **2.4.17**; this immediately implies that $- \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z}$ transforms all colimits in the first argument into the corresponding limits. Since a coend is a special kind of a colimit, and any $\mathcal{Y} : \mathcal{C}^{\mathcal{I}}$ can be expressed as a coend of special objects by (2.4.7.1), we obtain the following formula:

$$(\mathcal{Y} \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z}) \cong \int_{\beta\,:\,\mathcal{I}} \big([\mathcal{Y}(\beta)]^\beta \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z}\big). \tag{2.4.27.1}$$

The right-hand side of this formula is representable by (2.4.26.3) and by the assumption for $\mathcal{C}$ to be complete, i.e., to admit arbitrary small limits, in particular, ends. In this way we prove the existence of $(\mathcal{Y} \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z})$ for arbitrary $\mathcal{Y}$, $\mathcal{Z} : \mathcal{C}^{\mathcal{I}}$, and obtain an expression for this linear Hom in terms of ends in $\mathcal{C}^{\mathcal{I}}$. We can easily write out the individual components of $(\mathcal{Y} \multimap \mathcal{Z})$ by combining (2.4.27.1) with (2.4.26.3):

$$(\mathcal{Y} \multimap_{\mathcal{C}^{\mathcal{I}}} \mathcal{Z})\colon \alpha \rightsquigarrow \int_{\beta\,:\,\mathcal{I}} \big(\mathcal{Y}(\beta) \multimap_{\mathcal{C}} \mathcal{Z}(\alpha\otimes\beta)\big). \tag{2.4.27.2}$$

2.4.28. *The case of discrete $\mathcal{I}$ and additive $\mathcal{C}$.* Suppose that $\mathcal{I} = \Delta$ is discrete, i.e., a commutative monoid which will be written additively, and $\mathcal{C}$ is additive with $\Delta$-indexed products, for example, $\mathcal{C} = K\text{-}Mod$ for some commutative ring $K$. Then (2.4.27.2) reduces to the well-known formula for the "graded Hom" of $\Delta$-graded objects of $\mathcal{C}$:

$$(\mathcal{Y} \multimap \mathcal{Z})(\alpha) \cong \prod_{\beta\in\Delta} \mathbf{Hom}_{\mathcal{C}}\big(\mathcal{Y}(\beta), \mathcal{Z}(\alpha+\beta)\big). \tag{2.4.28.1}$$

2.4.29. *Monoids in $\mathcal{C}^{\mathcal{I}}$.* We claim that the *monoids in $\mathcal{C}^{\mathcal{I}}$ are exactly the lax monoidal functors $\mathcal{I} \to \mathcal{C}$* (cf. [Day]). When both $\mathcal{I}$ and $\mathcal{C}$ are symmetric monoidal, so the same holds for $\mathcal{C}^{\mathcal{I}}$, then the *commutative monoids in $\mathcal{C}^{\mathcal{I}}$ correspond to the lax monoidal functors $\mathcal{I} \to \mathcal{C}$ compatible with symmetries of $\mathcal{I}$ and $\mathcal{C}$.*

Indeed, a monoid structure on a functor $\mathcal{X} : \mathcal{C}^{\mathcal{I}}$ is a multiplication morphism $\mu\colon \mathcal{X} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{X} \to \mathcal{X}$ together with unit $\epsilon\colon \mathbf{1}_{\mathcal{C}_{\mathcal{I}}} \to \mathcal{X}$ satisfying the associativity and unit axiom. By definition, $\mu$ is essentially the same thing as a bilinear map $\mu' \in \mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}, \mathcal{X}; \mathcal{X})$, i.e., a collection of morphisms

$$\mu'_{\alpha,\beta}\colon \mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{X}(\beta) \to \mathcal{X}(\alpha \otimes_{\mathcal{I}} \beta). \tag{2.4.29.1}$$

Similarly, a unit $\epsilon\colon \mathbf{1}_{\mathcal{C}^{\mathcal{I}}} = [\mathbf{1}_{\mathcal{C}}]^{\mathbf{1}_{\mathcal{I}}} \to \mathcal{X}$ is the same thing as a morphism $\epsilon'\colon \mathbf{1}_{\mathcal{C}} \to \mathcal{X}(\mathbf{1}_{\mathcal{I}})$. The associativity and unit conditions on $\mu'_{\alpha,\beta}$ and $\epsilon'$ are exactly those of a lax monoidal functor structure on $\mathcal{X}\colon \mathcal{I} \to \mathcal{C}$.

**2.5. Functoriality of $\mathcal{C}^{\mathcal{I}}$ in $\mathcal{C}$.** Now we are interested in studying how the monoidal category $\mathcal{C}^{\mathcal{I}}$ varies with $\mathcal{C}$. Of course, any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\mathcal{I}}\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$. We want to know when we can expect $F^{\mathcal{I}}$ to be monoidal or lax monoidal, when it commutes with limits or colimits, or admits adjoints. We are going to apply these results to the scalar extension/restriction functors $\rho^*\colon K\text{-}Mod \to K'\text{-}Mod$ and $\rho_*\colon K'\text{-}Mod \to K\text{-}Mod$, where $\rho\colon K \to K'$ is a homomorphism of either classical commutative rings or generalized rings of [Du], i.e., commutative algebraic monads on *Sets*. Taking here $K = \mathbb{F}_\varnothing$ (corresponding to the identity monad $\mathrm{Id}_{Sets}$), we obtain free $K'$-module and forgetful functors $K'\text{-}Mod \rightleftarrows Sets$.

The results presented below seem a little *ad hoc* and unsystematic; a systematic point of view enabling one to understand and guess proper statements before proving them will be presented in **2.8**.

2.5.1. *$F^{\mathcal{I}}$ preserves limits or colimits whenever $F$ does.* It is easy to see that $F^{\mathcal{I}}\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ preserves some or all limits or colimits whenever $F\colon \mathcal{C} \to \mathcal{D}$ does, since the limits and colimits in the functor categories $\mathcal{C}^{\mathcal{I}}$ and $\mathcal{D}^{\mathcal{I}}$ are computed componentwise.

2.5.2. *$F^{\mathcal{I}}$ and $G^{\mathcal{I}}$ are adjoint whenever $F$ and $G$ are adjoint.* Similarly, if $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are adjoint functors, the same holds for $F^{\mathcal{I}}\colon \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{D}^{\mathcal{I}} : G^{\mathcal{I}}$. Indeed, if $\xi\colon \mathrm{Id}_{\mathcal{C}} \to G \circ F$ is an adjunction unit for $F$ and $G$, then $\xi^{\mathcal{I}}\colon \mathrm{Id}_{\mathcal{C}^{\mathcal{I}}} \to G^{\mathcal{I}} \circ F^{\mathcal{I}} = (G \circ F)^{\mathcal{I}}$ is given by

$$\xi^{\mathcal{I}}_{\mathcal{X}} := \xi \star \mathcal{X}\colon \mathcal{X} \to (G \circ F)^{\mathcal{I}}(\mathcal{X}) = G \circ F \circ \mathcal{X} \quad \text{for any } \mathcal{X}\colon \mathcal{I} \to \mathcal{C}, \tag{2.5.2.1}$$

and similarly, the counit $\eta^{\mathcal{I}}$ is given by $\eta^{\mathcal{I}}_{\mathcal{Y}} := \eta \star \mathcal{Y}$ for any $\mathcal{Y}\colon \mathcal{I} \to \mathcal{D}$, where $\eta\colon F \circ G \to \mathrm{Id}_{\mathcal{D}}$ is the adjunction counit of $\mathcal{C}$ and $\mathcal{D}$.

2.5.3. *Comparison of special objects in $\mathcal{C}^{\mathcal{I}}$ and $\mathcal{D}^{\mathcal{I}}$.* Let $M : \mathcal{C}$, $\alpha : \mathcal{I}$ be objects, and $F\colon \mathcal{C} \to \mathcal{D}$ a functor. Consider the special objects $[M]^\alpha : \mathcal{C}^{\mathcal{I}}$ and $[F(M)]^\alpha : \mathcal{D}^{\mathcal{I}}$ defined in **2.4.4**. Both $F^{\mathcal{I}}([M]^\alpha)$ and $[F(M)]^\alpha : \mathcal{D}^{\mathcal{I}}$ lie in $\mathcal{D}^{\mathcal{I}}$; we claim that they are related by a canonical morphism in $\mathcal{D}^{\mathcal{I}}$:

$$\theta_{M,\alpha}\colon \big[F(M)\big]^\alpha \to F^{\mathcal{I}}\big([M]^\alpha\big). \tag{2.5.3.1}$$

In order to construct $\theta_{M,\alpha}$, it suffices to construct

$$\theta^\flat_{M,\alpha}\colon F(M) \to (F^{\mathcal{I}}([M]^\alpha))(\alpha) = F\big(([M]^\alpha)(\alpha)\big)$$

by the universal property (2.4.4.1) of special objects. Such a morphism is easily obtained by applying functor $F$ to the canonical morphism

$$\mathrm{id}^\flat_{[M]^\alpha}\colon M \to ([M]^\alpha)(\alpha).$$

2.5.4. *$F^{\mathcal{I}}$ preserves special objects whenever $F$ preserves small coproducts.* We claim that *the morphism $\theta_{M,\alpha}$ of* (2.5.3.1) *is an isomorphism provided $F$ preserves small coproducts.* In this case one can write $F^{\mathcal{I}}([M]^\alpha) \cong [F(M)]^\alpha$ and say that $F^{\mathcal{I}}$ *preserves special objects whenever $F$ preserves small coproducts.* In order to prove this, recall the explicit formula (2.4.4.2) for the values of the special object $[M]^\alpha\colon \mathcal{I} \to \mathcal{C}$: we have $[M]^\alpha(\beta) \cong M^{(\mathrm{Hom}(\alpha,\beta))}$. Since $F$ preserves small coproducts, we see that both $F^{\mathcal{I}}([M]^\alpha)$ and $[F(M)]^\alpha$ map $\beta$ into $F(M)^{(\mathrm{Hom}(\alpha,\beta))}$, and therefore are canonically isomorphic.

2.5.5. *$F^{\mathcal{I}}$ is compatible with bilinear maps whenever $F$ is lax monoidal.* Now suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor between monoidal categories, i.e., for any $X, Y : \mathcal{C}$ we have a morphism

$$\gamma_{X,Y}\colon F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y),$$

functorial in $X$ and $Y$ and compatible with the constraints of the monoidal categories $\mathcal{C}$ and $\mathcal{D}$. In this case $F^{\mathcal{I}}$ *acts on bilinear maps in $\mathcal{C}^{\mathcal{I}}$.* More precisely, for any $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z} : \mathcal{C}^{\mathcal{I}}$, we have a canonical map

$$\widetilde{F}\colon \mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}, \mathcal{Y}; \mathcal{Z}) \to \mathrm{Bilin}_{\mathcal{D}^{\mathcal{I}}}(F^{\mathcal{I}}\mathcal{X}, F^{\mathcal{I}}\mathcal{Y}; F^{\mathcal{I}}\mathcal{Z}). \tag{2.5.5.1}$$

This map can be described as follows. A bilinear map $\Phi \in \mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}, \mathcal{Y}; \mathcal{Z})$ actually is a collection of morphisms $\Phi_{\alpha,\beta}\colon \mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta) \to \mathcal{Z}(\alpha \otimes \beta)$, natural in $\alpha$, $\beta : \mathcal{I}$. Applying $F$ to each $\Phi_{\alpha,\beta}$ yields a similar collection defining a bilinear map $\widetilde{F}(\Phi) \in \mathrm{Bilin}_{\mathcal{D}^{\mathcal{I}}}(F^{\mathcal{I}}\mathcal{X}, F^{\mathcal{I}}\mathcal{Y}; F^{\mathcal{I}}\mathcal{Z})$:

$$\begin{aligned} (F^{\mathcal{I}}\mathcal{X})(\alpha) \otimes_{\mathcal{D}} (F^{\mathcal{I}}\mathcal{Y})(\beta) &= F\big(\mathcal{X}(\alpha)\big) \otimes_{\mathcal{D}} F\big(\mathcal{Y}(\beta)\big) \\ &\xrightarrow{\gamma_{\mathcal{X}(\alpha),\mathcal{Y}(\beta)}} F\big(\mathcal{X}(\alpha) \otimes_{\mathcal{C}} \mathcal{Y}(\beta)\big) \\ &\xrightarrow{F(\Phi_{\alpha,\beta})} F\big(\mathcal{Z}(\alpha \otimes \beta)\big) = (F^{\mathcal{I}}\mathcal{Z})(\alpha \otimes \beta). \end{aligned} \tag{2.5.5.2}$$

2.5.6. *$F^{\mathcal{I}}$ is lax monoidal whenever $F$ is.* Recall that $\mathcal{X} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{Y}$ was defined in **2.4.15** as the object representing $\mathrm{Bilin}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X}, \mathcal{Y}; -)$, so (2.5.5.1) may be rewritten as

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\mathcal{X} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{D}^{\mathcal{I}}}(F^{\mathcal{I}}\mathcal{X} \otimes_{\mathcal{D}^{\mathcal{I}}} F^{\mathcal{I}}\mathcal{Y}, F^{\mathcal{I}}\mathcal{Z}). \tag{2.5.6.1}$$

Applying this to the identity morphism of $\mathcal{Z} := \mathcal{X} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{Y}$, we obtain a canonical morphism

$$\mu_{\mathcal{X},\mathcal{Y}}\colon F^{\mathcal{I}}\mathcal{X} \otimes_{\mathcal{D}^{\mathcal{I}}} F^{\mathcal{I}}\mathcal{Y} \to F^{\mathcal{I}}(\mathcal{X} \otimes_{\mathcal{C}^{\mathcal{I}}} \mathcal{Y}). \tag{2.5.6.2}$$

We also have a canonical homomorphism for units of monoidal categories:

$$\epsilon\colon \mathbf{1}_{\mathcal{D}^{\mathcal{I}}} = [\mathbf{1}_{\mathcal{D}}]^{\mathbf{1}_{\mathcal{I}}} \to \big[F(\mathbf{1}_{\mathcal{C}})\big]^{\mathbf{1}_{\mathcal{I}}} \to F^{\mathcal{I}}\big([\mathbf{1}_{\mathcal{C}}]^{\mathbf{1}_{\mathcal{I}}}\big) = F^{\mathcal{I}}(\mathbf{1}_{\mathcal{C}^{\mathcal{I}}}). \tag{2.5.6.3}$$

The first arrow here is induced by $\mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ coming from the lax monoidal functor structure on $F$; the second arrow is that of (2.5.3.1).

It is easy to see that (2.5.6.2) and (2.5.6.3) together define a lax monoidal structure on $F^{\mathcal{I}}$. The simplest way to see this is to observe that the action (2.5.5.1) of $F$ on $\mathcal{C}^{\mathcal{I}}$-bilinear maps extends immediately to polylinear maps, thus defining a polyfunctor $\bar{F}$ between polycategories $\mathcal{C}^{\mathcal{I}} \to \mathcal{D}^{\mathcal{I}}$ extending $F^{\mathcal{I}}$. Such a polyfunctor extension corresponds exactly to the lax monoidal structure on $F^{\mathcal{I}}$.

2.5.7. *If $F$ is monoidal and preserves small colimits, the same holds for $F^{\mathcal{I}}$.* We claim that *if $F$ is monoidal and preserves small colimits, the same holds for $F^{\mathcal{I}}$.* Indeed, we know already that $F^{\mathcal{I}}$ is lax monoidal by **2.5.6**, and preserves small colimits by **2.5.1**. We have to show that the morphisms (2.5.6.3) and (2.5.6.2) are isomorphisms for all $\mathcal{X}$, $\mathcal{Y} : \mathcal{C}^{\mathcal{I}}$. Since all functors involved commute with small colimits and any object of $\mathcal{C}^{\mathcal{I}}$ can be represented as a small colimit of special objects (cf. **2.4.5**), it suffices to check this for special $\mathcal{X} = [M]^\alpha$ and $\mathcal{Y} = [N]^\beta$, where our statement immediately follows from **2.5.4** and (2.4.19.2): both the source and the target of $\mu_{[M]^\alpha,[N]^\beta}$ turn out to be isomorphic to $[F(M) \otimes_{\mathcal{D}} F(N)]^{\alpha\otimes\beta}$. The isomorphism property in (2.5.6.3) is even simpler: the first morphism is induced by $\mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbf{1}_{\mathcal{C}})$, and the second one is an isomorphism by **2.5.4**.

2.5.8. *If $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a monoidal adjunction, the same holds for $F^{\mathcal{I}}\colon \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{D}^{\mathcal{I}} : G^{\mathcal{I}}$.* Combining together **2.5.2**, **2.5.6**, and **2.5.7**, we see that *whenever $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a monoidal adjunction* (so that $F$ is monoidal and preserves colimits, and $G$ is lax monoidal and preserves limits), *the same is true for $F^{\mathcal{I}}\colon \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{D}^{\mathcal{I}} : G^{\mathcal{I}}$.*

2.5.9. *Monad defined by $F^{\mathcal{I}} \dashv G^{\mathcal{I}}$.* Let $F \dashv G$ be adjoint functors, defining a monad $\Sigma := G \circ F$ on a category $\mathcal{C}$. Then $F^{\mathcal{I}} \dashv G^{\mathcal{I}}$ are also adjoint, and define the monad $\Sigma^{\mathcal{I}} = G^{\mathcal{I}} \circ F^{\mathcal{I}}$ on $\mathcal{C}^{\mathcal{I}}$.

2.5.10. *$\Sigma^{\mathcal{I}}$ is a monad on $\mathcal{C}^{\mathcal{I}}$ whenever $\Sigma$ is one on $\mathcal{C}$.* Furthermore, if $\Sigma = (\Sigma, \mu, \epsilon)$ is a monad on $\mathcal{C}$, then $\Sigma^{\mathcal{I}} = (\Sigma^{\mathcal{I}}, \mu^{\mathcal{I}}, \epsilon^{\mathcal{I}})$ is a monad on $\mathcal{C}^{\mathcal{I}}$. This can be checked directly, or deduced from the previous result by finding a couple of adjoint functors with composition equal to $\Sigma$. Such a couple always exists; one can take for $\mathcal{D}$ for example the Eilenberg–Moore category $\mathcal{C}^\Sigma$, or the Kleisli category $\mathcal{C}_\Sigma$.

2.5.11. *If $G\colon \mathcal{D} \to \mathcal{C}$ is monadic, the same holds for $G^{\mathcal{I}}\colon \mathcal{D}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{I}}$.* Now suppose that the functor $G\colon \mathcal{D} \to \mathcal{C}$ is monadic; in particular, it admits a left adjoint $F$. Then $G^{\mathcal{I}}$ admits a left adjoint $F^{\mathcal{I}}$; we claim that *$G^{\mathcal{I}}$ is monadic.* We know already that the monad corresponding to $G^{\mathcal{I}}$ is $\Sigma^{\mathcal{I}}$ if $\Sigma$ is the monad defined by $G$. Thus the statement about the property of $G^{\mathcal{I}}$ to be monadic may be restated as a natural equivalence of Eilenberg–Moore categories:

$$(\mathcal{C}^{\mathcal{I}})^{\Sigma^{\mathcal{I}}} \cong (\mathcal{C}^\Sigma)^{\mathcal{I}}, \quad \text{for any monad } \Sigma \text{ on } \mathcal{C}. \tag{2.5.11.1}$$

The latter statement can be checked directly: an object of $(\mathcal{C}^{\mathcal{I}})^{\Sigma^{\mathcal{I}}}$, i.e., a $\Sigma^{\mathcal{I}}$-model (or $\Sigma^{\mathcal{I}}$-algebra in another terminology) in $\mathcal{C}^{\mathcal{I}}$ is an object $\mathcal{X} : \mathcal{C}^{\mathcal{I}}$, i.e., a functor $\mathcal{X}\colon \mathcal{I} \to \mathcal{C}$, together with a morphism $\alpha\colon \Sigma^{\mathcal{I}}(\mathcal{X}) \to \mathcal{X}$, i.e., a natural transformation $\alpha\colon \Sigma \circ \mathcal{X} \to \mathcal{X}$, which must be compatible with monad multiplication $\mu^{\mathcal{I}}$ and unit $\epsilon^{\mathcal{I}}$. We see immediately that the commutative diagrams expressing this compatibility together with the naturality of $\alpha$ and other natural transformations involved amount to saying that each $\bigl(\mathcal{X}(\iota), \alpha_\iota\colon \Sigma\bigl(\mathcal{X}(\iota)\bigr) \to \mathcal{X}(\iota)\bigr)$, where $\iota : \mathcal{I}$ is a $\Sigma$-model in $\mathcal{C}$, i.e., an object of $\mathcal{C}^\Sigma$, and any $\mathcal{X}(\phi)$, $\phi\colon \iota \to \kappa$ in $\mathcal{I}$, is a $\Sigma$-model morphism, i.e., a morphism in $\mathcal{C}^\Sigma$. We obtain a functor $\mathcal{I} \to \mathcal{C}^\Sigma$, i.e., an object of $(\mathcal{C}^\Sigma)^{\mathcal{I}}$; this gives the required equivalence (even isomorphism) of categories.

2.5.12. *Kleisli categories.* A similar statement for Kleisli categories is also true and can be checked similarly:

$$(\mathcal{C}^{\mathcal{I}})_{\Sigma^{\mathcal{I}}} \cong (\mathcal{C}_\Sigma)^{\mathcal{I}}, \quad \text{for any monad } \Sigma \text{ on } \mathcal{C}. \tag{2.5.12.1}$$

**2.6. Application: normed and graded algebraic structures.** Let $\Sigma$ be a *finitary* monad on *Sets*, i.e., a monad commuting with filtered colimits (in other words, $\Sigma$ is $\omega$-accessible). In [Du] such monads were called *algebraic*, because they are in one-to-one correspondence with (finitary) algebraic theories, such as groups, monoids, Abelian groups, modules over a ring $K$, algebras over a commutative ring $K$ and so on. However, if one allows algebraic theories with operations of infinite arities, one has to admit that all *accessible* monads are "algebraic".

We can apply the results of **2.5.11** in this case. We obtain a monad $\Sigma^{\mathcal{I}}$ on the category $Sets^{\mathcal{I}}$ of $\mathcal{I}$-graded sets, and the corresponding Eilenberg–Moore category is isomorphic to $(Sets^{\Sigma})^{\mathcal{I}}$, the category of *$\mathcal{I}$-graded $\Sigma$-models.*

In particular, we obtain categories of $\mathcal{I}$-graded groups, Abelian groups, $K$-modules and so on, with (monadic) forgetful functors into the category $Sets^{\mathcal{I}}$ of $\mathcal{I}$-graded sets and their left adjoints, the free $\mathcal{I}$-graded $\Sigma$-model functor.

2.6.1. *Commutative case.* Now suppose that a finitary monad $\Sigma$ is *commutative* (cf. [Du, 3]), i.e., $\Sigma$ is a *generalized ring* in the terminology of [Du]. Then we have a natural symmetric monoidal structure $\otimes_\Sigma$ on the category of $\Sigma$-models $\Sigma\text{-}Mod = Sets^{\Sigma}$ such that the free $\Sigma$-model functor $L_\Sigma\colon Sets \to \Sigma\text{-}Mod$ is monoidal (and its right adjoint, the forgetful functor $\Gamma_\Sigma\colon \Sigma\text{-}Mod \to Sets$ is automatically lax monoidal). Applying **2.5.7** and **2.5.6**, we see that the "free $\mathcal{I}$-graded $\Sigma$-model functor" $L_\Sigma^{\mathcal{I}}\colon Sets^{\mathcal{I}} \to (\Sigma\text{-}Mod)^{\mathcal{I}}$ is monoidal, and its right adjoint, the forgetful functor $\Gamma_\Sigma^{\mathcal{I}}\colon (\Sigma\text{-}Mod)^{\mathcal{I}} \to Sets^{\mathcal{I}}$, is lax monoidal. This is applicable in particular to the categories of Abelian groups (with usual tensor product $\otimes = \otimes_{\mathbb{Z}}$) and of modules over a commutative ring $K$.

2.6.2. *Explicit description of $\Delta$-normed $\Sigma$-models.* Now suppose that $\mathcal{I} = \Delta$ is a partially ordered set, and that $\mathcal{X} : (\Sigma\text{-}Mod)^{\Delta}$ is a $\Delta$-graded $\Sigma$-model such that its underlying $\Delta$-graded set $\bar{\mathcal{X}} := \Gamma_{\Sigma^\Delta}(\mathcal{X})$ is projective. We know that $\operatorname{Proj}(Sets^{\Delta})$ is equivalent to the category $\mathcal{N}_\Delta$ of $\Delta$-normed sets (**2.1.1**); under this equivalence, $\bar{\mathcal{X}}$ corresponds to the set $X := \operatorname{colim}_\Delta \bar{\mathcal{X}}$ with a norm $|\cdot|_{\mathcal{X}}\colon X \to \Delta$ such that $\bar{\mathcal{X}}(\iota) = X_{\leq\iota} = \{x \in X : |x|_{\mathcal{X}} \leq \iota\}$ for any $\iota \in \Delta$.

Next, each $\bar{\mathcal{X}}(\iota) = X_{\leq\iota}$ is the underlying set of the $\Sigma$-model $\mathcal{X}(\iota)$, i.e., we must have a "$\Sigma$-structure" on each $X_{\leq\iota}$, compatible with embeddings $X_{\leq\iota} \to X_{\leq\kappa}$ whenever $\iota \leq \kappa$. Since $\Sigma$ is finitary, we know that a $\Sigma$-structure $\alpha\colon \Sigma(X_{\leq\iota}) \to X_{\leq\iota}$ on the set $X_{\leq\iota}$ is completely determined by its "application maps" $\alpha_n = \alpha_{n,\iota}\colon \Sigma(n) \times X_{\leq\iota}^n \to X_{\leq\iota}$, for all integers $n \geq 0$. The elements of $\Sigma(n) = \Sigma(\mathbf{n})$, $\mathbf{n} = \{1, 2, \ldots, n\}$, are usually called *$n$-ary operations of* $\Sigma$, and $\alpha_n(t, x_1, \ldots, x_n)$ is usually denoted by $[t]_{X_{\leq\iota}}(x_1, \ldots, x_n)$ or $t(x_1, \ldots, x_n)$, and called "the result of applying the operation $t$ to the $n$-tuple $x_1$, $\ldots$, $x_n$ of elements of $X_{\leq\iota}$". These "application maps" are related to the monadic action $\alpha\colon \Sigma(X_{\leq\iota}) \to X_{\leq\iota}$ as follows. A collection $x = (x_1, \ldots, x_n)$ is essentially a map $\mathbf{x}\colon \mathbf{n} \to X_{\leq\iota}$; then

$$[t]_{X_{\leq\iota}}(x_1, \ldots, x_n) = \big(\alpha \circ \Sigma(\mathbf{x})\big)(t). \tag{2.6.2.1}$$

More details can be found in [Du, 4].

In our case, for any $n \geq 0$, $t \in \Sigma(n)$, $\iota \in \Delta$, and $x_1$, $\ldots$, $x_n \in X$ such that $|x_i|_{\mathcal{X}} \leq \iota$ for all $i$, we obtain an element $t_\iota(x_1, \ldots, x_n)$ in $X$, of norm not exceeding $\iota$. Furthermore, the requirement that all embeddings $X_\iota \subset X_\kappa$, $\iota \leq \kappa$, be $\Sigma$-homomorphisms, means that $t_\kappa(x_1, \ldots, x_n) = t_\iota(x_1, \ldots, x_n)$ whenever $\kappa \geq \iota \geq |x_i|$ for all $i$.

In order to determine the $\Sigma$-model structure, one has to define $t_\iota$ only for a subset of "generating operations" of the monad $\Sigma$. For example, when $\Sigma = \Sigma_{\mathbb{Z}}$ is the free $\mathbb{Z}$-module monad (identified with the "classical" ring $\mathbb{Z}$ in [Du]), defining the category of Abelian groups, it is sufficient to determine addition $[+]_\iota\colon X_\iota \times X_\iota \to X_\iota$ corresponding to the element $[+] = (1,1) \in \Sigma_{\mathbb{Z}}(2) = \mathbb{Z}^2$, symmetry $[-]_\iota\colon X_\iota \to X_\iota$, corresponding to $[-] = (-1) \in \Sigma_{\mathbb{Z}}(1) = \mathbb{Z}$, and the zero $0_\iota \in X_\iota$, corresponding to $0 \in \Sigma_{\mathbb{Z}}(0) = 0$.

Let us say that a $\Delta$-normed set together with a $\Sigma^\Delta$-action on its image in $Sets^\Delta$ is a *$\Delta$-normed $\Sigma$-model.* We have obtained an explicit description of $\Delta$-normed $\Sigma$-models: it is a $\Delta$-normed set $X = (X, |\cdot|_{\mathcal{X}}\colon X \to \Delta)$, together with operations $t_\iota\colon X^n_{\leq\iota} \to X_{\leq\iota}$, defined for all $t \in \Sigma(n)$ and all $\iota \in \Delta$, compatible with all embeddings $X_{\leq\iota} \hookrightarrow X_{\leq\kappa}$, $\iota \leq \kappa$.

2.6.3. *The case of discrete $\Delta$.* When $\Delta$ is discrete, we arrive at a $\Delta$-graded set $X = (X, |\cdot|_{\mathcal{X}}\colon X \to \Delta)$, i.e., a $\Delta$-indexed collection of sets $(X_\iota)_{\iota\in\Delta}$, $X_\iota = \{x \in X : |x| = \iota\}$, with action of $n$-ary operations of $\Sigma$ defined only on the collections $(x_1, \dots, x_n)$ such that all $x_i$ have the same "degree" or "norm": $|x_1| = \cdots = |x_n|$, independently for each value of norm. In other words, we simply obtain a $\Delta$-indexed collection of $\Sigma$-models $X_\iota$, i.e., a functor $\Delta \to \Sigma\text{-}Mod$, something we should have expected.

2.6.4. *The case of filtered $\Delta$.* Another interesting case is that of *filtered* $\Delta$, for example linearly ordered $\Delta = [0,1]$ or $\Delta = \mathbb{R}_{\geq 0}$. In this case, the forgetful functor $\Gamma_\Sigma : \Sigma\text{-}Mod$ commutes with the "underlying object functor" $|-| = \operatorname{colim}_\Delta$, because it is a filtered colimit in this case, and $\Gamma_\Sigma$ commutes with filtered colimits whenever monad $\Sigma$ is finitary.

We see that the set $X = \operatorname{colim}_\Delta \mathcal{X}$ is equipped with the operations $t_X\colon X^n \to X$ for all $t \in \Sigma(n)$, apart from the norm $|\cdot|_{\mathcal{X}}\colon X \to \Delta$. They are related by the following condition: if all $|x_i|$ do not exceed $\iota$, then $t_X(x_1, \dots, x_n)$ has also norm of at most $\iota$. In other words, the subset $X_{\leq\iota} \subset X$ is a $\Sigma$-submodel of $X$, i.e., it is stable under all operations of the finitary monad $\Sigma$.

When $\Delta$ admits finite suprema, for example if $\Delta = \mathbb{R}_{\geq 0}$, then a $\Delta$-normed $\Sigma$-model can be described as follows: it is a $\Delta$-normed set $X$ together with a $\Sigma$-action on it, i.e., operations $t_X\colon X^n \to X$ for all $t \in \Sigma(n)$, $n \geq 0$, such that

$$\big|t_X(x_1, \dots, x_n)\big|_X \leq \sup_{1\leq i\leq n} |x_i|_X. \tag{2.6.4.1}$$

For example, if $\Sigma = \Sigma_{\mathbb{Z}}$, we obtain the $\mathbb{R}_{\geq 0}$-normed Abelian groups: these are Abelian groups $X$ with an $\mathbb{R}_{\geq 0}$-valued norm $|\cdot|$ such that $|x+y| \leq \max(|x|, |y|)$, $|-x| = |x|$ and $|0| = 0$, i.e., *the $\mathbb{R}_{\geq 0}$-normed Abelian groups are merely Abelian groups with non-Archimedian norms.* Similarly, the $\mathbb{R}_{\geq 0}$-normed left $K$-modules are $K$-modules $X$ with norms as above, with the additional condition $|\lambda x| \leq |x|$ for any $\lambda \in K$ and $x \in X$.

We will see later how to deal with "Archimedian norms" as well.

2.6.5. *$\Sigma$-models in arbitrary category with finite products $\mathcal{C}$.* Let $\Sigma$ be a finitary monad on *Sets*. Then the $\Sigma$-models (or "$\Sigma$-algebras") in *Sets* can be described in several equivalent ways. One of them is given by the Eilenberg–Moore category $Sets^\Sigma$ of the monad $\Sigma$. Another one, due to Lawvere, is given by considering $Funct_\Pi(\mathscr{T}_\Sigma, Sets)$, the category of all finite product-preserving functors from the corresponding "Lawvere theory" category $\mathscr{T}_\Sigma$ into *Sets*. Here $\mathscr{T}_\Sigma$ can be taken to be $(\underline{\mathbb{N}}_\Sigma)^{\mathrm{op}}$, where $\underline{\mathbb{N}}$ denotes the full subcategory of *Sets* consisting of standard finite sets $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, ..., $\mathbf{n} = \{1, 2, \dots, n\}$, ..., and $\underline{\mathbb{N}}_\Sigma$ is the full subcategory of the Kleisli category $Sets_\Sigma$ with object set equal to $\operatorname{Ob}\underline{\mathbb{N}}$.

This description enables us to define the category $\Sigma\text{-}Mod_{\mathcal{C}}$ of $\Sigma$-models in arbitrary category $\mathcal{C}$ with finite products as $Funct_\Pi\big((\underline{\mathbb{N}}_\Sigma)^{\mathrm{op}}, \mathcal{C}\big)$, and extend any finite product-preserving functor $Q\colon \mathcal{C} \to \mathcal{D}$ to a functor $\Sigma\text{-}Mod_{\mathcal{C}} \to \Sigma\text{-}Mod_{\mathcal{D}}$ by mapping $X\colon \underline{\mathbb{N}}_\Sigma^{\mathrm{op}} \to \mathcal{C}$ to $Q \circ X$. For example, $\Sigma\text{-}Mod_{Sets^{\mathcal{I}}} \cong (\Sigma\text{-}Mod)^{\mathcal{I}}$, and $\Sigma\text{-}Mod_{\mathcal{N}_{\mathcal{I}}}$ is equivalent to the category of "$\mathcal{I}$-normed $\Sigma$-models" discussed above in **2.6.1**.

2.6.6. *Tensor product of* $\mathbb{R}_{\geq 0}$*-normed Abelian groups.* Recall that for $\mathcal{I} = \Delta = \mathbb{R}_{\geq 0}$ the fully faithful functor $I \colon \mathcal{N}_\Delta \to \mathit{Sets}^\Delta$ preserves finite products and admits a finite product-preserving right adjoint $Q$ (cf. **2.2.27**). This means that, for any finitary monad $\Sigma$, we have the induced functor

$$Q_\Sigma \colon \Sigma\text{-}\mathit{Mod}^\Delta \to \Sigma\text{-}\mathit{Mod}_{\mathcal{N}_\Delta},$$

the right adjoint to the fully faithful functor $I_\Sigma$ induced by $I \colon \mathcal{N}_\Delta \to \mathit{Sets}^\Delta$. If $\Sigma$ is commutative, we have a natural tensor product $\otimes_\Sigma$ on $\Sigma\text{-}\mathit{Mod}^\Delta$, given by the Day convolution product of **2.4**. Using the functors $I_\Sigma$ and $Q_\Sigma$, we can define a tensor product on $\Delta$-normed $\Sigma$-models:

$$X \otimes_\Sigma Y := Q_\Sigma\big(I_\Sigma(X) \otimes_\Sigma I_\Sigma(Y)\big). \tag{2.6.6.1}$$

For example, if $X$ and $Y$ are two $\mathbb{R}_{\geq 0}$-normed Abelian groups, we have their "normed tensor product" $X \otimes Y = X \otimes_{\mathbb{Z}} Y$. Its underlying set coincides with the usual tensor product of Abelian groups (because $\Delta = \mathbb{R}_{\geq 0}$ is filtered), and the norm $|\cdot|_{X\otimes Y}$ is given by a non-Archimedean variant of the classical formula for norms on tensor products of vector spaces:

$$|z|_{X\otimes Y} = \inf_{z=\sum_i x_i \otimes y_i} \sup_i |x_i|_X \cdot |y_i|_Y. \tag{2.6.6.2}$$

This can be checked directly by using (2.4.21.1) and (2.2.26.1).

However, using the localization functor $Q$ in this fashion loses some information. For example, it is not even clear whether this tensor product of normed Abelian groups is associative. Derived versions of this construction are better in most respects, even if they lead to less familiar notions, cf. [DuN2, 3.6.30]

**2.7. Functoriality of $\mathcal{C}^{\mathcal{I}}$ in $\mathcal{I}$.** We would like to discuss now the less obvious functoriality of $\mathcal{C}^{\mathcal{I}}$ in $\mathcal{I}$. Of course, this involves left and right Kan extensions of functors $f \colon \mathcal{I} \to \mathcal{J}$.

2.7.1. *Notation for Kan extensions:* $f_!$, $f^*$, $f_*$. Given a functor $f \colon \mathcal{I} \to \mathcal{J}$ between small categories, we denote by $f^* \colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ the corresponding "restriction" or "pullback" functor:

$$f^* \colon \mathcal{X} \mapsto \mathcal{X} \circ f. \tag{2.7.1.1}$$

Its left and right adjoints, i.e., the left and right Kan extensions of $f$ will be denoted by $f_!$ and $f_* \colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}}$, respectively. Recall that $f_*$ always exists when $\mathcal{C}$ is complete, and $f_!$ when $\mathcal{C}$ is cocomplete.

2.7.2. *Explicit formula for* $f_!$. We would like to recall here the explicit formula for the values of $f_!\mathcal{X}$, for any $\mathcal{X} : \mathcal{C}^{\mathcal{I}}$:

$$(f_!\mathcal{X})(\gamma) = \operatorname*{inj\,lim}_{\alpha \,:\, \mathcal{I}_{/\gamma}} \mathcal{X}(\alpha). \tag{2.7.2.1}$$

Here $\mathcal{I}_{/\gamma}$ denotes the small category $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/\gamma}$ consisting of the couples $(\alpha, \phi)$, where $\alpha$ is an object of $\mathcal{I}$, and $\phi \colon f(\alpha) \to \gamma$ is a morphism in $\mathcal{J}$.

2.7.3. $f_!$ *preserves special objects.* Our next observation is that $f_!$ preserves special objects, cf. **2.4.4**. More precisely, if $M : \mathcal{C}$ and $\alpha : \mathcal{I}$, then

$$f_!\big([M]^\alpha\big) \cong [M]^{f(\alpha)}. \tag{2.7.3.1}$$

The shortest way to prove this is to recall that $[M]^\alpha = I_{\alpha,!}(M)$, where $I_\alpha\colon \mathbf{1} \to \mathcal{I}$ denotes the functor from the point category into $\mathcal{I}$ with image $\alpha$. By the transitivity of left Kan extensions, this implies

$$f_!([M]^\alpha) = f_! I_{\alpha,!}(M) = (f \circ I_\alpha)_!(M) = I_{f(\alpha),!}(M) = [M]^{f(\alpha)}.$$

2.7.4. *$f_!$ preserves free objects when $\mathcal{C} = Sets$.* A special case of formula (2.7.3.1) for $\mathcal{C} = Sets$, $M = \mathbf{1}$ reads

$$f_!(h^\alpha) = h^{f(\alpha)} \quad \text{for any } \alpha : \mathcal{I}. \tag{2.7.4.1}$$

Since $f_!$ preserves arbitrary colimits and in particular coproducts, we see that $f_!$ transforms free objects of $Sets^{\mathcal{I}}$, i.e., coproducts of the corepresentable functors, into free objects of $Sets^{\mathcal{J}}$:

$$f_!\Bigl(\bigsqcup_{\iota \in X} h^{\alpha_\iota}\Bigr) \cong \bigsqcup_{\iota \in X} h^{f(\alpha_\iota)}. \tag{2.7.4.2}$$

2.7.5. *Induced functor $f_!\colon \mathcal{N}_{\mathcal{I}} \to \mathcal{N}_{\mathcal{J}}$ on normed sets.* According to **2.2.14**,c), the free objects of $Sets^{\mathcal{I}}$ and $Sets^{\mathcal{J}}$ admit an equivalent description as $\mathcal{I}$- and $\mathcal{J}$-normed sets (cf. **2.2.11**). Therefore, restriction of the functor $f_!$ to $\mathrm{Free}(Sets^{\mathcal{I}})$ induces a functor $\bar{f}_!\colon \mathcal{N}_{\mathcal{I}} \to \mathcal{N}_{\mathcal{J}}$ between equivalent categories:

$$\begin{tikzcd} \mathcal{N}_{\mathcal{I}} \arrow[r, dashed, "\bar{f}_!"] \arrow[d, "I_{\mathcal{I}}"] & \mathcal{N}_{\mathcal{J}} \arrow[d, "I_{\mathcal{J}}"] \\ Sets^{\mathcal{I}} \arrow[r, "f_!"] & Sets^{\mathcal{J}}. \end{tikzcd} \tag{2.7.5.1}$$

2.7.6. *Explicit description of $f_!\colon \mathcal{N}_{\mathcal{I}} \to \mathcal{N}_{\mathcal{J}}$.* Let $\mathcal{X} = (X, |\cdot|_{\mathcal{X}}\colon X \to \mathrm{Ob}\,\mathcal{I})$ be an $\mathcal{I}$-normed set, so that $I_{\mathcal{I}}(\mathcal{X}) = \bigsqcup_{x \in X} h^{|x|_{\mathcal{X}}}$. Then (2.7.4.2) and (2.7.5.1) imply that $f_!\mathcal{X} = \bar{f}_!\mathcal{X}$ can be described explicitly as $f_!\mathcal{X} = (X, f \circ |\cdot|_{\mathcal{X}})$. In other words, $f_!\mathcal{X}$ has the same underlying set $X$ as $\mathcal{X}$, with new norm $|\cdot|_{f_!\mathcal{X}}$ given by

$$|x|_{f_!\mathcal{X}} = f(|x|_{\mathcal{X}}). \tag{2.7.6.1}$$

Since $f$ is a functor, this definition is compatible with the morphisms $\phi = (\bar{\phi}, \xi)\colon \mathcal{X} \to \mathcal{Y}$ of $\mathcal{I}$-normed sets.

2.7.7. *Special case: the poset map $f\colon \Delta \to \Delta'$.* This description of $f_!$ can be specialized further to the case of an increasing map of partially ordered sets $f\colon \Delta \to \Delta'$. In this case, $f_!$ transforms a $\Delta$-normed set into a $\Delta'$-normed set simply by composing its norm with $f$. This explains the notation $e_!$ used previously in **1.4.3**.

2.7.8. *Conditions for a nice behavior of $f^*\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$.* Our next goal is to determine when $f^*\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ behaves nicely with respect to special objects. The special of case $\mathcal{C} = Sets$ is important here: it turns out that $f^*$ has nice properties for general $\mathcal{C}$ whenever $f^*\colon Sets^{\mathcal{J}} \to Sets^{\mathcal{I}}$ preserves free objects, thus inducing a functor $\bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ on normed sets. We are going to study this situation in more detail, starting with some special cases.

2.7.9. *Special case: $f^*$ when $f$ admits a left adjoint $g$.* Suppose that $f\colon \mathcal{I} \to \mathcal{J}$ admits a left adjoint $g\colon \mathcal{J} \to \mathcal{I}$. Then the functor $f^*$ is canonically isomorphic to $g_!$, for example because the category $\mathcal{J}_{/\alpha} = \mathcal{J} \times_{\mathcal{I}} \mathcal{I}_{/\alpha}$, needed for computing $g_!\mathcal{Y}$ by (2.7.2.1), turns out to be isomorphic to the category $\mathcal{J}_{/f(\alpha)}$, admitting a final object $\mathrm{id}_{f(\alpha)}$. We can apply (2.7.3.1) to $g$ in this case:

$$f^*\big([M]^\gamma\big) \cong g_!\big([M]^\gamma\big) \cong [M]^{g(\gamma)}. \tag{2.7.9.1}$$

If $\mathcal{C} = \mathit{Sets}$, we also have $f^*(h^\gamma) \cong h^{g(\gamma)}$ for the corepresentable functors by (2.7.4.1), hence $f^*\colon \mathit{Sets}^{\mathcal{J}} \to \mathit{Sets}^{\mathcal{I}}$ *preserves corepresentable functors and free objects, provided $f$ admits a left adjoint.* We obtain an induced functor $\bar{f}^* = \bar{g}_!\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ in this case, the right adjoint to $\bar{f}_!$ of **2.7.6**.

It is worthwhile to remark that, conversely, *if $f^*$ transforms corepresentable functors into corepresentable functors, then $f$ admits a left adjoint $g$.* Indeed, if $f^*(h^\gamma) \cong h^\alpha$ for some $\alpha : \mathcal{I}$, then $\alpha$ satisfies the universal property required from $g(\gamma)$: $\mathrm{Hom}_{\mathcal{I}}(\alpha, \beta) = h^\alpha(\beta) \cong (f^*(h^\gamma))(\beta) = h^\gamma(f(\beta)) = \mathrm{Hom}_{\mathcal{J}}(\gamma, f(\beta))$.

2.7.10. *Application to continuously and discretely normed sets.* The above argument is applicable to the situation when we want to consider "continuously normed sets" from $\mathcal{N}_{\mathbb{R}_{\geq 0}}$ and compare them to "discretely normed sets" from $\mathcal{N}_{\mathbb{Z}\cup\{-\infty\}}$, appearing while we work over discrete valuation rings. In this case, we fix some real $\rho > 1$ and define a homomorphism

$$f_\rho \;:\; \Delta = \mathbb{Z} \cup \{-\infty\} \to \Delta' = \mathbb{R}_{\geq 0},$$

where

$$f_\rho(n) := \rho^n$$

(we prefer not to invert the order on $\mathbb{Z}$ here, so we use $\rho^n$, not $\rho^{-n}$). This increasing map $f_\rho\colon \Delta \to \Delta'$ admits a left adjoint $g_\rho\colon \Delta' \to \Delta$, given by $g_\rho(x) := \lceil \log_\rho x \rceil$. We obtain adjoint functors

$$f_{\rho,!}\colon \mathcal{N}_{\mathbb{Z}\cup\{-\infty\}} \rightleftarrows \mathcal{N}_{\mathbb{R}_{\geq 0}} \;:\; f_\rho^* = g_{\rho,!},$$

which transform discretely normed sets into continuously normed sets and conversely by composing their norms with $f_\rho$ or $g_\rho$.

2.7.11. *Special case: $f^*$ when $f$ is a sieve embedding.* Now suppose that $\mathcal{I} \subset \mathcal{J}$ is a *sieve* in $\mathcal{J}$, i.e., a full subcategory of $\mathcal{J}$ containing the source of any morphism from $\mathcal{J}$ whenever it contains its target, and $f\colon \mathcal{I} \to \mathcal{J}$ is the embedding of this sieve into $\mathcal{J}$. In particular, $f$ is fully faithful.

In this case,

$$\mathrm{Hom}_{\mathcal{J}}\big(\gamma, f(\alpha)\big) = \begin{cases} \mathrm{Hom}_{\mathcal{I}}(\gamma, \alpha) & \text{if } \gamma \in \mathrm{Ob}\,\mathcal{I}; \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.7.11.1}$$

This implies for $f^*\colon \mathit{Sets}^{\mathcal{J}} \to \mathit{Sets}^{\mathcal{I}}$ that

$$f^*(h^\gamma) = \begin{cases} h^\alpha & \text{if } f(\alpha) \cong \gamma; \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.7.11.2}$$

For general $\mathcal{C}$, this also implies, in view of (2.4.4.2),

$$f^*\big([M]^\gamma\big) = \begin{cases} [M]^\alpha & \text{if } f(\alpha) \cong \gamma; \\ 0_{\mathcal{C}^{\mathcal{I}}} & \text{otherwise.} \end{cases} \tag{2.7.11.3}$$

Here $0_{\mathcal{C}^{\mathcal{I}}}$ denotes the initial object of $\mathcal{C}^{\mathcal{I}}$.

2.7.12. *$\bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ for a sieve embedding $f$.* According to (2.7.11.2), $f^*$ transforms any corepresentable functor from $Sets^{\mathcal{J}}$ into either a corepresentable functor on $Sets^{\mathcal{I}}$ or the initial object $\varnothing$ of $Sets^{\mathcal{I}}$. This implies that $f^*$ transforms the free objects of $Sets^{\mathcal{J}}$ into free objects of $Sets^{\mathcal{I}}$, thus inducing a functor $\bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ on equivalent categories of normed objects.

This functor $\bar{f}^*$ can be described explicitly in terms of normed sets. Namely, if $\mathcal{Y} = (Y, |\cdot|_{\mathcal{Y}}\colon Y \to \operatorname{Ob}\mathcal{J})$ is an $\mathcal{J}$-normed set, then $\bar{f}^*\mathcal{Y}$ is given by subset $\bar{Y} = \{y \in Y : |y|_{\mathcal{Y}} \in \operatorname{Ob}\mathcal{I}\}$ together with the restriction of $|\cdot|_{\mathcal{Y}}$ to $\bar{Y} \subset Y$, considered as a map $\bar{Y} \to \operatorname{Ob}\mathcal{I}$. One can also say that $|\cdot|_{\bar{f}^*\mathcal{Y}}\colon \bar{Y} \to \operatorname{Ob}\mathcal{I}$ is obtained from $|\cdot|_{\mathcal{Y}}\colon Y \to \operatorname{Ob}\mathcal{J}$ by pulling back along $f\colon \operatorname{Ob}\mathcal{I} \to \operatorname{Ob}\mathcal{J}$.

2.7.13. *Special case: embedding $\Delta \subset \Delta'$ of downward-closed subset of a poset.* This is applicable in particular to the case when $f$ is an embedding of a downward closed subset $\Delta$ into a partially ordered set $\Delta'$, for example, the map $e : \Delta \to \Delta_\infty$ of **1.4.3**. In the latter case we recover the functor already denoted by $e^*$ in *loc.cit.,* a right adjoint to $e_!$.

Now we pass to the general case.

2.7.14. *Multiadjoint functors.* We say that a *functor $f\colon \mathcal{I} \to \mathcal{J}$ admits a left multiadjoint* (cf. [AR, 4.24]) if any of the following equivalent conditions holds:

a) Given any object $\gamma : \mathcal{J}$, one can find a (small) family $(\alpha_i)_{i\in S}$ of objects of $\mathcal{I}$ and morphisms $\phi_i\colon \gamma \to f(\alpha_i)$ having the following property: for any object $\beta : \mathcal{I}$ and morphism $\psi\colon \gamma \to f(\beta)$, there is exactly one index $i \in S$ and exactly one morphism $\chi\colon \alpha_i \to \beta$ such that $\psi = f(\chi) \circ \phi_i$.

b) For any object $\gamma : \mathcal{J}$, there is a small family $(\alpha_i)_{i\in S}$ of objects of $\mathcal{I}$, together with a functorial isomorphism in $\beta : \mathcal{I}$:

$$\operatorname{Hom}_{\mathcal{J}}\bigl(\gamma, f(\beta)\bigr) \cong \bigsqcup_{i\in S} \operatorname{Hom}_{\mathcal{I}}(\alpha_i, \beta). \tag{2.7.14.1}$$

c) For any $\gamma : \mathcal{J}$, one can find a small family $(\alpha_i)_{i\in S}$ of objects of $\mathcal{I}$ such that

$$f^*(h^\gamma) \cong \bigsqcup_{i\in S} h^{\alpha_i} \quad \text{in } Sets^{\mathcal{I}}. \tag{2.7.14.2}$$

d) The functor $f^*\colon Sets^{\mathcal{J}} \to Sets^{\mathcal{I}}$ transforms corepresentable functors from $Sets^{\mathcal{J}}$ into free functors from $Sets^{\mathcal{I}}$.

e) The functor $f^*\colon Sets^{\mathcal{J}} \to Sets^{\mathcal{I}}$ transforms $\operatorname{Free}(Sets^{\mathcal{J}})$ into $\operatorname{Free}(Sets^{\mathcal{I}})$.

The equivalence of these conditions is immediate, each condition being an easy reformulation of the preceding one.

2.7.15. *If $f$ admits a left adjoint, it admits a left multiadjoint.* Of course, if $f$ admits a left adjoint functor $g$, it admits a left multiadjoint in the sense of the previous definition. More precisely, if $f$ admits a left multiadjoint with the cardinality of the set $S$ of the above conditions a)–c) equal to one for any choice of $\gamma : \mathcal{J}$, then $f$ admits a left adjoint.

2.7.16. *If $f\colon \mathcal{I} \to \mathcal{J}$ is a sieve embedding, it admits a left multiadjoint.* Similarly, if $f\colon \mathcal{I} \to \mathcal{J}$ is a sieve embedding, formula (2.7.11.2) shows that condition **2.7.14**,c) is satisfied for any $\gamma : \mathcal{J}$, with $\operatorname{card} S$ equal to 1 or 0 depending on whether $\gamma$ belongs to the sieve $\mathcal{I}$ or not.

2.7.17. *If $f$ admits left multiadjoint, $f^*$ induces a functor $\bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$.* If $f$ admits left multiadjoint, condition **2.7.14**,e) shows that $f^*$ maps $\operatorname{Free}(Sets^{\mathcal{J}})$ into $\operatorname{Free}(Sets^{\mathcal{I}})$, hence it induces a functor $\bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ on the equivalent categories of normed sets, right adjoint to $\bar{f}_!$.

2.7.18. *Explicit description of $f^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$.* The functor $f^* = \bar{f}^*\colon \mathcal{N}_{\mathcal{J}} \to \mathcal{N}_{\mathcal{I}}$ is in general not so easy to describe in terms of normed sets. One can say that each point $y$ of $\mathcal{Y} = (Y, |\cdot|_{\mathcal{Y}})$ with norm $\gamma := |y|_{\mathcal{Y}}$ is transformed by $f^*$ into several points of $f^*\mathcal{Y}$ with norms $(\alpha_i)_{i\in S}$ of conditions a)–c) of **2.7.14**.

2.7.19. *Action of $f^*$ on special objects.* Now consider the case of general $\mathcal{C}$, still assuming that $f$ admits a left multiadjoint. Then $f^*\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$ transforms the special object $[M]^\gamma$ of $\mathcal{C}^{\mathcal{J}}$ into a coproduct of special objects of $\mathcal{C}^{\mathcal{I}}$:

$$f^*\big([M]^\gamma\big) = \coprod_{i\in S} [M]^{\alpha_i}. \tag{2.7.19.1}$$

Here the family $(\alpha_i)_{i\in S}$ is determined by $\gamma$ as in conditions a)–c) of **2.7.14**.

This formula follows immediately from (2.4.4.2) and (2.7.14.1).

2.7.20. *$f_!$ is (lax/oplax) monoidal whenever $f$ is (oplax/lax) monoidal.* Now suppose that $\mathcal{C}$, $\mathcal{I}$, and $\mathcal{J}$ are monoidal categories, and $f\colon \mathcal{I} \to \mathcal{J}$ is either monoidal, lax monoidal, or oplax monoidal. We claim that $f_!\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}}$ is monoidal, oplax monoidal, or lax monoidal, respectively, with respect to the Day monoidal structure on $\mathcal{C}^{\mathcal{I}}$ and $\mathcal{C}^{\mathcal{J}}$, discussed in **2.4**.

The easiest way to see this is with the aid of special objects. Choose arbitrary $M : \mathcal{C}$, $N : \mathcal{C}$, $\alpha : \mathcal{I}$ and $\beta : \mathcal{I}$; then by (2.7.3.1) and (2.4.19.2),

$$f_!\big([M]^\alpha \otimes_{\mathcal{C}^{\mathcal{I}}} [N]^\beta\big) \cong f_!\big([M \otimes_{\mathcal{C}} N]^{\alpha\otimes\beta}\big) \cong [M \otimes_{\mathcal{C}} N]^{f(\alpha\otimes\beta)}, \tag{2.7.20.1}$$

and

$$f_!\big([M]^\alpha\big) \otimes_{\mathcal{C}^{\mathcal{J}}} f_!\big([N]^\beta\big) \cong [M]^{f(\alpha)} \otimes_{\mathcal{C}^{\mathcal{J}}} [N]^{f(\beta)} \cong [M \otimes_{\mathcal{C}} N]^{f(\alpha)\otimes f(\beta)}. \tag{2.7.20.2}$$

Now, $[M \otimes N]^\gamma$ is contravariant in $\gamma : \mathcal{J}$; so, if $f$ is oplax, we have a morphism

$$u_{\alpha,\beta}\colon f(\alpha\otimes\beta) \to f(\alpha) \otimes f(\beta),$$

inducing a morphism from (2.7.20.2) to (2.7.20.1):

$$v_{[M]^\alpha,[N]^\beta}\colon f_!\big([M]^\alpha\big) \otimes_{\mathcal{C}^{\mathcal{J}}} f_!\big([N]^\beta\big) \to f_!\big([M]^\alpha \otimes_{\mathcal{C}^{\mathcal{I}}} [N]^\beta\big). \tag{2.7.20.3}$$

This is exactly what is needed to define a lax monoidal structure on $f_!\colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}}$, at least on special objects. The required associativity conditions are also easily checked by considering triples of special objects in $\mathcal{C}^{\mathcal{I}}$. Furthermore, $v_{[M]^\alpha,[N]^\beta}$ is an isomorphism whenever $u_{\alpha,\beta}$ is, making $f_!$, or rather its restriction to special objects, a monoidal functor whenever $f$ is one. Similarly, if $f$ is lax, we use the morphisms $v_{\alpha,\beta}\colon f(\alpha)\otimes f(\beta) \to f(\alpha\otimes\beta)$ to define morphisms $u_{[M]^\alpha,[N]^\beta}$ from (2.7.20.1) to (2.7.20.2), needed to construct an oplax monoidal structure on $f_!$.

Now we have to extend these results from special to arbitrary objects of $\mathcal{C}^{\mathcal{I}}$. This is best done with the aid of formula (2.4.7.1), providing a canonical expression of arbitrary object of $\mathcal{C}^{\mathcal{I}}$ as a colimit of special objects, since $f_!$, $\otimes_{\mathcal{C}^{\mathcal{I}}}$, and $\otimes_{\mathcal{C}^{\mathcal{J}}}$ all commute with colimits in each argument.

2.7.21. *Reformulation in terms of $\mathcal{I}^{\mathrm{op}}$ and $\mathcal{J}^{\mathrm{op}}$.* The previous result looks better if we replace $\mathcal{I}$ and $\mathcal{J}$ with the opposite categories. So, let $\mathcal{C}$ be a monoidal category, and let $f\colon \mathcal{I} \to \mathcal{J}$ be a lax, oplax, or monoidal functor. Then the left Kan extension functor $f_!\colon \mathcal{C}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{C}^{\mathcal{J}^{\mathrm{op}}}$ canonically is a lax/oplax/monoidal functor as well. This is applicable in particular to $\mathcal{C} = \mathit{Sets}$: then any lax/oplax/monoidal functor $f\colon \mathcal{I} \to \mathcal{J}$ extends to a functor $f_!\colon \widehat{\mathcal{I}} \to \widehat{\mathcal{J}}$ on presheaf categories having the same property.

2.7.22. *Statement for monoidal adjunctions.* Finally, let $f \dashv g$ be a monoidal adjunction between small monoidal categories $\mathcal{I}$ and $\mathcal{J}$, meaning that $f\colon \mathcal{I} \rightleftarrows \mathcal{J} \;:\; g$ are adjoint, $f$ is monoidal, $g$ is automatically lax monoidal, and the monoidal structure of $f$ is compatible with the lax monoidal structure of $g$ via the adjunction. Then, for any closed monoidal category $\mathcal{C}$, the left Kan extension functors $f_!\colon \mathcal{C}^{\mathcal{I}^{\mathrm{op}}} \rightleftarrows \mathcal{C}^{\mathcal{J}^{\mathrm{op}}} \;:\; g_!$ constitute a monoidal adjunction as well.

Indeed, the adjointness of $f_!$ and $g_!$ was already discussed in **2.7.9**; the monoidality of $f_!$ and the lax monoidality of $g_!$ was shown in **2.7.21**; finally, compatibility of (lax) monoidal structures on these functors with the adjunction is easily verified on special objects, generating the categories in question under colimits.

2.7.23. *Orthogonality with changes of $\mathcal{C}$.* We would like to remark that the above "functoriality" of $\mathcal{C}^{\mathcal{I}}$ with respect to $\mathcal{I}$ is "orthogonal" to the "functoriality" in $\mathcal{C}$, discussed in **2.5**. This means, for example, that for any $f\colon \mathcal{I} \to \mathcal{J}$ and $F\colon \mathcal{C} \to \mathcal{D}$ we have a commutative diagrams like

$$\begin{tikzcd} \mathcal{C}^{\mathcal{I}} \arrow[r, "F^{\mathcal{I}}"] \arrow[d, "f_!"] & \mathcal{D}^{\mathcal{I}} \arrow[d, "f_!"] \\ \mathcal{C}^{\mathcal{J}} \arrow[r, "F^{\mathcal{J}}"] & \mathcal{D}^{\mathcal{J}}. \end{tikzcd} \tag{2.7.23.1}$$

**2.8. Monoidal category $\mathcal{C}^{\mathcal{I}}$ as a vectoid product $\mathcal{C} \boxtimes \widehat{\mathcal{I}^{\mathrm{op}}}$.** We would like to sketch here a general point of view improving our understanding of the categories $\mathcal{C}^{\mathcal{I}}$ and their properties. While not indispensable for the exposition, we find it quite illuminating.

2.8.1. *Bicategory of cocomplete categories.* Our starting point is that the Yoneda embedding $h\colon \mathcal{I} \to \widehat{\mathcal{I}}$ of a small category into its presheaf category $\widehat{\mathcal{I}}$ describes $\widehat{\mathcal{I}}$ as "the category freely generated by $\mathcal{I}$ under colimits" (cf. [HT, 5.1.5.6]), or as "the cocomplete category freely generated by $\mathcal{I}$". More precisely, consider the bicategory of cocomplete categories, with cocomplete categories (i.e., categories admitting arbitrary (small) colimits) as objects and cocontinuous (i.e., colimit-preserving) functors as morphisms (and natural transformations as 2-morphisms). This bicategory admits a forgetful 2-functor into the category *Cat* of all categories; its left adjoint transforms $\mathcal{I}$ into $\widehat{\mathcal{I}}$. This is the meaning of the phrase "$\widehat{\mathcal{I}}$ is the cocomplete category freely generated by $\mathcal{I}$".

2.8.2. *Universal property of $h_{\mathcal{I}}\colon \mathcal{I} \to \widehat{\mathcal{I}}$.* The universal property of the Yoneda embedding $h_{\mathcal{I}}\colon \mathcal{I} \to \widehat{\mathcal{I}}$ for a small category $\mathcal{I}$ can be made explicit without reference to bicategories and adjoint 2-functors between them. Namely,

a) $h_{\mathcal{I}}\colon \mathcal{I} \to \widehat{\mathcal{I}}$ is a functor from $\mathcal{I}$ into a cocomplete category $\widehat{\mathcal{I}}$;
b) any functor $F\colon \mathcal{I} \to \mathcal{C}$ from $\mathcal{I}$ into a cocomplete category $\mathcal{C}$ can be factorized through $h_{\mathcal{I}}\colon \mathcal{I} \to \widehat{\mathcal{I}}$ up to isomorphism, i.e., given any $\mathcal{C}$ and $F\colon \mathcal{I} \to \mathcal{C}$ as above, there is a cocontinuous functor $\widetilde{F}\colon \widehat{\mathcal{I}} \to \mathcal{C}$ and a functorial isomorphism $\theta\colon \widetilde{F} \circ h_{\mathcal{I}} \xrightarrow{\sim} F$.
c) The above factorization $(\widetilde{F}, \theta)$ is essentially unique, i.e., if $(\widetilde{F}', \theta')$ is another couple with the same properties, then there is an isomorphism $\eta\colon \widetilde{F} \xrightarrow{\sim} \widetilde{F}'$ such that $\theta' \circ (\eta \star h_{\mathcal{I}}) = \theta$.

The proof of this statement is straightforward. One first writes arbitrary presheaf $X$ as a canonical colimit of representable presheaves:

$$X = \varinjlim_{\alpha : \mathcal{I}_{/X}} h_\alpha. \tag{2.8.2.1}$$

Then, since $\widetilde{F}$ is to be cocontinuous, and $\widetilde{F}(h_\alpha) \cong F(\alpha)$ for any $\alpha : \mathcal{I}$, we must have

$$\widetilde{F}(X) = \operatorname*{inj\,lim}_{\alpha : \mathcal{I}_{/X}} F(\alpha). \tag{2.8.2.2}$$

We can use this formula to *define* $F(X)$ because $\mathcal{C}$ is assumed to be cocomplete.

2.8.3. *Functoriality of* $\mathcal{I} \rightsquigarrow \widehat{\mathcal{I}}$. Since $\mathcal{I} \rightsquigarrow \widehat{\mathcal{I}}$ is a left adjoint 2-functor — or, equivalently, since $\widehat{\mathcal{I}}$ is obtained from $\mathcal{I}$ with the aid of some universal property — any functor $f \colon \mathcal{I} \to \mathcal{J}$ must induce an (essentially unique) colimit-preserving functor $f_! \colon \widehat{\mathcal{I}} \to \widehat{\mathcal{J}}$ such that $f_! \circ h_{\mathcal{I}} \cong h_{\mathcal{J}} \circ f$, i.e., $f_!(h_\alpha) \cong h_{f(\alpha)}$ for any $\alpha : \mathcal{I}$. This functor $f_!$ is nothing else than the left Kan extension of $f$, because this left Kan extension preserves colimits and satisfies $f_!(h_\alpha) \cong h_{f(\alpha)}$ as well.

This explains the ubiquity of left Kan extensions $f_!$ in this context, cf. **2.7.3** and several next items.

2.8.4. $\mathcal{I} \rightsquigarrow \widehat{\mathcal{I}}$ *preserves adjunction.* Since $\mathcal{I} \rightsquigarrow \widehat{\mathcal{I}}$ is a bifunctor, it has to transform adjoint pairs $f \dashv g$ in the 2-category of small categories into adjoint pairs $f_! \dashv g_!$ in the 2-category of cocomplete categories. This is what we have already observed and exploited in **2.7.9**, up to replacing some categories with opposites.

2.8.5. *Convolution product of cocomplete categories.* Given two cocomplete categories $\mathcal{A}$ and $\mathcal{B}$, we can consider componentwise cocontinuous (i.e., colimit-preserving separately in each argument) bifunctors $\Psi \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, with values in arbitrary cocomplete categories $\mathcal{C}$. Sometimes a universal couple $(\Psi, \mathcal{C})$ of this sort exists, which will be denoted by $\Phi_{\mathcal{A},\mathcal{B}} \colon \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$. This means that any componentwise bicontinuous $\Psi \colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ factorizes through $\Phi_{\mathcal{A},\mathcal{B}}$ up to a unique isomorphism, much like the universal property made explicit in **2.8.2**.

When such $\mathcal{A} \boxtimes \mathcal{B}$ exists, it is unique up to equivalence of categories; it will be called the *convolution product of* $\mathcal{A}$ *and* $\mathcal{B}$. Sometimes it is also called the *Day convolution product* (even though Day originally defined a monoidal structure on a functor category, cf. **2.4**); we also called it *vectoid product* in [DuV]. When $\mathcal{A}$ and $\mathcal{B}$ are (Grothendieck) topoi, $\mathcal{A} \boxtimes \mathcal{B}$ turns out to be the (2-categorical) direct product of these topoi.

The bifunctor $\Phi_{\mathcal{A},\mathcal{B}}$ is usually also denoted by $\boxtimes$, so that, if $X : \mathcal{A}$ and $Y : \mathcal{B}$ are two objects, $X \boxtimes Y := \Phi_{\mathcal{A},\mathcal{B}}(X, Y)$ is an object of $\mathcal{A} \boxtimes \mathcal{B}$. This makes $\boxtimes$ into a sort of "tensor product" for cocomplete categories and their objects.

2.8.6. *Existence of the convolution product.* Unfortunately, one cannot prove the existence of $\mathcal{A} \boxtimes \mathcal{B}$ for arbitrary cocomplete categories $\mathcal{A}$ and $\mathcal{B}$. One has to impose suitable finiteness conditions. The best choice is to require that $\mathcal{A}$ and $\mathcal{B}$ be *presentable* categories, i.e., cocomplete categories admitting a (small) set of $\lambda$-presentable generators for some (small) regular cardinal $\lambda$. We refer to [HT] or to [AR] (where this notion is known as a "locally presentable category") for technical details.

For our present purpose it will suffice to know that $\mathcal{A} \boxtimes \mathcal{B}$ exists and is presentable whenever $\mathcal{A}$ and $\mathcal{B}$ are presentable, and that $\widehat{\mathcal{I}}$ is presentable for any small category $\mathcal{I}$. Therefore, we can go on, replacing everywhere the 2-category of cocomplete categories (and cocontinuous functors) with the 2-category of presentable categories (and cocontinuous functors).

2.8.7. 2*-monoidal structure* $\boxtimes$ *is closed.* Notice that the convolution $\boxtimes$ is symmetric and "closed", in the sense that the category of cocontinuous functors $\mathit{Funct}^L(\mathcal{B}, \mathcal{C}) := \mathit{Funct}_{\mathrm{cocont}}(\mathcal{B}, \mathcal{C})$ is a cocomplete category for any cocomplete $\mathcal{B}$ and $\mathcal{C}$ (it is better to assume $\mathcal{B}$ presentable, so as to make this functor category locally small; when both $\mathcal{B}$ and $\mathcal{C}$ are presentable, this functor category is presentable as well), such that

$$\mathit{Funct}^L(\mathcal{A}\boxtimes\mathcal{B},\mathcal{C})\cong\mathit{Funct}^L\bigl(\mathcal{A},\mathit{Funct}^L(\mathcal{B},\mathcal{C})\bigr). \tag{2.8.7.1}$$

2.8.8. *Convolution product with Sets.* Notice that *Sets* is the "unit object" for the "2-monoidal structure $\boxtimes$ on presentable categories":

$$\mathit{Sets}\boxtimes\mathcal{C}\cong\mathcal{C}\quad\text{for any cocomplete }\mathcal{C}. \tag{2.8.8.1}$$

This can easily be deduced from (2.8.7.1) and obvious equivalence for the category of cocontinuous functors $\mathit{Sets}\to\mathcal{C}$:

$$\mathit{Funct}^L(\mathit{Sets},\mathcal{C})\cong\mathcal{C}. \tag{2.8.8.2}$$

2.8.9. *Convolution product with* $\widehat{\mathcal{I}}$. An important formula is

$$\widehat{\mathcal{I}}\boxtimes\mathcal{C}\cong\mathit{Funct}(\mathcal{I}^{\mathrm{op}},\mathcal{C})=\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}. \tag{2.8.9.1}$$

One can prove this statement for *presentable* $\mathcal{C}$ as follows. First, one considers the special case of $\mathcal{C}=\widehat{\mathcal{J}}$, discussed below in **2.8.10**. Then one represents an arbitrary presentable category $\mathcal{C}$ as a localization of a presheaf category $\widehat{\mathcal{J}}$ with respect to some small set of morphisms $S\subset\operatorname{Ar}\widehat{\mathcal{J}}$ (this is in fact an equivalent description of presentable categories), meaning that $\mathcal{C}$ is equivalent to the full reflective subcategory $S^\perp$ of $\widehat{\mathcal{J}}$ consisting of *$S$-local objects*, i.e., objects $X$ with the property that $\operatorname{Hom}_{\widehat{\mathcal{J}}}(-,X)$ transforms all morphisms from $S$ into bijections. Next, one shows that if $\mathcal{C}=S^\perp\subset\widehat{\mathcal{J}}$, then $\widehat{\mathcal{I}}\boxtimes\mathcal{C}$ can be obtained from $\widehat{\mathcal{I}}\boxtimes\widehat{\mathcal{J}}$ by localizing along $T:=\{h_\alpha\}\boxtimes S=\{h_\alpha\boxtimes\phi\,|\,\alpha:\mathcal{I},\,\phi\in S\}$; this is in fact part of the general proof of existence for the convolution product of arbitrary presentable categories. Then one shows that a functor $F\in\mathit{Funct}(\mathcal{I}^{\mathrm{op}},\widehat{\mathcal{J}})\cong\widehat{\mathcal{I}}\boxtimes\widehat{\mathcal{J}}$ is $(h_\alpha\boxtimes S)$-local (for fixed $\alpha$) if and only if its value $F(\alpha)$ is $S$-local in $\widehat{\mathcal{J}}$, i.e., belongs to $\mathcal{C}=S^\perp\subset\widehat{\mathcal{J}}$, hence $F$ is $T$-local if and only if it factorizes through $\mathcal{C}\subset\widehat{\mathcal{J}}$, i.e., belongs to $\mathit{Funct}(\mathcal{I}^{\mathrm{op}},\mathcal{C})\subset\mathit{Funct}(\mathcal{I}^{\mathrm{op}},\widehat{\mathcal{J}})$.

2.8.10. *Convolution product of two presheaf categories.* A special case of (2.8.9.1) is

$$\widehat{\mathcal{I}}\boxtimes\widehat{\mathcal{J}}\cong\widehat{\mathcal{I}\times\mathcal{J}}\quad\text{for any small }\mathcal{I}\text{ and }\mathcal{J}. \tag{2.8.10.1}$$

This formula can be proved directly by comparing universal properties of both sides:

$$\begin{aligned}\mathit{Funct}^L(\widehat{\mathcal{I}}\boxtimes\widehat{\mathcal{J}},\mathcal{C})&\cong\mathit{Funct}^L\bigl(\widehat{\mathcal{I}},\mathit{Funct}^L(\widehat{\mathcal{J}},\mathcal{C})\bigr)\\&\cong\mathit{Funct}\bigl(\mathcal{I},\mathit{Funct}(\mathcal{J},\mathcal{C})\bigr)\cong\mathit{Funct}(\mathcal{I}\times\mathcal{J},\mathcal{C})\cong\mathit{Funct}^L(\widehat{\mathcal{I}\times\mathcal{J}},\mathcal{C}).\end{aligned} \tag{2.8.10.2}$$

2.8.11. $\mathcal{I}\rightsquigarrow\widehat{\mathcal{I}}$ *is* 2*-monoidal.* This formula can be interpreted by saying that $\mathcal{I}\rightsquigarrow\widehat{\mathcal{I}}$ is a "2-monoidal functor" from the 2-category of small categories with Cartesian product into the 2-category of presentable categories with convolution product. In particular, it has to transform monoids in the first of these 2-categories, i.e., small monoidal categories $\mathcal{I}$, into monoids in presentable categories, i.e., presentable categories with componentwise cocontinuous monoidal structure.

2.8.12. $\widehat{\mathcal{I}}$ *is monoidal whenever* $\mathcal{I}$ *is.* We see that *if* $\mathcal{I}$ *is monoidal,* $\widehat{\mathcal{I}}$ *admits a componentwise cocontinuous monoidal structure such that*

$$h_\alpha\otimes_{\widehat{\mathcal{I}}}h_\beta\cong h_{\alpha\otimes_{\mathcal{I}}\beta}. \tag{2.8.12.1}$$

This explains what we already saw in **2.3.1** and **2.3.6**.

2.8.13. *Vectoids as commutative monoids under* $\boxtimes$. It is interesting to note that the *vectoids* of [DuV] are precisely the "commutative monoids" for the 2-monoidal structure $\boxtimes$ on the 2-category of presentable categories. This explains our alternative name "vectoid product" for the convolution product $\boxtimes$, since it induces the 2-direct product on the 2-category of vectoids.

2.8.14. *Special objects of* $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$. When $X$ runs over a set of generators of a presentable category $\mathcal{C}$, and $Y$ over generators of a presentable category $\mathcal{D}$, their products $X \boxtimes Y$ run over a set of generators of the convolution product $\mathcal{C} \boxtimes \mathcal{D}$.

We can specialize this to the case of $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$: the objects $h_\alpha \boxtimes M$, where $\alpha$ runs over $\mathcal{I}$ and $M$ over all objects or merely some set of generators of $\mathcal{C}$, constitute a family of generators for $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$.

Now under the equivalence $\widehat{\mathcal{I}} \boxtimes \mathcal{C} \cong \mathit{Funct}(\mathcal{I}^{\mathrm{op}}, \mathcal{C})$, the objects $h_\alpha \boxtimes M$ are transformed into "special objects"

$$[M]_\alpha \colon \beta \rightsquigarrow M^{(\mathrm{Hom}_{\mathcal{I}}(\beta,\alpha))}. \tag{2.8.14.1}$$

These are exactly the special objects of **2.4.4**, provided we replace $\mathcal{I}$ with the opposite category. This explains the appearance and importance of special objects in our considerations.

2.8.15. $\mathcal{C} \boxtimes \mathcal{D}$ *is cocontinuous monoidal whenever* $\mathcal{C}$ *and* $\mathcal{D}$ *are.* The tensor product of two monoids in a symmetric monoidal category is a monoid again, commutative if the two original monoids were. Applying this to the "2-monoidal structure" on the 2-category of presentable categories given by the convolution product, we see that if $\mathcal{C}$ and $\mathcal{D}$ are vectoids, i.e., monoidal presentable categories with tensor product preserving colimits in each argument, then $\mathcal{C} \boxtimes \mathcal{D}$ is a vectoid again.

2.8.16. *Monoidal structure on* $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$. In particular, whenever there is a monoidal structure $\otimes_{\mathcal{I}}$ on a small category $\mathcal{I}$, inducing componentwise cocontinuous monoidal structure $\otimes_{\widehat{\mathcal{I}}}$ on $\widehat{\mathcal{I}}$ by **2.8.12**, and a componentwise a cocontinuous monoidal structure $\otimes_{\mathcal{C}}$ on a presentable category $\mathcal{C}$, we obtain a componentwise cocontinuous monoidal structure $\otimes_{\widehat{\mathcal{I}} \boxtimes \mathcal{C}}$ on $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$, symmetric if the original monoidal structures were. Under the equivalence $\widehat{\mathcal{I}} \boxtimes \mathcal{C} \cong \mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$ of (2.8.9.1), this monoidal structure corresponds to a monoidal structure on $\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$, which is precisely the Day convolution product discussed in **2.4**. This is best seen on the level of special objects, cf. (2.4.19.2) and **2.8.14**.

This explains to a certain extent why the convolution product of presentable categories sometimes is also called the "Day convolution product", even though Day himself originally considered only the functor category case in [Day].

2.8.17. *Functoriality of* $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$ *in* $\mathcal{C}$. Notice that the construction

$$\mathcal{C} \rightsquigarrow \widehat{\mathcal{I}} \boxtimes \mathcal{C} \cong \mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$$

is 2-functorial in $\mathcal{C}$ for any fixed small category $\mathcal{I}$. In particular, any cocontinuous functor $F \colon \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\mathcal{I}^{\mathrm{op}}} \colon \mathcal{C}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{D}^{\mathcal{I}^{\mathrm{op}}}$, (lax) monoidal whenever $F$ is. Furthermore, this construction has to preserve adjoint pairs, so if $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ are adjoint functors, so are $F^{\mathcal{I}^{\mathrm{op}}}$ and $G^{\mathcal{I}^{\mathrm{op}}}$. This explains some of the results of **2.5**.

2.8.18. *Functoriality of* $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$ *in* $\mathcal{I}$. Similarly, the construction

$$\mathcal{I} \rightsquigarrow \widehat{\mathcal{I}} \boxtimes \mathcal{C} \cong \mathcal{C}^{\mathcal{I}^{\mathrm{op}}}$$

is 2-functorial in a small category $\mathcal{I}$ for a fixed presentable category $\mathcal{C}$. In particular, any functor $f \colon \mathcal{I} \to \mathcal{J}$ induces a cocontinuous functor $f_! \colon \widehat{\mathcal{I}} \to \widehat{\mathcal{J}}$ and then also a cocontinuous $f_! \boxtimes \mathcal{C} \colon \mathcal{C}^{\mathcal{I}^{\mathrm{op}}} \to \mathcal{C}^{\mathcal{J}^{\mathrm{op}}}$. The latter functor still merits to be denoted $f_!$, because it is a left

Kan extension of $f^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{J}^{\mathrm{op}}$. Again, this 2-functorial construction has to preserve adjoint pairs of functors, monoidality and lax monoidality of functors and so on, thus explaining most of the results of **2.7**.

2.8.19. *Orthogonality of the two functorialities of* $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$. One sees that the functoriality of $\widehat{\mathcal{I}} \boxtimes \mathcal{C}$ in $\mathcal{I}$ is "orthogonal" or "decoupled" from the functoriality of this convolution product in $\mathcal{C}$, giving rise to a large amount of commutative squares. This orthogonality was already observed in **2.7.23**.

*Continued in* [DuN2] *and* [DuN3].

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