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BINOMIALS WHOSE DILATIONS GENERATE $H^2(\mathbb{D})$

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Abstract. This note is about the completeness of the function families

$$\{z^n(\lambda - z^n)^N \,:\, n = 1, 2, \dots\}$$

in the Hardy space $H^2_0(\mathbb{D})$, and some related questions. It is shown that for $|\lambda| > R(N)$ the family is complete in $H^2_0(\mathbb{D})$ (and often is a Riesz basis of H^2_0), whereas for $|\lambda| < r(N)$ it is not, where both radii $r(N) \le R(N)$ tends to infinity and behave more or less as N (as $N \to \infty$). Several results are also obtained for more general binomials $\{z^n(1 - \frac{1}{\lambda} z^n)^\nu \,:\, n = 1, 2, \dots\}$ where $|\lambda| \ge 1$ and $\nu \in \mathbb{C}$.

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§1. The Dilation Completeness Problem (DCP)

The general *dilation completeness problem* (DCP) consists in the description of functions $f \in L^p_{\mathrm{loc}}[0, \infty)$ such that

$$E_f =: \operatorname{span}_{L^p(0,1)} g\big(f(nx)\big|(0,1) \,:\, n = 1, 2, 3, \dots\big) = L^p(0,1).$$

Recall that the famous Riemann Hypothesis (RH) on zeros of the ζ-function is closely related to the DCP for $p = 2$ and $f(x) = \frac{1}{x} - [\frac{1}{x}]$, $x > 0$: (RH) is equivalent to the inclusion $1 \in E_f$, and/or to the equality

$$\operatorname{span}_{L^2(0,1)} \big(f(sx)\big|(0,1) \,:\, s \ge 1\big) = L^2(0,1)$$

(see [Nym1950, Bae2003], or [Nik2012a, Chap. 6]). According to A. Wintner [Win1944] and A. Beurling [Beu1945], the following partial case of the DCP (2-periodic DCP for $p = 2$) is also related to some number theoretic questions (in Diophantine analysis): to determine odd 2-periodic functions $f \in L^2_{\mathrm{odd}}(-1,1)$ on $\mathbb{R}$ such that $E_f = L^2(0,1)$.

Since the functions $e_k = \sin(\pi k x)\sqrt{2}$, $k = 1, 2, \dots$ form an orthonormal basis in $L^2_{\mathrm{odd}}(-1,1)$ and the dilations $f \longmapsto f(nx)$ act as $e_k \longmapsto e_{nk}$ $(n, k = 1, 2, \dots)$, one can unitarily change the basis (e_k) for (z^k) $(k = 1, 2, \dots)$ on the Hardy space $H^2_0(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$,

$$H^2_0(\mathbb{D}) = \Big\{ f = \sum_{k \ge 1} \widehat{f}(k) z^k \,:\, \|f\|^2 =: \sum_{k \ge 1} |\widehat{f}(k)|^2 < \infty \Big\},$$

and get *the following equivalent form of the periodic DCP*:

to describe $f \in H^2_0(\mathbb{D})$ such that the dilations

$$T_n f = f(z^n)$$

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generate the whole space,

$$E_f =: \mathrm{span}_{H_0^2}(T_n f \,:\, n = 1, 2, \dots) = H_0^2;$$

such a function f will be called "*dilation cyclic*".

The most of very few facts and references known on the periodic DCP are gathered in [HLS1997, HLS1999, Mit2016, Nik2012a, Nik2012b, Nik2017]. B. Mityagin pointed (see for example [Mit2016]) to an applied interest of the dilation cyclicity of the binomials $f = z(\lambda - z)^N$ ($\lambda \in \mathbb{C}$, $N = 1, 2, \dots$) related to the Cohn–Lehmer–Schur algorithm (localization of zeros of polynomials). This special choice of f was already treated partially in [Nik2012b] and [Nik2017]. Below, we add new details to the study of these polynomials. Moreover, general binomial functions $f_{\nu,\lambda}$ are considered as well,

$$f_{\nu,\lambda} = z(\lambda - z)^\nu =: \lambda^\nu z\Big(1 - \frac{z}{\lambda}\Big)^\nu =: \lambda^\nu \sum_{k\geq 0} \binom{\nu}{k} \frac{z^{k+1}(-1)^k}{\lambda^k}, \quad |z| < |\lambda|,$$

where $\nu, \lambda \in \mathbb{C}$, and $\binom{\nu}{k} = \frac{\nu(\nu-1)\dots(\nu-k+1)}{k!}$ stands for a binomial coefficient. Clearly, $f_{\nu,\lambda} \in H_0^2(\mathbb{D})$ for all ν and $|\lambda| > 1$, or for positive integers $\nu = N$ and all $\lambda \in \mathbb{C}$, in which case

$$f_{N,\lambda} \ \text{ is a polynomial.}$$

We also treat *the Riesz basis question* for the families $(f_{\nu,\lambda}(z^n))_{n\geq 1}$, that is the question when every function $g \in H_0^2(\mathbb{D})$ can uniquely be developed in a norm convergent series

$$g = \sum_{n\geq 1} a_n f_{\nu,\lambda}(z^n),$$

with the norm $\|g\|^2$ comparable with $\sum_{n\geq 1} |a_n|^2$.

More notation. Given $\nu \in \mathbb{C}$, denote

$$\Omega(\nu) = \big\{\lambda \in \mathbb{C} \,:\, f_{\nu,\lambda} \ \text{ is dilation cyclic on } \ H_0^2(\mathbb{D})\big\}.$$

Except some particular values of ν (see below), at present, we cannot describe the shape of $\Omega(\nu)$. Instead, following B. Mityagin's suggestion (see also [Mit2016]), for $\nu \in \mathbb{N}$ we consider the radii $r(\nu)$ and $R(\nu)$,

$$r(\nu) = \sup\big\{r > 0 \,:\, |\lambda| < r \Rightarrow f_{\nu,\lambda} \ \text{ is NOT cyclic}\big\},$$
$$R(\nu) = \inf\big\{R > 0 \,:\, |\lambda| > R \Rightarrow f_{\nu,\lambda} \ \text{ IS cyclic}\big\}.$$

For $\nu \in \mathbb{C} \setminus \mathbb{N}$, we modify the definition writing $f_{\nu,\lambda} = \lambda^\nu z\big(1 - \frac{z}{\lambda}\big)^\nu$ and always requiring $|\lambda| \geq 1$.

In particular, $R(\nu) < \infty$ means that $\mathbb{C} \setminus \Omega(\nu)$ is a bounded set, and $r(\nu)$ is the radius of the largest zero centered disc (the annulus $1 \leq |\lambda| < r(\nu)$ for a noninteger ν) inscribed into $\mathbb{C} \setminus \Omega(\nu)$; for integers $\nu \in \mathbb{N}$,

$$\big\{\lambda \,:\, |\lambda| < r(\nu)\big\} \subset \mathbb{C} \setminus \Omega(\nu).$$

On the other hand, we verify on examples that *the set* $\mathbb{C} \setminus \Omega(\nu)$ *in general is not a disc.* Similarly, $\{\lambda \,:\, |\lambda| > R(\nu)\}$ is the largest disc centered at ∞ and included into $\Omega(\nu)$.

The idea of what follows is easy: if $|\lambda|$ is "large", then $f_{\nu,\lambda}$ is "close" to $\lambda^\nu z$ and the latter function is obviously cyclic, but if $|\lambda|$ is "small" and $\nu = N$ is a positive integer then $f_{\nu,\lambda}$ is "close" to z^{1+N} which is not dilation cyclic. In what follows, we precise these "large", "small" and "close" and verify that our "closeness" implies the corresponding (non)cyclicity properties for $f_{\nu,\lambda}$. Here are our statements — first, for integer exponents, and then for general complex exponents (the reader can see that the result for the latter case is less complete than for the former one).

Theorem 1. *Given a positive integer $N \in \mathbb{N}$, then*

$$N \leq R(N) \leq N \cdot \log_2 e,$$

and moreover, $(f_{N,\lambda}(z^n))_{n\geq 1}$ is a Riesz basis of $H_0^2(\mathbb{D})$ if $|\lambda| \geq N \cdot \log_2 e$. For the lower radius $r(N)$, we have

$$N^{1-\epsilon} \preceq r(N) \leq R(N),$$

for every ϵ, $0 < \epsilon < 1$, where for two sequences of positive reals, $a_N \preceq b_N$ (or $b_N \succeq a_N$) means $b_N > (1-\delta)a_N$ for $N > N(\delta)$, for every $\delta > 0$.

In addition, for real λ, $0 \leq \lambda \leq N \Rightarrow \lambda \in \mathbb{C} \setminus \Omega(N)$.

Theorem 2. *Let $\nu \in \mathbb{C}$, $|\lambda| > 1$.*

(1) *$\frac{1}{|\lambda|} \leq 1 - 2^{-1/|\nu|} \Rightarrow \lambda \in \Omega(\nu)$, and in particular, $|\lambda| \geq |\nu| \cdot 2\log_2 e \Rightarrow \lambda \in \Omega(\nu)$ for $|\nu| \geq \ln 2$, and hence*

$$r(\nu) \leq R(\nu) \leq |\nu| \cdot 2\log_2 e,$$

and also $|\lambda| \geq 1 + 2e^{-\ln 2/|\nu|} \Rightarrow \lambda \in \Omega(\nu)$ for $0 < |\nu| \leq 1$.

(2) *For real positive ν, $\nu \geq 2$, with an* even *entire part ($[\nu] \in 2\mathbb{N}$) and for real λ, we have*

$$1 \leq \lambda < \nu \Rightarrow \lambda \in \mathbb{C} \setminus \Omega(\nu) \quad (f_{\nu,\lambda} \ \text{ is noncyclic}),$$

and hence

$$\nu \leq R(\nu).$$

(3) *For real negative ν ($\nu < 0$) such that*

$$\sum_{l\geq 1} \left| \binom{\nu}{2^l - 1} \right| > 1$$

let $a(\nu) = \sup\{a > 1 \,:\, \sum_{l\geq 1} \left|\binom{\nu}{2^l-1}\right| a^{-2^l+1} > 1\}$. Then

$$(-a(\nu), -1) \subset \mathbb{C} \setminus \Omega(\nu), \quad \text{and} \quad R(\nu) \geq a(\nu).$$

In particular, $a(-1/2) > 1{,}005$ and $a(\nu) > |\nu|$ for $|\nu| \geq 1$, so that $R(\nu) \geq |\nu|$ for $\nu \in (-\infty, -1]$ (for example, $a(-1) \geq 1.4662\ldots$).

In particular, $\mathbb{C} \setminus \Omega(\nu)$ is always bounded. An interesting case of "Lambert series" should also be mentioned, where $\nu = -1$ and $f_{-1,\lambda} = \frac{z}{\lambda - z}$ is the Cauchy kernel, see 5.3(2) and a remark to 5.5 below.

The Riesz basis property of the dilations $(f_{\nu,\lambda}(z^n))_{n\geq 1}$ usually happens when we replace an inequality giving a sufficient completeness condition for its strong form; below, we specify the corresponding properties in each case separately.

§2. Recalling some known facts

All known results on periodic DCP use the language of the Bohr lift to the Hilbert multidisc. An infinite-dimensional Hilbert multi-disc $\mathbb{D}_2^\infty$ is defined as

$$\mathbb{D}_2^\infty = \big\{\zeta = (\zeta_k)_{k\geq 1} \in \ell^2 \,:\, |\zeta_k| < 1 (\forall k)\big\}.$$

A holomorphic function theory on $\mathbb{D}_2^\infty$ is sketched in [Hil1909]. The Hardy space on $\mathbb{D}_2^\infty$ can be defined as the space of absolutely convergent power series

$$H^2(\mathbb{D}_2^\infty) =: \Big\{F = \sum_{\alpha \in \mathbb{Z}_+(\infty)} c_\alpha(F)\zeta^\alpha \,:\, \|F\|_2^2 = \sum_{\alpha \in \mathbb{Z}_+(\infty)} |c_\alpha(F)|^2 < \infty\Big\},$$

where the multiindex α runs over the set

$$\mathbb{Z}_+(\infty) = \bigcup_{k\geq 1} \mathbb{Z}_+^k$$

of all finitely supported sequences of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_s, 0, 0, \dots)$, and $\zeta^\alpha = \zeta_1^{\alpha_1} \dots \zeta_s^{\alpha_s}$ ($\zeta \in \mathbb{D}_2^\infty$). The *Bohr transform* (introduced in [Boh1913] in the framework of Dirichlet series) is a unitary map $U\colon H_0^2(\mathbb{D}) \to H^2(\mathbb{D}_2^\infty)$,

$$U : f = \sum_{n\geq 1} \widehat{f}(n) z^n \longmapsto Uf(\zeta) = \sum_{n\geq 1} \widehat{f}(n)\zeta^{\alpha(n)}, \quad \zeta \in \mathbb{D}_2^\infty,$$

where the multiindex $\alpha(n) = (\alpha_1, \dots, \alpha_s, 0, \dots)$ is defined by the prime decomposition of an integer n,

$$n = p_1^{\alpha_1} \dots p_s^{\alpha_s}, \quad \alpha_j \in \mathbb{Z}_+,$$

$p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$ are naturally ordered primes. Clearly, a function $f \in H_0^2(\mathbb{D})$ is a polynomial if and only if Uf is a polynomial on $\mathbb{D}_2^\infty$ (finite linear combination of monomials $\zeta \longmapsto \zeta^\alpha, \zeta \in \mathbb{D}_2^\infty$).

The use of the Bohr transform is based on the following obvious facts.

A. *U is unitary $H_0^2(\mathbb{D}) \to H^2(\mathbb{D}_2^\infty)$ and transforms the dilations (T_n) into a multiplication semigroup $M_\zeta = (M_{\zeta^\alpha})_{\alpha\in\mathbb{Z}_+(\infty)}$:*

$$(UT_nU^{-1})f(\zeta) = \zeta^{\alpha(n)}f(\zeta) \quad (\zeta \in \mathbb{D}_2^\infty, \quad f \in H^2(\mathbb{D}_2^\infty)).$$

B. $E \in \mathrm{Lat}(T_n) \Leftrightarrow UE \in \mathrm{Lat}(M_\zeta)$, *where* $\mathrm{Lat}(A)$ *stands for the collection* (*lattice*) *of all A-invariant subspaces.*

In particular, a function $f \in H_0^2$ is (T_n)-cyclic if and only if Uf is M_ζ-cyclic (*i.e., the weighted polynomials $\{pUf : p$ is a polynomial on $\mathbb{D}_2^\infty\}$ are dense in $H^2(\mathbb{D}_2^\infty)$*).

Here are some known results on the dilation cyclicity of a function $f \in H_0^2(\mathbb{D})$.

(a) The condition $Uf(\zeta) \neq 0$ ($\forall\zeta \in \mathbb{D}_2^\infty$) is *necessary* for (T_n)-cyclicity ([Beu1945]).

(b) If f is *a polynomial*, condition (a) is *also sufficient* ([NGN1970]).

(c) If the Fourier spectrum $\sigma(f) = \{k \in \mathbb{Z}_+ \setminus \{0\} : \widehat{f}(k) \neq 0\}$ is finitely generated (in the multiplicative semigroup $\mathbb{Z}_+ \setminus \{0\}$) and Uf (which depends on finitely many variables $\zeta_1, \dots, \zeta_s$ only) is holomorphic on a larger disc $(1+\epsilon)\mathbb{D}_2^\infty$, then condition (a) *is also sufficient* ([Nik2012b]).

(d) If $\sigma(f)$ has only one generator, $\sigma(f) = \{N^k : k \geq 0\}$ and hence $f(z) = \sum_{k\geq 0} a_k z^{N^k}$, then the necessary and sufficient condition for the (T_n)-cyclicity of f is that $\sum_{k\geq 0} a_k z^k$ is a Beurling outer function ([Nik2012b]).

(e) If $f \neq 0$ and $\mathrm{Re}(Uf(\zeta)) \geq 0$ ($\forall\zeta \in \mathbb{D}_2^\infty$), then f is (T_n)-cyclic ([Nik2012b]); in particular, f is cyclic if

$$|\widehat{f}(1)| \geq \sum_{n\geq 2} |\widehat{f}(n)|$$

([HLS1997, HLS1999]). Moreover, if $|\widehat{f}(1)| > \sum_{n\geq 2} |\widehat{f}(n)|$ then $(T_nf)_{n\geq 1}$ is *a Riesz basis* in $H_0^2(\mathbb{D})$ (see [GN1968/1969]).

(f) If $Uf(\zeta) \neq 0$ ($\forall\zeta \in \mathbb{D}_2^\infty$) and there exists $\epsilon > 0$ such that $(Uf)^{1+\epsilon}$, $(Uf)^{-\epsilon} \in H^2(\mathbb{D}_2^\infty)$, then f is (T_n)-cyclic (real powers of Uf can be confidently, defined [Nik2017]).

Unfortunately, there are no visible relations between the values $f(z)$ on the unit disc $z \in \mathbb{D}$ and the values $Uf(\zeta)$ on the multidisc $\zeta \in \mathbb{D}_2^\infty$ (except the trivial $Uf(0) = \frac{f(z)}{z}(0)$, and $\lim_{z\to 1} f(z) = \lim_{t\to 1} Uf((t^k)_{k\geq 1})$, $|t| < 1$, for functions f having $\sum_{n\geq 1} |\widehat{f}(n)| < \infty$).

§3. Polynomials $f_{N,\lambda}$ with small integers N

Following 2(b), for a polynomial $f \in H_0^2(\mathbb{D})$, the dilations T_nf, $n \geq 1$ are complete in $H_0^2(\mathbb{D})$ if and only if $Uf(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}_2^\infty$.

3.1. N = 1. Here $f_{1,\lambda} = f =: z(\lambda - z)$, $Uf(\zeta) = \lambda - \zeta_1$, $\zeta = (\zeta_1, \zeta_2, \dots)$, and hence $\Omega(1) = \mathbb{C} \setminus \mathbb{D} = \{\lambda : |\lambda| \geq 1\}$, $r(1) = R(1) = 1$.

3.2. N = 2. Here $f_{2,\lambda} = f =: z(\lambda - z)^2 = \lambda^2 z - 2\lambda z^2 + z^3$, $Uf(\zeta) = \lambda^2 - 2\lambda\zeta_1 + \zeta_2$, $\zeta = (\zeta_1, \zeta_2, \dots)$, and hence

$$Uf(\zeta) \neq 0 \ (\forall \zeta \in \mathbb{D}_2^\infty) \Leftrightarrow 0 \notin \lambda^2 + 2\lambda\mathbb{D} + \mathbb{D}$$

$$\Leftrightarrow 0 \notin \lambda^2 + (2|\lambda| + 1)\mathbb{D} \Leftrightarrow |\lambda^2| \geq (2|\lambda| + 1) \Leftrightarrow |\lambda| \geq 1 + \sqrt{2},$$

and finally

$$\Omega(2) = \{\lambda : |\lambda| \geq 1 + \sqrt{2}\}, \quad r(2) = R(2) = 1 + \sqrt{2}.$$

3.3. N = 1; $\mathbb{C}\setminus\Omega(3)$ is not a disc. Here $f_{3,\lambda} = f =: z(\lambda - z)^3 = \lambda^3 z - 3\lambda^2 z^2 + 3\lambda z^3 - z^4$,

$$Uf(\zeta) = \lambda^3 - 3\lambda^2\zeta_1 + 3\lambda\zeta_2 - \zeta_1^2, \quad \zeta = (\zeta_1, \zeta_2, \dots),$$

and hence

$$Uf(\zeta) = p_\lambda(\zeta_1) + 3\lambda\zeta_2 \neq 0 \quad (\forall \zeta \in \mathbb{D}_2^\infty) \Leftrightarrow 0 \notin (p_\lambda(\mathbb{D}) + 3\lambda\mathbb{D}),$$

where

$$p_\lambda(\zeta_1) = \lambda^3 - 3\lambda^2\zeta_1 - \zeta_1^2,$$

or

$$\Omega(3) = \{\lambda \in \mathbb{C} \setminus \{0\} : |p_\lambda(\zeta_1)| \geq 3|\lambda|, \quad \forall \zeta_1 \in \mathbb{D}\}.$$

It is easy to see that

$$1.73 < \sqrt{3} \leq r(3) \leq R(3) < 3.85$$

(indeed, since $|p_\lambda(\zeta_1)| \geq |\lambda|^3 - 3|\lambda|^2 - 1$ and $t^3 - 3t^2 - 3t - 1 > 0$ for $t > 3.85$, we have $R(3) < 3.85$, and on the other hand, $Uf(0, -\lambda^2/3) = 0$, which gives $r(3) \geq \sqrt{3} > 1.73$).

In order to see that $\Omega(3)$ is not the complement of a disc, observe that for $\lambda > 0$, $Uf(0,0) = \lambda^3 > 0$, and hence the condition $Uf(1,-1) = \lambda^3 - 3\lambda^2 - 3\lambda - 1 \geq 0$ is necessary for $\lambda \in \Omega(3)$ (because $Uf(1,-1) < 0$ implies that there exists t, $0 < t < 1$, such that $Uf(t,-t) = 0$). But it is also sufficient (by 2(e)), and so

$$\Omega(3) \cap \mathbb{R}_+ = \big\{\lambda > 0 : \lambda^3 - 3\lambda^2 - 3\lambda - 1 \geq 0\big\}.$$

For $\lambda < 0$, we have $Uf(0,0) = \lambda^3 < 0$, and a *necessary condition* for $\lambda \in \Omega(3)$ consists in $Uf(-1,-1) = p_\lambda(-1) - 3\lambda \leq 0$ (by a similar reason). In particular, $p_\lambda(-1) < 0$, and since

$$p_\lambda(\zeta_1) = \lambda^3 - (3\lambda^2/2 + \zeta_1)^2 + (3\lambda^2/2)^2 = \lambda^3 + (3\lambda^2/2)^2 - q(\zeta_1)^2,$$

where $q(\zeta_1) = 3\lambda^2/2 + \zeta_1$, it implies that $\lambda^3 + (3\lambda^2/2)^2 - q(-1)^2 < 0$. For real $\lambda \in \mathbb{R}\cap\Omega(3)$, the image $q(\mathbb{D})$ is a disc with the diameter $(3\lambda^2/2 - 1, 3\lambda^2/2 + 1)$ and for $\lambda < 0$, by the previous observation, $\lambda^3 + (3\lambda^2/2)^2 - (3\lambda^2/2 - 1)^2 < 0$, which entails the formula

$$\min_{|\zeta_1|\leq 1} |p_\lambda(\zeta_1)| = \operatorname{dist}\big(\lambda^3 + (3\lambda^2/2)^2, q(\mathbb{D})^2\big)$$

$$= (3\lambda^2/2 - 1)^2 - \lambda^3 - (3\lambda^2/2)^2 = 1 - 3\lambda^2 - \lambda^3.$$

Thus,

$$\Omega(3) \cap \mathbb{R}_- \subset \big\{\lambda < 0 : \min_{|\zeta_1|\leq 1} |p_\lambda(\zeta_1)| \geq 3|\lambda|\big\} = \big\{\lambda < 0 : 1 - 3\lambda^2 - \lambda^3 \geq -3\lambda\big\}$$

$$= \big\{ -t : t^3 - 3t^2 - 3t + 1 \geq 0\big\}.$$

Conversely, if $-t = \lambda < 0$ satisfies the last condition, then $3\lambda^2 + \lambda^3 - 1 \leq 3\lambda < 0$, which implies $\lambda^3 + (3\lambda^2/2)^2 - (3\lambda^2/2 - 1)^2 < 0$, and hence, as above, $\min_{|\zeta_1|\leq 1} |p_\lambda(\zeta_1)| = 1 - 3\lambda^2 - \lambda^3 \geq -3\lambda$, which in turn means that $\lambda \in \Omega(3) \cap \mathbb{R}_-$. Conclusion:

$$-(\Omega(3) \cap \mathbb{R}_-) = \{t : t^3 - 3t^2 - 3t + 1 \geq 0\},$$

which set is larger than $\Omega(3)\cap\mathbb{R}_+$, and so $\mathbb{C}\setminus\Omega(3)$ is not a disc: $\mathbb{R}_+\cap\Omega(3) \neq -(\mathbb{R}_-\cap\Omega(3))$ (the left-hand side is smaller).

§4. General polynomials $f_{N,\lambda}$ $(N \in \mathbb{N})$

Below, one can observe that our sufficient cyclicity statements (i.e., upper estimates for $R(N)$) usually hold for general complex λ, whereas the non-cyclicity ones (i.e., lower estimates for $R(N)$) sometimes depend on specific properties of $Uf_{N,\lambda}$ for real λ (see also 5.4 below).

4.1. *A sufficient condition for cyclicity:* $|\lambda| \geq N \cdot \log_2 e \Rightarrow \lambda \in \Omega(N)$, *and moreover,* $(f_{N,\lambda}(z^n))_{n\geq 1}$ *is a Riesz basis of* $H_0^2(\mathbb{D})$. *Therefore,* $R(N) \leq N \cdot \log_2 e$.

Proof. Indeed, the claim in the title follows directly from 2(e): since

$$f_{N,\lambda} = \sum_{k=0}^{N} \binom{N}{k} (-1)^k z^{k+1} \lambda^{N-k}$$

and

$$\sum_{k=1}^{N} \binom{N}{k} |\lambda|^{N-k} = (|\lambda|+1)^N - |\lambda|^N,$$

2(e) implies that the condition $(|\lambda|+1)^N - |\lambda|^N \leq |\lambda|^N$, that is $1+\frac{1}{|\lambda|} \leq 2^{1/N}$, is sufficient for $\lambda \in \Omega(N)$. The result follows by using the inequality $2^{1/N} - 1 > \frac{\ln 2}{N}$. □

4.2. *For* $N \geq 2$ *and real* λ, $0 \leq \lambda < N \Rightarrow \lambda \in \mathbb{C} \setminus \Omega(N)$ *(noncyclic case), and hence,*

$$N \leq R(N) \leq N \cdot \log_2 e.$$

Proof. Indeed, $Uf_{N,\lambda}(0) = \lambda^N > 0$, and assuming $s \in \mathbb{N}$ $(s \geq 2)$ such that $2^s \leq N+1 < 2^{s+1}$, we obtain for $0 < t < 1$,

$$Uf_{N,\lambda}(t,0,0,\dots) = \lambda^N + \sum_{k=1}^{s} \binom{N}{2^k - 1} \lambda^{N-2^k+1} (-1)^{2^k-1} t^k < \lambda^N - N\lambda^{N-1} t.$$

Since $N > \lambda$, there exists $0 < t_0 < 1$ making Uf negative $Uf(t_0, 0, \dots) < 0$, and by continuity, we find t, $0 < t < t_0$ such that $Uf(t, 0, \dots) = 0$. Hence, f is not cyclic in H_0^2. □

4.3. Lemma. *Let* $N \geq 1$ *be an integer and* q *a prime such that* $\sqrt{N+1} < q \leq N+1$. *If* $|\lambda| < \binom{N}{q-1}^{1/(q-1)}$, *the function* $f_{N,\lambda} = z(\lambda - z)^N$ *is NOT* (T_n)*-cyclic, and hence*

$$r(N) \geq \max \left\{ \binom{N}{q-1}^{1/(q-1)} : q \text{ prime}, \ \sqrt{N+1} < q \leq N+1 \right\};$$

in fact, the maximum is attained on the least prime q *in the interval* $(\sqrt{N+1}, N+1]$, *since the function* $k \longmapsto \binom{N}{k}^{1/k}$ *is monotone decreasing.*

Proof. For the notation convenience, we replace λ by $-\lambda$ and will treat the function

$$f = z(\lambda + z)^N.$$

We have $f = \sum_{k=0}^{N} \binom{N}{k} z^{k+1} \lambda^{N-k}$ and $Uf = \sum_{k=0}^{N} \binom{N}{k} \zeta^{\alpha(k+1)} \lambda^{N-k}$. Denoting $q = p_s$ the minimal prime in $(\sqrt{N+1}, N+1]$, we obtain

$$Uf(0, \dots, \zeta_s, \dots, 0) = \lambda^N + \binom{N}{q-1} \lambda^{N-q+1} \zeta_s,$$

where the last (linear) function obviously vanishes at $\zeta_s = -\lambda^{q-1}/\binom{N}{q-1}$, $|\zeta_s| < 1$. From 2(a) above, it follows that $(T_n f)_{n\geq 1}$ is not complete in $H_0^2(\mathbb{D})$.

The expression $\binom{N}{k}^{1/k} = \left(\frac{(N-k+1)\ldots N}{k!}\right)^{1/k}$ is monotone decreasing in k since the nominator decreases (the arithmetic mean $a_k =: \frac{1}{k}\sum_{j=0}^{k-1}\log(N-j)$ reduces to the mean a_{k+1} when we add the least term $\log(N-k)$), and the denominator $(k!)^{1/k}$ increases (for the same reason). □

We also need the following elementary lemma.

4.4. Lemma. *Let reals k and l be such that $0 < k < l$ and $l \to \infty$. Then*

$$\binom{l}{k}^{1/k} \sim e\frac{l}{k} \text{ as } \lim\left(\frac{k}{l} + \frac{\log l}{k}\right) = 0.$$

Proof. Since $\binom{l}{k} = \frac{\Gamma(l+1)}{\Gamma(k+1)\Gamma(l-k+1)}$ and $\Gamma(l+1) \sim (l/e)^l\sqrt{2\pi l}$, we have

$$\begin{aligned}\frac{1}{k}\log\binom{l}{k} - \log\frac{l}{k} &= \frac{l}{k}\log\frac{l}{e} - \frac{l-k}{k}\log\frac{l-k}{e} - \log\frac{k}{e} - \log\frac{l}{k} + o(1)\\ &= \frac{l}{k}\log\frac{l}{l-k} + \log\frac{l-k}{e} - \log\frac{k}{e} - \log\frac{l}{k} + o(1)\\ &= -\frac{l}{k}\log\left(1-\frac{k}{l}\right) + \log\frac{l-k}{k} - \log\frac{l}{k} + o(1)\\ &= -\frac{l}{k}\log\left(1-\frac{k}{l}\right) + \log\left(1-\frac{k}{l}\right) + o(1) = 1 + o(1).\end{aligned}$$ □

Remark. Observe that $a_N \sim b_N$ implies both $b_N \succeq a_N$ and $a_N \succeq b_N$. It is known also that $\frac{l}{k} < \binom{l}{k}^{1/k} < \frac{el}{k}$ for all $0 < k < l$.

Below, we will make use of some results on the prime gaps. The simplest one is the so-called Bertrand postulate (claimed 1845 by J. Bertrand, and proved by P. Chebyshev in 1852) saying that there is always a prime in the interval $(n, 2n)$, $n = 2, 3, \ldots$. The following known result is much more involved: there exists $A > 0$ such that for every $a \geq A$, the interval $[a, a + a^{21/40}]$ contains at least one prime integer $q = q(a)$ (see [BHP2001]). For our purposes, a simpler form of this will be enough: *there is a function $\epsilon(a) > 0$, $\lim_{a\to\infty}\epsilon(a) = 0$, such that any interval $[a, a(1+\epsilon(a))]$ contains at least one prime integer.*

4.5. Corollary. **(1)** *Asymptotically, as $N \to \infty$,*

$$r(N) \succeq eN^{1/2}$$

(*recall that $a_N \succeq b_N$ means $a_N > (1-\epsilon)b_N$ for $N > N(\epsilon)$, for every $\epsilon > 0$).*

Proof. Indeed, we simply combine Lemmas 4.3 and 4.4 with a weak form of Baker–Harman–Pintz' result quoted in Remark to Lemma 4.4 applied for $a = N^{1/2}$: take a prime q in $(N^{1/2}, N^{1/2}(1+o(1))]$ and apply 4.4 for $l = N$, $k = N^{1/2}(1+o(1))$:

$$r(N) \geq \binom{N}{q-1}^{1/(q-1)} \succeq e\frac{N}{N^{1/2}(1+o(1))} > e(N^{1/2}(1+o(1))) \succeq eN^{1/2}.$$ □

(2) *For rather small values of N, we have*
i) *for $N = 1, 2$, $r(N) \geq N$;*
ii) *for $2 \leq N \leq 7$, $r(N) \geq N/\sqrt{3}$;*
iii) *for $N = q^2 - 2$, where q is a prime, $r(N) > q - 1$.*

Proof. Indeed, a prime number q will serve in Lemma 4.3 for an integer N if $\sqrt{N+1} < q \le N+1$. For $q=2$ we have $\sqrt{N+1} < 2 \le N+1$, so $N=1,2$ and $\binom{N}{q-1}^{1/(q-1)} = N$.

For $q=3$ we have $\sqrt{N+1} < 3 \le N+1$, so $2 \le N \le 7$ and $\binom{N}{2}^{1/2} = N\big(\frac{1}{2}(1-\frac{1}{N})\big)^{1/2} \ge \frac{N}{\sqrt{3}}$, $N \ge 3$.

A prime q serves for all N with $\sqrt{N+1} < q \le N+1$ and gives $|\lambda| < \binom{N}{q-1}^{1/(q-1)}$ as a sufficient condition for noncyclicity; as the function $N \longmapsto \binom{N}{q-1}^{1/(q-1)}$ is monotone increasing, the best result will be for $N = q^2-2$; using $\binom{N}{q-1}^{1/(q-1)} > \binom{(q-1)^2}{q-1}^{1/(q-1)}$ and

$$\binom{m^2}{m} = \prod_{k=1}^{m} \frac{m^2-m+k}{k} = \prod_{k=1}^{m}\Big(1+\frac{m^2-m}{k}\Big) > m^m,$$

we get $|\lambda| \le q-1$ as a sufficient condition of the noncyclicity for $N = q^2-2$. □

4.6. *Proof of Theorem* 1. The first claim of Theorem 1 is already proved (see items 4.1–4.2 above), as well as a weakened form of the second one (Corollary 4.5). In order to prove Theorem 1 in full, fix positive integers m, $m \ge 1$ and N_m such that the interval $((N+1)^{1/m+1}, (N+1)^{1/m}]$ contains at least one prime for every $N > N_m$ (such N_m exists, for instance, by the Bertrand–Chebyshev theorem (1852), we can find a prime in any interval $(n, 2n)$, $n = 2, 3, \dots$). Let

$$p = p(N,m), \quad p \in \big((N+1)^{1/m+1}, (N+1)^{1/m}\big]$$

be the least such prime (so that, by Bertrand–Chebyshev,

$$p \in \big((N+1)^{1/m+1}, 2((N+1)^{1/m+1}), N > N_m\big)$$

and let $p = p_s$ be its place in the primes ordering.

As above, we consider the function $f = z(\lambda+z)^N = f_{N,-\lambda}$ having the Bohr transform

$$Uf = \sum_{k=0}^{N} \binom{N}{k} \zeta^{\alpha(k+1)} \lambda^{N-k}.$$

The case of $m=1$ was considered already in Lemma 4.3 and Corollary 4.5, see the proof of that lemma. Suppose $m > 1$. Since $p^m \le N+1$ and $p^{m+1} > N+1$, we have

$$Uf(0,\dots,\zeta_s,\dots,0) = \lambda^N + \binom{N}{p-1}\zeta_s \lambda^{N-p+1} + \sum_{k=2}^{m}\binom{N}{p^k-1}\zeta_s^k \lambda^{N-p^k+1}.$$

By Rouché's theorem, there will be at least one zero of this function in the disc $|\zeta_s| < 1$ if

$$\binom{N}{p-1}|\lambda|^{N-p+1} > |\lambda|^N + \sum_{k=2}^{m}\binom{N}{p^k-1}|\lambda|^{N-p^k+1},$$

in particular, if the following conditions are fulfilled simultaneously:

(1) $\binom{N}{p-1}|\lambda|^{N-p+1}\frac{1}{m} > |\lambda|^N$,

(2) $\binom{N}{p-1}|\lambda|^{N-p+1}\frac{1}{m} > \binom{N}{p^k-1}|\lambda|^{N-p^k+1}$ for every k, $2 \le k \le m$.

Condition (1) means that

$$|\lambda| < \binom{N}{p-1}^{1/p-1} \frac{1}{m^{1/p-1}} \sim e\frac{N}{p} \quad \text{as} \quad N \to \infty$$

(see Lemma 4.4). Similarly, each of conditions (2) means

$$|\lambda|^{p^k-p} > m\binom{N}{p^k-1}\Big/\binom{N}{p-1}, \quad k = 2, \dots, m.$$

Define $\rho(N,m,k)$ by

$$\rho(N,m,k)=\left(m\binom{N}{p^k-1}\Big/\binom{N}{p-1}\right)^{1/p^k-p},\quad k=2,\dots,m.$$

We have $m^{1/p^k-p}\sim 1$ (as $N\to\infty$), and $\binom{N}{p^k-1}^{1/p^k-1}\sim e\frac{N}{p^k}$, and hence

$$\frac{1}{p^k-1}\log\binom{N}{p^k-1}=1+\log\frac{N}{p^k}+o(1)$$

(as $N\to\infty$). Now, since $p^k\ge p>N^{1/m+1}$, we obtain

$$\begin{aligned}\frac{1}{p^k-p}\log\binom{N}{p^k-1}-\frac{1}{p^k-1}\log\binom{N}{p^k-1}&=\frac{p-1}{(p^k-1)(p^k-p)}\log\binom{N}{p^k-1}\\&=\frac{p-1}{p^k-p}\log\frac{N}{p^k}+o(1)=o(1),\end{aligned}$$

(as $N\to\infty$), and finally

$$\binom{N}{p^k-1}^{1/p^k-p}\sim e\frac{N}{p^k},$$

for every k, $2\le k\le m$.

For the denominator, by the same Lemma 4.4, $\binom{N}{p-1})^{1/p-1}\sim e\frac{N}{p}$ and since

$$\frac{1}{p^{k-1}}\ge\frac{p-1}{p^k-p}\ge\frac{1}{mp^{k-1}}\quad\text{and}\quad p^{k-1}\ge p>N^{1/m+1},$$

we get

$$\binom{N}{p-1}^{1/p^k-p}\sim 1\quad\text{as}\quad N\to\infty.$$

Finally, conditions (2) are equivalent to

$$|\lambda|>\rho(N,m,k),\ \ \rho(N,m,k)\sim e\frac{N}{p^k}\quad\text{as}\quad N\to\infty\quad\text{for all}\quad k,\quad k=2,\dots,m.$$

Therefore, given an integer $m\ge 2$ and a number $\epsilon>0$, the annulus

$$A(N,m,\epsilon)=:\left\{\lambda\in\mathbb{C}\,:\,(1+\epsilon)e\frac{N}{p^2}<|\lambda|<(1-\epsilon)e\frac{N}{p},\quad p=p(N,m)\right\}$$

is included in the noncyclicity set $\mathbb{C}\setminus\Omega(N)$ for all N, $N>N(m,\epsilon)$.

For $m=1$, we have $r(N)\succeq eN^{1/2}$ (Corollary 4.5), and hence the disc $A(N,1,\epsilon)=:$ $\{\lambda\in\mathbb{C}\,:\,|\lambda|<(1-\epsilon)eN^{1/2}\}$ is contained in $\mathbb{C}\setminus\Omega(N)$ for sufficiently large N (we can also argue as before, using that in this case, Uf contains two terms only, $Uf(0,\dots,\zeta_s,\dots,0)=\lambda^N+\binom{N}{p-1}\zeta_s\lambda^{N-p+1}$).

Consider now two consecutive annuli $A(N,m,\epsilon)$ and $A(N,m+1,\epsilon)$. The upper radius of the former $R_{N,m}=(1-\epsilon)e\frac{N}{p}$, $p=p(N,m)$ satisfies

$$R_{N,m}>(1-\epsilon)e\frac{N}{2(N+1)^{1/m+1}},$$

and the lower radius of the latter one $r_{N,m+1}=(1+\epsilon)e\frac{N}{p^2}$, $p=p(N,m+1)$ satisfies

$$r_{N,m+1}\le(1+\epsilon)e\frac{N}{(N+1)^{2/m+2}}.$$

Since $\frac{1}{2(N+1)^{1/m+1}}>\frac{1}{(N+1)^{2/m+2}}$ for every $m\ge 1$ and sufficiently large N, it follows that

$$R_{N,m}>r_{N,m+1}\ \textit{forsufficientlysmall}\ \epsilon>0,\ \textit{all}\ m\ge 1\ \textit{and}\ N>N(m,\epsilon).$$

This means that, given an integer $l \geq 2$, the consecutive annuli $A(N, m, \epsilon)$ and $A(N, m+1, \epsilon)$, $1 \leq m < l$, overlap (for sufficiently small $\epsilon > 0$ and sufficiently large N), and so

$$\bigcup_{m=1}^{l} A(N, m, \epsilon) = \left\{\lambda : |\lambda| < (1-\epsilon)e\frac{N}{p(N,l)}\right\} \subset \mathbb{C} \setminus \Omega(N)$$

for all N, $N > N(l, \epsilon)$, or due to the choice of $p(N, l)$ (Corollary 4.5 is also used)

$$\left\{\lambda : |\lambda| < (1-\epsilon)\frac{e}{2}N^{1-\frac{1}{l+1}}\right\} \subset \mathbb{C} \setminus \Omega(N).$$

The result ($r(N) \succeq \frac{e}{2}N^{1-\epsilon}$) follows with $l + 1 > 1/\epsilon$. □

Remark. Of course, the series of asymptotic estimates stated in the theorem is simply equivalent to $r(N) > N^{1-\epsilon}$ for $N > N(\epsilon)$, for every $\epsilon > 0$.

§5. General binomials $f_{\nu,\lambda}$

This section contains a few analogues of the results of Sections 3–4 for binomial functions $f_{\nu,\lambda}$ that are not polynomials.

We assume everywhere that $\nu \in \mathbb{C} \setminus \mathbb{N}$, $|\lambda| \geq 1$. A binomial function

$$f_{\nu,-\lambda} = \lambda^{\nu} z(1 + z/\lambda)^{\nu},$$
$$(1 + z/\lambda)^{\nu} = \sum_{k\geq 0} \binom{\nu}{k} \frac{z^k}{\lambda^k},$$

is in $H_0^2(\mathbb{D})$ if and only if either 1) $|\lambda| > 1$ and $\nu \in \mathbb{C}$ (arbitrary), or 2) $|\lambda| = 1$ and $\mathrm{Re}(\nu) > -1/2$. The binomial coefficients

$$\binom{\nu}{k} = \frac{\nu(\nu-1)\dots(\nu-k+1)}{k!}$$

have powerlike behaviour (since $\nu \notin \mathbb{N}$),

$$\binom{\nu}{k} = \frac{a(\nu,k)}{k^{\nu+1}}, \quad 0 < \inf_{k\geq 1} |a(\nu,k)| < \sup_{k\geq 1} |a(\nu,k)| < \infty.$$

5.1. A sufficient condition for cyclicity: $\sum_{k\geq 1} \left|\binom{\nu}{k}\right| \frac{1}{|\lambda|^k} \leq 1 \Rightarrow f_{\nu,\lambda}$ *is dilation cyclic* ($\lambda \in \Omega(\nu)$)*; the strong inequality implies the Riesz basis property for* $f_{\nu,\lambda}(z^n)$, $n \geq 1$.

Proof. Indeed, this follows from 2(e). □

The following is a direct analogue of the claim of 4.1 but for an arbitrary exponent $\nu \in \mathbb{C}$.

5.2. Corollary. $\frac{1}{|\lambda|} \leq 1 - 2^{-1/|\nu|} \Rightarrow \lambda \in \Omega(\nu)$ (*and even the Riesz basis property in the case of a strong inequality*), *and in particular,*

– *for* $|\nu| \geq \ln 2 \,:\, |\lambda| \geq |\nu| \cdot 2\log_2 e \Rightarrow \lambda \in \Omega(\nu)$ (*and the Riesz basis property*), *and hence*

$$r(\nu) \leq R(\nu) \leq |\nu| \cdot 2\log_2 e,$$

– *for* $0 < |\nu| \leq 1 \,:\, |\lambda| \geq 1 + 2e^{-\ln 2/|\nu|} \Rightarrow \lambda \in \Omega(\nu)$ (*and the Riesz basis property*), *and hence*

$$r(\nu) \leq R(\nu) \leq 1 + 2e^{-\ln 2/|\nu|}.$$

(*The latter can be of interest for* $|\nu| \to 0$).

Proof. Indeed, since

$$\left|\binom{\nu}{k}\right| = \left|\frac{\nu(\nu-1)\dots(\nu-k+1)}{k!}\right| \leq \left|\frac{(-|\nu|)(-|\nu|-1)\dots(-|\nu|-k+1)}{k!}\right| = (-1)^k\binom{-|\nu|}{k},$$

we have

$$\sum_{k\geq 1}\left|\binom{\nu}{k}\right|\frac{1}{|\lambda|^k} \leq \sum_{k\geq 1}(-1)^k\binom{-|\nu|}{k}\frac{1}{|\lambda|^k} = \left(1-\frac{1}{|\lambda|}\right)^{-|\nu|} - 1,$$

and hence the condition

$$\left(1-\frac{1}{|\lambda|}\right)^{-|\nu|} - 1 \leq 1,$$

i.e., $1 - \frac{1}{|\lambda|} \geq 2^{-1/|\nu|}$, is sufficient for $\lambda \in \Omega(\nu)$.

Since $1 - e^{-t} > t/2$ for $0 < t \leq 1$, the condition $\frac{\ln 2}{2|\nu|} \geq \frac{1}{|\lambda|}$ implies $\lambda \in \Omega(\nu)$ for $\frac{\ln 2}{|\nu|} \leq 1$ (and even the Riesz basis property).

Similarly, for $0 < x =: e^{-\ln 2/|\nu|} \leq 1/2$, we have $\frac{1}{1-x} \leq 1 + 2x$, and hence $1 + 2x \leq |\lambda|$ implies $\lambda \in \Omega(\nu)$. □

Remark. Of course, one can replace $2e^{-\ln 2/|\nu|}$ in the last claim of 5.2 by a larger linear function (if the corresponding linearized condition seems to be more transparent): since $|\nu|/\ln 2 \geq 2e^{-\ln 2/|\nu|}$ (for $\frac{\ln 2}{|\nu|} \geq 1$),

$$|\lambda| \geq 1 + \frac{|\nu|}{\ln 2} \Rightarrow \lambda \in \Omega(\nu).$$

5.3. Special cases. Here, we list several linearized forms of statement 5.2 giving sufficient conditions for the completeness of the family $(f_{\nu,\lambda}(z^n))_{n\geq 1}$ (and its Riesz basis property when we replace the corresponding inequality for its strong form).

(1) $\mathrm{Re}(\nu) \geq -1$, *in which case*

$$A(\nu) =: \sup_{k\geq 1}\left|\binom{\nu}{k}\right| < \infty.$$

Then $|\lambda| \geq A(\nu) + 1 \Rightarrow \lambda \in \Omega(\nu)$.

Proof. Indeed, applying 5.1, we use $\sum_{k\geq 1}\left|\binom{\nu}{k}\right|\frac{1}{|\lambda|^k} \leq \frac{A(\nu)}{|\lambda|-1} \leq 1$ whenever the condition $|\lambda| \geq A(\nu) + 1$ holds. □

Remark. The quantity $A(\nu)$, at least for real positive ν tending to ∞, is of the order $A(\nu) \sim \frac{\Gamma(\nu+1)}{\Gamma(\nu/2+1)^2} \sim 2^{\nu+1/2}/\sqrt{\pi\nu}$, and so a sufficient cyclicity condition we are getting from $|\lambda| \succeq 2^{\nu+1/2}/\sqrt{\pi\nu}$ (as $\nu \to \infty$) is much worse than those of 5.2.

(2) Lambert series. An interesting *partial case of* (1) occurs when $\nu = -1$, the Cauchy kernels

$$f_{-1,\lambda} = \frac{z}{\lambda}\left(1 - \frac{z}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{k\geq 0}\frac{z^{k+1}}{\lambda^k}, \quad |\lambda| > 1.$$

Here, $A(\nu) = 1$, *and so,*

$$|\lambda| \geq 2 \Rightarrow \lambda \in \Omega(-1).$$

Moreover, as in the general situation of 2(e), condition $|\lambda|>2$ implies that $\left(\frac{z^n}{1-z^n/\lambda}\right)_{n\geq 1}$ is (not only complete but) *a Riesz basis in* $H_0^2(\mathbb{D})$, that is every function $f \in H_0^2(\mathbb{D})$ is the sum of a norm convergent series

$$f = \sum_{n\geq 1} a_n \frac{z^n}{\lambda - z^n},$$

and the square norm $\|f\|_2^2$ is comparable to $\sum_{n\geq 1} |a_n|^2$. Series of this type are called (generalized) *Lambert series* (the "classical" ones corresponds to $\lambda = 1$ for which case $f_{-1,\lambda}$ is out of H^2); they are important for the theory of arithmetic functions, see [Apl2010].

Remark. What happens for $1 < |\lambda| < 2$ is not completely clear but for some negative ν (in particular, for $\nu = -1$) the set $\mathbb{C} \setminus \Omega(\nu)$ contains an interval $\{\lambda : -a < \lambda < -1\}$ where $1 < a < 2$, see 5.5 below.

(3) $\mathrm{Re}(\nu) > -1/2$, *in which case*

$$B(\nu)^2 =: \sum_{k\geq 1} \left|\binom{\nu}{k}\right|^2 < \infty.$$

Then $|\lambda|^2 \geq B(\nu)^2 + 1 \Rightarrow \lambda \in \Omega(\nu)$, *and hence* $R(\nu) \leq (B(\nu)^2 + 1)^{1/2}$.

Proof. Indeed, $\sum_{k\geq 1} \left|\binom{\nu}{k}\right| \frac{1}{|\lambda|^k} \leq \frac{B(\nu)}{(|\lambda|^2-1)^{1/2}} \leq 1$ whenever the condition holds. □

Remark. For real $\nu > 0$, the formula

$$B(\nu)^2 + 1 = \int_{\mathbb{T}} |(1+z)^\nu|^2 \, dm = \frac{1}{2\pi} \int_0^{2\pi} 4^\nu \cos^{2\nu}(x/2) \, dx = \binom{2\nu}{\nu} \sim \frac{4^\nu}{\sqrt{\pi\nu}}$$

holds as $\nu \to \infty$. It leads to a slightly worse (as compared to the case (2) above) sufficient condition for (T_n)-cyclicity: $|\lambda| \succeq \frac{2^\nu}{(\pi\nu)^{1/4}}$ (as $\nu \to \infty$).

(4) $\mathrm{Re}(\nu) > 0$, *in which case*

$$C(\nu) =: \sum_{k\geq 1} \left|\binom{\nu}{k}\right| < \infty.$$

Then $|\lambda| \geq C(\nu) \Rightarrow \lambda \in \Omega(\nu)$, *and hence* $R(\nu) \geq C(\nu)$.

Proof. Indeed, $\sum_{k\geq 1} \left|\binom{\nu}{k}\right| \frac{1}{|\lambda|^k} \leq \frac{C(\nu)}{|\lambda|} \leq 1$ whenever the condition holds. □

In the following special case, one can claim much a better condition.

(5) The case of real ν, $0 < \nu \leq 1$. *In this case, for every* ν *and* λ, $|\lambda| \geq 1$, $f_{\nu,\lambda}$ *is dilation cyclic* (*and even has the Riesz basis property in the case of a strong inequality* $|\lambda| > 1$), *so that* $\Omega(\nu) = \{\lambda : |\lambda| \geq 1\}$, $R(\nu) = 1$.

Proof. Indeed, the power series for $f_{\nu,\lambda}$ is absolutely convergent for $|z| \leq 1$ and all λ, $|\lambda| \geq 1$, and hence, regarding $f_{\nu,\lambda}\left(\frac{\lambda}{|\lambda|}\right)$, we observe that

$$\left(1 - \frac{1}{|\lambda|}\right)^\nu = 1 - \sum_{k\geq 1} \left|\binom{\nu}{k}\right| \frac{1}{|\lambda|^k} \geq 0,$$

and the claim follows from 2(e) above. □

Remark. The last inequality, of course, implies $Uf_{\nu,\lambda}(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}_2^\infty$. However, zeros on the boundary can appear which does not discard the completeness: for example, for $\lambda = 1$, $Uf_{\nu,1}(1, 1, \dots) = f_{\nu,1}(1) = 0$.

It is curious to note that taking above $\lambda = 1$ we get $\sum_{k\geq 1} \left|\binom{\nu}{k}\right| = 1$ for all $0 < \nu < 1$.

5.4. A lower estimate for $R(\nu)$, ν real. Here we present a partial analogue of 4.2 (above) for positive noninteger ν.

Claim. For a real ν, $\nu \geq 2$ with an *even* entire part ($[\nu] \in 2\mathbb{N}$) and for real λ, we have

$$1 \leq \lambda < \nu \Rightarrow \lambda \in \mathbb{C} \setminus \Omega(\nu) \quad (f_{\nu,\lambda} \textit{ isnoncyclic}),$$

and hence

$$\nu \leq R(\nu) \leq \nu \cdot 2 \log_2 e.$$

Proof. Indeed, let $f = z(1 - z/\lambda)^\nu = \sum_{k\geq 0} \binom{\nu}{k} \frac{(-1)^k z^{k+1}}{\lambda^k}$, and as in 4.2, observe that $Uf(0) = 1 > 0$, and for $0 < t < 1$

$$\begin{aligned} Uf(t, 0, 0, \dots) &= 1 + \sum_{k\geq 1} \binom{\nu}{2^k - 1} (1/\lambda)^{2^k - 1} (-1)^{2^k - 1} t^k \\ &= 1 - \frac{\nu}{\lambda} t - \sum_{k\geq 2} \binom{\nu}{2^k - 1} (1/\lambda)^{2^k - 1} t^k. \end{aligned}$$

It is clear that $\binom{\nu}{2^k-1} \geq 0$ for $2^k - 2 \leq [\nu]$, and for $2^k - 2 > [\nu]$ we have also $\binom{\nu}{2^k-1} = \frac{\nu(\nu-1)\dots(\nu-[\nu])(\nu-[\nu]-1)\dots(\nu-2^k+2)}{(2^k-1)!} > 0$ because $[\nu]$ is even. Since $\frac{\nu}{\lambda} > 1$, it follows that there exists $0 < t_0 < 1$ making Uf negative $Uf(t_0, 0, \dots) < 0$, and by continuity, we find t, $0 < t < t_0$ such that $Uf(t, 0, \dots) = 0$. Hence, f is not cyclic in H_0^2. □

Remark. The asymptotic formula for $\binom{\nu}{k}^{1/k}$ from Lemma 4.4 is available for all complex ν (since the asymptotics for the Gamma function $\Gamma(z+1)$ is known in complex domain), but their consequences similar to these of 4.3, 4.5 and 4.6 (giving quite sharp lower estimates for $r(\nu)$) meet at the moment some computational problems. The author hopes to return to the question later.

5.5 *Proof of Theorem* 2. Items (1)–(2) of the theorem were already proved in 5.2 and 5.4. As above, let

$$f = z(1 - z/\lambda)^\nu = \sum_{k\geq 0} \binom{\nu}{k} \frac{(-1)^k z^{k+1}}{\lambda^k},$$

and as in 4.2, observe that $Uf(0) = 1 > 0$, and for $0 < t < 1$

$$Uf(t, 0, 0, \dots) = 1 + \sum_{k\geq 1} \binom{\nu}{2^k - 1} (1/\lambda)^{2^k - 1} (-1)^{2^k - 1} t^k.$$

Since $\nu < 0$, we have $\binom{\nu}{2^k-1}(-1)^{2^k-1} = \left|\binom{\nu}{2^k-1}\right|$, and for real negative λ, $\lambda < -1$, $1/\lambda^{2^k-1} = -1/|\lambda|^{2^k-1}$, so that

$$Uf(t, 0, 0, \dots) = 1 - \sum_{k\geq 1} \left| \binom{\nu}{2^k - 1} \right| \frac{t^k}{|\lambda|^{2^k-1}}.$$

By the hypothesis, $1 - \sum_{k\geq 1} \left|\binom{\nu}{2^k-1}\right| \frac{1}{|\lambda|^{2^k-1}} < 0$ for every $\lambda \in (-a(\nu), 1)$. (In fact, $a(\nu)$ is a unique solution to the equation $\sum_{k\geq 1} \left|\binom{\nu}{2^k-1}\right| \frac{1}{|\lambda|^{2^k-1}} = 1$ since the sum $\sum_{k\geq 1} \left|\binom{\nu}{k}\right| \frac{1}{|\lambda|^k}$ is a finite and strictly monotone decreasing function of $|\lambda|$ having limit 0 at infinity; clearly, $a(\nu) > |\nu|$ for $\nu \in (-\infty, -1]$). Hence, there exists t, $0 < t < 1$ such that $Uf_{\nu,\lambda}(t, 0, 0, \dots) = 0$, and hence $\lambda \in \mathbb{C} \setminus \Omega(\nu)$.

For $\nu = -1/2$, we have already $\sum_{k=1}^3 \left|\binom{\nu}{2^k-1}\right| (1/1.005)^{2^k-1} > 1$, and similarly for an estimate of $a(-1)$. □

Remark. Recall that for the "Lambert series" case, $\nu = -1$, see 5.3(2) above, for $|\lambda| \geq 2$ we have already the cyclicity case for $f_{-1,\lambda}$ (and the Riesz basis property for $|\lambda| > 2$), but for $\lambda \in (-a, -1)$, $a = 1.46627 \cdots < 2$, we have noncyclic Cauchy type kernels $f_{-1,\lambda}$.

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