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A MOVING LEMMA FOR MOTIVIC SPACES

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Abstract. The following moving lemma is proved. Let k be a field and X be a quasi-projective variety. Let Z be a closed subset in X and let U be the semi-local scheme of finitely many closed points on X. Then the natural morphism $U \to X/(X-Z)$ of Nisnevich sheaves is $\mathbf{A}^1$-homotopic to the constant morphism of $U \to X/(X-Z)$ sending U to the distinguished point of $X/(X-Z)$.

§1. Introduction

There is a well-known inaccuracy in Morel's book [M]. He used the so-called geometric presentation Gabber's lemma over finite fields. However there is no published proof of that lemma. Let us point out that Morel needed only a consequence of that lemma, rather than the lemma itself. The indicated consequence is exactly Theorem 2.1, which we prove in the present paper. We do not prove and do not use Gabber's geometric presentation lemma. Instead we use a result proved in [Pan].

§2. The main result

Let k be a field. In particular, k can be a finite field. We prove the following useful geometric theorem

Theorem 2.1. (*A moving lemma*) *Let X be a k-smooth quasi-projective irreducible k-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in X. Let $U = \operatorname{Spec}(\mathcal{O}_X, x_1, x_2, \ldots, x_n)$. Let $U \xrightarrow{\text{can}} X$ be the inclusion and $X \xrightarrow{p} X/(X-Z)$ the factor morphism to the Nisnevich sheaf $X/(X-Z)$. Let $* = (X-Z)/(X-Z) \in X/(X-Z)$ be the distinguished point of $X/(X-Z)$. Given a closed subset $Z \subset X$ there is a Nisnevich sheaf morphism $\Phi_t \colon \mathbb{A}^1 \times U \to X/(X-Z)$ such that $\Phi_0 \colon U \to X/(X-Z)$ is the composite morphism $U \xrightarrow{\text{can}} X \xrightarrow{p} X/(X-Z)$ and $\Phi_1 \colon U \to X/(X-Z)$ takes U to the distinguished point $* \in X/(X-Z)$.*

To prove this result, recall the following result [Pan, Theorem 5.1].

Theorem 2.2. *Let X be an affine k-smooth irreducible k-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in X. Let $U = \operatorname{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$. Given a nonzero function $f \in k[X]$ vanishing at each the point x_i, there is a diagram of the form*

(1)
$$\begin{tikzcd}
\mathbf{A}^1 \times U \arrow[dr, "\mathrm{pr}_U"'] & \mathcal{X} \arrow[l, "\sigma"'] \arrow[r, "q_X"] \arrow[d, "q_U"'] & X \\
& U \arrow[u, bend right, "\Delta"'] \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

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with an irreducible scheme $\mathcal{X}$, a smooth morphism q_U, a finite surjective morphism σ and an essentially smooth morphism q_X, and a function

$$f' \in q_X^*(f)k[\mathcal{X}],$$

which enjoys the following properties:

(a) *if $\mathcal{Z}'$ is a closed subscheme of $\mathcal{X}$ defined by the principal ideal (f'), the morphism $\sigma|_{\mathcal{Z}'}\colon \mathcal{Z}' \to \mathbf{A}^1 \times U$ is a closed embedding and the morphism $q_U|_{\mathcal{Z}'}\colon \mathcal{Z}' \to U$ is finite*;

(a') $q_U \circ \Delta = id_U$, *and* $q_X \circ \Delta = \mathrm{can}$, *and* $\sigma \circ \Delta = i_0$;

(b) *σ is étale in a neighborhood of $\mathcal{Z}' \cup \Delta(U)$;*

(c) $\sigma^{-1}(\sigma(\mathcal{Z}')) = \mathcal{Z}' \coprod \mathcal{Z}''$ *scheme theoretically and* $\mathcal{Z}'' \cap \Delta(U) = \varnothing$;

(d) $\mathcal{D}_0 := \sigma^{-1}(\{0\} \times U) = \Delta(U) \coprod \mathcal{D}'_0$ *scheme theoretically and* $\mathcal{D}'_0 \cap \mathcal{Z}' = \varnothing$;

(e) *for* $\mathcal{D}_1 := \sigma^{-1}(\{1\} \times U)$ *one has* $\mathcal{D}_1 \cap \mathcal{Z}' = \varnothing$;

(f) *there is a monic polynomial* $h \in \mathcal{O}[t]$ *such that* $(h) = \mathrm{Ker}[\mathcal{O}[t] \xrightarrow{\bar{\ }\circ\sigma^*} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')]$.

We need also to a corollary to the last theorem, namely Corollary 2.3 below. The corollary was proved in [Pan, Corollary 5.3]. To state it, note that using items (b) and (c) of Theorem 2.2 one can find an element $g \in I(\mathcal{Z}'')$ such that

(1) $(f') + (g) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,
(2) $\mathrm{Ker}((\Delta)^*) + (g) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$,
(3) $\sigma_g = \sigma|_{\mathcal{X}_g}\colon \mathcal{X}_g \to \mathbf{A}^1_U$ is étale.

Corollary 2.3 (Corollary to Theorem 2.2). *The function f' in Theorem* 2.2, *the polynomial h in item* (f) *of that theorem, the morphism $\sigma\colon \mathcal{X} \to \mathbf{A}^1_U$, and the function $g \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ defined above enjoy the following properties*:

(i) *the morphism $\sigma_g = \sigma|_{\mathcal{X}_g}\colon \mathcal{X}_g \to \mathbf{A}^1 \times U$ is étale*;

(ii) *the data $(\mathcal{O}[t], \sigma_g^*\colon \mathcal{O}[t] \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g, h)$ satisfy the hypotheses of* [C-T/O, Proposition 2.6], *i.e.*, *$\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ is a finitely generated as an $\mathcal{O}[t]$-algebra, the element $(\sigma_g)^*(h)$ is not a zero-divisor in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g$ and*

$$\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g / h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_g;$$

(iii) $(\Delta(U) \cup \mathcal{Z}) \subset \mathcal{X}_g$ *and* $\sigma_g \circ \Delta = i_0\colon U \to \mathbf{A}^1 \times U$;

(iv) $\mathcal{X}_{gh} \subseteq \mathcal{X}_{gf'} \subseteq \mathcal{X}_{f'} \subseteq \mathcal{X}_{q_X^*(f)}$;

(v) $\mathcal{O}[t]/(h) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/(f')$ *and* $h\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = (f') \cap I(\mathcal{Z}'')$ *and* $(f') + I(\mathcal{Z}'') = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Proof of Theorem 2.1. Replacing X with an open subset and replacing Z with a closed subset $Z' \subset X$ containing Z, we may and will assume that X is an affine k-smooth quasi-projective irreducible k-variety and Z is the vanishing locus of a nonzero function $\mathrm{f} \in k[X]$ vanishing at each the point x_i, where $i = 1, 2, \ldots, n$. Apply now Theorem 2.3 and Corollary 2.3 to the affine variety X, the semilocal scheme $U = \mathrm{Spec}(\mathcal{O}_{X,\{x_1,x_2,\ldots,x_n\}})$ and the function $f \in k[X]$. This way we get the étale morphism $\sigma_g = \sigma|_{\mathcal{X}_g}\colon \mathcal{X}_g \to \mathbf{A}^1_U$. Consider the following Cartesian square with the map σ_g from Corollary 2.3.

$$\begin{array}{ccc} \mathcal{X}_{gh} & \xrightarrow{\mathrm{inc}} & \mathcal{X}_g \\ {\scriptstyle\sigma_{gh}}\downarrow & & \downarrow{\scriptstyle\sigma_g} \\ (\mathbf{A}^1 \times U)_h & \xrightarrow{\mathrm{inc}} & \mathbf{A}^1 \times U. \end{array} \tag{2}$$

Items (i) and (ii) show that the square (2) is an elementary **distinguished** square in the category of smooth U-schemes in the sense of [MV, Definition 3.1.3]. The distinguished elementary square (2) defines a motivic space isomorphism $\mathcal{X}_g/\mathcal{X}_{gh} \xleftarrow{\sigma} \mathbf{A}^1_U/(\mathbf{A}^1 \times U)_h$

(merely a Nisnevich sheaf isomorphism), hence there is a composite morphism of motivic spaces of the form

$$\Phi_t\colon \mathbf{A}^1_U \to \mathbf{A}^1_U/(\mathbf{A}^1\times U)_h \xrightarrow{\sigma^{-1}} \mathcal{X}_g/\mathcal{X}_{gh} \to \mathcal{X}_g/\mathcal{X}_{q^*_X(\mathrm{f})} \xrightarrow{q} X/X_{\mathrm{f}}.$$

Let $i_0\colon 0\times U \to \mathbf{A}^1_U$ be the natural morphism. By properties (a') and (d) from Theorem 2.2 the morphism $\Phi_0 := \Phi\circ i_0$ is equal to the one

$$U \xrightarrow{\mathrm{can}} X \xrightarrow{p} X/X_{\mathrm{f}},$$

where $p\colon X \to X/X_{\mathrm{f}}$ is the canonical morphism. By item (e) of Theorem 5.1, the morphism $\Phi_1 := \Phi\circ i_1\colon U \to X/X_f$ is the constant morphism to the distinguished point $*$ of $X/(X-Z)$. □

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