Алгебра и анализ
Том 29 (2017), № 6

St. Petersburg Math. J.
Vol. 29 (2018), No. 6, Pages 997–1006
https://doi.org/10.1090/spmj/1524
Article electronically published on September 4, 2018

THE MAXWELL OPERATOR WITH PERIODIC COEFFICIENTS IN A CYLINDER

N. FILONOV AND A. PROKHOROV

Abstract. The object of study is the Maxwell operator in a three-dimensional cylinder with coefficients periodic along the axis of the cylinder. It is proved that for cylinders with circular and rectangular cross-section the spectrum of the Maxwell operator is absolutely continuous.

Introduction

Let $U \subset \mathbb{R}^2$ be a bounded domain, and let $\Pi = U \times \mathbb{R}$ be a three-dimensional cylinder. Let ε and μ be two scalar functions describing the dielectric and magnetic permeabilities of the medium that fills the cylinder Π. We will always assume that these functions are bounded and separated away from zero:

$$0 < \varepsilon_0 \le \varepsilon(x) \le \varepsilon_1, \quad 0 < \mu_0 \le \mu(x) \le \mu_1, \quad x \in \bar{\Pi}. \tag{0.1}$$

The Maxwell operator (see, e.g., [1]) acts by the formula

$$\mathcal{M}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i\varepsilon^{-1}\operatorname{curl} v \\ -i\mu^{-1}\operatorname{curl} u \end{pmatrix} \tag{0.2}$$

on the pairs of vector-valued functions $\{u, v\}$ defined on Π and satisfying the divergence free condition

$$\operatorname{div}(\varepsilon u) = \operatorname{div}(\mu v) = 0 \tag{0.3}$$

and the perfect conductivity condition on the boundary of the cylinder:

$$u_\tau\big|_{\partial\Pi} = 0, \quad v_\nu\big|_{\partial\Pi} = 0; \tag{0.4}$$

the subscripts τ and ν denote the tangent and normal components of a vector, respectively. The functions u and v have the meaning of the electric and magnetic field components in the cylinder. The exact definition of the operator $\mathcal{M}$ is given below in §1. The Maxwell operator is selfadjoint in a suitable Hilbert space. We are interested in the structure of its spectrum in the case where the coefficients ε and μ are periodic along the axis of the cylinder,

$$\varepsilon(x + e_3) = \varepsilon(x), \quad \mu(x + e_3) = \mu(x), \quad x \in \bar{\Pi}. \tag{0.5}$$

It is well known that for operators with periodic coefficients the spectrum has a band structure (see, e.g., [8]), no singular continuous component (see also [5]) is present, and the eigenvalues can only be of infinite multiplicity (degenerate bands). Our goal is to establish the absolute continuity of the spectrum of the Maxwell operator or, equivalently, the absence of eigenvalues.

2010 *Mathematics Subject Classification.* Primary 35Q61; secondary 35J10.
Key words and phrases. Maxwell operator, periodic coefficients, absolute continuity of the spectrum.
The first author was supported by RNF (grant no. 17-11-01126). The second author was supported by RFBR (grant no. 14-01-00760).

©2018 American Mathematical Society

In the paper [9] by A. Morame, the absolute continuity of the Maxwell operator was established in the entire space $\mathbb{R}^3$ in the case of scalar coefficients $\varepsilon, \mu \in C^\infty$ periodic with respect to a three-dimensional lattice. T. A. Suslina simplified the proof of Morame, which made it possible to relax the smoothness conditions to

$$\varepsilon, \mu \in W^2_{3/2,\mathrm{loc}}. \tag{0.6}$$

Moreover, Suslina proved the absolute continuity of the Maxwell operator in a layer $[0, a] \times \mathbb{R}^2$ under the same condition (0.6), see [10] (formally speaking, in [10] it was required that $\varepsilon, \mu \in W^2_{3/2,\mathrm{loc}} \cap W^1_{p,\mathrm{loc}}$, $p > 3$, but the proof works in the case of $p = 3$, and then (0.6) suffices, by the embedding theorem $W^2_{3/2} \subset W^1_3$). In the paper [3] the absolute continuity of the Maxwell operator in $\mathbb{R}^3$ was established in the case where the coefficients are periodic along certain directions and tend to constants super-exponentially fast in the remaining directions (the so-called soft waveguide).

We show that in the case of cylinders with circular and rectangular cross-section the spectrum of the Maxwell operator is absolutely continuous. We also assume that the coefficients ε and μ are scalar functions (which corresponds to an isotropic medium), rather than matrix-valued functions, and are sufficiently smooth. The question as to what happens if at least one of these conditions fails is open even in the case of $\mathbb{R}^3$. It is known that if we lift both conditions, that is, consider an operator with nonsmooth matrix coefficients, then the spectrum *can* have eigenvalues of infinite multiplicity (see [2]).

In all three papers [9, 10] and [3] it was proved that an eigenvalue of the Maxwell operator (0.2)–(0.4) can occur only if there is an eigenvalue of some Schrödinger operator $-\Delta + V$. The function V is periodic, (6×6)–matrix-valued, and, generally speaking, nonselfadjoint. In the case of a layer, the Robin boundary condition arises with periodic coefficients that are combinations of ε, μ, and their derivatives. We follow the same pattern. In this paper we show that in the case of a cylinder with quite an arbitrary cross-section, the question of the absolute continuity of the Maxwell operator also reduces to the question of the absence of eigenvalues for some Schrödinger operator in a cylinder with the Robin boundary condition. Moreover, some terms related to the curvature of the boundary are added to the coefficients in the boundary condition. The result on the absolute continuity of the Maxwell operator is obtained only for cylinders with rectangular or circular cross-section, because currently the corresponding theorems for the Schrödinger operator are obtained only for such cylinders, see [6] and [7].

§1. Formulation of results

Let ε, μ be real scalar functions in a cylinder $\Pi = U \times \mathbb{R}$; we assume conditions (0.1) and (0.5). We introduce the Hilbert space

$$J = \left\{(u; v) \in L_2(\Pi, \mathbb{C}^3, \varepsilon dx) \oplus L_2(\Pi, \mathbb{C}^3, \mu dx) : \operatorname{div}(\varepsilon u) = \operatorname{div}(\mu v) = 0,\ v_\nu|_{\partial\Pi} = 0\right\}. \tag{1.1}$$

The conditions on u and v are understood in the sense of the integral identities

$$\operatorname{div}(\varepsilon u) = 0 \ \Leftrightarrow\ \int_\Pi \langle \varepsilon u, \nabla\eta\rangle\, dx = 0, \quad \eta \in H^1_0(\Pi, \mathbb{C}),$$
$$\operatorname{div}(\mu v) = 0,\ v_\nu|_{\partial\Pi} = 0 \ \Leftrightarrow\ \int_\Pi \langle \mu v, \nabla\theta\rangle\, dx = 0, \quad \theta \in H^1(\Pi, \mathbb{C}),$$

where H^1, H^1_0 are Sobolev spaces; $H^1_0(\Pi)$ is the closure of the set $C_0^\infty(\Pi)$ in $H^1(\Pi)$. In the space J we consider the Maxwell operator

$$\mathcal{M} = \begin{pmatrix} 0 & i\varepsilon^{-1}\operatorname{curl} \\ -i\mu^{-1}\operatorname{curl} & 0 \end{pmatrix} \tag{1.2}$$

on the domain

$$\mathrm{Dom}\,\mathcal{M} = \left\{(u;v) \in J \,:\, \operatorname{curl} u,\, \operatorname{curl} v \in L_2(\Pi, \mathbb{C}^3),\, u_\tau\big|_{\partial\Pi} = 0\right\}. \tag{1.3}$$

The boundary condition is again defined in terms of an integral identity:

$$u_\tau\big|_{\partial\Pi} = 0 \;\Leftrightarrow\; \int_\Pi \langle \operatorname{curl} u, z\rangle\, dx = \int_\Pi \langle u, \operatorname{curl} z\rangle\, dx \text{ for all } z \in L_2(\Pi, \mathbb{C}^3) \text{ with } \operatorname{curl} z \in L_2(\Pi, \mathbb{C}^3).$$

It is well known (see [1]) that the operator $\mathcal{M}$ defined in this way is selfadjoint.

We introduce the notation $\widetilde{L_p}(\Pi)$ for the space of periodic functions f,

$$f(x + e_3) = f(x),$$

such that $f \in L_p(\Pi \cap B_R)$ for any R, where B_R is a ball of radius R. The symbols $\widetilde{W_p^l}(\Pi)$ and $\widetilde{L_p}(\partial\Pi)$ have a similar meaning.

The next result was established in a previous work of the authors, see [4, Theorem 1.4].

Theorem 1.1. *Let $\Pi = U \times \mathbb{R}$, where the cross-section U is a convex bounded domain on the plane. Suppose that the coefficients ε, μ satisfy* (0.1) *and $\varepsilon, \mu \in \widetilde{W_3^1}(\Pi)$. Then the set* $\mathrm{Dom}\,\mathcal{M}$ *admits an equivalent description*:

$$\mathrm{Dom}\,\mathcal{M} = \left\{(u;v) \in H^1(\Pi, \mathbb{C}) \,:\, \operatorname{div}(\varepsilon u) = \operatorname{div}(\mu v) = 0,\, u_\tau|_{\partial\Pi} = 0,\, v_\nu|_{\partial\Pi} = 0\right\};$$

here the boundary conditions can be understood in the sense of traces.

Thus, the "weak" Maxwell operator coincides with the "strong" Maxwell operator.

Remark 1.2. The claim of Theorem 1.1 remains valid if the convexity condition on the domain U is replaced by the smoothness condition $\partial U \in W_p^2$, $p > 2$.

We introduce the space

$$\widehat{H}^1(\Pi) = \left\{\Phi \in H^1(\Pi, \mathbb{C}^6) \,:\, N\Phi\big|_{\partial\Pi} = 0\right\},$$

where

$$N = \begin{pmatrix} 0 & -\nu_3 & \nu_2 & 0 & 0 & 0 \\ \nu_3 & 0 & -\nu_1 & 0 & 0 & 0 \\ -\nu_2 & \nu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 \end{pmatrix},$$

$\nu(x)$ is the normal vector at the point $x \in \partial\Pi$.

We introduce an additional restriction on the cylinder Π. Let $q \geq 3/2$, $r \geq 2$.

Condition $(A(q,r))$**.** The domain Π is such that the conditions

$$\Phi \in \widehat{H}^1(\Pi), \quad V \in \widetilde{L_q}(\Pi, \mathrm{Mat}(6 \times 6, \mathbb{C})), \quad \Sigma \in \widetilde{L_r}(\partial\Pi, \mathrm{Mat}(6 \times 6, \mathbb{C}))$$

and the integral identity

$$\int_\Pi \left(\langle \partial_j \Phi, \partial_j \Psi\rangle_{\mathbb{C}^6} + \langle V\Phi, \Psi\rangle_{\mathbb{C}^6}\right) dx + \int_{\partial\Pi} \langle \Sigma\Phi, \Psi\rangle_{\mathbb{C}^6}\, dS(x) = 0 \quad \text{for all} \quad \Psi \in \widehat{H}^1(\Pi) \tag{1.4}$$

imply $\Phi \equiv 0$.

Hereinafter, summation over repeated indices from 1 to 3 is implied.

Remark 1.3. The identity (1.4) means that the function Φ lies in the kernel of the matrix Schrödinger operator $-\Delta + V$ in the cylinder Π with the boundary conditions

$$\Phi_\tau^{(1)}\big|_{\partial\Pi} = 0,\; \Phi_\nu^{(2)}\big|_{\partial\Pi} = 0,\; \left(\partial_\nu \Phi^{(1)} + (\Sigma\Phi)^{(1)}\right)_\nu\big|_{\partial\Pi} = 0,\; \left(\partial_\nu \Phi^{(2)} + (\Sigma\Phi)^{(2)}\right)_\tau\big|_{\partial\Pi} = 0,$$

where $\Phi^{(1)} = (\Phi_1, \Phi_2, \Phi_3)$, $\Phi^{(2)} = (\Phi_4, \Phi_5, \Phi_6)$. Thus, the condition $A(q,r)$ means that any Schrödinger operator with potentials V and Σ in the classes under consideration admits no eigenvalues.

We now state the main result.

Theorem 1.4. *Let U be a bounded convex plane domain with piecewise C^2-smooth boundary. Suppose that the cylinder $\Pi = U \times \mathbb{R}$ satisfies the condition $A(q, r)$ for some $q \geq 3/2$, $r \geq 2$. Let the scalar coefficients ε, μ satisfy condition* (0.1), *and let*

$$\varepsilon, \mu \in \widetilde{W}_p^2(\Pi), \qquad p = \max\left(q, \frac{3r}{2+r}\right).$$

Then the spectrum of the Maxwell operator defined by formulas (1.1)–(1.3) *is absolutely continuous.*

Theorem 1.4 is conditional: *if the periodic Schrödinger operator with the Robin boundary condition in a cylinder has no eigenvalues, then the spectrum of the Maxwell operator in the same cylinder is absolutely continuous.* The question of the absence of eigenvalues for the Schrödinger operator in a cylinder with an arbitrary cross-section remains open. I. Kachkovskiĭ established this for cylinders with rectangular cross-section and with circular cross-section.

Theorem 1.5 ([6])**.** *Let U be a rectangle on the plane. Then the condition $A(3/2, r)$ with any $r > 2$ is fulfilled for the cylinder $U \times \mathbb{R}$.*

Theorem 1.6 ([7])**.** *Let U be a circle on the plane. Then the condition $A(2, 4)$ is fulfilled for the cylinder $U \times \mathbb{R}$.*

Now Theorem 1.4 implies the following.

Corollary 1.7. *Suppose U is a rectangle, the coefficients ε, μ satisfy* (0.1), *and $\varepsilon, \mu \in \widetilde{W}_p^2(\Pi)$ with $p > 3/2$. Then the spectrum of the Maxwell operator* (1.1)–(1.3) *is absolutely continuous.*

Corollary 1.8. *Suppose U is a disk, the coefficients ε, μ satisfy* (0.1), *and $\varepsilon, \mu \in \widetilde{W}_2^2(\Pi)$. Then the spectrum of the Maxwell operator* (1.1)–(1.3) *is absolutely continuous.*

Remark 1.9. Theorem 1.4 remains true also for the case of a nonconvex cross-section U with C^2-smooth boundary ∂U, see Remark 1.2.

Remark 1.10. For simplicity, Theorem 1.4 is stated for the *cylinder* Π. From the proof it will be clear that a similar result holds true for a periodic *waveguide* (a domain with a variable cross-section whose change along the x_3-axis is also periodic) with C^2-smooth boundary.

§2. Lemmas

Here and in what follows it is assumed that the cylinder Π satisfies the conditions of Theorem 1.4.

Lemma 2.1. a) *Suppose $a \in H^1(\Pi, \mathbb{C}^3)$ and $\varphi \in \widetilde{W}_3^1(\Pi) \cap L_\infty(\Pi)$. Then*

$$\operatorname{curl}(\varphi a) = \varphi \operatorname{curl} a + [\nabla\varphi, a] \in L_2(\Pi, \mathbb{C}^3),$$

where $[\,.\,,\,.\,]$ stands for the cross product of three-dimensional vectors.

b) *Suppose $a, d \in H^1(\Pi, \mathbb{C}^3)$ and $b \in \widetilde{W}_{3/2}^1(\Pi)$. Then*

$$\langle \operatorname{curl}[a, b], d\rangle = \langle a \operatorname{div} b - b \operatorname{div} a - \langle a, \nabla\rangle b + \langle b, \nabla\rangle a, d\rangle \in L_1(\Pi, \mathbb{C}^3).$$

These identities are well known, and the L_2- or L_1-integrability follows from the embedding theorems.

Lemma 2.2. *Suppose* $d \in H^1(\Pi, \mathbb{C}^3)$ *and* $c \in H^1(\Pi, \mathbb{C}^3)$ *or* $c = [a, b]$, *where* $a \in H^1(\Pi, \mathbb{C}^3)$, $b \in \widetilde{W}^1_{3/2}(\Pi)$. *Then*

$$\int_\Pi \langle \operatorname{curl} c, d\rangle \, dx = \int_\Pi \langle c, \operatorname{curl} d\rangle \, dx + \int_{\partial\Pi} \langle c, [d, \nu]\rangle \, dS.$$

Here ν *is the outer unit normal to the boundary.*

This lemma is also well known for smooth functions. The convergence of all integrals again follows from the embedding theorems.

Lemma 2.3. *Suppose that* $v \in H^1(\Pi, \mathbb{C}^3)$, μ *satisfies* (0.1), *and* $\mu \in \widetilde{W}^2_{3/2}(\Pi)$. *Then*

$$\operatorname{div}(\mu^{-1}\nabla\mu)v + \mu^{-2}|\nabla\mu|^2 v - \langle v, \nabla\rangle(\mu^{-1}\nabla\mu) = W_0(\mu)v,$$

where

$$W_0(\mu)_{jk} = \mu^{-1}\Delta\mu\delta_{jk} + \mu^{-2}\partial_j\mu\partial_k\mu - \mu^{-1}\partial_j\partial_k\mu. \tag{2.1}$$

This lemma is verified by direct computation.

On the boundary of the cylinder Π, we introduce a function κ as follows: for $x = (x_1, x_2, x_3) \in \partial U \times \mathbb{R}$, the value $\kappa(x)$ is equal to the curvature of the curve ∂U at the point (x_1, x_2). In other words, $\kappa(x)$ is the mean curvature (the sum of the principal curvatures) of the surface $\partial\Pi$ at the point x.

Lemma 2.4. *Let* $a, b \in C^1(\bar{\Pi}, \mathbb{C}^3)$. *On the boundary* $\partial\Pi$ *we consider the function*

$$I(x) \equiv \nu_k(x)a_k(x)\partial_j\bar{b}_j(x) - \nu_j(x)a_k(x)\partial_k\bar{b}_j(x), \tag{2.2}$$

where $\nu(x)$ *is the outer unit normal. If the boundary conditions*

$$a_\nu\big|_{\partial\Pi} = b_\nu\big|_{\partial\Pi} = 0 \tag{2.3}$$

or

$$a_\tau\big|_{\partial\Pi} = b_\tau\big|_{\partial\Pi} = 0, \tag{2.4}$$

are satisfied, then

$$I(x) = \kappa(x)\langle P_{e_3^\perp} a(x), b(x)\rangle,$$

where $P_{e_3^\perp}$ *is the projection onto the plane orthogonal to the axis of the cylinder.*

Proof. 1) Consider the case where (2.3) is true. We have

$$I(x) = -\nu_j a_k \partial_k \bar{b}_j = \bar{b}_j a_k \partial_k \nu_j$$

because $\langle \nu, b\rangle = 0$ on $\partial\Pi$, and this identity admits differentiation along the tangent vector a. It is also clear that $\partial_3\nu_j = 0$. Thus, $I(x) = \langle \partial_{\tilde{a}}\nu, b\rangle$, where $\tilde{a} = P_{e_3^\perp} a$. The condition $a_\nu = 0$ implies $\tilde{a} = e^{i\theta}|\tilde{a}|\tau$, where $\theta(x) \in \mathbb{R}$, $\tau_1 = \nu_2$, $\tau_2 = -\nu_1$. By the Frenet–Serret formulas,

$$\partial_{\tilde{a}}\nu = e^{i\theta}|\tilde{a}|\partial_\tau\nu = e^{i\theta}|\tilde{a}|\kappa(x)\tau.$$

Therefore, $I(x) = \kappa(x)\langle \tilde{a}, b\rangle$.

2) Now consider the case of (2.4). We have $a = e^{i\theta}|a|\nu$ and

$$\nu_j\nu_k\partial_k + \tau_j\tau_k\partial_k = \partial_j, \quad j = 1, 2,$$

where $\tau_1 = \nu_2$, $\tau_2 = -\nu_1$, $\tau_3 = \nu_3 = 0$. Hence,

$$I(x) = e^{i\theta}|a|\left(\nu_k\nu_k\partial_j - \nu_j\nu_k\partial_k\right)\bar{b}_j = e^{i\theta}|a|\tau_j\tau_k\partial_k\bar{b}_j,$$

where we have taken into account the fact that $b_3 = 0$. Furthermore, $\partial_\tau(\tau_j\bar{b}_j) = 0$ because $\tau_j\bar{b}_j = 0$ along $\partial\Pi$. Therefore, again by the Frenet–Serret formulas, we have

$$I(x) = -e^{i\theta}|a|\bar{b}_j\partial_\tau\tau_j = e^{i\theta}|a|\bar{b}_j\kappa(x)\nu_j = \kappa(x)\langle a, b\rangle. \qquad \square$$

Lemma 2.5. *Suppose that* $a, b \in H^1(\Pi, \mathbb{C}^3)$ *and the boundary conditions* (2.3) *or* (2.4) *are satisfied. Then*

$$\int_\Pi \langle \operatorname{curl} a, \operatorname{curl} b\rangle\, dx = \int_\Pi \langle \partial_j a, \partial_j b\rangle\, dx - \int_\Pi \langle \operatorname{div} a, \operatorname{div} b\rangle\, dx + \int_{\partial\Pi} \kappa(x)\langle P_{e_3^\perp} a(x), b(x)\rangle\, dS.$$

Proof. It suffices to prove the claim for smooth functions a, b. For such functions, it is well known that

$$\int_\Pi \langle \operatorname{curl} a, \operatorname{curl} b\rangle\, dx = \int_\Pi \langle \partial_j a, \partial_j b\rangle\, dx - \int_\Pi \langle \operatorname{div} a, \operatorname{div} b\rangle\, dx + \int_{\partial\Pi} I(x)\, dS(x),$$

where $I(x)$ is as in (2.2). It remains to refer to the preceding lemma. □

Lemma 2.6. *Suppose that* $a, b \in H^1(\Pi, \mathbb{C}^3)$, μ *satisfies* (0.1), *and* $\mu \in \widetilde{W}^2_{3/2}(\Pi)$. *Then*

$$\int_\Pi \langle \partial_j(\mu a), \partial_j(\mu^{-1} b)\rangle\, dx = \int_\Pi \langle \partial_j(\mu^{1/2} a), \partial_j(\mu^{-1/2} b)\rangle\, dx - \int_\Pi \mu^{-1}\partial_j\mu\langle \partial_j a, b\rangle\, dx$$
$$- \int_\Pi \left((4\mu^2)^{-1}|\nabla\mu|^2 + (2\mu)^{-1}\Delta\mu\right)\langle a, b\rangle\, dx + \int_{\partial\Pi} (2\mu)^{-1}\partial_\nu\mu\langle a, b\rangle\, dS.$$

Proof. We differentiate the products $\mu^{1/2}(\mu^{1/2}a)$ and $\mu^{-1/2}(\mu^{-1/2}b)$:

$$\int_\Pi \langle \partial_j(\mu a), \partial_j(\mu^{-1} b)\rangle\, dx = \int_\Pi \langle \partial_j(\mu^{1/2} a), \partial_j(\mu^{-1/2} b)\rangle\, dx + \int_\Pi \partial_j(\mu^{1/2})\langle a, \partial_j(\mu^{-1/2} b)\rangle\, dx$$
$$+ \int_\Pi \partial_j(\mu^{-1/2})\langle \partial_j(\mu^{1/2} a), b\rangle\, dx + \int_\Pi \partial_j(\mu^{1/2})\partial_j(\mu^{-1/2})\langle a, b\rangle\, dx$$
$$=: J_1 + J_2 + J_3 + J_4.$$

In the second term we integrate by parts:

$$J_2 = -\int_\Pi \langle \partial_j(\partial_j(\mu^{1/2})a), \mu^{-1/2} b\rangle\, dx + \int_{\partial\Pi} \nu_j\partial_j(\mu^{1/2})\langle a, \mu^{-1/2} b\rangle\, dS$$
$$= -\int_\Pi \left(((2\mu)^{-1}\Delta\mu - (4\mu^2)^{-1}|\nabla\mu|^2)\langle a, b\rangle + (2\mu)^{-1}\partial_j\mu\langle \partial_j a, b\rangle\right) dx$$
$$+ \int_{\partial\Pi} (2\mu)^{-1}\partial_\nu\mu\langle a, b\rangle\, dS.$$

Furthermore,

$$J_3 = \int_\Pi \left(-\frac{|\nabla\mu|^2}{4\mu^2}\langle a, b\rangle - \frac{\partial_j\mu}{2\mu}\langle \partial_j a, b\rangle\right) dx, \qquad J_4 = -\int_\Pi \frac{|\nabla\mu|^2}{4\mu^2}\langle a, b\rangle\, dx.$$

Summarizing, we get the result. □

§3. Integration by parts

Theorem 3.1. *Suppose that* $v, f \in H^1(\Pi, \mathbb{C}^3)$, $\operatorname{div}(\mu v) = 0$, ε *and* μ *satisfy* (0.1), *and* $\varepsilon, \mu \in \widetilde{W}^2_{3/2}(\Pi)$. *Then*

$$\int_\Pi \varepsilon^{-1}\langle \operatorname{curl} v, \operatorname{curl} f\rangle\, dx = \int_\Pi \langle \operatorname{curl}(\mu v), \operatorname{curl}((\varepsilon\mu)^{-1} f)\rangle\, dx$$
$$- \int_\Pi \langle \operatorname{curl} v, \mu[\nabla(\varepsilon\mu)^{-1}, f]\rangle\, dx + \int_\Pi \varepsilon^{-1}\langle W_0(\mu)v, f\rangle\, dx \tag{3.1}$$
$$+ \int_\Pi (\varepsilon\mu)^{-1}\partial_j\mu\langle \partial_j v, f\rangle\, dx + \int_{\partial\Pi} (\varepsilon\mu)^{-1}\langle [\nabla\mu, v], [f, \nu]\rangle\, dS,$$

where $W_0(\mu)$ *is as in* (2.1).

Proof. We have

$$\begin{aligned}\int_\Pi \varepsilon^{-1}\langle \operatorname{curl} v, \operatorname{curl} f\rangle\, dx &= \int_\Pi \big\langle \mu \operatorname{curl} v, \operatorname{curl}((\varepsilon\mu)^{-1} f) - [\nabla(\varepsilon\mu)^{-1}, f]\big\rangle\, dx \\ &= -\int_\Pi \big\langle \mu \operatorname{curl} v, [\nabla(\varepsilon\mu)^{-1}, f]\big\rangle\, dx + \int_\Pi \big\langle \operatorname{curl}(\mu v), \operatorname{curl}((\varepsilon\mu)^{-1} f)\big\rangle\, dx \\ &\quad - \int_\Pi \big\langle [\nabla\mu, v], \operatorname{curl}((\varepsilon\mu)^{-1} f)\big\rangle\, dx,\end{aligned}$$

where we have used Lemma 2.1 twice.

In the last summand, we apply Lemma 2.2:

$$\begin{aligned}&- \int_\Pi \big\langle [\nabla\mu, v], \operatorname{curl}((\varepsilon\mu)^{-1} f)\big\rangle\, dx \\ &\qquad = -\int_\Pi \big\langle \operatorname{curl}[\nabla\mu, v], (\varepsilon\mu)^{-1} f\big\rangle\, dx + \int_{\partial\Pi} \langle [\nabla\mu, v], [(\varepsilon\mu)^{-1} f, \nu]\rangle\, dS.\end{aligned}$$

In the first term on the right-hand side, we apply Lemma 2.1 b), taking into account the fact that $\operatorname{div}(\mu v) = 0$:

$$\begin{aligned}&- \int_\Pi \big\langle \operatorname{curl}[\mu^{-1}\nabla\mu, \mu v], (\varepsilon\mu)^{-1} f\big\rangle\, dx \\ &\qquad = \int_\Pi \big\langle \mu v \operatorname{div}(\mu^{-1}\nabla\mu) + \langle \mu^{-1}\nabla\mu, \nabla\rangle(\mu v) - \langle \mu v, \nabla\rangle(\mu^{-1}\nabla\mu), (\varepsilon\mu)^{-1} f\big\rangle\, dx.\end{aligned}$$

It remains to use Lemma 2.3. □

Recall that our goal is to transform the Maxwell operator to a Schrödinger operator. The terms on the right-hand side of (3.1) that do not involve the derivatives v and f, as well as the second summand involving $\operatorname{curl} v$, are suitable for this purpose. The first term on the right-hand side of (3.1) is transformed with the help of Lemmas 2.5 and 2.6 to

$$\langle \partial_j(\mu^{1/2} v), \partial_j(\varepsilon^{-1}\mu^{-1/2} f)\rangle,$$

which corresponds to the Laplace operator. The fourth term on the right-hand side of (3.1), which involves $\langle \partial_j v, f\rangle$, cancels exactly. This explains the choice of the exponents of ε and μ in the expression $\langle \partial_j(\mu^{1/2} v), \partial_j(\varepsilon^{-1}\mu^{-1/2} f)\rangle$.

Lemma 3.2. *Suppose that* $v, f \in H^1(\Pi, \mathbb{C}^3)$, $\operatorname{div}(\mu v) = 0$, *and* $v_\nu\big|_{\partial\Pi} = 0$, $f_\nu\big|_{\partial\Pi} = 0$. *Let* ε *and* μ *satisfy* (0.1), *and let* $\varepsilon, \mu \in \widetilde{W}^2_{3/2}(\Pi)$. *Then*

$$\begin{aligned}\int_\Pi \varepsilon^{-1}\langle \operatorname{curl} v, \operatorname{curl} f\rangle\, dx &= \int_\Pi \langle \partial_j(\mu^{1/2} v), \partial_j(\varepsilon^{-1}\mu^{-1/2} f)\rangle\, dx - \int_\Pi \langle \operatorname{curl} v, \mu[\nabla(\varepsilon\mu)^{-1}, f]\rangle\, dx \\ &\quad + \int_\Pi \varepsilon^{-1}\langle W(\mu) v, f\rangle\, dx + \int_{\partial\Pi} \big\langle \big(\varepsilon^{-1}\kappa(x) P_{e_3^\perp} - (2\varepsilon\mu)^{-1}\partial_\nu\mu\big) v, f\big\rangle\, dS.\end{aligned}$$

Here

$$W_{jk}(\mu) = \big((2\mu)^{-1}\Delta\mu - (4\mu^2)^{-1}|\nabla\mu|^2\big)\,\delta_{jk} + \mu^{-2}\partial_j\mu\partial_k\mu - \mu^{-1}\partial_j\partial_k\mu. \tag{3.2}$$

Proof. By Lemma 2.5, since $\operatorname{div}(\mu v) = 0$, we have

$$\int_\Pi \langle \operatorname{curl}(\mu v), \operatorname{curl}((\varepsilon\mu)^{-1} f)\rangle\, dx = \int_\Pi \langle \partial_j(\mu v), \partial_j((\varepsilon\mu)^{-1} f)\rangle\, dx + \int_{\partial\Pi} \kappa(x)\varepsilon^{-1}\langle P_{e_3^\perp} v, f\rangle\, dS.$$

Using Lemma 2.6 with $a = v$, $b = \varepsilon^{-1} f$, we get

$$\begin{aligned}\int_\Pi \langle \partial_j(\mu v), \partial_j((\varepsilon\mu)^{-1} f)\rangle\,dx = \int_\Pi \langle \partial_j(\mu^{1/2} v), \partial_j(\varepsilon^{-1}\mu^{-1/2} f)\rangle\,dx \\ - \int_\Pi (\varepsilon\mu)^{-1}\partial_j\mu\langle \partial_j v, f\rangle\,dx \\ - \int_\Pi \big((4\varepsilon\mu^2)^{-1}|\nabla\mu|^2 + (2\varepsilon\mu)^{-1}\Delta\mu\big)\langle v, f\rangle\,dx \\ + \int_{\partial\Pi} (2\varepsilon\mu)^{-1}\partial_\nu\mu\langle v, f\rangle\,dS.\end{aligned} \tag{3.3}$$

Finally, the condition $v_\nu|_{\partial\Pi} = 0$ implies

$$\langle [\nabla\mu, v], [f, \nu]\rangle = \langle [\nu, [\nabla\mu, v]], f\rangle = -\partial_\nu\mu\langle v, f\rangle.$$

Therefore, the last integral on the right-hand side of (3.1) is equal to

$$\int_{\partial\Pi} (\varepsilon\mu)^{-1}\langle [\nabla\mu, v], [f, \nu]\rangle\,dS = -\int_{\partial\Pi} (\varepsilon\mu)^{-1}\partial_\nu\mu\langle v, f\rangle\,dS. \qquad \square$$

Lemma 3.3. *Suppose that* $u, w \in H^1(\Pi, \mathbb{C}^3)$, $\operatorname{div}(\varepsilon u) = 0$, *and* $u_\tau\big|_{\partial\Pi} = 0$, $w_\tau\big|_{\partial\Pi} = 0$. *Let* ε *and* μ *satisfy* (0.1) *and* $\varepsilon, \mu \in \widetilde{W}^2_{3/2}(\Pi)$. *Then*

$$\begin{aligned}\int_\Pi \mu^{-1}\langle \operatorname{curl} u, \operatorname{curl} w\rangle\,dx = \int_\Pi \langle \partial_j(\varepsilon^{1/2} u), \partial_j(\varepsilon^{-1/2}\mu^{-1} w)\rangle\,dx - \int_\Pi \langle \operatorname{curl} u, \varepsilon[\nabla(\varepsilon\mu)^{-1}, w]\rangle\,dx \\ + \int_\Pi \mu^{-1}\langle W(\varepsilon) u, w\rangle\,dx + \int_{\partial\Pi} \left(\mu^{-1}\kappa(x) + (2\varepsilon\mu)^{-1}\partial_\nu\varepsilon\right)\langle u, w\rangle\,dS,\end{aligned}$$

where the matrix $W(\varepsilon)$ *is given by formula* (3.2).

Proof. We apply Theorem 3.1 with $v = u$, $f = w$, and with ε and μ interchanged. We get

$$\begin{aligned}\int_\Pi \mu^{-1}\langle \operatorname{curl} u, \operatorname{curl} w\rangle\,dx = \int_\Pi \langle \operatorname{curl}(\varepsilon u), \operatorname{curl}((\varepsilon\mu)^{-1} w)\rangle\,dx - \int_\Pi \langle \operatorname{curl} u, \varepsilon[\nabla(\varepsilon\mu)^{-1}, w]\rangle\,dx \\ + \int_\Pi \mu^{-1}\langle W_0(\varepsilon) u, w\rangle\,dx + \int_\Pi (\varepsilon\mu)^{-1}\partial_j\varepsilon\langle \partial_j u, w\rangle\,dx.\end{aligned}$$

The last integral on the right-hand side of (3.1) is zero because $w_\tau|_{\partial\Pi} = 0$.

Again, we apply Lemma 2.5 to the first integral on the right-hand side, taking into account the fact that $\operatorname{div}(\varepsilon u) = 0$:

$$\int_\Pi \langle \operatorname{curl}(\varepsilon u), \operatorname{curl}((\varepsilon\mu)^{-1} w)\rangle\,dx = \int_\Pi \langle \partial_j(\varepsilon u), \partial_j((\varepsilon\mu)^{-1} w)\rangle\,dx + \int_{\partial\Pi} \kappa(x)\mu^{-1}\langle u, w\rangle\,dS.$$

By Lemma 2.6 with $a = u$, $b = \mu^{-1} w$, arguing as in (3.3), we obtain

$$\begin{aligned}\int_\Pi \langle \partial_j(\varepsilon u), \partial_j((\varepsilon\mu)^{-1} w)\rangle\,dx = \int_\Pi \langle \partial_j(\varepsilon^{1/2} u), \partial_j(\varepsilon^{-1/2}\mu^{-1} w)\rangle\,dx \\ - \int_\Pi (\varepsilon\mu)^{-1}\partial_j\varepsilon\langle \partial_j u, w\rangle\,dx \\ - \int_\Pi \left((4\varepsilon^2\mu)^{-1}|\nabla\varepsilon|^2 + (2\varepsilon\mu)^{-1}\Delta\varepsilon\right)\langle u, w\rangle\,dx \\ + \int_{\partial\Pi} (2\varepsilon\mu)^{-1}\partial_\nu\varepsilon\langle u, w\rangle\,dS,\end{aligned}$$

and the claim follows. $\square$

§4. Proof of Theorem 1.4

It suffices to prove the absence of eigenvalues. Suppose that $\{u, v\}$ is an eigenfunction of the Maxwell operator, that is,

$$-i\mu^{-1}\operatorname{curl} u = \lambda v, \quad \operatorname{div}(\varepsilon u) = 0, \quad u_\tau\big|_{\partial\Pi} = 0, \tag{4.1}$$

$$i\varepsilon^{-1}\operatorname{curl} v = \lambda u, \quad \operatorname{div}(\mu v) = 0, \quad v_\nu\big|_{\partial\Pi} = 0. \tag{4.2}$$

Suppose $w \in H^1(\Pi, \mathbb{C}^3)$, $w_\tau|_{\partial\Pi} = 0$. Using (4.1), (4.2), and Lemma 2.2, we obtain

$$\int_\Pi \mu^{-1}\langle \operatorname{curl} u, \operatorname{curl} w\rangle\, dx = i\lambda \int_\Pi \langle v, \operatorname{curl} w\rangle\, dx = i\lambda \int_\Pi \langle \operatorname{curl} v, w\rangle\, dx = \lambda^2 \int_\Pi \varepsilon\langle u, w\rangle\, dx. \tag{4.3}$$

Similarly, if $f \in H^1(\Pi, \mathbb{C}^3)$, $f_\nu|_{\partial\Pi} = 0$, then

$$\int_\Pi \varepsilon^{-1}\langle \operatorname{curl} v, \operatorname{curl} f\rangle\, dx = -i\lambda \int_\Pi \langle u, \operatorname{curl} f\rangle\, dx = -i\lambda \int_\Pi \langle \operatorname{curl} u, f\rangle\, dx = \lambda^2 \int_\Pi \mu\langle v, f\rangle\, dx. \tag{4.4}$$

In the middle identities in (4.3) and (4.4) we have used Lemma 2.2 and the conditions $w_\tau|_{\partial\Pi} = 0$ and $u_\tau|_{\partial\Pi} = 0$. We add identities (4.3) and (4.4), substituting the expressions from Lemmas 3.2 and 3.3 for the left-hand sides:

$$\begin{aligned}
&\int_\Pi \big(\langle \partial_j(\varepsilon^{1/2}u), \partial_j(\varepsilon^{-1/2}\mu^{-1}w)\rangle + \langle \partial_j(\mu^{1/2}v), \partial_j(\varepsilon^{-1}\mu^{-1/2}f)\rangle - \langle \operatorname{curl} u, \varepsilon[\nabla(\varepsilon\mu)^{-1}, w]\rangle \\
&\qquad - \langle \operatorname{curl} v, \mu[\nabla(\varepsilon\mu)^{-1}, f]\rangle + \mu^{-1}\langle W(\varepsilon)u, w\rangle + \varepsilon^{-1}\langle W(\mu)v, f\rangle\big)\, dx \\
&\qquad + \int_{\partial\Pi} \Big(\big(\mu^{-1}\kappa(x) + (2\varepsilon\mu)^{-1}\partial_\nu\varepsilon\big)\langle u, w\rangle + \big\langle \big(\varepsilon^{-1}\kappa(x)P_{e_3^\perp} - (2\varepsilon\mu)^{-1}\partial_\nu\mu\big)v, f\big\rangle\Big)\, dS \\
&= \lambda^2 \int_\Pi \big(\varepsilon\langle u, w\rangle + \mu\langle v, f\rangle\big)\, dx.
\end{aligned}$$

Since $\operatorname{curl} u$ and $\operatorname{curl} v$ in the first integral on the left-hand side can be expressed with the help of (4.1) and (4.2), the last identity can be rewritten as

$$\int_\Pi \big(\langle \partial_j\Phi, \partial_j\Psi\rangle_{\mathbb{C}^6} + \langle V\Phi, \Psi\rangle_{\mathbb{C}^6}\big)\, dx + \int_{\partial\Pi} \langle \Sigma\Phi, \Psi\rangle_{\mathbb{C}^6}\, dS = 0, \tag{4.5}$$

where

$$\Phi = \begin{pmatrix} \varepsilon^{1/2}u \\ \mu^{1/2}v \end{pmatrix}, \quad \Psi = \begin{pmatrix} \varepsilon^{-1/2}\mu^{-1}w \\ \varepsilon^{-1}\mu^{-1/2}f \end{pmatrix}, \quad V = \begin{pmatrix} W(\varepsilon) - \varepsilon\mu\lambda^2 I_3 & -2i\lambda F \\ 2i\lambda F & W(\mu) - \varepsilon\mu\lambda^2 I_3 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & -\partial_3(\varepsilon\mu)^{1/2} & \partial_2(\varepsilon\mu)^{1/2} \\ \partial_3(\varepsilon\mu)^{1/2} & 0 & -\partial_1(\varepsilon\mu)^{1/2} \\ -\partial_2(\varepsilon\mu)^{1/2} & \partial_1(\varepsilon\mu)^{1/2} & 0 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} (\kappa(x) + (2\varepsilon)^{-1}\partial_\nu\varepsilon)I_3 & 0 \\ 0 & \kappa(x)P_{e_3^\perp} - (2\mu)^{-1}\partial_\nu\mu I_3 \end{pmatrix}.$$

The conditions $\varepsilon, \mu \in \widetilde{W}_q^2$ for $q \geq 3/2$ and condition (0.1) yield $W(\varepsilon), W(\mu) \in \widetilde{L}_q$ and $V \in \widetilde{L}_q(\Pi)$.

From $\varepsilon, \mu \in \widetilde{W}^2_{\frac{3r}{2+r}}$ it follows that $\partial_\nu\varepsilon, \partial_\nu\mu, \Sigma \in \widetilde{L}_r(\partial\Pi)$. Since (4.5) is valid for any $\Psi \in \widehat{H}^1(\Pi)$, the condition $A(q, r)$ shows that $\Phi \equiv 0$. This means that $u \equiv v \equiv 0$ and the point spectrum of the Maxwell operator is empty. Theorem 1.4 is proved.

References

[1] M. Sh. Birman and M. Z. Solomyak, *The selfadjoint Maxwell operator in arbitrary domains*, Algebra i Analiz **1** (1989), no. 1, 96–110; English transl., Leningrad Math. J. **1** (1990), no. 1, 99–115. MR1015335

[2] M. N. Demchenko, *On the nonuniqueness of the continuation of the solution of the Maxwell system*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **393** (2011), 80–100; English transl., J. Math. Sci. (N.Y.) **185** (2012), no. 4, 554–566. MR2870205

[3] N. Filonov and F. Klopp, *Absolutely continuous spectrum for the isotropic Maxwell operator with coefficients that are periodic in some directions and decay in others*, Comm. Math. Phys. **258** (2005), 75–85. MR2166840

[4] A. Prohorov and N. Filonov, *Regularity of electromagnetic fields in convex domains*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **425** (2014), 55–85; English transl., J. Math. Sci. (N.Y.) **210** (2015), no. 6, 793–813.

[5] N. Filonov and A. V. Sobolev, *Absence of the singular continuous component in the spectrum of analytic direct integrals*, Zap. Nauch. Sem. S.-Peterburg. Otdel. Math. Inst. Steklov. (POMI) **318** (2004), 298–307; English transl., J. Math. Sci. (N.Y.) **136** (2006), no. 2, 3826–3831. MR2120804

[6] I. Kachkovskiĭ, *Nonexistence of eigenvalues for the periodic Schrödinger operator with singular potential in a rectangular cylinder*, Funktsional. Anal. i Prilozhen. **47** (2013), no. 2, 27–37; English transl., Funct. Anal. Appl. **47** (2013), no. 2, 104–112. MR3113867

[7] ———, *Nonexistence of eigenvalues in the spectrum of some Schrödinger operators with periodic coefficients*, Kand. Diss., POMI RAN, SPb., 2013. (Russian)

[8] P. Kuchment, *Floquet theory for partial differential equations*, Oper. Theory Adv. Appl., vol. 60, Birkhauser Verlag, Basel, 1993. MR1232660

[9] A. Morame, *The absolute continuity of the spectrum of Maxwell operator in a periodic media*, J. Math. Phys. **41** (2000), no. 10, 7099–7108. MR1781426

[10] T. A. Suslina, *Absolute continuity of the spectrum of the periodic Maxwell operator in a layer*, Zap. Nauchn. Sem. S.-Peterburg. Mat. Inst. Steklov. (POMI) **288** (2002), 232–255; English transl., J. Math. Sci. (N.Y.) **123** (2004), no. 6, 4654–4667. MR1923552

St. Petersburg Branch, Steklov Institute of Mathematics, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russia; St. Petersburg State University

Email address: filonov@pdmi.ras.ru

Department of Physics, St. Petersburg State University, Ul′yanovskaya ul. 3, Petrodvorets, 198504, St. Petersburg, Russia; Indiana University-Purdue, University Indianapolis, 402N. Blackford st., LD 270, Indianapolis, IN, 46220, USA

Email address: andruprokhorov@mail.ru

Received 14/MAY/2016

Translated by AUTHORS