Алгебра и анализ
Том 29 (2017), № 6

St. Petersburg Math. J.
Vol. 29 (2018), No. 6, Pages 1031–1043
https://doi.org/10.1090/spmj/1526
Article electronically published on September 4, 2018

FOUR-DIMENSIONAL GRAPH-MANIFOLDS WITH FUNDAMENTAL GROUPS QUASIISOMETRIC TO FUNDAMENTAL GROUPS OF ORTHOGONAL GRAPH-MANIFOLDS

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Abstract. A topological invariant called the *type* of a graph-manifold, which takes natural values, is introduced. For a 4-dimensional graph-manifold whose type does not exceed two it is proved that its universal cover is bi-Lipschitz equivalent to a universal cover of an orthogonal graph-manifold (for arbitrary Riemannian metrics on graph-manifolds).

§1. Introduction

The main result of this paper (see Theorem 1.1) establishes a bi-Lipschitz equivalence of universal covers for some classes of 4-dimensional graph-manifolds. This result is motivated by the problem of finding asymptotic invariants of graph-manifolds, in particular, the asymptotic dimension ($\operatorname{asdim}\pi_1(M)$) and the linearly-controlled asymptotic dimension ($\ell\text{-}\operatorname{asdim}\pi_1(M)$) of their fundamental groups. Theorem 1.1 allows us to reduce finding these dimensions for a wide class of graph-manifolds to the results of [9].

In the 3-dimensional case, $\dim M = 3$, the problem of finding asymptotic dimensions were solved in [6]. For the case where $\dim M \geq 4$, the asymptotic dimensions $\operatorname{asdim}\pi_1(M)$ and $\ell\text{-}\operatorname{asdim}\pi_1(M)$ were found only for graph-manifolds of a special type, called the orthogonal graph-manifolds. Namely, for the orthogonal graph-manifolds in [9] it was proved that

$$\operatorname{asdim}\pi_1(M) = \ell\text{-}\operatorname{asdim}\pi_1(M) = \dim M.$$

The definition of these invariants can be found, e.g., in [4, 5, 9].

In the 3-dimensional case, the orthogonal graph-manifolds are analogs of the so-called flip graph-manifolds, for which gluings between blocks are especially simple. In accordance with [7], the fundamental group of any closed 3-dimensional graph-manifold is quasiisometric to the fundamental group of a flip graph-manifold, therefore, the fundamental groups of any closed 3-dimensional graph-manifolds are pairwise quasiisometric. In higher dimensions this is not true. In this paper we introduce a topological invariant, $\operatorname{type} M$, of the graph-manifold M, which takes natural values. In any dimension greater than 3 it is not difficult to construct a graph-manifold of any type. However, for the 4-dimensional orthogonal graph-manifolds, the type always does not exceed 2. The main result of this paper is as follows.

Theorem 1.1. *If the type of a 4-dimensional graph-manifold M does not exceed two,* $\operatorname{type} M \leq 2$, *then its universal cover is bi-Lipschitz equivalent to the universal cover of some orthogonal graph-manifold* (*for any Riemannian metrics on graph-manifolds*).

2010 *Mathematics Subject Classification.* Primary 51L20.
Key words and phrases. Graf-manifolds, quasiisometries.
Supported by the Program 01 of the Presidium of RAN "Fundamental Mathematics and its Applications" under grant PRAS-18-01, and by RFBR, grant no. 17-01-00128a.

Corollary 1.2. *For the fundamental group of any* 4*-dimensional graph-manifold* M *with* $\operatorname{type} M \le 2$*, there is a quasiisometric embedding into the product of* 4 *metric trees, and, consequently,* $\operatorname{asdim} \pi_1(M) = \ell\text{-asdim}\, \pi_1(M) = 4$*, where* asdim *and* ℓ-asdim *are the asymptotic and linearly-controlled asymptotic dimensions.*

This corollary yields a simple and easily verifiable sufficient condition that allows us to calculate $\operatorname{asdim} \pi_1(M)$ and $\ell\text{-asdim}\, \pi_1(M)$.

Moreover, the class $\mathcal{GM}_2$ of graph-manifolds with $\operatorname{type} M \le 2$ is much wider (see §5) than the class of orthogonal graph-manifolds. Next, it is highly doubtful that any graph-manifold of class $\mathcal{GM}_2$ has a finite cover by an orthogonal graph-manifold.

As an important additional result we give a criterion of orthogonality for the 4-dimensional graph-manifolds whose blocks have type 2, see Theorem 5.2. As a consequence, we obtain a wide class of nonorthogonal 4-dimensional graph-manifolds whose type is equal to 2 (see Corollary 5.3).

The proof of Theorem 1.1 consists of two steps. An important role is played by the intersection number and the secondary intersection number, which in the general case can be arbitrary positive integers (see Subsection 2.5). For orthogonal graph-manifolds, these numbers are equal to 1. At the first step, in §3, we pass to a finite cover of the graph-manifold M to build a 4-dimensional graph-manifold N whose intersection numbers and secondary intersection numbers are equal to 1. Since the universal covers of graph-manifolds M and N coincide, we only need to prove Theorem 1.1 for the graph-manifold N.

At the second step, we "reglue" the graph-manifold N to an orthogonal graph-manifold without changing the bi-Lipschitz type of its universal cover. The procedure of regluing is described in §4. It is a generalization of the procedure used in [7] for 3-dimensional graph-manifolds.

Corollary 1.2 follows from the result of [9] saying that the fundamental group of any n-dimensional orthogonal graph-manifold M can be quasiisometrically embedded into the product of n metric trees, and, consequently,

$$\operatorname{asdim} \pi_1(M) = \ell\text{-asdim}\, \pi_1(M) = n.$$

§2. Preliminaries

2.1. Graph-manifolds.

Definition 1. Let $n \ge 3$. *A higher-dimensional graph-manifold* is a closed, oriented, n-dimensional manifold M that is glued from finitely many blocks M_v, $M = \bigcup_{v \in V} M_v$, such that the following conditions (1)–(3) are satisfied.

(1) Each block M_v is a trivial bundle of $(n-2)$-dimensional tori T^{n-2} over a compact, oriented surface Φ_v with boundary (the surface must be different from the disk and the annulus);
(2) the manifold M is glued from blocks M_v, $v \in V$, by diffeomorphisms between the boundary components (we do not exclude the case of gluing the boundary components of a single block);
(3) the gluing diffeomorphisms do not identify the homotopy classes of fiber tori.

Such graph-manifolds for $n \ge 4$ were introduced in [3].

With each graph-manifold M, the graph G dual to its block decomposition is associated. Thus, the set of blocks of a graph-manifold coincides with the set of vertices V of the graph G, and the set of pairs of glued blocks coincides with the set of edges E of G. The set of all directed edges of G will be denoted by W.

The orthogonal graph-manifolds defined in [9] are only distinguished in the class of graph-manifolds by the condition in item 3 on gluing diffeomorphisms. They are obtained as follows.

For each vertex $v \in \mathrm{V}$, we fix a trivialization of the fibration $M_v \to \Phi_v$, that is, we represent the block $M_v = \Phi_v \times S^1 \times \cdots \times S^1$ as the product where S^1 occurs $n-2$ times. Thus, for each edge $w = \{vv'\}$ adjacent to a vertex v, we have a trivialization of the boundary torus $T_w = S^1 \times S^1 \times \cdots \times S^1$, ($(n-1)$ times) of the block M_v that corresponds to the edge w.

In the same way, for each edge $-w$ going in the opposite direction, we have a trivialization of the boundary torus $T_{-w} = S^1 \times S^1 \times \cdots \times S^1$ of the block $M_{v'}$.

We fix an order on the set of all factors of the trivialization, and define a gluing diffeomorphism of the tori T_w and T_{-w} by some permutation $\mathfrak{s}_w$ of factors of the trivialization that does not identify the boundary components Φ_v and $\Phi_{v'}$.

Note that this map is a well-defined gluing, because the permutations $\mathfrak{s}_w$, and $\mathfrak{s}_{-w}$ are selected to be mutually inverse. Also, the map η_w does not identify the homotopy classes of fiber tori.

In this case, for edges w and $-w$ going in opposite directions, the permutations $\mathfrak{s}_w$ and $\mathfrak{s}_{-w}$ are selected to be mutually inverse, i.e., $\mathfrak{s}_{-w} \circ \mathfrak{s}_w = \mathrm{id}$.

In other words, a graph-manifold is orthogonal if and only if there exists a trivialization of all blocks such that the gluing maps are determined by permutations of the factors as described above. The disadvantage of this definition is that it does not allow one to verify whether a given graph-manifold is orthogonal or not. It depends on the choice of trivializations of the blocks, which is not unique. For another choice of trivializations of gluing blocks the graph-manifold in question may cease to be orthogonal. In §5 we present a criterion of orthogonality for some class of 4-dimensional graph-manifolds. This criterion does not depend on the choice of trivializations.

2.2. W-structure. The main tool for working with graph-manifolds is the so-called W-structure, first described in the 3-dimensional case in the papers [1, 2] by Waldhausen. For the n-dimensional case, the definition of a W-structure was given in [3]. For the reader's convenience, we give these definitions here.

Let G be the graph of a graph-manifold M. For a vertex $v \in \mathrm{V}$, by ∂v we mean the set of all directed edges adjacent to v.

With each directed edge $w \in \mathrm{W}$, we associate the homology group $L_w = H_1(T_w;\mathbb{Z}) \simeq \mathbb{Z}^{n-1}$ of the gluing torus T_w, and with each vertex $v \in V$ we associate the homology group $F_v = H_1(T_v;\mathbb{Z}) \simeq \mathbb{Z}^{n-2}$ of the fiber T_v of the block M_v.

Moreover, if $w \in \partial v$, then F_v embeds in L_w as a maximal subgroup F_w.

We call the group $F_v \simeq F_w$ *a fiber group.* Each orientation of the graph-manifold M fixes the corresponding orientations of each block of M, and, thus, the corresponding orientations of the groups L_w, $w \in \mathrm{W}$. The orientations of the groups L_w and L_{-w} are opposite.

The gluing of blocks is described by an isomorphism $\widehat{g}_w \colon L_{-w} \to L_w$ such that

$$\widehat{g}_{-w} = \widehat{g}_w^{-1}; \tag{1}$$

$$\widehat{g}_w(F_{-w}) \neq F_w. \tag{2}$$

For each edge $w \in \partial v$, the choice of a trivialization of each block M_v, as well as a trivialization of the fiber, fixes a basis of the group L_w (up to the choice of the signs of its elements) so that the corresponding subset of elements forms a basis of the group F_w.

Such bases are said to be *selected.*

We describe the set of selected bases of the groups $L_w, w \in \mathrm{W}$, in terms of their transformation groups. Let $f_v = (f_v^1, \dots, f_v^{n-2})$ be a basis of the group F_v.

We choose a basis (z_w, f_w) of the group L_w so that $f_w = f_v$, and there exists a trivialization $M_v = \Phi_v \times T^{n-2}$ such that the set $\{z_w \mid w \in \partial v\}$ corresponds to the boundary of the surface Φ_v.

In this case, the basis f_v determines an orientation of the fiber F_v, and the basis (z_w, f_w) yields some orientation of the group L_w.

The group of transformations of these bases consists of matrices of the form

$$h_w = \begin{pmatrix} \varepsilon_v & 0 \\ n_w & \sigma_v \end{pmatrix},$$

where $\varepsilon_v = \pm 1$, $n_w \in \mathbb{Z}^{n-2}$, $\sigma_v \in GL(n-2, \mathbb{Z})$, and on bases it acts as follows:

$$(z_w, f_w) \cdot h_w = (z_w \cdot \varepsilon_w + f_w \cdot n_w, f_w \cdot \sigma_v).$$

We require that for each vertex $v \in V$ the following conditions be fulfilled:

$$\varepsilon_v \cdot \det \sigma_v = 1; \tag{3}$$

$$\sum_{w \in \partial v} n_w = 0. \tag{4}$$

It is easily seen that the set $\mathcal{H}$ of matrices of the form

$$h = \bigoplus_{w \in \mathrm{W}} h_w$$

satisfying conditions (3) and (4) is a group. Condition (3) means that each basis (z_w, f_w) agrees with the fixed orientation of the group L_w, and condition (4) means that these bases correspond to some trivialization of the block M_v.

The *W-structure* associated with a graph-manifold M is a collection of groups $\{L_w \mid w \in \mathrm{W}\}$, satisfying conditions (1) and (2) and the set of their bases of the form $\Theta = (z, f) \cdot \mathcal{H}$, where (z, f) is the set of bases mentioned above and $\det g_w^{z,f} = -1$ for each directed edge $w \in \mathrm{W}$.

The last condition means that the isomorphism $\widehat{g}_w : L_{-w} \to L_w$ reverses orientation. The elements $(z, f) \in \Theta$ are called a *Waldhausen basis*.

For a fixed Waldhausen basis (z, f), the gluing isomorphism is described by the matrix

$$g_w = g_w^{z,f} = \begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix},$$

where

$$(z_{-w}, f_{-w}) = (z_w, f_w) \cdot g_w \tag{5}$$

(it is assumed that the groups L_{-w} and L_w are identified by the isomorphism $\widehat{g}_w$). Here $a_w \in \mathbb{Z}$. The row b_w and the column c_w consist of $n-2$ integers. The matrix d_w is an integral matrix of size $(n-2) \times (n-2)$.

Remark 1. A graph-manifold M is orthogonal if and only if on each block of M there is a trivialization such that for each directed edge $w \in \mathrm{W}$ its induced bases (z_w, f_w) and (z_{-w}, f_{-w}) of the groups L_w and L_{-w} differ only by permutation of elements, and, perhaps, by putting signs at vectors.

2.3. Fiber subspaces and intersection of lattices. In what follows, we shall use subgroups of groups isomorphic to $\mathbb{Z}^n$. In this case, the maximal subgroups will play an important role. For brevity, we shall call them *lattices*.

The lattice in a group G isomorphic to $\mathbb{Z}^n$ is a maximal subgroup H that is isomorphic to $\mathbb{Z}^k$ for some $k \le n$. That is, this is a subgroup for which there is no another subgroup $H' < G$ isomorphic to $\mathbb{Z}^k$ and such that $H < H'$. By the *dimension* of the lattice we mean the number k.

Remark 2. The intersection $G_1 \cap G_2$ of two lattices G_1 and G_2 is a lattice, because if $\gamma^m \in G_i$, $m \neq 0$, then $\gamma \in G_i$, $i = 1, 2$.

For each edge $|w| \in \mathrm{E}$ we denote by $P_{|w|}$ the intersection of the lattices F_w and F_{-w}. This lattice in $L_{|w|}$ will be called the *intersection lattice* for the edge w.

Definition 2. For any edges w, $w' \in \partial v$, we say that the lattices $P_{|w|}$ and $P_{|w'|}$ are *parallel* if and only if they are the same in the group F_v.

In this case, for brevity, we also say that the edges w and w' *are parallel*.

Definition 3. The lattices $P_{|w|}$ viewed as subgroups of the group F_v, $w \in \partial v$, are called the *intersection lattices* for the vertex v.

2.4. Type of a block and graph-manifolds. Let M_v be a block of a graph-manifold M that corresponds to a vertex v.

Definition 4. The *type* of the vertex v (or of the block M_v) is the maximal number of pairwise nonparallel edges $w \in \partial v$.

We denote the type of the vertex (of the block) by $\operatorname{type} v$ ($\operatorname{type} M_v$). The *type of a graph-manifold* M is the maximal type of its blocks, $\operatorname{type} M := \max_{v \in \mathrm{V}} \operatorname{type} v$.

Remark 3. The type of a block, and consequently the type of the graph-manifold, do not depend on the choice of a Waldhausen basis. This means that they are topological invariants of the graph-manifold.

In this paper we consider only graph-manifolds of dimension 4, and we are interested in blocks having type 1 or 2. For each block M_v of type 1, we denote a unique intersection lattice of the vertex v by P_v^1. For each block M_v of type 2, we denote the corresponding intersection lattices by P_v^1 and P_v^2.

In what follows, unless otherwise stated, by a graph-manifold we mean a 4-dimensional graph-manifold.

2.5. Intersection number and secondary intersection number. Recall the definitions of some invariants of W-structures, as described in [3].

Using condition (2), we see that any integer string b_w is nonzero. Therefore, the greatest common divisor $i_w \geq 1$ of its elements is well defined.

Definition 5. The number i_w is called the *intersection number* of the W-structure on the edge w.

Geometrically, i_w is the number of intersection components of $T_w \cap T_{-w} \subset T_{|w|}$.

Since $i_w = i_{-w}$, i.e., the intersection number is independent of the edge direction, we can introduce the intersection number of an undirected edge $e = |w|$, as $i_e = i_w = i_{-w}$.

Let F_e be a smallest subgroup of the group L_e that contains F_w and F_{-w}, $F_e = \langle F_w,\ F_{-w} \rangle$.

Lemma 2.1. *The subgroup index* $(L_e : F_e)$ *is equal to the intersection number* i_e *of the edge* e.

Proof. The group F_e is generated by the elements f_w, f_{-w}, $F_e = \langle f_w,\ f_{-w} \rangle$, while $L_e = \langle z_w,\ f_w \rangle$.

Condition (5) shows that the elements $b_w^1 \cdot z_w$, $b_w^2 \cdot z_w$, $\ldots$, $b_w^{n-2} \cdot z_w$ belong to the group F_e, where $b_w = (b_w^1, \ldots, b_w^{n-2})$.

Since the intersection number i_e is equal to the greatest common divisor of the numbers b_w^1, b_w^2, $\ldots$, b_w^{n-2}, we have $\alpha \cdot z_w \in F_e$ if and only if α is divisible by i_e. Therefore, $(L_e : F_e) = i_e$. □

For each block M_v of type 2, the intersection lattices $P_v^1, P_v^2 \subset F_v$ generate a subgroup $P_v \simeq \mathbb{Z}^2$, $P_v = \langle P_v^1, P_v^2 \rangle$ (the smallest subgroup in F_v containing P_v^1 and P_v^2) in the group F_v.

Definition 6. We call the group P_v the *group of fiber intersections* for the block M_v.

The subgroup P_v is not necessarily maximal, so it may fail to be a lattice in $F_v \simeq \mathbb{Z}^2$.

Definition 7. The index j_v of the subgroup P_v in the group F_v is called the *secondary intersection number* at the vertex v.

For each block M_v of type 1, we choose $P_v = F_v$ and $j_v = 1$. (Since $P_v^1 \simeq \mathbb{Z}$ and $P_v \simeq \mathbb{Z}^2$, we have $P_v \neq P_v^1$.)

§3. Unwinding of intersection numbers up to 1.

In this section we prove that for any graph-manifold there is a finite sheet cover by a graph-manifold with all intersection numbers and all secondary intersection numbers equal to 1.

Lemma 3.1. *For any graph-manifold M there is a finite sheet cover by a graph-manifold N for which all intersection numbers are equal to* 1*, and also all secondary intersection numbers are equal to* 1.

Proof. For each vertex $v \in \mathrm{V}$ we consider a cover $r_v \colon T^2 \to T^2$ corresponding to a subgroup $P_v < F_v = \pi_1(T^2)$ of the fundamental group of the fiber torus (for a vertex of type 1 such a cover is trivial). The degree of this cover is equal to the secondary intersection number j_v at v.

Consider the surface Φ_v; we construct an orbifold Φ'_v as follows. For each edge $w \in \partial v$, we glue a disk D_w with a conic point with an angle of $2\pi/j_u$, where u is the other end of the edge w, to the corresponding component of the boundary of Φ_v.

Since the surface with boundary Φ_v is different from the disk and the ring, the orbifold Φ'_v is a good compact 2-dimensional orbifold without boundary, and therefore (see [8, Theorem 2.5]), there is a closed surface Ψ'_v and a finite cover $p'_v \colon \Psi'_v \to \Phi'_v$.

Let n_v be the degree of this cover. We denote the product $\prod_{v\in \mathrm{V}} n_v$ by N, and the product $\prod_{v\in \mathrm{V}} j_v$ by J.

From the surface Ψ'_v we cut out the preimage $(p')_v^{-1}(\bigcup_{w\in\partial v} D_w)$ of the glued disks, obtaining a surface Ψ_v with boundary that covers the surface Φ_v with boundary with finite degree. We denote the corresponding cover by $p_v \colon \Psi_v \to \Phi_v$.

Let $N_v = \Psi_v \times T^2$. These manifolds will also be called blocks. We define the cover

$$q_v \colon N_v \to M_v = \Phi_v \times T^2,$$

as the product of the covers $p_v \colon \Psi_v \to \Phi_v$ and $r_v \colon T^2 \to T^2$.

Note that in the block N_v the group P_v plays the role of a fiber group.

Let w be an edge from a vertex v to a vertex u, and let γ_w be a boundary component of the surface Φ_v corresponding to w. On each component of the preimage of the torus $T_w = \gamma_w \times T^2$ the cover q_v is a product of covers and is determined by the subgroup $A_w = \langle P_v,\ P_u \rangle = B_w \times P_v$ of the group L_w, where B_w is a subgroup of the group $\pi_1(\gamma_w) \simeq \mathbb{Z}$ with subgroup index j_u.

Thus, the group A_w has the subgroup index $(L_w : A_w) = j_u \cdot j_v$ in the group L_w.

Now we describe the gluings of blocks. Let w be an edge from a vertex v to a vertex u. Let T_v be a boundary component of the block N_v, and let T_u be a boundary component of the block N_u. Let T_u and T_v cover the torus $T_{|w|}$. A gluing $g'_w \colon T_v \to T_u$ is determined by an isomorphism of the groups $H_1(T_v;\mathbb{Z})$ and $H_1(T_u;\mathbb{Z})$.

For each of these groups we have an isomorphism with the group A_w induced by the covers q_v and q_u, respectively. This determines the required gluing. Since the lattices $P_u, P_v < A_w$ are different, such a gluing satisfies condition (3) of Definition 1.

Note that for the edge w from a vertex u to a vertex v of the graph G, we have $P_{|w|} = F_w \cap F_{-w} = P_u \cap P_v < L_w$.

Therefore, for the edge w' that corresponds to the gluing of the blocks N_u and N_v, we have $P_{|w'|} = P_{|w|}$. Consequently, the group P_v plays the role of the fiber intersection group for the block N_v. This means that the secondary intersection number of the block N_v is equal to $(P_v : P_v) = 1$.

By Lemma 2.1, the intersection number on the edge w' is equal to the subgroup index $(A_w : \langle P_u, P_v\rangle)$ of the subgroup $\langle P_u, P_v\rangle$ in the group A_w, i.e., it is equal to 1.

For each vertex $v \in \mathrm{V}$ we consider $N/n_v \cdot J/j_v$ copies of the block N_v.

For each edge $w \in \mathrm{W}$ ($e = |w|$) between $u, v \in \mathrm{V}$, we have $N/n_u \cdot J/j_u$ copies of the block N_u and $N/n_v \cdot J/j_v$ copies of the block N_v.

The block N_u has $(n_u \cdot j_u)/(j_u \cdot j_v) = n_u/j_v$ boundary components covering the torus T_e, and the block N_v has n_v/j_u boundary components covering the torus T_e.

Then all copies of the block N_u have $N/n_u \cdot J/j_u \cdot n_u/j_v = (N \cdot J)/(j_u \cdot j_v)$ boundary components covering the torus T_e.

All copies of the block N_v have one and the same number of boundary components covering the torus T_e.

Having some correspondence between these copies, we glue each boundary component of a copy of the block M_u with the corresponding boundary component of a copy of the block M_v by some gluing homeomorphism g'_w.

We obtain a graph-manifold M' that is an $(N \cdot J)$-sheeted cover of M; all secondary intersection numbers of M' are equal to 1. □

Applying Lemma 3.1 to the graph-manifold M, we arrive at a graph-manifold N for which all intersection numbers are equal to 1, and also all secondary intersection numbers are equal to 1. Moreover, the fundamental groups $\pi_1(N)$ and $\pi_1(M)$ are quasiisometric.

§4. Proof of Theorem 1.1

For the reader's convenience, here we present Lemma 2.4 from [7]. This lemma plays an important role in the proof of Theorem 1.1.

Lemma 4.1. *Let S be a smooth compact manifold with strictly negative curvature and totally-geodesic boundary. Denote by $\widetilde{S}$ the universal cover of S. Let α be a closed smooth 1-form on ∂S. Denote by α' the pull-back of α to $\partial\widetilde{S}$.*

Then there exists a smooth Lipschitz function $h\colon \widetilde{S} \to \mathbb{R}$ satisfying $dh\big|_{\partial\widetilde{S}} = \alpha'$.

Let M be a 4-dimensional graph-manifold with type at most 2.

Passing to a finite cover, we may assume that all intersection numbers of M are equal to 1 and also all secondary intersection numbers are equal to 1.

Since all secondary intersection numbers of M are equal to 1, we can choose a Waldhausen basis $\{(z_w, f_w) \mid w \in \partial v, v \in \mathrm{V}\}$ such that for each block M_v of type 2 we have $f_v^1 \in P_v^1$ and $f_v^2 \in P_v^2$.

Moreover for each block of type 1 we can choose $f_v^1 \in P_v^1$.

For each edge $w \in \mathrm{W}$ from v to u, the gluing $\widehat{g}_{-w}$ of blocks M_v and M_u is given by bases (z_w, f_w) and (z_{-w}, f_{-w}) of the lattice $L_{|w|}$.

In other words, the matrix g_{-w} is obtained by decomposing the basis (z_w, f_w) over the basis (z_{-w}, f_{-w}).

We may assume that $P_u^1 = P_w = P_v^1$, where $P_w = F_w \cap F_{-w}$. Then, in this notation, $f_w^1 = \pm f_{-w}^1$. Moreover, since the intersection numbers are equal to 1 and $f_w^1 = \pm f_{-w}^1$, formula (5) shows that $f_{-w}^2 - z_w \in F_w$.

We denote the vector $f_{-w}^2 - z_w$ by δ_w.

Stepwise, changing gluings on the edges, we construct an orthogonal graph-manifold N whose universal cover $\widetilde{N}$ is bi-Lipschitz equivalent to the universal cover $\widetilde{M}$ of the graph-manifold M.

We fix an edge $w \in \mathrm{W}$. Let it connect vertices v and u. The new gluing $\widehat{g}'_{-w}$ of the blocks M_v and M_u will be defined via the basis $z'_w = z_w + \delta_w$, $f'_w = f_w$.

Thus, the isomorphism $\widehat{g}'_{-w}$ is obtained from the isomorphism $\widehat{g}_{-w}$ by translation by the vector δ_w along the first coordinate. That is, such a gluing identifies the vectors f_w^1 and f_{-w}^1, as well as the vectors z_w and f_{-w}^2.

Since the lattices P_v, $v \in \mathrm{V}$, and F_e, $e \in \mathrm{E}$ (see Definitions 5, and 7) do not change under such a modification of gluings, the intersection number and the secondary intersection number will remain unchanged.

Cutting the graph-manifold M along the torus $T_{|w|}$, and then gluing along it with the gluing $\widehat{g}'_{-w}$, we obtain the graph-manifold N.

Lemma 4.2. *The universal covers of the graph-manifold M and N are bi-Lipschitz homeomorphic.*

Proof. The graph-manifolds M and N have a common graph G, and hence, the Bass–Serre tree of M coincides with that of N.

Moreover, for each vertex $v' \in \mathrm{V}$ the blocks $M_{v'}$ and $N_{v'}$ are isomorphic.

The universal cover $\widetilde{M}$ of the graph-manifold M is divided into blocks dual to the Bass-Serre tree T_M, each of which is the universal cover of some block of the graph-manifold M.

We call the blocks that cover the block M_v the *distinguished* blocks.

The universal cover $\widetilde{N}$ of the graph-manifold N is also divided into blocks. Since the Bass–Serre trees of these graph-manifolds coincide, the blocks of the manifold $\widetilde{N}$ are copies of the blocks of the manifold $\widetilde{M}$.

The manifold $\widetilde{M}$ differs from the manifold $\widetilde{N}$ only by gluings along boundary components of distinguished blocks. The blocks that correspond to the distinguished blocks will also be called distinguished blocks.

Now we prove that the universal covers $\widetilde{M}$ and $\widetilde{N}$ are bi-Lipschitz homeomorphic. We construct a map $\widetilde{M} \to \widetilde{N}$ in the following way: each nondistinguished block of $\widetilde{M}$ is mapped identically to the corresponding nondistinguished block of $\widetilde{N}$. For each distinguished block $\widetilde{M}_v = \widetilde{\Phi}_v \times \mathbb{R}^2$, our map induces a map of the boundary of this block to the boundary of the corresponding distinguished block $\widetilde{N}_v = \widetilde{\Phi}_v \times \mathbb{R}^2$. This map is identical on each boundary component that does not correspond to the edge w. On the boundary component $\ell_w \times \mathbb{R}^2$ corresponding to the edge w, this map is an affine map $A_w \colon \ell_w \times \mathbb{R}^2 \to \ell_w \times \mathbb{R}^2$ that corresponds to the map

$$h_w = (g'_{-w})^{-1} \circ g_{-w} \colon H_1(T_{|w|}; \mathbb{Z}) \to H_1(T_{|w|}; \mathbb{Z}).$$

The map A_w is determined up to an integral shift in the second factor.

We expand the vector δ_w in the basis (f_w^1, f_w^2) of the space F_w, $\delta_w = \gamma_1 f_w^1 + \gamma_2 f_w^2$.

In the basis (z_w, f_w^1, f_w^2) the map h_w is given by the formulas

$$h_w(z_w) = z_w - \delta_w = z_w - \gamma_1 f_w^1 - \gamma_2 f_w^2, \quad h_w(f_w^1) = f_w^1, \quad h_w(f_w^2) = f_w^2.$$

Consider a coordinate system (x, y, z) on the boundary component $\ell_w \times \mathbb{R}^2$; we assume that the line $y = z = 0$ corresponds to the direction z_w, the line $x = z = 0$ corresponds to the direction f_w^1, and the line $x = y = 0$ corresponds to the direction f_w^2.

In this coordinate system, the map h_w corresponds to the class $\mathcal{A}_w$ of maps

$$A\colon \ell_w \times \mathbb{R}^2 \to \ell_w \times \mathbb{R}^2$$

of the form

$$A(x, y, z) = (x, y - \gamma_1 \cdot x + c_1, z - \gamma_2 \cdot x + c_2), \quad (c_1, c_2) \in \mathbb{Z}^2.$$

Consider the function $\varphi_1\colon \partial\widetilde{\Phi}_v \to \mathbb{R}$ equal to $-\gamma_1$ on the components that correspond to the edge w and equal to 0 on the other components. This function determines a closed 1-form on the boundary of the compact surface Φ_v with boundary.

Similarly, the function $\varphi_2\colon \partial\widetilde{\Phi}_v \to \mathbb{R}$ equal to $-\gamma_2$ on the components that correspond to the edge w and to 0 on the other components determines a closed 1-form on the boundary of the compact surface Φ_v with boundary.

From Lemma 4.1 we know that there exists a smooth Lipschitz function $h_1\colon \widetilde{\Phi}_v \to \mathbb{R}$ satisfying $dh_1\big|_{\partial\widetilde{\Phi}_v} = \varphi_1$.

Similarly, there exists a smooth Lipschitz function $h_2\colon \widetilde{\Phi}_v \to \mathbb{R}$ with $dh_2\big|_{\partial\widetilde{\Phi}_v} = \varphi_2$.

In other words, the restrictions of the functions h_1 and h_2 to the boundary components of the surface $\widetilde{\Phi}_v$ are affine functions.

By construction, on each boundary component σ of the block M_v, the homeomorphism $\widehat{h}\colon \widetilde{M}_v \to \widetilde{N}_v$ given by the formula $\widehat{h}(x, y, z) = (x, y + h_1(x), z + h_2(x))$ differs from some map of class $\mathcal{A}_w$ by a bounded vector $(c_1^\sigma, c_2^\sigma) \in \mathbb{R}^2$. We consider Lipschitz functions $h_1', h_2'\colon \widetilde{\Phi}_v \to \mathbb{R}$ with support in a sufficiently small neighborhood of the boundary $\partial\widetilde{\Phi}_v$ and such that on each boundary component σ of the block M_v we have $h_1 = c_1^\sigma$ and $h_2 = c_2^\sigma$.

Then the difference $h(x, y, z) = \widehat{h}(x, y, z) - (0, h_1'(x), h_2'(x))$ is a bi-Lipschitz homeomorphism, as required. □

For the graph-manifold N, for the edge $-w$ opposite to the edge w of the graph G, we act as above to construct a gluing $\widehat{g}_w'\colon L_{-w} \to L_w$ that identifies the vectors f_w^1 and f_{-w}^1 and the vectors z_{-w} and f_w^2.

This gluing does not change the vectors z_w and f_{-w}^2.

Cutting the graph-manifold N along the torus $T_{|w|}$, and then gluing along it with the gluing $\widehat{g}_w'$, we obtain a graph-manifold N' whose gluing along the edge $|w|$ is orthogonal.

From Lemma 4.2 it follows that the universal covers of the graph-manifolds M and N' are bi-Lipschitz homeomorphic.

Applying the above operation step-by-step to all opposing pairs of edges $(w, -w)$ of the graph G, we pass from the graph-manifold M to a graph-manifold N. The universal cover of N is bi-Lipschitz homeomorphic to the universal cover of M.

This proves Theorem 1.1. □

§5. A criterion of orthogonality

In this section, we present a criterion of orthogonality for the 4-dimensional graph-manifolds such that the type of each vertex is equal to 2. As a consequence, we construct an example of a 4-dimensional graph-manifold that is not orthogonal, all blocks of which have type 2, and the intersection numbers and secondary intersection numbers are equal to 1.

We recall the definition (see [3]) of the charge map of a graph-manifold (for the case of graph-manifolds M of arbitrary dimension, $\dim M = n$).

Below, we pass to homology groups with real coefficients, keeping the same notation. In particular, we denote $F_w \otimes_{\mathbb{Z}} \mathbb{R}$ by F_w and $L_{|w|} \otimes_{\mathbb{Z}} \mathbb{R}$ by $L_{|w|}$.

For each directed edge w of the graph G of the graph-manifold M, the gluing matrix g_w gives rise to a map $D_w : F_{-w} \to F_w$ such that $D_w(f_{-w}p_w) = f_w d_w p_w$, where $p_w \in \mathbb{R}^{n-2}$ is a column of reals. In other words, the map D_w is defined in the bases f_{-w}, and f_w by the submatrix d_w of the matrix g_w. This map can be interpreted as a projection of the space F_{-w} onto the space F_w along the vector z_w.

In particular, the map D_w is the identity map at the intersection $F_{-w} \cap F_w$.

For each directed edge w of the graph G of the graph-manifold M, we fix an orientation of the space F_w.

Let $u_w = f_w^1 \wedge f_w^2 \wedge \cdots \wedge f_w^{n-2}$. Identifying the spaces L_{-w} and L_w via the map g_w, we obtain a space $L_{|w|}$ with the couple of oriented subspaces F_w and F_{-w}.

Under these conditions, we have the canonical intersection orientation $u_{w\cap -w}$ on the subspace $F_w \cap F_{-w}$ (see [3]).

Definition 8. The *charge map* of the vertex $v \in V$ is the restriction

$$K_v \colon Q_v \to F_v$$

of the map

$$\bigoplus_{w\in\partial v} \frac{1}{i_w} D_w \colon \bigoplus_{w\in\partial v} F_{-w} \to F_v$$

to the subspace Q_v, where $Q_v \subset \bigoplus_{w\in\partial v} F_{-w}$ consists of all vectors $q_v = \bigoplus_{w\in\partial v} q_{-w}$, such that there exists a number $\alpha \in \mathbb{R}$ with $q_{-w} \wedge u_{w\cap -w} = \alpha \cdot u_{-w}$ for each $w \in \partial v$.

This subspace does not depend on the choice of the Waldhausen basis (z, f), and for its dimension we have $\dim Q_v = (n-3)|\partial v| + 1$ (for the details see [3]). Note that the subspace

$$A_v = \{q_v = \bigoplus_{w\in\partial v} q_{-w} \mid q_{-w} \wedge u_{w\cap -w} = 0\} \subset Q_v$$

is a hyperplane in Q_v, $\dim A_v = (n-3)|\partial v|$.

In the 3-dimensional case, $n = 3$, the map $K_v \colon Q_v \to F_v$ is a linear map of 1-dimensional spaces, and therefore it is uniquely determined by a rational number k_v, the charge of the vertex v.

Although the charge map in higher dimensions is not a number, nevertheless, we can speak about the vanishing of the vertex charge.

Definition 9. *We say that the charge of a vertex $v \in V$ vanishes* if and only if the kernel K_v of the charge map is not included in the subspace $A_v \subset Q_v$, $\ker K_v \not\subset A_v$.

In this case we write $k_v = 0$.

Remark 4. In the 3-dimensional case we have $\dim F_v = \dim Q_v = 1$ and $A_v = \{0\}$, so that the condition $k_v = 0$ is equivalent to $\ker K_v = Q_v$, i.e., $\ker K_v \not\subset A_v$; then our definition coincides with the regular definition of $k_v = 0$.

Let M be a 4-dimensional graph-manifold all blocks of which have type 2. For each vertex v of the graph G of M, and for each edge w from v to u, there are two intersection lattices in the fiber lattice F_u of the block M_u. We denote by $\bar{P}_u$ one of them, namely, the one that is not an intersection lattice for the edge w.

Also, we denote $\bar{P}_u \otimes_{\mathbb{Z}} \mathbb{R}$ by J_{-w}.

Definition 10. We define *the subspace of intersection vectors* in the space Q_v by the formula

$$B_v := Q_v \cap \bigoplus_{w\in\partial v} J_{-w}.$$

Remark 5. From the definition it follows that the subspace of intersection vectors for the vertex v does not depend on the Waldhausen basis and, consequently, is a topological invariant of the graph-manifold M.

Lemma 5.1. *For any orthogonal graph-manifold M and any vertex v of its graph G, we have $B_v \subset \ker K_v$.*

Moreover, the charge of any vertex of M vanishes.

Proof. By Remark 1, on each block of the graph-manifolds M we can choose a Waldhausen basis so that, for each edge $w \in \mathrm{W}$, the bases (z_w, f_w) and (z_{-w}, f_{-w}) of the groups L_w and L_{-w} differ only by a permutation of elements, and, perhaps, the signs before vectors. Fix a vertex v.

By orthogonality, for each edge $w \in \partial v$ the subspace J_{-w} is generated by the vector z_w. Therefore, $B_v \subset \ker K_v$. Moreover, we can choose a sign $\varepsilon_w = \pm 1$ before the vector z_w so that $\varepsilon_w \cdot z_w \wedge u_{w \cap -w} = 1 \cdot u_{-w}$.

Thus, $q_v = \bigoplus_{w \in \partial v} \varepsilon_w \cdot z_w \in Q_v$, and, at the same time, $q \notin A_v$.

We see that $K_v(q_v) = 0$, and, consequently $k_v = 0$. $\square$

Remark 6. Lemma 2.1 and Definition 7 show that the intersection numbers of any edge and the secondary intersection numbers of any vertex of an orthogonal graph-manifold are equal to 1.

This means that the fact that an intersection number or a secondary intersection number is not equal to 1 is an obstruction to the orthogonality of graph-manifolds. However, even in the class of graph-manifolds all blocks of which have type 2 and whose intersection numbers and the secondary intersection numbers are equal to 1, there exist nonorthogonal graph-manifolds. Below (see Corollary 5.3) we give an example of such a graph-manifold.

Theorem 5.2. *Let M be a graph-manifold all blocks of which have type* 2*. The graph-manifold M is orthogonal if and only if the following three conditions are satisfied*:

1) *the intersection number of each edge is equal to* 1;
2) *the secondary intersection number of each vertex is equal to* 1;
3) *the subspace of intersection vectors of each vertex is contained in the kernel of the charge map $B_v \subset \ker K_v$.*

Proof. If M is orthogonal, then Lemma 2.1, Definition 7, and Lemma 5.1 show that conditions 1)–3) are satisfied.

Conversely, since the secondary intersection numbers are equal to 1, it follows that we can choose a Waldhausen basis $\{(z_w, f_w) \mid w \in \partial v, v \in \mathrm{V}\}$ such that for every block M_v we have $f_v^1 \in P_v^1$ and $f_v^2 \in P_v^2$.

For each edge $w \in \mathrm{W}$ from the vertex v to the vertex u, the gluing $\widehat{g}_{-w}$ of the blocks M_v and M_u is given by the bases (z_w, f_w) and (z_{-w}, f_{-w}) of the space $L_{|w|}$.

In other words, the matrix g_{-w} of this gluing is obtained by expanding the elements of the basis (z_w, f_w) in the basis (z_{-w}, f_{-w}). We may assume that $P_u^1 = P_w = P_v^1$, where $P_w = F_w \cap F_{-w}$. Then, in this notation, $f_w^1 = \pm f_{-w}^1$. Moreover, since the intersection numbers are equal to 1 and $f_w^1 = \pm f_{-w}^1$, from formula (5) we have $f_{-w}^2 - z_w \in F_w$.

Condition (3) implies the relation

$$q = \bigoplus_{w \in \partial v} f_{-w}^2 \in \ker K_v.$$

This means that $\sum_{w \in \partial v} D_w(f_{-w}^2) = 0$. On the other hand, since $f_{-w}^2 - z_w \in F_w$, we have $D_w(f_{-w}^2) = f_{-w}^2 - z_w$.

Denote $f^2_{-w} - z_w$ by n_w. Properties (3) and (4) and the identity $\sum_{w\in\partial v} n_w = 0$ show that there exists a Waldhausen basis $\{(\bar{z}_w, \bar{f}_w) \mid w \in \partial v, v \in \mathrm{V}\}$ such that $\bar{z}_w = z_q + n_w$ and $\bar{f}_w = f_w$ for any directed edge of the graph G.

For such a basis the following conditions are satisfied: $\bar{f}^1_w = \bar{f}^1_{-w}$, $\bar{f}^2_{-w} = \bar{z}_w$ and $\bar{f}^2_w = \bar{z}_{-w}$.

This means that the manifold M is orthogonal. □

Corollary 5.3. *There exists a* 4*-dimensional nonorthogonal graph-manifold all blocks of which have type* 2*, all intersection numbers are equal to* 1*, and all secondary intersection numbers are equal to* 1*.*

Proof. As a graph G of a graph-manifold M we take a cycle of length $k \geq 3$.

We enumerate its vertices: $v_1, \dots, v_k$.

For each vertex v_i, $i = 1, \dots, k$, we consider a block $M_i = \Phi \times T^2$, where Φ is a torus with 2 boundary components. We glue the graph-manifold from blocks so that for each edge $w \in \mathrm{W}$ the corresponding gluing matrix is equal to

$$g_w = g^{z,f}_w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

From the definition (see Subsection 2.1), it follows that the resulting graph-manifold is orthogonal. Consequently, by Theorem 5.2, for each vertex $v \in \mathrm{V}$ the set of intersection vectors is included in the kernel of the charge map, $B_v \subset \ker K_v$.

Consider the edge w from the vertex v_2 to the vertex v_3.

Replacing the gluing on this edge by the gluing

$$\bar{g}_w = \bar{g}^{z,f}_w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

we obtain a graph-manifold M' all blocks of which have type 2. By Lemma 2.1 and Definition 7, all intersection numbers of M' are equal to 1, and all secondary intersection numbers of M' are equal to 1. On the other hand, the charge map of the vertex v_1 does not change. At the same time, the spaces of intersection vectors are different. Indeed, consider the edges w_k and w_2 from v_1 to v_k and v_2, respectively. Then the new space of intersection vectors B'_{v_1} is obtained from the previous one by translation by the vector $0 + f^1_{-w_2} \in F_{-w_k} \oplus F_{-w_2}$. This means that $K_v(q) = f^1_{v_1}$, where $q = f^2_{-w_k} \oplus f^2_{-w_2} \neq 0$.

Therefore, B'_{v_1} does not lie in the kernel $\ker K_{v_1}$. By Theorem 5.2, the graph-manifold M' is not orthogonal. □

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Received 7/SEP/2017

Translated by THE AUTHOR