ON A CLASS OF FUNCTIONAL–DIFFERENTIAL OPERATORS SATISFYING THE KATO CONJECTURE

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Dedicated to the 130th anniversary of Vladimir Ivanovich Smirnov’s birth

Abstract. Second order elliptic differential-difference operators with degeneration in a cylinder are considered. It is proved that these operators satisfy the Kato conjecture about the square root of an operator.

Introduction

The theory of fractional powers of closed maximal accretive operators was considered by many authors, see [5,8,11,13,15,29,32]. This theory is closely related to the theory of semigroups of linear operators and the theory of operator-differential equations. In the paper [11], it was proved that if \( B \) is a closed maximal accretive operator in a Hilbert space \( H \), then \( \mathcal{D}(B^{\alpha}) = \mathcal{D}(B^{*\alpha}) \) for \( 0 \leq \alpha < 1/2 \). In the same paper, examples of maximal accretive operators satisfying \( \mathcal{D}(B^{\alpha}) \neq \mathcal{D}(B^{*\alpha}) \) for \( 1/2 < \alpha \leq 1 \) were constructed. The question about the identity

\[ \mathcal{D}(B^{1/2}) = \mathcal{D}(B^{*1/2}) \]

was formulated in the form of an open problem, which was named in the literature the Kato problem on the square root of an operator or Kato’s conjecture. A positive solution of this problem, even for a specific class of closed maximal accretive operators, allows one to give an explicit characterization of the space of initial conditions of the Cauchy problem for the corresponding operator-differential equation for which a strong solution exists, see, e.g., [28]. Sufficient conditions for identity (1) were obtained in [7,12,17,31] and other papers. It was proved by Lions [17] that the Kato conjecture is true for strongly elliptic differential operators with smooth coefficients considered in a bounded domain with a smooth boundary. In the same paper, an example of a maximal accretive operator that does not satisfy Kato’s conjecture was constructed. An example of a regularly accretive operator \( B \) for which (1) fails was constructed in [19]. Later, mathematicians tried to find new classes of operators satisfying Kato’s conjecture. The proof of the Kato conjecture for strongly elliptic differential operators with bounded measurable coefficients (see [2,14]) was a great progress in this direction. In this case, the principal difficulty is the following. In contrast to strongly elliptic differential operators with infinitely differentiable coefficients, generalized solutions of the corresponding boundary value problems may fail to be smooth. Consequently, the domain of an operator with measurable coefficients cannot be described explicitly.

A new class of operators satisfying the Kato conjecture consists of elliptic functional-differential operators in a bounded domain \( Q \subset \mathbb{R}^n \), arising in problems of the plasma
physics, nonlinear optics, elasticity theory, and the theory of multidimensional diffusion processes, see [6, 25, 27]. The interpolation formula

\[ [D(B), H]_{1/2} = [D(B^*), H]_{1/2}, \]

equivalent to Kato’s conjecture, was obtained in [28] for a certain class of strongly elliptic differential-difference operators with the Dirichlet condition on the boundary. Application of the results of [17] to abstract regularly accretive operators made it possible to prove (see [27]) formula (1) for strongly elliptic functional-differential operators with arbitrary smooth nondegenerate transformations of variables and Dirichlet conditions. Besides the differential-difference operators [9, 10, 25, 27], such operators include, for example, functional-differential operators with dilation and compression of arguments, whose theory was constructed in [20, 22]. It is interesting to note that the development of the theory of functional-differential equations with dilation and compression of an argument is also associated with the name of Kato, see [14].

1.1. We consider some properties of sectorial operators and sectorial forms (see [13, Chapter V, §3 and Chapter VI, §§1,2]) that will be needed in what follows.

Let \( H \) be a Hilbert space with inner product \((u, v)\) \((u, v \in H)\).

A closed linear operator \( B : H \supset D(B) \rightarrow H \) is called accretive if \( \text{Re}(Bu, u) \geq 0 \) for all \( u \in D(B) \). A linear operator \( B : H \supset D(B) \rightarrow H \) is called \( m \)-accretive if for every \( \lambda \) with \( \text{Re} \lambda > 0 \), there exists a bounded inverse operator \((B + \lambda I)^{-1} : H \rightarrow H \) such that \( \|B + \lambda I\|^{-1} \leq (\text{Re} \lambda)^{-1} \). Here \( I \) is the identity operator on \( H \). We introduce the set \( \Theta(B) = \{(Bu, u) : u \in D(B), \|u\| = 1\} \) called the numerical range of the operator \( B \).

Remark 1.1. If the operator \( B : H \supset D(B) \rightarrow H \) is \( m \)-accretive, then \( B \) is a closed maximal accretive operator, \( D(B) \) is dense in \( H \), and \( \Theta(B) \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq 0\} \).

We say that a linear operator \( B : H \supset D(B) \rightarrow H \) is quasi-\( m \)-accretive if \( B + \alpha I \) is \( m \)-accretive for some \( \alpha \in \mathbb{R} \). An operator \( B \) is said to be sectorial if there exists

\[ [D(B), H]_{1/2} = [D(B^*), H]_{1/2}, \]

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0 \leq \theta < \pi/2 \text{ and } \gamma \in \mathbb{R} \text{ such that } \Theta(B) \subset \{ \lambda \in \mathbb{C} : |\arg(\lambda - \gamma)| \leq \theta \}. \text{ The numbers } \theta \text{ and } \gamma \text{ are called the half-angle and the vertex of the sectorial operator } B. \text{ The numbers } \theta \text{ and } \gamma \text{ are not uniquely determined by } B. \text{ The operator } B: H \supset \mathcal{D}(B) \rightarrow H \text{ is called } m\text{-sectorial if it is sectorial and quasi-}m\text{-accretive.}

We consider a sesquilinear form \( b[u,v] \) with domain \( \mathcal{D}(b) \subset H \). Denote by \( \Theta(b) = \{ b[u] : u \in \mathcal{D}(b), \|u\| = 1 \} \) the numerical range of the form \( b[u] = b[u,u] \). A form \( b[\cdot,\cdot] \) is said to be sectorial if there exists \( \theta, 0 \leq \theta < \pi/2 \), and \( \gamma \in \mathbb{R} \) such that \( \Theta(b) \subset \{ \lambda \in \mathbb{C} : |\arg(\lambda - \gamma)| \leq \theta \} \). The number \( \gamma \) is called the vertex of the sectorial form, while the number \( \theta \) is called the half-angle of the sectorial form. The numbers \( \theta \) and \( \gamma \) are not uniquely determined by \( b \). A sectorial form \( b \) is said to be closed if the conditions \( u_k \in \mathcal{D}(b), \|u_k - u\| \rightarrow 0 \), and \( b[u_k - u_m] \rightarrow 0 \) as \( k,m \rightarrow \infty \) imply \( u \in \mathcal{D}(b) \) and \( b[u_k - u] \rightarrow 0 \) as \( k \rightarrow \infty \).

Let \( b \) be a sectorial form. We introduce the inner product in \( H_p = \mathcal{D}(b) \) defined by the formula

\[
(u,v)_p = p(u,v) + (1-\gamma)(u,v), \quad (u,v) \in \mathcal{D}(b),
\]

where \( \gamma \) is the vertex of the form \( b \), \( p[u,v] = (b[u,v] + \overline{b[v,u]})/2 \). A sectorial form \( b \) is closed in \( H \) if and only if the pre-Hilbert space \( H_p \) is complete (see [13, Chapter VI, §1], Theorem 1.11).

1.2. We formulate the Kato problem for accretive and regularly accretive operators.

Let \( B: H \supset \mathcal{D}(B) \rightarrow H \) be an \( m \)-accretive operator. By Theorem 3.35 in [13, Chapter V, §3], there exists a unique \( m \)-accretive square root \( B^{1/2} \) of \( B \) such that \( (B^{1/2})^2 = B \). In addition, \( B^{1/2} \) is an \( m \)-sectorial operator whose numerical range is contained in the sector \( |\arg \zeta| \leq \pi/4 \).

In the paper [11] Kato formulated the following problem: “Is formula (1) true for an \( m \)-accretive operators \( B \) acting in a Hilbert space \( H \)?”

Let \( b[u,v] \) be a densely defined closed sectorial sesquilinear form in a Hilbert space \( H \) with vertex \( \gamma \geq 0 \). By the first representation theorem (see [13, Chapter VI, §2], Theorem 2.1), there exists an \( m \)-sectorial operator \( B \) associated with the form \( b[u,v] \) such that \( \mathcal{D}(B) \subset \mathcal{D}(b) \) and \( b[u,v] = (Bu,v) \) for all \( u \in \mathcal{D}(B) \) and \( v \in \mathcal{D}(b) \).

**Definition 1.1.** An operator \( B \), associated with a densely defined closed sectorial form \( b[\cdot, \cdot] \) in \( H \), is called regularly accretive if \( \gamma > 0 \).

In the papers by Kato [11,13] the following open problem was formulated: “Is the formula

\[
\mathcal{D}(B^{1/2}) = \mathcal{D}(B^{*1/2}) = \mathcal{D}(b)
\]

true for regularly accretive operators?”

In the paper [17], the following definition of regular acrretivity was given.

Let a Hilbert space \( V \) be continuously and densely embedded in \( H \), and let \( b[\cdot, \cdot] \) be a sesquilinear form defined and continuous on \( \mathcal{D}(b) = V \) that satisfies the inequality

\[
\Re b[u] \geq c_1 \|u\|^2, \quad u \in V.
\]

By Theorem 1.11 in [13, Chapter VI, §1], Theorem 2.1 in [13, Chapter VI, §2], and inequality (1.2), there exists an \( m \)-sectorial operator \( B \) associated with the form \( b[\cdot, \cdot] \) such that \( \mathcal{D}(B) \subset \mathcal{D}(b) \) and \( b[u,v] = (Bu,v) \) for all \( u \in \mathcal{D}(B) \) and \( v \in \mathcal{D}(b) \). The operator \( B \) constructed in [17] is said to be regularly accretive.

We show that Definition 1.1 is equivalent to the definition of regular acrretivity from [17].
Therefore, the sectorial form $b_{332}$ we introduce additional notation. Below we formulate some auxiliary results that we need in what follows. To do this, we introduce additional notation.

Let $H_1$ and $H_0$ be Hilbert spaces such that $H_1$ is dense in $H_0$ and continuously embedded in $H_0$. For every $ψ ∈ H_0$ and all $t > 0$, we define the functional

$$K(t; ψ; H_1, H_0) = \inf_{ψ = ψ_0 + ψ_1} \left( ||ψ_1||_{H_1}^2 + t^2 ||ψ_0||_{H_0}^2 \right)^{1/2}, \quad ψ_1 ∈ H_1, ψ_0 ∈ H_0.$$

We introduce the interpolation space

$$[H_1, H_0]_{1/2} = \left\{ ψ ∈ H_0 : \int_0^∞ t^{-2} K^2(t; ψ; H_1, H_0) dt < ∞ \right\}$$

with the norm

$$||ψ||_{[H_1, H_0]_{1/2}} = \left( ||ψ||_{H_0}^2 + \int_0^∞ t^{-2} K^2(t; ψ; H_1, H_0) dt \right)^{1/2}.$$

The following lemma is a consequence of Theorem 3.1 in [17] and Theorems 3.2, 15.1 in [18] Chapter I].

**Lemma 1.1.** Let $B : H ⊃ D(B) → H$ be a regularly accretive operator associated with a densely defined closed sectorial sesquilinear form $b[·, ·]$ in a Hilbert space $H$. Then

$$(1.3) \quad [D(B), H]_{1/2} = D(B^{1/2}) \quad \text{and} \quad [D(B^*) , H]_{1/2} = D(B^{*1/2}).$$

The following statement holds.

**Lemma 1.2** (see Corollary 5.1 in [17] and the corollary to Lemma 1 in [12]). Suppose that the hypotheses of Lemma 1.1 are satisfied and the following embeddings are valid: $D(b) ⊂ D(B^{1/2})$, $D(b) ⊂ D(B^{*1/2})$ or $D(B^{1/2}) ⊂ D(b)$, $D(B^{*1/2}) ⊂ D(b)$. Then (1.1) is true.
§2. Difference operators in a cylinder

In this section, we give some auxiliary results related to properties of difference operators with degeneration in a cylinder. The proofs can be found in [26, §2].

2.1. Consider the difference operator $R: L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ of the form

$$Ru(x) = \sum_{j=-k}^{k} b_j u(x_1 + j, x_2, \ldots, x_n),$$

where $k \in \mathbb{N}$, $b_j \in \mathbb{C}$.

Now we define the difference operator in the domain $Q = (0, d) \times G$, where $G \subset \mathbb{R}^{n-1}$ is a bounded domain (with boundary $\partial G$ of class $C^\infty$ provided that $n \geq 3$). To do this, we note that the operator $R$ is nonlocal. Namely, translations in the first variable may send points $x = (x_1, x_2, \ldots, x_n) \in Q$ to the points $(x_1 + i, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus Q$. That is why it is natural to introduce the bounded difference operator $R_Q = P_Q R I_Q : L_2(Q) \to L_2(Q)$, where $I_Q : L_2(Q) \to L_2(\mathbb{R}^n)$ is the operator that extends functions in $L_2(Q)$ outside $Q$ by zero, while $P_Q : L_2(\mathbb{R}^n) \to L_2(Q)$ is the operator of restriction of functions from $L_2(\mathbb{R}^n)$ to $Q$. Consequently, the operator $R_Q$ acts on the function $u(x)$ as follows. First, we extend this function by zero outside $Q$, then we apply to this extension the difference operator $R$ acting in the entire space $\mathbb{R}^n$, and finally, we consider the restriction of the function $R_Q u(x)$ to $Q$.

Without loss of generality, we assume that $d = k + \theta$, where $0 < \theta \leq 1$.

Obviously, the operators $R_Q, R_Q^* : L_2(Q) \to L_2(Q)$ are bounded. Moreover,

$$R_Q^* = P_Q R^* I_Q,$$
$$R^* u(x) = \sum_{j=-k}^{k} \bar{b}_j u(x_1 + j, x_2, \ldots, x_n).$$

Denote by $L_2(\bigcup_{l=1}^{N} Q_{s,l})$ the subspace of functions in $L_2(Q)$ vanishing outside the set $\bigcup_{l=1}^{N} Q_{s,l}$, where $s = 1, 2$ if $0 < \theta < 1$; $s = 1$ if $\theta = 1$; $N = k + 1$ if $s = 1$; $N = k$ if $s = 2$; $Q_{1l} = (l-1, l-1+\theta) \times G$ ($l = 1, \ldots, k+1$) and $Q_{2l} = (l-1+\theta, l) \times G$ ($l = 1, \ldots, k$). We denote by $\mathcal{R}$ the set of subdomains $\{Q_{s,l}\}$ and call it a partition of the domain $Q$.

We introduce the isometric isomorphism of Hilbert spaces

$$U_s : L_2\left(\bigcup_{l=1}^{N} Q_{s,l}\right) \to L_2^N(Q_{s,l})$$

defined by

$$(U_s u)_l(x) = u(x_1 + l - 1, x_2, \ldots, x_n), \quad l = 1, \ldots, N, \quad x \in Q_{s,l},$$

where $L_2^N(Q_{s,l}) = \prod_{l=1}^{N} L_2(Q_{s,l})$.

Remark 2.1. $R_{Q_s} = U_s R_Q U_s^{-1} : L_2^N(Q_{s,l}) \to L_2^N(Q_{s,l})$ is the operator of multiplication by an $N \times N$ matrix of the form

$$R_s = \begin{pmatrix}
    b_0 & b_1 & b_2 & \cdots & b_{N-1} \\
    b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    b_{-N+1} & b_{-N+2} & b_{-N+3} & \cdots & b_0
\end{pmatrix}.$$

Lemma 2.1. If $0 < \theta < 1$, then $\sigma(R_Q) = \sigma(R_1) \cup \sigma(R_2)$; if $\theta = 1$, then $\sigma(R_Q) = \sigma(R_1)$. Each point of the spectrum $\sigma(R_Q)$ has infinite multiplicity.

Lemma 2.2. If a function $\varphi \in C(\overline{Q})$ is 1-periodic in the $x_1$ variable in $\overline{Q}$, then $R_Q(\varphi u) = \varphi R_Q u$ for all $u \in L_2(Q)$.
2.2. Now, we consider the properties of difference operators \( R_Q : L_2(Q) \to L_2(Q) \) with nontrivial kernel. Such difference operators will be called degenerate.

**Lemma 2.3.** \( L_s^N(Q_{s1}) = \mathcal{N}(R_{Qs}) \oplus \mathcal{R}(R_{Qs}^*) \); \( L_s^N(Q_{s1}) = \mathcal{N}(R_{Qs}^*) \oplus \mathcal{R}(R_{Qs}) \), where \( \mathcal{N} (\cdot) \) and \( \mathcal{R} (\cdot) \) are the kernel and range of an operator, respectively.

Denote \( A_Q = (R_Q + R_Q^*)/2 \), \( B_Q = (R_Q - R_Q^*)/2i \). Obviously, \( R_Q = A_Q + iB_Q \). The operators \( A_Q \) and \( B_Q \) are called the *real* and imaginary parts of \( R_Q \), respectively. We put \( A_{Qs} = U_sA_QU_s^{-1} \) and \( B_{Qs} = U_sB_QU_s^{-1} \). By Remark 2.1 the operators \( A_{Qs}, B_{Qs} : L_s^N(Q_{s1}) \to L_s^N(Q_{s1}) \) are operators of multiplication by the matrices \( A_s = (R_s + R_s^*)/2 \) and \( B_s = (R_s - R_s^*)/2i \), respectively. Denote by \( P^R, P^R^*, P^A, P^B : L_2(Q) \to L_2(Q) \), and \( P^R_s, P^R^*_s, P^A_s, P^B_s : L_s^N(Q_{s1}) \to L_s^N(Q_{s1}) \) the operators of orthogonal projection onto the spaces \( \mathcal{R}(R_Q), \mathcal{R}(R_Q^*), \mathcal{R}(A_Q), \mathcal{R}(B_Q) \) and \( \mathcal{R}(R_{Qs}), \mathcal{R}(A_{Qs}), \mathcal{R}(B_{Qs}) \), respectively.

**Lemma 2.4.** \( L_2(Q) = \mathcal{N}(R_Q) \oplus \mathcal{R}(R_Q^*) \), \( L_2(Q) = \mathcal{N}(R_Q^*) \oplus \mathcal{R}(R_Q) \).

A bounded selfadjoint operator \( A \) from a Hilbert space \( H \) to \( H \) is said to be nonnegative if for every \( u \in H \) we have \((Au,u)_H \geq 0\).

**Lemma 2.5.** Let \( \mathcal{N}(A_s) \subset \mathcal{N}(B_s) \) (s = 1, 2 if \( \theta < 1 \); s = 1 if \( \theta = 1 \)). Then \( \mathcal{N}(A_Q) \subset \mathcal{N}(B_Q) \) and for every function \( u \in L_2(Q) \) we have

\[
\|B_Q u\|_{L_2(Q)} \leq c_1 \|A_Q u\|_{L_2(Q)},
\]

where \( c_1 > 0 \) does not depend on \( u \).

If, in addition, the matrix \( A_1 \) is nonnegative, then the operator \( A_Q \) is nonnegative. Moreover, \( \mathcal{N}(R_Q) = \mathcal{N}(R_Q^*) = \mathcal{N}(A_Q) \) and \( \mathcal{R}(R_Q) = \mathcal{R}(R_Q^*) = \mathcal{R}(A_Q) \).

**Example 2.1.** Let \( Ru(x) = u(x_1, x_2) + u(x_1 + 1, x_2) + u(x_1 - 1, x_2) \), and let \( Q = (0, 2) \times (0, 1) \). Then the partition \( \mathcal{R} \) of the domain \( Q \) consists of one class of subdomains \( Q_{11} = (0, 1) \times (0, 1) \), \( Q_{12} = (1, 2) \times (0, 1) \) so that \( h_{11} = (0, 0), h_{12} = (1, 0) \). The matrix \( R_1 \) has the form

\[
R_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

By Lemma 2.2 \( \sigma(R_Q) = \sigma(R_1) = \{0\} \cup \{2\} \). Evidently,

\[
\mathcal{N}(R_Q) = \{u \in L_2(Q) : -u(x_1, x_2) = u(x_1 + 1, x_2) \text{ if } x \in Q_{11}\},
\]

\[
\mathcal{R}(R_Q) = \{u \in L_2(Q) : u(x_1, x_2) = u(x_1 + 1, x_2) \text{ if } x \in Q_{11}\}.
\]

**Remark 2.2.** The operator \( P^R_s : L_s^N(Q_{s1}) \to L_s^N(Q_{s1}) \) is the operator of multiplication by a matrix (which is denoted by \( P^R_s \)). Furthermore, multiplication by the matrix \( P^R_s \) in the complex space \( \mathbb{C}^N \) is an operator of orthogonal projection in \( \mathbb{C}^N \) onto the range of the matrix \( R_s \).

**Lemma 2.6.** For any \( u \in L_2(Q) \), we have

\[
P^R u = \sum_{s=1,2} U_s^{-1} P^R_s U_s P_s u \text{ if } 0 < \theta < 1, \quad P^R u = U_1^{-1} P^R U_1 u \text{ if } \theta = 1.
\]

2.3. Now we consider properties of the difference operators \( R_Q \) in Sobolev spaces.

Denote by \( W^k_2(Q) \), \( k \in \mathbb{N} \), the Sobolev space of complex-valued functions with the norm

\[
\|u\|_{W^k_2(Q)} = \left( \sum_{|\alpha| \leq k} \int_Q |D^\alpha u(x)|^2 dx \right)^{1/2},
\]
where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i \frac{\partial}{\partial x_j}$. Denote by $W_2^{k-1/2}(\Gamma)$ the space of traces on $\Gamma$ with the norm

$$
\|u\|_{W_2^{k-1/2}(\Gamma)} = \inf_u \|u\|_{W_2^k(Q)}, \quad u \in W_2^k(Q), \ u|_\Gamma = v,
$$

where $\Gamma \subset \bar{Q}$ is an $(n-1)$-dimensional smooth manifold. Different ways of introducing equivalent norms in Sobolev spaces of nonintegral order can be found, for example, in [18].

We denote by $\mathring{W}_2^{1}(Q)$ the closure of the linear manifold $C_0^\infty(Q)$ (of compactly supported functions infinitely differentiable in $Q$) in the space $W_2^1(Q)$. It can be shown that, by analogy with a domain with a smooth boundary, the following property holds for the domain $Q = (0, d) \times G$:

$$
\mathring{W}_2^{1}(Q) = \{ u \in W_2^1(Q) : u|_{\partial Q} = 0 \}.
$$

**Lemma 2.7.** The operator $R_Q$ maps continuously $\mathring{W}_2^{1}(Q)$ to $W_2^1(Q)$, so that for all $u \in \mathring{W}_2^{1}(Q)$ we have

$$
(R_Q u)_{x_j} = R_Q u_{x_j}, \quad j = 1, \ldots, n.
$$

**Lemma 2.8.** Given $s$, let $Q'_{s_j}$ be open connected sets such that $Q'_{s_1} \subset Q_{sl}$ and $Q'_{s_l} = Q'_{s_1} + h_{sl}$ ($1 \leq l \leq N = N(s)$). Then for all $u \in L_2(Q)$ such that $R_Q u \in W_2^k(Q'_{s_j})$ ($j = 1, \ldots, N$), we have $P^{R^*} u \in W_2^k(Q_{sl})$ ($l = 1, \ldots, N$) and

$$
\|P^{R^*} u\|_{W_2^k(Q'_{s_j})} \leq c_2 \sum_{j=1}^{N} \|R_Q u\|_{W_2^k(Q'_{s_j})},
$$

where $c_2 > 0$ does not depend on $u$; $s = 1, 2$ if $\theta < 1$; $s = 1$ if $\theta = 1$.

### §3. Differential-difference operators with degeneration

#### 3.1. Consider an unbounded differential-difference operator

$$
L_R: L_2(Q) \supset D(L_R) \rightarrow L_2(Q)
$$

with the domain

$$
D(L_R) = C_0^\infty(Q),
$$

acting by the formula

$$
L_R u = LR_Q u, \quad u \in D(L_R).
$$

Here, $R_Q = P_Q RI_Q; R: L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ is the difference operator defined by

$$
R u(x) = \sum_{j=-k}^{k} b_j u(x_1 + j, x_2, \ldots, x_n),
$$

$k \in \mathbb{N}, b_j \in \mathbb{C}$, the operators $I_Q: L_2(Q) \rightarrow L_2(\mathbb{R}^n)$ and $P_Q: L_2(\mathbb{R}^n) \rightarrow L_2(Q)$ were defined in Subsection 2.1; the operator $L$ has the following form:

$$
L u(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u(x)}{\partial x_j},
$$

$a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^n)$ are real, 1-periodic functions in the $x_1$ variable. Suppose, in addition, that $Q = (0, d) \times G$, where $d = k + \theta, 0 < \theta \leq 1$, $k$ is a natural number, $G \subset \mathbb{R}^{n-1}$ is a bounded domain (with the boundary $\partial G \in C^\infty$ for $n \geq 3$).

We assume that the following conditions are satisfied:

**Condition 1.** $\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j > 0$ for all $x \in \bar{Q}$ and $0 \neq \xi \in \mathbb{R}^n$. 


**Condition 2.** \( \text{Re} b_0 > 0 \).

**Condition 3.** \( \det A_1 = 0 \) and \( \mathcal{N}(A_1) \subset \mathcal{N}(B_1) \).

**Condition 4.** The matrix \( A_1 \) is nonnegative.

**Remark 3.1.** If \( A_1 \geq 0 \) and \( \det A_2 = 0 \), then \( \det A_1 = 0 \) in view of the extreme property of the eigenvalues. Moreover, obviously, \( A_2 \geq 0 \).

**Lemma 3.1.** Suppose that conditions [1–4] are fulfilled. Then there exists a constant \( c_1 > 0 \) such that

\[
\text{Re}(LRQ u, u)_{L_2(Q)} \geq c_1 \| \nabla RQ u \|_{L_2(Q)}^2
\]

for all \( u \in C_0^\infty(Q) \).

**Proof.** By Lemma 2.7 and the second part of Lemma 2.5, we have

\[
\| \nabla RQ u \|_{L_2(Q)}^2 = \sum_{|\alpha|=1} \| RQ D^\alpha u \|_{L_2(Q)}^2
\]

\[
\leq k_1 \sum_{\alpha} \| P^A D^\alpha u \|_{L_2(Q)}^2 \leq k_2 \sum_{\alpha} \| A Q D^\alpha u \|_{L_2(Q)}^2
\]

for all \( u \in C_0^\infty(Q) \).

From Lemma 2.6, it follows that

\[
U_s P_s P^A = P^A U_s P_s.
\]

Since \( A Q s \) is the operator of multiplication by the matrix \( A_s \), taking Lemmas 2.4 and 2.7 into account and using (3.6) and the fact that the operators \( P_s \) and \( A Q \) commute, we obtain

\[
\sum_{\alpha} \| A Q D^\alpha u \|_{L_2(Q)}^2 = \sum_{\alpha} \| D^\alpha A Q u \|_{L_2(Q)}^2 = \sum_{\alpha} \sum_{s,l} \| D^\alpha P_s A Q P^A u \|_{L_2(Q_{s,l})}^2
\]

\[
= \sum_{\alpha} \sum_{s} \| D^\alpha A_s P_s P^A u \|_{L_2^N(Q_{s,l})}^2
\]

\[
\leq k_3 \sum_{\alpha} \sum_{s} \| D^\alpha U_s P_s P^A u \|_{L_2^N(Q_{s,l})}^2
\]

\[
= k_3 \sum_{\alpha} \sum_{s} \| P^A U_s P_s D^\alpha u \|_{L_2^N(Q_{s,l})}^2.
\]

Here \( s \) takes the values 1 or 2 if \( \theta < 1 \), and \( s \) takes the value 1 if \( \theta = 1 \).

By Condition 4 the matrices \( A_s \) are nonnegative. Thus, Lemma 2.9 implies the inequality

\[
\sum_{\alpha} \sum_{s} \| P^A U_s P_s D^\alpha u \|_{L_2^N(Q_{s,l})}^2
\]

\[
\leq k_4 \sum_{\alpha} \sum_{s} \left( A_s P^A U_s P_s D^\alpha u, P^A U_s P_s D^\alpha u \right)_{L_2^N(Q_{s,l})}
\]

\[
= k_4 \sum_{\alpha} \sum_{s} \left( A_s U_s P_s D^\alpha u, U_s P_s D^\alpha u \right)_{L_2^N(Q_{s,l})}.
\]

By Condition 1 the formula \( A Q s = U_s A_Q U_s^{-1} \), the fact that \( P_s \) and \( A Q \) commute, Lemmas 2.7 and 2.9, and the property for the coefficients \( a_{ij}(x) \) to be 1-periodic in the
\[ x_1 \text{ variable, we have} \]
\[
\sum_{\alpha} \sum_{s} \left( A_s U_s P_s D^\alpha u, U_s P_s D^\alpha u \right)_{L^2_2(Q_{x_1})} = \sum_{\alpha} \sum_{s} \left( \sqrt{A_s} U_s P_s D^\alpha u, \sqrt{A_s} U_s P_s D^\alpha u \right)_{L^2_2(Q_{x_1})} \leq k_5 \sum_{|\alpha|=1} \sum_{s} \left( a_{\alpha} \sqrt{A_s} U_s P_s D^\alpha u, \sqrt{A_s} U_s P_s D^\alpha u \right)_{L^2_2(Q_{x_1})} = k_5 \sum_{\alpha} \left( a_{\alpha} \sqrt{A_s} U_s P_s D^\alpha u, P^{s} D^\alpha u \right)_{L^2_2(Q)} = k_5 \Re \left( LRQ u, u \right)_{L^2_2(Q)},
\]

Inequalities (3.5), (3.7)–(3.9) imply (3.4).

\[ \square \]

Lemma 3.2. Let conditions (a) be satisfied. Then there exists a constant \( c_2 > 0 \) such that for every \( u, v \in C^\infty_0(Q) \) we have
\[
\| (LRQ u, v)_{L^2_2(Q)} \| \leq c_2 \| \nabla RQ u \|_{L^2_2(Q)} \| \nabla RQ v \|_{L^2_2(Q)}.
\]

Proof. Since the operators of multiplication by 1-periodic functions in the \( x_1 \) variable commute with \( R_Q \), integrating by parts and using Lemmas 2.4, 2.5, and 2.7, we obtain
\[
\| (LRQ u, v)_{L^2_2(Q)} \| = \left| \sum_{|\alpha|=1} \left( R_Q a_{\alpha}^{s} D^\beta u, P^{s} D^\alpha v \right)_{L^2_2(Q)} \right| \leq k_1 \sum_{\alpha, \beta} \| R_Q D^\beta u \|_{L^2_2(Q)} \| R_Q D^\alpha v \|_{L^2_2(Q)} \leq k_2 \| \nabla RQ u \|_{L^2_2(Q)} \| \nabla RQ v \|_{L^2_2(Q)}
\]
for all \( u, v \in C^\infty_0(Q) \).

\[ \square \]

We introduce the unbounded operator \( L^+_R : L^2_2(Q) \supset D(L^+_R) \rightarrow L^2_2(Q) \) with domain \( D(L^+_R) = C^\infty_0(Q) \), acting by the formula
\[
L^+_R u = LR^*_Q u, \quad u \in D(L^+_R).
\]

From Lemmas 3.1 and 3.2 we obtain the following statement.

Lemma 3.3. Suppose that conditions (a) are satisfied. Then the operators \( L_R \) and \( L^+_R \) defined by formulas (3.1) and (3.11) are sectorial.

3.2. Denote by \( G \) a sectorial operator densely defined in a Hilbert space \( H \). We consider a sesquilinear form \([u, v] = (G u, v)\) with the domain \( D(g) = D(G) \). By Theorem 1.27 in [13] Chapter VI, \( \S 1 \) there exists a closure \( t \) of the form \( g \). Denote by \( T = T_t \) the \( m \)-sectorial operator associated with the form \( t \), see Theorem 2.1 in [13] Chapter VI, \( \S 2 \). Since \( D(G) \) is the kernel of the form \( t \), by Theorem 2.1 in [13] Chapter VI, \( \S 2 \), we have \( G \subset T \). The operator \( T \) is called the Friedrichs extension of \( G \).

Now, we construct the Friedrichs extension of the differential-difference operator \( L_R \) and study properties of this extension. Note that the existence of such an extension is guaranteed by the sectoriality of the operator \( L_R \), which follows from Lemma 3.3.

We consider the sesquilinear forms
\[
g_R[u, v] = (L_R u, v)_{L^2_2(Q)}, \quad u, v \in C^\infty_0(Q),
\]
\[
g^+_R[u, v] = (L^+_R u, v)_{L^2_2(Q)}, \quad u, v \in C^\infty_0(Q).
\]
From Lemma 3.3 it follows that the forms $g_R$ and $g_R^+$ are sectorial. Therefore, by Theorem 1.27 in [13, Chapter VI, §1], they have closures in $L_2(Q)$ which will be denoted by $l_R$ and $l_R^+$, respectively. It is obvious that

$$g_R[u, v] = g_R[v, u], \quad u, v \in C_0^\infty(Q).$$

Hence, $l_R = l_R^+$.

Since the operators of multiplication by 1-periodic functions in the $x_1$ variable and $R_Q$ commute, by Lemmas 2.7 and 3.1 one can introduce the inner product in $C_0^\infty(Q)$ by the formula

$$\langle u, v \rangle_{l_R} = \langle L A_Q u, v \rangle_{L_2(Q)} + \langle u, v \rangle_{L_2(Q)}$$

$$= \langle g_R [u, v] + g_R^+[u, v] \rangle_2 + \langle u, v \rangle_{L_2(Q)}, \quad u, v \in C_0^\infty(Q).$$

(3.12)

Denote by $H_{l_R}$ the set of elements in $L_2(Q)$ for which there exists a sequence $\{u_p\} \subset C_0^\infty(Q)$ such that

$$\lim_{p \to \infty} \|u_p - u\|_{L_2(Q)}, \quad \lim_{p, q \to \infty} \|u_p - u_q\|_{l_R} = 0.$$

We introduce the inner product in $H_{l_R}$ as follows:

$$\langle u, v \rangle_{l_R} = \lim_{p \to \infty} l_R[u_p, v_p],$$

where $u_p, v_p \in C_0^\infty(Q)$, $u_p \to u$, $v_q \to v$ in $L_2(Q)$ and $\lim_{p, q \to \infty} \|u_p - u_q\|_{l_R} = 0$, $\lim_{p, q \to \infty} \|v_p - v_q\|_{l_R} = 0$. This inner product does not depend on the choice of the sequence $u_p$, while the space with the corresponding norm is complete and $H_{l_R} = \mathcal{D}(l_R) = \mathcal{D}(l_R^*)$, see Theorems 1.11 and 1.17 in [13, Chapter VI, §1].

Now we construct the Friedrichs extensions of the operators $L_R$ and $L_R^+$ and consider their spectral properties.

We introduce the unbounded operators

$$\mathcal{L}_R: L_2(Q) \supset \mathcal{D}(\mathcal{L}_R) \to L_2(Q)$$

and

$$\mathcal{L}_R^+: L_2(Q) \supset \mathcal{D}(\mathcal{L}_R^+) \to L_2(Q)$$

acting by the formulas

$$\mathcal{L}_R u = L R_Q u, \quad u \in \mathcal{D}(\mathcal{L}_R) = \{u \in H_{l_R} : \mathcal{L}_R u \in L_2(Q)\},$$

$$\mathcal{L}_R^+ u = L R_Q^+ u, \quad u \in \mathcal{D}(\mathcal{L}_R^+) = \{u \in H_{l_R} : \mathcal{L}_R^+ u \in L_2(Q)\}.$$
We introduce the concept of a generalized solution of the boundary value problem for an elliptic differential-difference equation with degeneration.

Consider the equation

\[ LRu(x) = f(x), \quad x \in Q, \]

with the boundary condition

\[ u(x) = 0, \quad x \in \mathbb{R}^n \setminus Q, \]

where \( f \in L_2(Q) \).

The nonclassical form of the boundary condition (3.17) is related to the fact that the translations \( x \mapsto x + h \) in the difference operator \( R \) can send points of the domain \( Q \) to the set \( \mathbb{R}^n \setminus Q \). Therefore, the boundary conditions must be specified not only for the boundary \( \partial Q \), but also outside the domain \( Q \). For this purpose, we consider the operator \( L_R = LR_Q \), where \( R_Q = P_Q R I_Q \), and then we construct a Friedrichs extension \( L_R \) of \( L_R \).
**Definition 3.1.** A function \( u \in \mathcal{D}(L_R) \) is called a generalized solution of problem \((3.16), (3.17)\) if it satisfies the operator equation

\[
L_Ru = f.
\]  

By Lemma 3.4, problem \((3.16), (3.17)\) can have a generalized solution \( u \in L_2(Q) \) not even belonging to the space \( W^1_2(Q) \). Nevertheless, the function \( R_Qu \) already has an appropriate smoothness, i.e., \( R_Qu \in W^2_2(Q) \) (see Theorem 3.2).

To prove the theorem on the smoothness of generalized solutions, we need the following auxiliary result.

**Lemma 3.5.** Suppose that conditions \(2, 4\) are satisfied. Then the first row of the matrix \( R_1 \) is a linear combination of last \( k \) rows with coefficients \( \gamma_j, j = 2, \ldots, k + 1 \), while the last row of the matrix \( R_1 \) is a linear combination of the first \( k \) rows with coefficients \( \bar{\gamma}_{k+2-j}, j = 1, \ldots, k \).

See the proof in [26, §7].

**Theorem 3.2.** Suppose that conditions \(2, 4\) are satisfied. Let \( u \in \mathcal{D}(L_R) \) be a solution of equation \((3.18)\), and let \( f \in L_2(Q) \). Then \( R_Qu \in W^2_2(Q) \) and \( P^R_u \in W^2_2(Q) \) (if \( \theta = 1 \), then \( s = 1, l = 1, \ldots, k + 1 \); if \( \theta < 1 \), then \( s = 1, 2, l = 1, \ldots, k + 1 \) for \( s = 1 \) and \( l = 1, \ldots, k \) for \( s = 2 \)).

**Proof.** 1. By Lemma 3.4 \( w = R_Qu \in W^1_2(Q) \). Therefore, the traces of the function \( w \) on the manifolds \( \{0\} \times G, \{d\} \times G \) and \( [0, d] \times \partial G \) are defined. We prove that \( w \) obeys the nonlocal conditions

\[
\begin{align*}
(w|_{x_1 = 0} & = \sum_{j=1}^{k} \gamma_{j+1} w|_{x_1 = j}, \quad w|_{x_1 = d} = \sum_{j=1}^{k} \bar{\gamma}_{j+1} w|_{x_1 = d-j}, \\
(w|_{[0, d] \times \partial G} & = 0,
\end{align*}
\]  

where the \( \gamma_j \) are complex numbers.

Evidently, \( R_Qu|_{[0, d] \times \partial G} = 0 \) for any function \( v \in C^\infty_0(Q) \). Hence, by Lemma 3.1 and the continuity of the operator of trace acting from \( W^1_2(Q) \) to \( L_2([0, d] \times \partial G) \), the last relation among \((3.19)\) is true. We prove the first two relations.

We define an isometric isomorphism \( U_1: L_2(\bigcup_{k=1}^{k+1} Q_{11}) \to L^{k+1}_2(Q_{11}) \) by the formula

\[
(U_1 u)_l(x) = u(x_1 + l - 1, x_2, \ldots, x_n), \quad x \in Q_{11},
\]  

where \( l = 1, \ldots, k + 1; Q_{11} = (0, \theta) \times G \). From the commutativity of the operators \( P_i \) and \( R_Q \) and Lemma 3.5 it follows that

\[
(U_1 P_1 R_Q u)_{k+1}(x) = \sum_{j=1}^{k} \bar{\gamma}_{j+1} (U_1 P_1 R_Q u)_{k+1-j}(x), \quad x \in Q_{11},
\]  

where the \( \gamma_j \) are complex coefficients.

Similarly, we have

\[
(U_1 P_1 R_Q u)_{k+1}(x) = \sum_{j=1}^{k} \bar{\gamma}_{j+1} (U_1 P_1 R_Q u)_{k+1-j}(x), \quad x \in Q_{11}.
\]
Taking (3.20) and (3.22) into account, we obtain

\[(R_Q u)(x) = \sum_{j=1}^{k} \gamma_{j+1}(R_Q u)(x_1 + j, x_2, \ldots, x_n), \quad x \in Q_{11},\]

\[(R_Q u)(x) = \sum_{j=1}^{k} \tilde{\gamma}_{j+1}(R_Q u)(x_1 - j, x_2, \ldots, x_n), \quad x \in Q_{1,k+1}.\]

Formulas (3.23) and (3.24) imply the first two relations of (3.19).

2. By Lemma 2.5 \(R(Q') = R(Q)\). Hence, by Lemma 2.8 to prove the theorem, it suffices to show that \(w = R_Q u \in W_2^1(Q)\).

We introduce a function \(\eta \in C_0^\infty(\mathbb{R})\) so that \(0 \leq \eta(x_1) \leq 1, \eta(x_1) = 1\) for \(x_1 \in (-\delta, \delta)\), \(\eta(x_1) = 0\) for \(x_1 \notin (-2\delta, 2\delta)\), where \(0 < \delta < \theta/4\). Now we define a function \(\xi\) as follows:

\[\xi(x) = \eta(x_1) \sum_{j=1}^{k} \gamma_{j+1} w(x_1 + j, x_2, \ldots, x_n) \quad \text{if } x \in (0, 2\delta) \times G,\]

\[\xi(x) = \eta(x_1 - d) \sum_{j=1}^{k} \tilde{\gamma}_{j+1} w(x_1 - j, x_2, \ldots, x_n) \quad \text{if } x \in (d - 2\delta, d) \times G,\]

\[\xi(x) = 0 \quad \text{if } x \in [2\delta, d - 2\delta] \times G.\]

Since \(\partial Q \in C^\infty\) and \(|w|_{[0,d] \times \partial G} = 0\), by the theorem on the smoothness of generalized solutions of elliptic differential equations with homogeneous Dirichlet conditions, \(w \in W_2^1((\varepsilon, d - \varepsilon) \times G)\) for any \(\varepsilon > 0\) in the vicinity of a smooth piece of boundary. Hence, \(\xi \in W_2^1(Q)\). Therefore, \(L(w - \xi) \in L_2(Q)\), and by (3.19) and the definition of \(\xi\), we have \(w - \xi \in W_2^1(Q)\). Now we note that the theorem on the smoothness of generalized solutions near the entire boundary is applicable to second order differential equations in a cylindrical domain \(Q\) (see Theorem 10.1 in [16, Chapter III, §10]). Hence, \(w - \xi \in W_2^1(Q)\), i.e., \(w \in W_2^1(Q)\).

Now we consider the domain of the sesquilinear form \(D(L_R) = H_{L_R}\).

**Lemma 3.6.** Suppose that conditions (3.19) are satisfied. Then

\[H_{l_R} = \{ u \in L_2(Q) : R_Q u \in W_2^1(Q), R_Q u|_{[0,d] \times \partial G} = 0 \},\]

i.e.,

\[R_Q H_{l_R} = \{ w \in W_2^1(Q) \cap R_Q (L_2(Q)) : w|_{[0,d] \times \partial G} = 0 \}.\]

A proof can be found in [26, §7].

Lemma 3.6 and Theorem 3.2 imply the following result.

**Corollary 3.1.** Let conditions (3.19) be satisfied. Then

\[D(L_R) = \{ u \in L_2(Q) : R_Q u \in W_2^1(Q), R_Q u|_{[0,d] \times \partial G} = 0 \}.\]

Our next statement follows from Lemma 3.6 and the proof of Theorem 3.2.

**Corollary 3.2.** Let conditions (3.19) be satisfied, and let \(w \in R_Q (H_{l_R})\). Then \(w\) obeys the nonlocal boundary conditions (3.19).

**Example 3.1.** Let \(Q = (0, 3) \times (0, 1)\). Suppose that \(L\) is the differential operator (3.3) satisfying condition (3.1) and \(R\) is the difference operator defined by the formula

\[Ru(x) = (1 + i)u(x) + (b + i)(u(x_1 + 1, x_2) + u(x_1 - 1, x_2)) + (1 + i)(u(x_1 + 2, x_2) + u(x_1 - 2, x_2)),\]
where \( b \in \mathbb{R}, \, 0 < |b| < 1 \). The partition of the domain \( Q \) consists of one class of subdomains \( Q_l = (l - 1, l) \times (0, 1) \) \((l = 1, 2, 3)\). The matrices \( R_1, A_1, \) and \( B_1 \) have the form

\[
R_1 = \begin{pmatrix} 1 + i & b + i & 1 + i \\ b + i & 1 + i & b + i \\ 1 + i & b + i & 1 + i \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & b & 1 \\ b & 1 & b \\ 1 & b & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

Obviously, \( \text{Re}(1 + i) = 1 > 0 \), det \( A_1 = 0 \), \( \mathcal{N}(A_1) \subset \mathcal{N}(B_1) \), and \( A_1 \geq 0 \). Thus, conditions \([1][2]\) are satisfied. Then by Lemma 3.6 and Corollary 3.1, \( H_{lr} = \{ u \in L_2(Q) : RQu \in W_2^1(Q), \, RQu|_{\partial Q} = 0 \} \) and \( \mathcal{D}(\mathcal{L}_R) = \{ u \in L_2(Q) : RQu \in W_2^2(Q), \, RQu|_{\partial Q} = 0 \} \).

\section{The Kato conjecture for elliptic differential-difference operators with degeneration}

\subsection{4.1.} We introduce the unitary spaces \( H_{R^k}^s = \{ u \in L_2(Q) : RQu \in W_2^s(Q) \} \) and \( H_{R^s}^s = \{ u \in L_2(Q) : RQu \in W_2^s(Q) \} \) with inner products

\[
(1) \quad (u_1, v_1)_{H_{R}^s} = (u_1, v_1)_{L_2(Q)} + (RQu_1, RQv_1)_{W_2^s(Q)},
\]

\[
(2) \quad (u_2, v_2)_{H_{R^s}^s} = (u_2, v_2)_{L_2(Q)} + (RQu_2, RQu^*v_2)_{W_2^s(Q)},
\]

where \( s \in \mathbb{N} \).

\textbf{Lemma 4.1.} The spaces \( H_{R}^s \) and \( H_{R^s}^s \) are Hilbert spaces.

\textbf{Proof.} We show, for example, that the space \( H_{R}^s \) is complete. Consider a Cauchy sequence \( \{ u_k \} \subset H_{R}^s \), i.e., \( \| u_k - u_m \|_{H_{R}^s} \to 0 \) as \( k, m \to \infty \). Then \( \| u_k - u_m \|_{L_2(Q)} \to 0 \) and \( \| RQu_k - RQu_m \|_{W_2^s(Q)} \to 0 \) as \( k, m \to \infty \). Since the spaces \( L_2(Q) \) and \( W_2^s(Q) \) are complete, there exist functions \( u \in L_2(Q) \) and \( w \in W_2^s(Q) \) such that \( \| u_k - u \|_{L_2(Q)} \to 0 \) and \( \| RQu_k - w \|_{W_2^s(Q)} \to 0 \) as \( k \to \infty \). From the continuity of the operator \( RQ : L_2(Q) \to L_2(Q) \), it follows that \( \| RQu_k - RQu \|_{L_2(Q)} \to 0 \) as \( k \to \infty \). By the uniqueness of the limit, \( w = RQu \), i.e., \( \| u_k - u \|_{H_{R}^s} \to 0 \) as \( k \to \infty \), where \( u \in H_{R}^s \).

Denote by \( H_{R,0}^1 \) and \( H_{R^s,0}^1 \) the closures of the linear manifold \( C_0^\infty(Q) \) in the spaces \( H_{R}^s \) and \( H_{R^s}^s \), respectively.

\textbf{Lemma 4.2.} \( H_{R,0}^1 = H_{R^s,0}^1 = H_{lr} \), and the norms \( \| \cdot \|_{H_{R}^1}, \| \cdot \|_{H_{R^s}^1}, \) and \( \| \cdot \|_{H_{R}^1} \) are equivalent on \( H_{lr} \).

\textbf{Proof.} We show, for example, that \( H_{R,0}^1 = H_{lr} \) and the norms \( \| \cdot \|_{H_{R}^1} \) and \( \| \cdot \|_{H_{R^s}^1} \) are equivalent. To do this, it suffices to verify that

\[
(3) \quad k_1 \| u \|_{H_{R}^1} \leq \| u \|_{H_{R^s}^1} \leq k_2 \| u \|_{H_{R}^1},
\]

for all \( u \in C_0^\infty(Q) \). By Lemmas 3.2, 2.7, 2.4 and 2.5 for each \( u \in C_0^\infty(Q) \) we have

\[
\| u \|_{H_{R}^1}^2 = (LQu, u)_{L_2(Q)} + \| u \|_{L_2(Q)}^2 \leq k_3 \left( \sum_i \| AQ_{u_x} \|_{L_2(Q)}^2 + \| u \|_{L_2(Q)}^2 \right)
\]

\[
\leq k_4 \left( \sum_i \| P_{u_x} \|_{L_2(Q)}^2 + \| u \|_{L_2(Q)}^2 \right) \leq k_5 \left( \sum_i \| R_{u_x} \|_{L_2(Q)}^2 + \| u \|_{L_2(Q)}^2 \right)
\]

\[
\leq k_5 \left( \| R_{u_x} \|_{W_2^1(Q)}^2 + \| u \|_{L_2(Q)}^2 \right) = k_5 \| u \|_{H_{R}^1}^2.
\]
On the other hand, by Lemmas 3.1, 2.2, and 2.5 for each \( u \in C_0^\infty(Q) \) we obtain
\[
\|u\|^2_{L_R} \geq c_1 \sum_i \|A_i u_x, L_{2}(Q)\|^2 + \|u\|^2_{L_{2}(Q)} \geq k_6 \left( \sum_i \|P_i u_x, L_{2}(Q)\|^2 + \|u\|^2_{L_{2}(Q)} \right)
\]
\[
\geq k_7 \left( \sum_i \|R_i u_x, L_{2}(Q)\|^2 + \|u\|^2_{L_{2}(Q)} \right)
\]
\[
\geq k_8 \left( \|R_i u\|^2_{W_{2}(Q)} + \|u\|^2_{L_{2}(Q)} \right) = k_8 \|u\|^2_{H_R^1}.
\]

\[\Box\]

**Theorem 4.1.** Suppose that conditions 1-4 are satisfied. Then
\[
\mathcal{D}(L_R + I)^{1/2} = \mathcal{D}(L_R^* + I)^{1/2} = H_{R}.\]

**Proof.** By Lemma 4.2 the operators \( L_R + I \) and \( L_R^* + I \) are regularly accretive operators associated with the densely defined closed sectorial form \( l_R[u, v] + (u, v)_{L_{2}(Q)} \) in the Hilbert space \( L_{2}(Q) \) satisfying the inequality
\[
Re(l_R[u, u] + (u, u)_{L_{2}(Q)}) \geq k_2 \|u\|^2_{H_R^1}
\]
for all \( u \in \mathcal{D}(l_R) \), where \( \mathcal{D}(l_R) = H_{R, 0} = H_{R^*, 0} \).

Hence, by Lemma 1.3
\[
[\mathcal{D}(L_R + I), L_{2}(Q)]_{1/2} = \mathcal{D}(l_R)^{1/2},
\]
\[
[\mathcal{D}(L_R^* + I), L_{2}(Q)]_{1/2} = \mathcal{D}(l_R^*)^{1/2}.
\]

Consequently, by Lemmas 1.2 and 4.2 to complete the proof it suffices to show that
\[
[\mathcal{D}(L_R + I), L_{2}(Q)]_{1/2} \subset H_{R, 0},
\]
\[
[\mathcal{D}(L_R^* + I), L_{2}(Q)]_{1/2} \subset H_{R^*, 0},
\]
where the embeddings are continuous.

We prove, for example, (4.6). From Theorem 3.2 and Lemmas 4.2 and 3.6 it follows that
\[
\mathcal{D}(L_R + I) = \mathcal{D}(L_R) = \{ u \in H_{R, 0}^1 : R_i u \in W_{2}^2(Q) \}.
\]

Since the operator \( L_R : L_2(Q) \supset \mathcal{D}(L_R) \rightarrow L_2(Q) \) is closed, the space \( \mathcal{D}(L_R) \) with the inner product
\[
(u, v)_{\mathcal{D}(L_R)} = (u, v)_{L_2(Q)} + (L_R u, L_R v)_{L_2(Q)}, \quad u, v \in \mathcal{D}(L_R),
\]
is a Hilbert space.

Clearly, one can introduce an equivalent inner product in \( \mathcal{D}(L_R) \) as follows:
\[
(u, v)_{\mathcal{D}(L_R)}' = (u, v)_{L_2(Q)} + (R_i u, R_i v)_{W_2^2(Q)}, \quad u, v \in \mathcal{D}(L_R).
\]

In what follows, we denote the equivalent inner product (4.9) and the corresponding norm in \( \mathcal{D}(L_R) \) without a prime mark.

Now we prove that
\[
[\mathcal{D}(L_R), L_2(Q)]_{1/2} \subset H_{R, 0}^1.
\]

For any \( \psi \in L_2(Q) \) and \( t > 0 \), we have
\[
K(t, \psi ; \mathcal{D}(L_R), L_2(Q)) = \inf_{\psi = \psi_1 + \psi_0} \left( \|\psi_1\|^2_{\mathcal{D}(L_R)} + t^2 \|\psi_0\|^2_{L_{2}(Q)} \right)^{1/2}
\]
\[
= \inf_{\psi = \psi_1 + \psi_0} \left( \|\psi_1\|^2_{L_2(Q)} + \|R_i \psi_1\|^2_{W_2^2(Q)} + t^2 \|\psi_0\|^2_{L_{2}(Q)} \right)^{1/2},
\]
\[
\psi_1 \in \mathcal{D}(L_R), \quad \psi_0 \in L_2(Q).
\]
Hence, in view of the boundedness of the operator \( R_Q: L^2(Q) \to L^2(Q) \) and the embedding \( D(L_R) \subset H^1_{R_0} \), we have
\[
K(t, \psi; D(L_R), L^2(Q))
\geq k_1 \inf_{\psi=\psi_0+\psi_1} \left( \| R_Q \psi_1 \|_{L^2_{R_0}(Q)}^2 + t^2 \| R_Q \psi_0 \|_{L^2(Q)}^2 \right)^{1/2}
\geq k_1 \inf_{R_Q \psi=\psi_0+\psi_1} \left( \| \varphi_1 \|_{L^2_{R_0}(Q)}^2 + t^2 \| \varphi_0 \|_{L^2(Q)}^2 \right)^{1/2}
= k_1 K(t, R_Q \psi; W^2_{R_0}(Q), L^2(Q)), \quad \varphi_1 \in W^2_{R_0}(Q), \quad \varphi_0 \in L^2(Q).
\] (4.10)

If \( \psi \in [D(L_R), L^2(Q)]_{1/2} \), then
\[
\int_0^\infty t^{-2} K^2(t, \psi; D(L_R), L^2(Q)) \, dt < \infty.
\]
Consequently, by (4.10),
\[
\int_0^\infty t^{-2} K^2(t, R_Q \psi; W^2_{R_0}(Q), L^2(Q)) \, dt < \infty.
\]
Therefore, by Theorem 9.6 in [18] Chapter 1, \( R_Q \psi \in W^1_{R_0}(Q) \) and
\[
\| R_Q \psi \|_{W^1_{R_0}(Q)} \leq k_2 \| \psi \|_{[D(L_R), L^2(Q)]_{1/2}},
\]
(4.11)
\[
\| \psi \|_{L^2(Q)} \leq \| \psi \|_{[D(L_R), L^2(Q)]_{1/2}},
\]
(4.12)
i.e., \( \psi \in H^1_{R_0} \). Thus, we have proved that \([D(L_R), L^2(Q)] \subset H^1_{R_0} \), where the embedding is continuous.

It remains to prove that \([D(L_R), L^2(Q)]_{1/2} \subset H^1_{R,0} \). Let \( \varphi \in [D(L_R), L^2(Q)]_{1/2} \),
Then by Theorem 1.6 in [30], for any \( m \in \mathbb{N} \) there exists \( \varphi_{1m} \in D(L_R) \) such that
\[
\| \varphi - \varphi_{1m} \|_{[D(L_R), L^2(Q)]_{1/2}} < 1/m.
\]
(4.13)

By taking \((4.11)-(4.13)\) into account, we get
\[
\| \varphi - \varphi_{1m} \|_{H^1_{R_0}} \leq k_3/m,
\]
(4.14)
where \( k_3 > 0 \) does not depend on \( m \).

Since \( D(L_R) \subset H^1_{R,0} = H^1_{R,0} \), for any \( m \in \mathbb{N} \) there exists \( \varphi_{2m} \in C^\infty_0(Q) \) such that
\[
\| \varphi_{1m} - \varphi_{2m} \|_{H^1_{R_0}} < 1/m.
\]
(4.15)

From \((4.14)\) and \((4.15)\), it follows that
\[
\| \varphi - \varphi_{2m} \|_{H^1_{R_0}} < (k_3 + 1)/m.
\]
Hence, \( \varphi_{2m} \to \varphi \) in \( H^1_{R_0} \) as \( m \to \infty \). Therefore, \( \varphi \in H^1_{R,0} = H^1_{R_0} \). We have proved (4.6). The embedding (4.7) can be proved similarly. \( \square \)

**Example 4.1.** Consider the differential-difference operator \( L_R \) introduced in Example 3.1. It was shown in Example 3.1 that this operator satisfies conditions [14]. Denote by \( L_R^* \) the Friedrichs extension of this operator. By Theorem 4.1
\[
D((L_R^* + I)^{1/2}) = D((L_R^* + I)^{1/2}) = H^1_{R,0}.
\]
Hence, the operator \( L_R + I \) satisfies Kato’s conjecture.
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