Алгебра и анализ
Том 30 (2018), № 5

St. Petersburg Math. J.
Vol. 30 (2019), No. 5, Pages 869–875
https://doi.org/10.1090/spmj/1573
Article electronically published on July 26, 2019

AN ANALOG OF THE SOBOLEV INEQUALITY ON A STRATIFIED SET

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Abstract. On a set $\Omega \subset \mathbb{R}^d$ that is formed from a finite collection of manifolds that adjoin each other in a special way (i.e., on a stratified set) an exact analog of the Sobolev inequality is proved, in which the exponents are expressed via the characteristics of the intrinsic geometric structure of Ω. The result is applied in order to prove the solvability of the Dirichlet problem for the p-Laplacian.

§1. Stratified sets

A stratified set is defined to be a connected subset of the Euclidean space $\mathbb{R}^d$ that is the union of a finite family S of disjoint connected submanifolds σ_{kj} (without boundary) called strata in what follows:

$$\Omega = \bigcup_{\sigma_{kj} \in S} \sigma_{kj}.$$

Each stratum σ_{kj} has compact closure in $\mathbb{R}^d$, the first index in the stratum designation shows its dimension, and the second shows its number among the strata of the dimension determined by the first index. It is assumed that the strata adjoin each other by the type of a cell complex, that is, the boundary of each stratum consists of some strata of the family S and each intersection $\bar{\sigma}_{kl} \cap \bar{\sigma}_{mn}$ of the closures of the strata in $\mathbb{R}^d$ either is empty or is a union of some strata from S. Furthermore, the relation $\sigma_{kj} \succ \sigma_{ml}$ between two strata means that $\sigma_{ml} \subset \partial\sigma_{kj}$; in this case we say that these strata adjoin each other.

Besides the assumptions about the mutual order of the strata described above, we also impose restrictions concerning the smoothness of contiguity that are specific for stratified sets.

The traditional requirements called Whitney regularity conditions can be found in the book [1]. For our purposes a certain modification of these conditions will be convenient. Namely, for each pair of strata

$$\sigma_{kj}, \sigma_{mi} \succ \sigma_{kj}$$

and a point $X \in \sigma_{kj}$ we assume the existence of a ball $B_\epsilon(X)$ in $\mathbb{R}^d$ and a diffeomorphism Φ of the ball into a ball centered at $\Phi(X) = 0$ that transforms fragments of these strata that are inside $B_\epsilon(X)$ to a fragment of the wedge of dimension m and, respectively, to a fragment of a k-dimensional subspace (which is a subset of the wedge edge) that are in $\Phi(B_\epsilon(X))$ (local rectification). Here the wedge K is an open set in $\mathbb{R}^d$ that, with every pair of points, contains every their linear combination with positive coefficients, and its edge is the intersection $\bar{K} \cap (-\bar{K})$. On the following picture the image of the stratum σ_{kj}

2010 *Mathematics Subject Classification.* 35K35, 35A05.
Key words and phrases. Stratified set, Sobolev inequality.
The work is partially supported by the grant of the Ministry of Education and Science of Republic of Kazakhstan AP05130222.

has the same dimension as the edge of the wedge, but in general this is not necessarily the case.

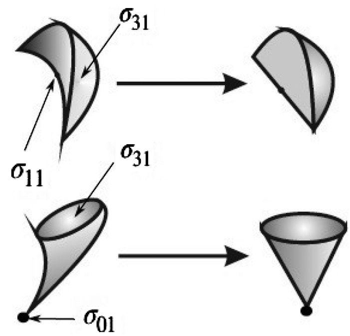

FIGURE 1. Local rectification.

Notice that the conditions formulated above guarantee that the cone condition is fulfilled on each stratum. This fact is used below in Lemmas 1 and 2.

We define a measure μ on Ω as follows. We say that a subset ω in Ω is μ-measurable if each intersection $\omega \cap \sigma_{kj}$ is measurable with respect to the k-dimensional Lebesgue measure on σ_{kj}. Obviously, all such subsets form a σ-algebra in Ω, and the function μ defined by the formula

$$\mu(\omega) = \sum_{\sigma_{kj} \in S} \mu_k(\sigma_{kj} \cap \omega), \tag{1}$$

in which μ_k is the usual k-dimensional Lebesgue measure on σ_{kj}, satisfies all properties of a measure. The Lebesgue intergal of μ-measurable functions f defined in the standard way reduces to the sum

$$\int_\omega f\, d\mu = \sum_{\sigma_{kj} \in S} \int_{\sigma_{kj} \cap \omega} f\, d\mu_k.$$

In what follows we will write μ instead of μ_k hoping that this will not lead to misunderstanding.

§2. SPACES OF FUNCTIONS ON A STRATIFIED SET

A set Ω is assumed to be represented as the union of two disjoint parts Ω_0 and $\partial\Omega_0 = \Omega \setminus \Omega_0$. Any open connected subset Ω is allowed to be taken as Ω_0 if it consists of some strata and the closure of Ω_0 coincides with Ω. Here and in what follows, all topological terms are related to the topology on Ω induced by the standard topology of the enclosing space $\mathbb{R}^d$. It is not difficult to observe that the subset $\partial\Omega_0$, which was defined formally, is in fact the boundary of Ω_0 in the specified topology. For instance, one can always take $\Omega_0 = \Omega$. In this paper this possibility is excluded. Thus, in the sequel we fix a nontrivial (that is, $\partial\Omega_0$ is not empty) representation of Ω as the pair $\{\Omega_0, \partial\Omega_0\}$. If Ω consists of more than one stratum, such a representation always exists.

Next, we give a list of spaces of functions that will be needed.

- $C(\Omega_0)$ is the set of all continuous functions $f \colon \Omega_0 \to \mathbb{R}$.
- $C_0(\Omega_0)$ is the subset of $C(\Omega_0)$ that consists of functions that vanish near the boundary $\partial\Omega_0$.
- $C^1(\Omega_0)$ is the set of functions in $C(\Omega_0)$ that have continuously differentiable restrictions to each stratum $\sigma_{kj} \subset \Omega_0$ (the restriction of f to the stratum σ_{kj} is denoted further by f_{kj}), and it is assumed that the gradient ∇f_{kj} admits

an extension by continuity to every stratum $\sigma_{k-1,l} \subset \Omega_0$ adjoining σ_{kj} whose dimension is less exactly by one, if it exists at all.

- $C_0^1(\Omega_0) = C^1(\Omega_0) \cap C_0(\Omega_0)$.
- $C^2(\Omega_0)$ is the subspace of $C^1(\Omega_0)$ that consists of all functions whose restrictions to every stratum from Ω_0 are twice continuously differentiable.
- $\mathring{W}_\mu^{1,p}(\Omega)$ (or $W_\mu^{1,p}(\Omega_0)$) is the Sobolev space with respect to the measure μ, it is defined as the completion of the space $C_0^1(\Omega_0)$ (respectively, of $C^1(\Omega_0)$) in the norm

$$\|u\|_{W_\mu^{1,p}} = \left(\int_{\Omega_0} |u|^p \, d\mu + \int_{\Omega_0} |\nabla u|^p \, d\mu \right)^{\frac{1}{p}}.$$

§3. Auxilliary inequalities

The key role in the proof of the Sobolev inequality on a stratified set is played by the inequalities from the two following lemmas.

Lemma 1. *Let $\sigma_{kj} \succ \sigma_{ml}$. If $k - m < p < k$ and $1 \le q \le \frac{mp}{k-p}$, or $p \ge k$ and $q \ge 1$, then for $u \in W_\mu^{1,p}(\Omega_0)$ the inequality*

$$\int_{\sigma_{ml}} |u|^q d\mu \le C \left[\left(\int_{\sigma_{kj}} |u|^p \, d\mu \right)^{q/p} + \left(\int_{\sigma_{kj}} |\nabla u|^p \, d\mu \right)^{q/p} \right] \tag{2}$$

is true with a constant C that does not depend on u. For $p = 1$ the inequality (2) *is fulfilled if $m = k - 1$ and $q = 1$.*

Our definition of the Sobolev space $W_\mu^{1,p}(\Omega_0)$ differs from the classical one, but it is not difficult to observe that the restriction of a function from this space to each stratum is a classical Sobolev function, and hence the inequality presented above does not require a proof; it can be found, e.g., in [2, Theorem 5.4] or [3, Subsection 2, §8, Chapter 1]. The cone condition, as remarked in §1, is fulfilled automatically.

The following assertion does not require a proof either; it is an easy consequence of Lemma 1 and a fact (the lemma about an equivalent normalization) that is presented, e.g., in [3, Section 9, Chapter 1], [4, Section 5, Chapter 5].

Lemma 2. *In the assumptions of Lemma* 1 *we have the inequality*

$$\int_{\sigma_{kj}} |u|^q d\mu \le C \left[\left(\int_{\sigma_{kj}} |\nabla u|^p \, d\mu \right)^{q/p} + \left(\int_{\sigma_{ml}} |u|^p \, d\mu \right)^{q/p} \right]. \tag{3}$$

§4. Statement and proof of the Sobolev inequality

The exponents of summability in the Sobolev inequality on a stratified set are determined by two numbers $d(\Omega)$ and $D(\Omega_0)$. The former, $d(\Omega)$, equals the maximal dimension of the strata in Ω. Now we pass to the definition of the latter.

Let $\sigma_{kj} \subset \Omega_0$. From our definitions it obviously follows that there exists of a connected chain of strata

$$\sigma_{k_1 j_1}, \sigma_{k_2 j_2}, \dots, \sigma_{k_m j_m} \tag{4}$$

satisfying the following requirements:

- $\sigma_{k_1 j_1} = \sigma_{kj}$;
- all strata except the last one are subsets of Ω_0, while $\sigma_{k_m j_m} \subset \partial\Omega_0$;
- for each index $i < m$ either $\sigma_{k_i j_i} \succ \sigma_{k_{i+1} j_{i+1}}$, or $\sigma_{k_{i+1} j_{i+1}} \succ \sigma_{k_i j_i}$.

The maximal of all numbers $|k_i - k_{i+1}|$ for the given chain will be called the number of connectedness of the chain, and the minimal number among the numbers of connectedness of all chains that connect σ_{kj} with a certain boundary stratum will be denoted by $D(\sigma_{kj})$. Finally, the number of connectedness $D(\Omega_0)$ of the pair $\{\Omega_0, \partial\Omega_0\}$ will be defined as the maximal number among the $D(\sigma_{kj})$ for all strata σ_{kj} in Ω_0. By virtue of this definition, for every stratum $\sigma_{kj} \subset \Omega_0$ there exists a connected chain joining it with the boundary, and whose number of connectedness does not exceed $D(\Omega_0)$.

Now we are ready to formulate the main result of this paper.

Theorem 1. *If $D(\Omega_0) < p < d(\Omega)$ and $1 \le q \le \frac{pd(\Omega)}{d(\Omega)-p}$, or $p \ge d(\Omega)$ and $1 \le q < \infty$, then*

$$\left(\int_\Omega |u|^q \, d\mu\right)^{1/q} \le C\left(\int_{\Omega_0} |\nabla u|^p \, d\mu\right)^{1/p} \tag{5}$$

for all $u \in \mathring{W}^{1,p}_\mu(\Omega)$ with constant C that does not depend on u. Moreover, the inequality (5) is fulfilled in the case where $D(\Omega_0) = p = 1$ and $1 \le q \le \frac{d(\Omega)}{d(\Omega)-1}$.

Remark 1. In the case of $p > d(\Omega)$, inequality (5) is true for $q = \infty$ if its left-hand side is viewed as the essential supremum with respect to the stratified measure μ.

Proof. The proof will be given in the case of $D(\Omega_0) < p < d(\Omega)$ and $1 \le q \le \frac{pd(\Omega)}{d(\Omega)-p}$, or $p \ge d(\Omega)$ and $1 \le q < \infty$, and we hope that the reader will easily be able to modify the arguments for the remaining cases.

Fix an arbitrary stratum $\sigma_{kj} \subset \Omega_0$ and some connected chain of strata (4) joining it with a certain boundary stratum. We can think that the dimensions of the adjacent strata differ by at most $D(\Omega_0)$.

By virtue of the classical embedding theorem for Sobolev spaces applied to the restriction of the function u to the stratum σ_{kj}, we have

$$\int_{\sigma_{k_1 j_1}} |u|^q d\mu \le C\left[\left(\int_{\sigma_{k_1 j_1}} |u|^p \, d\mu\right)^{q/p} + \left(\int_{\sigma_{k_1 j_1}} |\nabla u|^p \, d\mu\right)^{q/p}\right] \tag{6}$$

for q satisfying the inequalities $1 \le q \le \frac{k_1 p}{k_1 - p}$ in the case of $p < k_1$, or the inequalities $p \le q < \infty$ in the case where $p \ge k_1$. In order to avoid misunderstandings, recall that $\sigma_{k_1 j_1} = \sigma_{kj}$ (see the definition of the chain (4)).

On the right-hand side, we estimate the first summand containing $|u|^p$. In the case of $k_2 > k_1$, on the basis of inequality (2) of Lemma 1 (for the exponent q on the left-hand side equal to p), we conclude that

$$\int_{\sigma_{k_1 j_1}} |u|^p d\mu \le C_1\left[\int_{\sigma_{k_2 j_2}} |u|^p \, d\mu + \int_{\sigma_{k_2 j_2}} |\nabla u|^p \, d\mu\right]. \tag{7}$$

The applicability of this lemma is guaranteed by the inequality $p > D(\Omega_0)$. If $k_2 < k_1$ (notice that the coincidence of the dimensions of the adjacent elements of the chain is impossible because the strata can adjoin each other only on the boundary of one of them), then from inequality (3) of Lemma 2 we obtain

$$\int_{\sigma_{k_1 j_1}} |u|^p d\mu \le C_1\left[\int_{\sigma_{k_2 j_2}} |u|^p \, d\mu + \int_{\sigma_{k_1 j_1}} |\nabla u|^p \, d\mu\right]. \tag{8}$$

Substituting (7) into (6), we arrive at the estimate

$$\int_{\sigma_{k_1 j_1}} |u|^q d\mu \le C_2\left[\left(\int_{\sigma_{k_2 j_2}} |u|^p \, d\mu + \int_{\sigma_{k_2 j_2}} |\nabla u|^p \, d\mu\right)^{q/p} + \left(\int_{\sigma_{k_1 j_1}} |\nabla u|^p \, d\mu\right)^{q/p}\right]$$

with constant a C_2 that can be expressed via C a C_1 in an obvious way. Using the inequality

$$(a+b)^\alpha \le 2^{\alpha-1}(a^\alpha + b^\alpha), \quad \text{where} \quad \alpha = \frac{q}{p} \ge 1,$$

we can reshape the last inequality to the form

$$\int_{\sigma_{k_1j_1}} |u|^q\, d\mu \le C_3\left[\left(\int_{\sigma_{k_2j_2}} |u|^p\, d\mu\right)^{q/p} + \left(\int_{\sigma_{k_2j_2}} |\nabla u|^p\, d\mu\right)^{q/p} + \left(\int_{\sigma_{k_1j_1}} |\nabla u|^p\, d\mu\right)^{q/p}\right].$$

Similarly, this inequality can be obtained from (8).

The rest of the proof is almost obvious; moving along the chain, one should deal with the integral of $|u|^p$ that appeared every time on the previous step, which will give us the integral of the same function on the next stratum of the chain instead. Finally, we arrive at the inequality of the form (recall that $\sigma_{k_1j_1} = \sigma_{kj}$)

$$\int_{\sigma_{kj}} |u|^q d\mu \le C_{kj}\left[\left(\int_{\sigma_{k_mj_m}} |u|^p\, d\mu\right)^{q/p} + \sum \left(\int_{\sigma_{k_ij_i}} |\nabla u|^p\, d\mu\right)^{q/p}\right].$$

Since $\sigma_{k_mj_m}$ lies on the boundary $\partial\Omega_0$, taking into account the fact that u vanishes on $\partial\Omega_0$ (in the sense that $u \in \overset{\circ}{W}{}^{1,p}_{\mu}(\Omega)$), applying the inequality $\sum(a_i^\alpha) \le (\sum a_i)^\alpha$ (with $\alpha = q/p$) and extending summation from the strata of the chain to all strata in Ω_0, we obtain

$$\int_{\sigma_{kj}} |u|^q\, d\mu \le C_{kj} \sum \left(\int_{\sigma_{k_ij_i}} |\nabla u|^p\, d\mu\right)^{q/p} \le C_{kj}\left(\int_{\Omega_0} |\nabla u|^p\, d\mu\right)^{q/p}.$$

Rewrite the last inequality in the form

$$\left(\int_{\sigma_{kj}} |u|^q\, d\mu\right)^{1/q} \le \widehat{C}_{kj}\left(\int_{\Omega_0} |\nabla u|^p\, d\mu\right)^{1/p} \tag{9}$$

and recall that the range for q is bounded from above by the number $\frac{kp}{k-p}$ in the case of $p < k$, or is unbounded (otherwise).

Summing up the inequalities of the type (9) over all strata σ_{kj} in Ω_0, we arrive at inequality (5). Moreover, $1 \le q < \infty$ if p does not exceed the dimension of each stratum from Ω_0; otherwise,

$$1 \le q \le \min \frac{kp}{k-p} = \frac{pd(\Omega)}{d(\Omega) - p},$$

where the minimum is taken over all strata σ_{kj} satisfying the inequality $k > p$. □

§5. Application to the p-Laplacian on the stratified set

5.1. Divergence and Laplacian. Here we define the analogs of the standard classical operators, divergence and Laplacian, on the stratified set. A more detailed discussion of these notions can be found in the last chapter of the book [5].

We begin with the definition of the divergence operator on tangent vector fields on Ω_0. A vector field $\vec{F}$ on Ω_0 is an arbitrary mapping $F\colon \Omega_0 \to \mathbb{R}^d$. Recall that $\mathbb{R}^d$ is the enclosing Euclidean space containing the stratified set. The vector field $\vec{F}$ is said to be tangent to Ω_0 if for any stratum $\sigma_{kj} \subset \Omega_0$ and a point $X \in \sigma_{kj}$ we have $F(X) \in T_X\sigma_{kj}$. Here $T_X\sigma_{kj}$ is the notation for the usual tangent space to σ_{kj} at the point X.

As in the classical case, we define the divergence of the tangent vector field $\vec{F}$ at the point X as the density of the flow generated by this field with respect to the stratified measure μ. For this aim, take a certain smooth closed surface S in $\mathbb{R}^d$ encircling X and transversal to the strata from Ω_0, and denote by $\partial\omega$ the intersection of this surface with

Ω_0, accepting the notation ω for the part of Ω_0 located inside S. The set $\partial\omega$ itself is a stratified set if its strata are declared to be the intersections of the strata of Ω_0 with the surface S. Define the flow $\Phi_{\vec{F}}(\partial\omega)$ of the tangent vector field $\vec{F}$ through $\partial\omega$ as the sum of the flows $\vec{F}$ through the strata of $\partial\omega$. Associating the flow $\Phi_{\vec{F}}(\partial\omega)$ with the measure ω, we can define the divergence as the limit of the ratio:

$$(\nabla \cdot \vec{F})(X) = \lim_{\omega \to X} \frac{\Phi_{\vec{F}}(\partial\omega)}{\mu(\omega)}.$$

Here the notation $\omega \to X$ means that ω constringes to the point X. It is not difficult to show that for sufficiently smooth fields the divergence is expressed by the following formula (more details can be found in [5]):

$$\nabla \cdot \vec{F}(X) = \nabla_k \cdot \vec{F}(X) + \sum_{\sigma_{k+1,i} \succ \sigma_{kj}} F(X + 0 \cdot \nu_i) \cdot \nu_i. \tag{10}$$

Here ∇_k denotes the classical k-dimensional divergence on σ_{kj}, $F(X + 0 \cdot \nu_i)$ is the extension by continuity to the point X of the restriction of $\vec{F}$ to the stratum $\sigma_{k+1,i}$, and $\vec{\nu}_i$ is the unit normal vector to σ_{kj} at the point X directed inside of $\sigma_{k+1,i}$. Denote by $\vec{C}^1(\Omega_0)$ the set of tangent vector fields $\vec{F}$ that are continuously differentiable in each stratum σ_{kj} from Ω_0 and admit an extension by continuity from σ_{kj} to each stratum $\sigma_{k-1,l} \subset \Omega_0$ adjacent to σ_{kj} and whose dimension is less exactly by one, if such strata exist at all. For vector fields $\vec{F} \in \vec{C}^1(\Omega_0)$ we accept formula (10) as a definition, and the preceding arguments were presented to show that this definition is natural. In particular, the gradient of a function $u \in C^2(\Omega_0)$ belongs to $\vec{C}^1(\Omega_0)$, which allows us to define the Laplacian $\Delta u = \nabla \cdot (\nabla u)$ for such functions, as well as the more general operator $L_q(u) = \nabla \cdot (q\nabla u)$ for $q\nabla u \in \vec{C}^1(\Omega_0)$.

One can show (see details in [5]) that the following analog of the second Green formula is true:

$$\int_{\Omega_0} vL_q(u)\,d\mu = -\int_{\Omega_0} q\nabla v \cdot \nabla u\,d\mu \tag{11}$$

for $v \in C_0^1(\Omega_0)$ and $u \in C^2(\Omega_0)$. In the next subsection we apply this formula for defining an analog of the p-Laplacian on the set Ω_0.

5.2. Solvability of the Dirichlet problem for the p-Laplacian. We begin with the definition of the p-Laplacian on a stratified set. Like in the classical case, the basis of this definition is the following variational problem:

$$\Phi(u) = \int_{\Omega_0} |\nabla u|^p\,d\mu \longrightarrow \min,$$

in which the minimum is searched on the set of functions $C^1(\Omega_0)$ satisfying the Dirichlet condition $u\big|_{\partial\Omega_0} = \phi$; we denote this set by $C_\phi^1(\Omega_0)$. If a function $u \in C_\phi^1(\Omega_0)$ minimizes the functional Φ, then

$$\int_{\Omega_0} |\nabla u|^{p-2}\nabla u \cdot \nabla h\,d\mu = 0 \tag{12}$$

for any function $h \in C_0^1(\Omega_0)$. Putting $v = h$ and $q = |\nabla u|^{p-2}$ in the Green formula (11), we obtain

$$\int_{\Omega_0} \nabla \cdot (|\nabla u|^{p-2}\nabla u)h\,d\mu = 0,$$

again for all functions $h \in C_0^1(\Omega_0)$. From this we obtain the analog of the Euler–Lagrange equation

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$$

for functions $u \in C^2(\Omega_0)$. Clearly, the outer symbol ∇ denotes the operator of "stratified" divergence.

The obtained analog of the Sobolev inequality allows us to easily ensure the weak solvability of the following boundary problem with p-Laplacian:

$$\begin{aligned} \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \\ u\big|_{\partial\Omega_0} = \phi. \end{aligned} \tag{13}$$

The weak solution of the problem (13) is regarded as the function u satisfying (12) for all $h \in C_0^1(\Omega)$ and such that $u - \phi \in \mathring{W}_\mu^{1,p}(\Omega)$. We suppose that ϕ belongs to $W_\mu^{1,p}(\Omega)$.

Theorem 2. *Under the assumption $p > D(\Omega_0)$, problem* (13) *is weakly solvable.*

As we see, by using our terminology (stratified measure, stratified divergence, etc.), we managed to reduce the statement of the problem to the standard form, which, if we recall that we have an analog of the Sobolev inequality, makes standard also the proof of the above theorem (details can be found, e.g., in [4, §3, Chapter 7]). The statement of the theorem contains only the number $D(\Omega_0)$ of global connectedness of the pair $\{\Omega_0, \partial\Omega_0\}$, because in the proof we use only the partial case of the Sobolev inequality corresponding to $p = q$.

Acknowledgment

The authors are grateful to A. I. Nazarov for fruitful discussions on topics considered in this paper.

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Received 14/JUN/2017

Translated by V. V. KAPUSTIN