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RECENT TOPICS OF ARRANGEMENTS OF HYPERPLANES

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1. Introduction

First of all, we define an arrangement of hyperplanes.

Definition 1.1 (**Arrangement of hyperplanes**). Let V be an ℓ-dimensional vector space over a field $\boldsymbol{K}$. An affine subspace of dimension $(\ell - 1)$ in V is called a **hyperplane**. An **arrangement of hyperplanes** is a finite set of hyperplanes.

In this article, a field $\boldsymbol{K}$ is $\boldsymbol{R}$ or $\boldsymbol{C}$ in most of the cases.

For example, a set of finite points on a one-dimensional real line and a set of finite points on a one-dimensional complex plane are the simplest examples of an arrangement of hyperplanes. Therefore, "the planting-tree arithmetic" that n points on a line divide the line into $(n + 1)$ intervals and $(n - 1)$ of them are finite intervals is a result belonging to the theory of arrangement of hyperplanes. (Actually, as mentioned later, T. Zaslavsky generalized "the planting-tree arithmetic" to the higher dimensional cases in 1975.) In this sense, the history of arrangement of hyperplanes is as old as the recorded history itself. In this article, we will present the modern study of arrangement of hyperplanes by dividing it into the following three areas:

(A) Free arrangements and free multiarrangements (including Coxeter arrangements),

(B) Topology of complex complements,

(C) Hypergeometric integrals associated with an arrangement.

We can safely say that almost all the recent research topics related to arrangements of hyperplanes are in these three areas. We would like to convey to the readers the interesting relationships among them. Tracing these three areas back to the 1970s, we arrive at three Japanese mathematicians' (**Kazuhiko Aomoto, Akio Hattori, Kyoji Saito**) pioneering achievements.

The organization of this article is as follows: First, the situation of the research in arrangement of hyperplanes in the 1970s and these three Japanese mathematicians' works are described in Chapter 2. In Chapter 3, we describe (A) free arrangements and free multiarrangements in detail. The Coxeter arrangements, which are representative free arrangements of hyperplanes, are also dealt with in Chapter 3. In Chapter 4, we argue (B) topology of complex complements. Especially, we explain the "combinatorial determinativeness" and the "minimality". The theory of Aomoto's hypergeometric functions which belongs to (C) is closely related with the local system cohomology and homology on complex complements. We describe principal results derived from the theory in Chapter 5. We give the readers fair

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warning here: since it is impossible to describe the above three areas exhaustively due to limitations of the space and our ability, the topic (A) will be the main theme of this article. For the topics we do not cover in this article, especially the topics prior to the 1990s, the readers are advised to see [51, 52] among others.

2. The 1970s

The modern study of arrangement of hyperplanes began with focusing on its combinatorial properties first and abstracting them. To put it concretely, it was the use of the Möbius function on the intersection poset. G.-C. Rota [56] was the first person to define and study the Möbius function on a poset. In 1975, T. Zaslavsky [92] generalized "the planting-tree arithmetic" by using the Möbius function as the following Theorem 2.2.

Definition 2.1 (**Intersection poset**)**.** Let $\mathcal{A}$ be an arrangement of hyperplanes in an ℓ-dimensional vector space V. Define the **intersection poset** by

$$L(\mathcal{A}) := \left\{ \bigcap_{H\in\mathcal{B}} H \ \mid\ \mathcal{B} \subseteq \mathcal{A}, \bigcap_{H\in\mathcal{B}} H \neq \emptyset \right\},$$

where we agree that $V = \bigcap_{H\in\emptyset} H$ when $\mathcal{B}$ is the empty set. Define a **partial order** on $L(\mathcal{A})$ by

$$Y_1 \geq Y_2 \Longleftrightarrow Y_1 \subseteq Y_2.$$

The **Möbius function** μ is defined as a function satisfying

$$\mu : L(\mathcal{A}) \longrightarrow \boldsymbol{Z}, \qquad \mu(V) = 1, \qquad \mu(X) = -\sum_{\substack{Y\in L(\mathcal{A})\\ Y<X}} \mu(Y).$$

(Then $\mu(H) = -1$ for $H \in \mathcal{A}$.) Furthermore, we define the **rank function**

$$r : L(\mathcal{A}) \longrightarrow \boldsymbol{Z}_{\geq 0}$$

by $r(Y) = \operatorname{codim}_V Y$. (Then $r(V) = 0$ and $r(H) = 1$ for $H \in \mathcal{A}$.) Call $r(\mathcal{A}) = \max_{X\in L(\mathcal{A})} r(X)$ the **rank** of $\mathcal{A}$. The **Poincaré polynomial** $\pi(\mathcal{A}, t)$ is defined by

$$\pi(\mathcal{A}, t) := \sum_{X\in L(\mathcal{A})} \mu(X)\ (-t)^{r(X)} = \sum_{X\in L(\mathcal{A})} |\mu(X)|\ t^{r(X)}.$$

(It is known that the sign of $\mu(X)$ is equal to $(-1)^{r(X)}$.) Although it is essentially the same as the Pouncaré polynomial, the following polynomial $\chi(\mathcal{A}, t)$ is called the **characteristic polynomial** of $\mathcal{A}$:

$$\chi(\mathcal{A}, t) := \sum_{X\in L(\mathcal{A})} \mu(X)\ t^{\dim X}.$$

All of the objects defined in Definition 2.1 above depend only on $L(\mathcal{A})$. In this sense, they are combinatorial objects. From now on, express the **complement** of $\mathcal{A}$ as

$$M(\mathcal{A}) := V \setminus \bigcup_{H\in\mathcal{A}} H.$$

When $V = \boldsymbol{C}^\ell$, $\pi(\mathcal{A}, t)$ is equal to the Poincaré polynomial of $M(\mathcal{A})$ (Orlik-Solomon [50]). When $V = \boldsymbol{R}^\ell$, $M(\mathcal{A})$ is disconnected (except the case that $\mathcal{A}$ is empty) and decomposed into finite connected components. They are called the **chambers**.

Express the set of chambers by $\mathcal{C}(\mathcal{A})$. There are two types of chambers, namely a bounded (finite volume) chamber and an unbounded chamber. The set of bounded chambers is denoted by $\mathbf{b}\mathcal{C}(\mathcal{A})$.

Theorem 2.2 (T. Zaslavsky [92])**.** *Let $V = \boldsymbol{R}^\ell$. Then the number of chambers and bounded chambers are given by the following formulas:*

$$|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1) = \sum_{X \in L(\mathcal{A})} |\mu(X)|, \quad |\mathbf{b}\mathcal{C}(\mathcal{A})| = |\pi(\mathcal{A}, -1)| = \left| \sum_{X \in L(\mathcal{A})} \mu(X) \right|.$$

We can easily prove Theorem 2.2 by considering a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ defined as follows. The idea of a triple is useful.

Definition 2.3 (**Triple**)**.** Let $\mathcal{A}$ be a non-empty arrangement of hyperplanes in V. Fix $H_0 \in \mathcal{A}$. Put

$$\mathcal{A}' := \mathcal{A} \setminus \{H_0\}, \quad \mathcal{A}'' := \{H_0 \cap K \mid K \in \mathcal{A}', H_0 \cap K \neq \emptyset\};$$

then $\mathcal{A}'$ and $\mathcal{A}''$ are arrangements of hyperplanes in V and H_0 respectively. We call $\mathcal{A}'$ the **deletion** of $\mathcal{A}$ (with respect to H_0) and $\mathcal{A}''$ the **restriction** to H_0. We sometimes express $\mathcal{A}''$ as $\mathcal{A}^{H_0}$ with specifying H_0.

There are many formulas for a triple. For example

$$\begin{aligned} M(\mathcal{A}') &= M(\mathcal{A}) \cup M(\mathcal{A}'') \quad \text{(disjoint)}, \\ \pi(\mathcal{A}, t) &= \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t), \\ |\mathcal{C}(\mathcal{A})| &= |\mathcal{C}(\mathcal{A}')| + |\mathcal{C}(\mathcal{A}'')| \quad (\text{when } V = \boldsymbol{R}^\ell), \\ |\mathbf{b}\mathcal{C}(\mathcal{A})| &= |\mathbf{b}\mathcal{C}(\mathcal{A}')| + |\mathbf{b}\mathcal{C}(\mathcal{A}'')| \quad (\text{when } V = \boldsymbol{R}^\ell), \end{aligned}$$

etc. Here note that $M(\mathcal{A}'')$ is the complement in H_0. By these formulas, we can verify Theorem 2.2 using an induction on $|\mathcal{A}|$. In the case of dimension one, since $\pi(\mathcal{A}, t) = 1 + |\mathcal{A}|t$, Theorem 2.2 is nothing but "the planting-tree arithmetic".

Around 1975 when Theorem 2.2 was published, the "modern study" of arrangements of hyperplanes was initiated in a broad context of fields such as algebraic geometry, topology, and analysis (hypergeometric integrals). The time was ripe, and the "modern study" of arrangement of hyperplanes began.

Kazuhiko Aomoto started the full-scale study of the multivariable hypergeometric functions in his paper [12] in 1973. It was the first paper about what is called Aomoto-Gelfand's theory nowadays. "Un théorème du type de Matsushima-Murakami" appearing in the title of this paper is a vanishing theorem for a cohomology with coefficients in a local system[1] on a certain kind of symmetric space. In this paper, Aomoto proved the vanishing theorem for a cohomology with coefficients in a local system on the complement $M(\mathcal{A})$ of an arrangement of hyperplanes $\mathcal{A}$ of rank ℓ in an ℓ-dimensional affine complex space. Namely, for a generic local system $\mathcal{L}$, any $H^p(M(\mathcal{A}), \mathcal{L})$ is equal to zero if $p \neq \ell$. Then the dimension of $H^\ell(M(\mathcal{A}), \mathcal{L})$ is equal to the absolute value of the Euler characteristic of $M(\mathcal{A})$, that is, $\pi(\mathcal{A}, -1)$. (In the case that $\mathcal{A}$ is defined on the real number field, the number is equal to $|\mathbf{b}\mathcal{C}(\mathcal{A})|$ by Theorem 2.2.) The reason why the vanishing theorem is related to the hypergeometric functions or the hypergeometric integrals is that

[1] See [42], [26] for the general theory of local system.

Aomoto's hypergeometric integrals are defined by the twisted de Rham pairing of the local system cohomology and the local system homology

$$H^p(M(\mathcal{A}), \mathcal{L}) \times H_p(M(\mathcal{A}), \mathcal{L}^\vee) \longrightarrow \boldsymbol{C}$$

($\mathcal{L}^\vee$ is the dual system of $\mathcal{L}$). In the case of Gaussian hypergeometric functions which are the prototypes of hypergeometric functions, the corresponding arrangement of hyperplanes $\mathcal{A}$ is the three hyperplanes (points) $0, 1, x^{-1}$ ($x \notin \{0, 1, \infty\}$) in $\boldsymbol{C}$. Since $\chi(M(\mathcal{A})) = -2$, $H^0(M(\mathcal{A}), \mathcal{L}) = 0$ and $\dim H^1(M(\mathcal{A}), \mathcal{L}) = 2$ when $\mathcal{L}$ is generic. A basis for the cohomology $H^1(M(\mathcal{A}), \mathcal{L})$ can be given as two rational 1-forms with poles at $0, 1, x^{-1}$ by the twisted de Rham correspondence, and a basis for the homology $H_1(M(\mathcal{A}), \mathcal{L}^\vee)$ can be given as two twisted cycles. Studies of arrangement of hyperplanes in the framework above (for example, the study of the Aomoto complex or general conditions for the vanishing theorem to hold) has been active since Aomoto. We will describe a part of the study in Chapter 5.

On the other hand, in the paper [41] in 1975, **Akio Hattori** studied in depth the complement of an arrangement of hyperplanes in general position in $\boldsymbol{C}^\ell$, and explicitly constructed a homotopy equivalent complex by gluing tori. By the construction, he verified the complex complement is not $K(\pi, 1)$ and that the fundamental group is abelian. Such an explicit construction was a harbinger of the Salvetti complex by Salvetti [63]. As mentioned clearly in [41], this study was inspired by the study of the complement of an arrangement of hyperplanes in general position which appeared in the study of Aomoto's hypergeometric integrals. The mutual exchanges between analysis and topology like this are innumerable in the history of the study of arrangement of hyperplanes.

Here, turning the clock back, we mention the following important topological studies, from the 1960s and 1970s, of the complement of a complex reflection arrangement such as the Coxeter arrangement. As for the complement of the complex Coxeter arrangement of the type A, Fadell and others verified in [36] that it is fiber type (as defined later), therefore $K(\pi, 1)$. Moreover, Arnold [15] showed its Poincaré polynomial factors over the integers as $(1+t)(1+2t)\dots(1+(\ell-1)t)$. After that, Brieskorn [20] generalized the result by Arnold to Coxeter arrangements of the types other than the type A. These factorizations are the origin of the factorization theorem (Theorem 3.4). As for the $K(\pi, 1)$ property of the complex complement of a real arrangement of hyperplanes, Deligne's result [27] is especially important.

Also in the 1970s, **Kyoji Saito** introduced two important modules for the divisors which are studied in algebraic geometry, that is, the module of logarithmic differential 1-forms with poles along the divisor and the module of logarithmic vector fields tangent to the divisor. Although the comprehensive theory was published in [59] in 1980, the study itself dates back to the 1970s (for example, [58]). In this article, we give the definition restricted to the case of arrangements of hyperplanes (in algebraic category).

Definition 2.4 (**Logarithmic differential form and logarithmic vector field**). Let $\mathcal{A}$ be an arrangement of hyperplanes in V. Let S be an algebra of polynomial functions over V. The **defining polynomial** of $\mathcal{A}$ is given by

$$Q := \prod_{H \in \mathcal{A}} \alpha_H \in S$$

(where $\alpha_H = 0$ is an equation defining H). Then we define the following three S-modules:

$$\Omega^p(\mathcal{A}) := \{\omega \mid \omega \text{ is a rational differential } p\text{-form, } Q\omega \text{ and } Q(d\omega) \text{ are both regular}\}, \quad (1 \le p \le \ell)$$

$$\mathrm{Der}_V := \{\delta \mid \delta : S \longrightarrow S \text{ is a } \mathbf{K}\text{-linear derivation}\},$$

$$D(\mathcal{A}) := \{\delta \in \mathrm{Der}_V \mid \delta(\alpha_H) \in \alpha_H S \quad (\forall H \in \mathcal{A})\}.$$

An element of $\Omega^p(\mathcal{A})$ is called a **logarithmic differential p-form** with poles along $\mathcal{A}$, and an element of $D(\mathcal{A})$ is called a **logarithmic derivation** or a **logarithmic vector field**.

Definition 2.5 (**Free arrangement of hyperplanes**)**.** We say that an arrangement of hyperplanes $\mathcal{A}$ is a **free arrangement of hyperplanes**, or simply a **free arrangement**, when $\Omega^1(\mathcal{A})$ is a free S-module. Moreover, when we replace $\Omega^1(\mathcal{A})$ by $D(\mathcal{A})$, we still get an equivalent definition.

The Coxeter arrangements are the most important examples of free arrangements of hyperplanes. We will discuss them in detail in Section 3.6. The free arrangements are special kinds of free divisors. The theory of the free divisors is a by-product of Kyoji Saito's theory of the primitive forms. It was shown that the discriminant of the base space of a general family of semi-universal deformations of an isolated hypersurface singularity is a free divisor, and the theory of primitive forms [60] was developed.

3. Free arrangement and free multiarrangement

The freeness of an arrangement of hyperplanes sets strict limitations on the characteristic polynomials and the number of chambers, because of the factorization theorem (Theorem 3.4) and Zaslavsky's formula (Theorem 2.2). In other words, the "freeness" attaches algebraic and geometric meanings to the characteristic polynomials and the number of chambers. Thus we may say that the combinatorial structure of an arrangement of hyperplanes is controlled by the "freeness". Furthermore, the concept of freeness gives a nice framework in which the algebraic and geometric methods are applicable to the pure combinatorial problems. For example, the settlement of Edelman-Reiner's conjecture [33, 86] about the freeness of the generalized Catalan arrangements and the generalized Shi arrangements is positioned in such a framework. The combinatorial results about the characteristic polynomial and the number of chambers obtained in [86] contain results which cannot be proved by only purely combinatorial consideration at this writing.

In this chapter, we will survey the flow of research activities which started from the day when the study of free arrangement of hyperplanes was born to this day via the settlement of Edelman-Reiner's conjecture.

3.1. **Early studies.**
The following criterion for the freeness of $D(\mathcal{A})$ is well-known:

Theorem 3.1 (**K.Saito** [59])**.** *Let $\delta_1, \dots, \delta_\ell \in D(\mathcal{A})$, and put the vector fields $\delta_i = \sum_{j=1}^{\ell} f_{ij}(\partial/\partial x_j)$. Then $D(\mathcal{A}) = S \cdot \delta_1 \oplus \cdots \oplus S \cdot \delta_\ell$ if and only if* $\det(f_{ij})$ *is a non-zero constant multiple of Q.*

This theorem is called "Saito's criterion". It is a nice criterion in the sense that it immediately determines whether a given candidate of a basis is actually a basis or not. Even if we do not have any basis candidate, the criterion is still important theoretically.

We say that an arrangement of hyperplanes is **central** when every hyperplane $H \in \mathcal{A}$ passes through the origin $0 \in V$, namely the defining polynomial Q is a homogeneous polynomial. Then the Euler vector field $E = \sum_{i=1}^{\ell} x_i(\partial/\partial x_i) \in D(\mathcal{A})$ always belongs to the module of logarithmic vector fields. In Chapter 3, we assume that all arrangements are central. When a central arrangement $\mathcal{A}$ is free, $D(\mathcal{A})$ has a basis consisting of homogeneous vector fields[2] $\delta_1, \dots, \delta_\ell$. Then the tuple of the degrees of a basis $(\deg \delta_1, \dots, \deg \delta_\ell)$ is called the **exponents** denoted by $\exp(\mathcal{A})$. The exponents are independent of the choice of a basis. Furthermore, as mentioned later, if we assume the arrangement $\mathcal{A}$ is free, then the exponents are determined combinatorially (depending only on the structure of $L(\mathcal{A})$).

The freeness of an arrangement of hyperplanes is defined algebraically, however we can sometimes show the freeness by a combinatorial way. In combinatorial arguments of an arrangement of hyperplanes, focusing on the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a typical method. The following theorem is called the **Addition-Deletion theorem** which describes how the freeness of the triple behaves.

Theorem 3.2 ([75]). *Any two of the following statements with respect to the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ imply the third:*

(i) *$\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$.
(ii) *$\mathcal{A}'$ is free with* $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-1}, d_\ell - 1)$.
(iii) *$\mathcal{A}''$ is free with* $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-1})$.

(This formulation is given by P. Cartier [21]*.)*

An arrangement is called **inductively free** if it can be proved to be free by inductively applying Theorem 3.2. More accurately, we define the set of inductively free arrangements as the smallest class $\mathcal{IF}$ of arrangements of hyperplanes satisfying the following properties:

(1) the empty arrangement $\mathcal{A} = \emptyset$ is contained in $\mathcal{IF}$ for any dimension,
(2) if $\mathcal{A}', \mathcal{A}'' \in \mathcal{IF}$ and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A} \in \mathcal{IF}$.

Many free arrangements are inductively free, but there exist free arrangements which are not inductively free. Combining Theorem 3.2 and the inductive formulas of characteristic polynomial, $\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(t - d_i)$ holds for an inductively free arrangement $\mathcal{A}$.

The **fiber type** arrangements form an important class from the topological viewpoint. Let $X \in L(\mathcal{A})$ be a subspace satisfying $r(X) = r(\mathcal{A}) - 1$. Put $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$. Then we say that $\mathcal{A}$ **has a fiber structure** in X-direction when the projection to X-direction

$$M(\mathcal{A}) \longrightarrow M(\mathcal{A}_X/X)$$

is a topological fiber bundle (the fiber is $\boldsymbol{K}(\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C})$ minus $|\mathcal{A} \setminus \mathcal{A}_X|$ points). Here $\mathcal{A}_X/X$ is the arrangement of hyperplanes in V/X defined by $\mathcal{A}_X$. The above condition is equivalent to the combinatorial condition that there exists a hyperplane $H \in \mathcal{A}$ including $X \cup Y$ for any $Y \in L(\mathcal{A})$, $\operatorname{codim} Y \geq 2$. We say that $\mathcal{A}$ is **fiber type** when there exists a sequence (flag) of subspaces $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq$

[2] A vector field $\delta = \sum_i f_i(\partial/\partial x_i)$ is homogeneous if the coefficients f_i are homogeneous polynomial and $\deg f_1 = \cdots = \deg f_\ell$. Moreover, define the degree of δ by $\deg \delta = \deg f_i$. This integer is larger than the degree as a differential operator by one.

$X_{r-1} \subsetneq X_r = V$ ($r = \operatorname{rank} \mathcal{A}$) such that $\mathcal{A}_{X_k}/X_k$ has a fiber structure in X_{k+1}/X_k-direction. This implies that $\mathcal{A}$ can be constructed as a pagoda of fiber structures. The fiber type arrangements were introduced from a geometric viewpoint. It was, however, shown later ([78]) that $\mathcal{A}$ is fiber type if and only if the intersection poset $L(\mathcal{A})$ is a supersolvable lattice ([70, 71]) introduced by Stanley. The fiber type arrangements form an important class of free arrangements. It was shown that the fiber type arrangements are inductively free hence they are free arrangements. Also the complements of complex fiber-type arrangements are $K(\pi, 1)$ spaces because of their fiber structures and their fundamental groups are (iterated) semi-direct products of free groups.

Example 3.3. Let $\mathcal{A}(A_{\ell-1})$ be the arrangement of hyperplanes defined by the polynomial $Q(x_1, \ldots, x_\ell) = \prod_{1 \le i < j \le \ell}(x_j - x_i)$. This arrangement is called the Coxeter arrangement of the type $A_{\ell-1}$ and is a fiber type arrangement. In fact, the complement of $\mathcal{A}(A_{\ell-1})$ is

$$M(\mathcal{A}(A_{\ell-1})) = \{(x_1, \ldots, x_\ell) \in \boldsymbol{K}^\ell \mid x_i \neq x_j\},$$

hence if we put $X = \{x_2 = x_3 = \cdots = x_\ell\}$, then the projection to X-direction is given as

$$M(\mathcal{A}(A_{\ell-1})) \longrightarrow M(\mathcal{A}(A_{\ell-2})) : \ (x_1, x_2, \ldots, x_\ell) \longmapsto (x_2, \ldots, x_\ell).$$

This map is a fiber bundle whose fiber is $\boldsymbol{K} \setminus \{x_2, \ldots, x_\ell\}$ and has a fiber structure to the X-direction. We can verify that $\mathcal{A}(A_{\ell-1})$ is fiber type by the induction on the dimension ℓ. Therefore it is an inductively free arrangement, in particular, it is a free arrangement. We can explicitly construct a basis for the module of logarithmic vector fields $D(\mathcal{A}(A_{\ell-1}))$ as follows. Put $\delta_k = \sum_{i=1}^{\ell} x_i^k(\partial/\partial x_i)$, and $\delta_0, \delta_1, \ldots, \delta_{\ell-1} \in D(\mathcal{A}(A_{\ell-1}))$. Since the determinant of the coefficient matrix is equal to Q (Vandermonde's formula), it follows from Saito's criterion that $\mathcal{A}(A_{\ell-1})$ is a free arrangement.

Even if an arrangement is fiber type or inductively free, it is hard in general to construct an explicit basis for $D(\mathcal{A})$. However, for the Coxeter arrangements including the type $A_{\ell-1}$, we can construct a basis by an invariant-theoretical method (Section 3.6).

Finally we will explain the factorization theorem of characteristic polynomials for free arrangements.

Theorem 3.4 ([76])**.** *Let $\mathcal{A}$ be a free arrangement with* $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$*. Then the characteristic polynomial of $\mathcal{A}$ factors into linear polynomials over $\boldsymbol{Z}$ as*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(t - d_i).$$

The theorem asserts that the exponents of a free arrangement $\mathcal{A}$ are the roots of the characteristic polynomial. Thus the exponents of a free arrangement are combinatorially determined. Later, this result was generalized to the formula (Solomon-Terao's formula [68]) which describes the characteristic polynomial $\chi(\mathcal{A}, t)$ by using the Hilbert series of $\Omega^p(\mathcal{A})$ ($p = 0, 1, \ldots, \ell$).

As Theorem 3.4 above shows, the freeness of $\mathcal{A}$ imposes strong constraints on the combinatorial structures of $L(\mathcal{A})$. Since there are plenty of arrangements which are free and $K(\pi, 1)$ such as fiber type arrangements and the Coxeter arrangements, it was conjectured that all free arrangements are $K(\pi, 1)$ in an early study of free arrangements. Furthermore, it was also conjectured that the restriction $\mathcal{A}''$ is free for a free arrangement $\mathcal{A}$, but a counterexample is known for each of them now ([32, 31]). On the other hand, the conjecture "the freeness of $\mathcal{A}$ is characterized by only the structure of the intersection poset $L(\mathcal{A})$ (for a fixed field $\boldsymbol{K}$)" (Terao's conjecture [77]) is still open when the dimension is three or more.

3.2. **The freeness of multiarrangements by Ziegler.** From the definition, a logarithmic vector field $\delta \in D(\mathcal{A})$ of an arrangement of hyperplanes $\mathcal{A}$ is tangent to each hyperplane in $\mathcal{A}$. Therefore, for a hyperplane $H \in \mathcal{A}$, δ induces the tangent vector field $\delta|_H$ on H, and it belongs to $D(\mathcal{A}^H)$. However, in general, a hyperplane $K \in \mathcal{A}^H$ in the restriction is contained in several hyperplanes in $\mathcal{A}$. Put $\mathcal{A}_K = \{H, H', H'', \dots\}$. Since the δ is tangent to all hyperplanes $H, H', H'', \dots$, it is natural to consider that $\delta|_H$ is tangent to K **with a multiplicity**.[3] Ziegler's theory of multiarrangements of hyperplanes formulates this natural and important idea. In general, for an arrangement $\mathcal{A}$ and a map $m : \mathcal{A} \longrightarrow \boldsymbol{Z}_{\geq 0}$, the pair $(\mathcal{A}, m)$ is called an **arrangement of hyperplanes with a multiplicity** (or simply a **multiarrangement**). For a multiarrangement, $Q(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$ is called a defining polynomial of $(\mathcal{A}, m)$. First, we will define the vector fields of a multiarrangement.

Definition 3.5. Let $(\mathcal{A}, m)$ be a multiarrangement and fix a defining polynomial $\alpha_H \in V^*$ for each hyperplane $H \in \mathcal{A}$. Then we define $D(\mathcal{A}, m)$ as follows:

$$D(\mathcal{A}, m) = \{\delta \in \mathrm{Der}_V \mid \delta(\alpha_H) \in (\alpha_H)^{m(H)} S, \ \forall H \in \mathcal{A}\}.$$

When a multiplicity m is the constant map $m(H) = 1$ ($\forall H \in \mathcal{A}$), then $D(\mathcal{A}, m) = D(\mathcal{A})$ (Definition 2.4). Therefore $D(\mathcal{A}, m)$ is a generalization of the module of vector fields $D(\mathcal{A})$. A module $D(\mathcal{A}, m)$ is a reflexive module, thus when $\dim V = 2$ the module is automatically free. We can also formulate the freeness and the exponents for multiarrangements, and Saito-Ziegler's criterion ([93]) similar to Theorem 3.1 holds for them.

Multiarrangements naturally appear when we consider the restriction of an arrangement of hyperplanes. More specifically, we can consider the following multiplicity m^H for the restriction $\mathcal{A}^H$ of $\mathcal{A}$ to $H \in \mathcal{A}$:

$$m^H(K) = |\{H' \in \mathcal{A} \mid H \cap H' = K\}|, \ (K \in \mathcal{A}^H).$$

The following result by Ziegler is the most fundamental result with respect to free arrangements and free multiarrangements. It was the starting point of the recent characterizations of the freeness by using multiarrangements in Section 3.3.

Theorem 3.6 (Ziegler [93])**.** *Let $\mathcal{A}$ be a free arrangement with exponents $(1, d_2, \dots, d_\ell)$.[4] Then, for a hyperplane $H_0 \in \mathcal{A}$, $(\mathcal{A}^{H_0}, m^{H_0})$ is a free multiarrangement with exponents $(d_2, \dots, d_\ell)$.*

It is not so hard to verify the theorem. The outline is as follows. First, we fix a coordinate $(x_1, \dots, x_\ell)$ so that $H_0 = \{x_\ell = 0\}$. Consider the submodule $D_0(\mathcal{A}) := \{\delta \in D(\mathcal{A}) \mid \delta x_\ell = 0\}$ of $D(\mathcal{A})$. In other words, $D_0(\mathcal{A})$ is the set

[3] More accurately, the Euler vector field is tangent to K with only one multiplicity. Ziegler proved that the restrictions of the vector fields **except the Euler vector field** may be tangent to each hyperplane with a multiplicity more than one.

[4] When $\mathcal{A} \neq \emptyset$, since the Euler vector field is always in $D(\mathcal{A})$, we may suppose that $d_1 = 1$.

of all elements in $D(\mathcal{A})$ which are spanned by $(\partial/\partial x_1), \ldots, (\partial/\partial x_{\ell-1})$. Moreover, geometrically, each element of $D_0(\mathcal{A})$ is a vector field parallel to H_0. Then we can see that $D(\mathcal{A}) = S \cdot E \oplus D_0(\mathcal{A})$, and $\delta|_{x_\ell=0} \in D(\mathcal{A}^{H_0}, m^{H_0})$ for the restriction of $\delta \in D_0(\mathcal{A})$ to H_0. (Here E stands for the Euler vector field.) When $\mathcal{A}$ is a non-empty free arrangement, we can choose a basis $E, \delta_2, \ldots, \delta_\ell$ for $D(\mathcal{A})$ so that $\delta_i \in D_0(\mathcal{A})$, $(i = 2, \ldots, \ell)$; then it follows from Saito-Ziegler's criterion that the restrictions $\delta_2|_{H_0}, \ldots, \delta_\ell|_{H_0}$ to H_0 form a basis for $D(\mathcal{A}^{H_0}, m^{H_0})$.

At first, Ziegler [93] introduced the theory of multiarrangements as an approach to the conjecture (later a counterexample was given) which states that "any restriction of a free arrangement is free". However, the theory later played an extremely important role in proving the freeness of generalized Catalan/Shi arrangements (Edelman-Reiner's conjecture) [33, 86]. Ziegler's result asserts that "$\mathcal{A}$ is free $\Rightarrow$ $(\mathcal{A}^{H_0}, m^{H_0})$ is a free multiarrangement". Conversely, it is quite natural to ask whether we can prove the freeness of $\mathcal{A}$ by using Ziegler's result. Accordingly, we may apply the following strategy to show the freeness of $\mathcal{A}$:

(a) Show the freeness of the restriction $(\mathcal{A}^{H_0}, m^{H_0})$, and then construct a basis $\delta_2, \ldots, \delta_\ell \in D(\mathcal{A}^{H_0}, m^{H_0})$.
(b) Extend δ_i to an element of $D(\mathcal{A})$.

In regard to (a), as mentioned below, basic tools have been put in place, and various methods for determining the freeness/non-freeness have been established. Above all things, the Coxeter arrangements have been studied in depth by using invariant-theoretical methods. On the other hand, no general method to execute (b) has been found so far. In the case of the Shi arrangements associated with root systems, the search for the above-mentioned general method is currently in progress ([74, 73, 39]).

The proof of Theorem 3.6 tells us that, if $\mathcal{A}$ is free, then the restriction map

$$\rho : D_0(\mathcal{A}) \longrightarrow D(\mathcal{A}^{H_0}, m^{H_0}),\ \delta \longmapsto \delta|_{x_\ell=0},$$

is surjective. Conversely, if we verify that $D(\mathcal{A}^{H_0}, m^{H_0})$ is free and ρ is surjective, then we may obtain a free basis for $D(\mathcal{A})$ by taking the inverse image of a basis for $D(\mathcal{A}^{H_0}, m^{H_0})$, adding the Euler vector field, and applying Saito's criterion.

Corollary 3.7. *Let $\mathcal{A}$ be an arrangement of hyperplanes. Fix a hyperplane $H_0 \in \mathcal{A}$. If $(\mathcal{A}^{H_0}, m^{H_0})$ is free with exponents $(d_2, \ldots, d_\ell)$ and the restriction map $\rho : D_0(\mathcal{A}) \longrightarrow D(\mathcal{A}^{H_0}, m^{H_0})$ is surjective, then $\mathcal{A}$ is free with exponents $(1, d_2, \ldots, d_\ell)$.*

Therefore if we can show that

(b') the restriction map $\rho : D_0(\mathcal{A}) \longrightarrow D(\mathcal{A}^{H_0}, m^{H_0})$ is surjective

instead of the "natural strategy" (b) above, then we may prove the freeness of $\mathcal{A}$ (however, in this case, we cannot obtain an explicit basis for $D(\mathcal{A})$). In the next section, we will give several conditions for ensuring this surjectivity.

3.3. **Characterizations of the freeness by using a multiarrangement.** We have described several conclusions obtained from the freeness of arrangements of hyperplanes so far. The following three properties play important roles in characterizing the freeness. When $\mathcal{A}$ is a free arrangement with exponents $(d_1, d_2, \ldots, d_\ell)$ with $d_1 = 1$,

(1) the characteristic polynomial factors, and is expressed as

$$\chi(\mathcal{A},t)=\prod_{i=1}^{\ell}(t-d_i)$$

(Theorem 3.4),

(2) for $X \in L(\mathcal{A})$, $X \neq 0$, the localization $\mathcal{A}_X = \{H \in \mathcal{A}_X \mid X \subset H\}$ with respect to X is also free (the locally freeness),

(3) the restriction $(\mathcal{A}^H, m^H)$ is a free multiarrangement with exponents $(d_2,\ldots,d_\ell)$ (Theorem 3.6 by Ziegler).

None of these three properties implies the freeness of $\mathcal{A}$. As for (1), an example of three-dimensional non-free arrangement is known whose characteristic polynomial factors (Stanley, Kung [47]). When $\ell = 3$, the localization $\mathcal{A}_X$ and the restriction $(\mathcal{A}^H, m^H)$ are both of rank 2, thus they are free. Therefore, (2) and (3) are always satisfied for any arrangement when $\ell = 3$ (however, of course, even if $\ell = 3$, there exists an arrangement which is not free).

In an interesting twist, by combining two of the above properties, we can characterize the freeness of arrangements of hyperplanes.

Theorem 3.8 ([87])**.** *Let $\mathcal{A}$ be an arrangement of hyperplanes in a three-dimensional vector space, and $\chi(\mathcal{A},t) = (t-1)(t^2 - b_1 t + b_2)$ the characteristic polynomial. Since the restriction $(\mathcal{A}^H, m^H)$ for $H \in \mathcal{A}$ is of rank 2, it is free. Put its exponents (d_2, d_3).*

(1) $\operatorname{Coker}(\rho : D_0(\mathcal{A}) \longrightarrow D(\mathcal{A}^H, m^H))$ *is finite dimensional, and the dimension is equal to $b_2 - d_2 d_3$. In particular, the inequality $b_2 \geq d_2 d_3$ holds.*

(2) *$\mathcal{A}$ is free if and only if the inequality $b_2 = d_2 d_3$ holds.*

Theorem 3.9 ([86])**.** *Let $\ell \geq 4$, $\mathcal{A}$ be an arrangement of hyperplanes in a vector space of dimension ℓ, and fix a hyperplane $H_0 \in \mathcal{A}$. Then a necessary and sufficient condition for $\mathcal{A}$ to be a free arrangement with exponents $(1, d_2, \ldots, d_\ell)$ is given by the following two conditions:*

(i) *The restriction $(\mathcal{A}^{H_0}, m^{H_0})$ is free with exponents $(d_2,\ldots,d_\ell)$, and*

(ii) *$\mathcal{A}$ is locally free along H_0. That is to say, $\mathcal{A}_X$ is free for $X \in L(\mathcal{A}), X \subset H_0, X \neq 0$.*

The proofs of Theorem 3.8 and Theorem 3.9 are not so hard if we regard both of them as a splitting problem of when a coherent sheaf on projective space splits into a direct sum of line bundles.[5] To prove them, we use the idea introduced by Mustaţă-Schenck [48] (also [67]), that is, considering the reflexive sheaf $\widetilde{D(\mathcal{A})}$ on $\mathbf{P}^{\ell-1}$ obtained by the sheafification of the graded module $D(\mathcal{A})$. In general, a graded module M over a polynomial ring $\boldsymbol{C}[x_0,\ldots,x_n]$ induces an $\mathcal{O}_{\mathbf{P}^n}$-module $\widetilde{M}$ over $\mathbf{P}^n$. Conversely, we can obtain a graded module $M = \Gamma_*(\widetilde{M})$ over a polynomial ring from $\mathcal{O}_{\mathbf{P}^n}$-module over $\mathbf{P}^n$. If $\widetilde{M}$ splits into a direct sum of line bundles in this correspondence, then M is a free module. The following splitting criterion called Horrocks' theorem in splitting problems of vector bundles has been well known among the researchers of vector bundles. Theorem 3.9 is proved by generalizing Horrocks' theorem to reflexive sheaves ([86, 9]).

[5]Theorem 3.8 was obtained from evaluating the dimension of the cokernel of the restriction map ρ by using Solomon-Terao's formula in [87]. We can prove Theorem 3.8 (2) purely as a result with respect to vector bundles. For example, see [34, Corollary 2.12].

Theorem 3.10 (**Horrocks** [49]). *Let $n \geq 3$ and fix a hyperplane $H \subset \mathbf{P}^n$. A vector bundle E on $\mathbf{P}^n$ splits as $E = \mathcal{O}_{\mathbf{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^n}(d_r)$ if and only if the restriction of E to H splits as $E|_H = \mathcal{O}_H(d_1) \oplus \cdots \oplus \mathcal{O}_H(d_r)$.*

We may apply Horrocks' criterion to prove Theorem 3.9 as follows: the condition (ii) of Theorem 3.9 implies that $\widetilde{D(\mathcal{A})}$ is locally free around a neighborhood of $H_0 \subset \mathbf{P}^{\ell-1}$, thus $\widetilde{D(\mathcal{A})}|_{H_0} = D(\widetilde{\mathcal{A}^{H_0}}, m^{H_0})$ follows. In brief, the restriction of the sheaf $\widetilde{D(\mathcal{A})}$ coincides with the sheafification of the module of logarithmic vector fields of a multiarrangement $(\mathcal{A}^{H_0}, m^{H_0})$. Here we can see that $D(\widetilde{\mathcal{A}^{H_0}, m^{H_0}})$ splits from the hypothesis (i), and it follows from Horrocks' criterion for $\widetilde{D(\mathcal{A})}$ to split.

Furthermore, we will introduce a characterization found recently. This may look like a natural generalization of **Theorem 3.8**. However, in the proof, **Theorem 3.9** and the general theory (Section 3.4) of the characteristic polynomials of multiarrangements by Abe-Terao-Wakefield [6] are used essentially.

Theorem 3.11 ([11]). *Let $\ell \geq 4$, $\mathcal{A}$ be an arrangement of hyperplanes in an ℓ-dimensional vector space, and fix a hyperplane $H_0 \in \mathcal{A}$. Assume that the restriction $(\mathcal{A}^{H_0}, m^{H_0})$ is free with exponents $(d_2, \ldots, d_\ell)$. Set the characteristic polynomial of $\mathcal{A}$ as*

$$\chi(\mathcal{A}, t) = (t-1)(t^{\ell-1} - b_1 t^{\ell-2} + b_2 t^{\ell-3} - \cdots + (-1)^{\ell-1} b_{\ell-1});$$

then

(i) $b_2 \geq \sum_{2 \leq i \leq j \leq \ell} d_i d_j$.
(ii) *$\mathcal{A}$ is free if and only if the equality in (i) holds.*

From this result, we verify that if the freeness (as well as its exponents) of the restriction $(\mathcal{A}^H, m^H)$ is known, then the freeness of $\mathcal{A}$ is characterized by only its combinatorial structure. This suggests that the assumption that "the freeness of the restriction" is a strong constraint.

3.4. **General theory of the freeness of multiarrangements.** As described in Section 3.3, one of the motivations to study the freeness of multiarrangements is that the freeness of an ℓ-dimensional arrangement $\mathcal{A}$ is characterized by the freeness (the exponents) of $(\ell-1)$-dimensional multiarrangement $(\mathcal{A}^H, m^H)$ and its combinatorial structure. The studies of free multiarrangements can be split into two categories: One is to generalize known results of (non-)free multiarrangements. The other is to investigate peculiar properties of free multiarrangements. For the latter, the studies in the cases of dimension two have been advanced; we will describe it in Section 3.5. For the former, the characteristic polynomials for multiarrangements are defined by [6, 7], and the Addition-Deletion theorem (Theorem 3.2) is generalized to multiarrangements. Using these results, we may determine the freeness/non-freeness for many examples of multiarrangements. For example, as applications, the (non-) freeness is completely determined for the following two extreme cases:

(1) When an arrangement is generic, $(\mathcal{A}, m)$ is not free for any multiplicity $m : \mathcal{A} \longrightarrow \boldsymbol{Z}_{>0}$ ([6, 89]).
(2) If $(\mathcal{A}, m)$ is free for any multiplicity $m : \mathcal{A} \longrightarrow \boldsymbol{Z}_{>0}$, then $\mathcal{A}$ is a direct product of arrangements of dimension less than or equal to two ([8]).

When we fix $\mathcal{A}$, the free multiplicities and the non-free multiplicities coexist except in these extreme cases. The only case of the (non-) freeness being completely determined other than the cases above is the case of deleting one hyperplane from the arrangement of the type A_3 ([2]).

3.5. **General theory of two-dimensional multiarrangements.** As mentioned in the previous section, it can be said that there is essentially as much difficulty in showing the freeness of an ℓ-dimensional arrangement as in showing the freeness of an $(\ell-1)$-dimensional multiarrangement. For example, in order to show the freeness of a three-dimensional arrangement, it is sufficient to determine the exponents of a two-dimensional multiarrangement. The two-dimensional arrangements are always free, however, in general, it is difficult to determine the exponents of a multiarrangement even if the dimension is two, due to phenomena like the following:

Example 3.12. Let $t \in \boldsymbol{C} \setminus \{0, -1\}$, and consider the two-dimensional multiarrangement $x^3y^3(x-y)^1(tx+y)^1$. When $t = 1$, then $\delta_1 = x^3\partial_x + y^3\partial_y$, $\delta_2 = x^5\partial_x + y^5\partial_y$ form a basis for $D(\mathcal{A}, m)$ with exponents $(3, 5)$. However, when $t \neq 1$, then $\delta_1 = (tx+y)(x^3\partial_x + y^3\partial_y)$, $\delta_2 = (x-y)(t^2x^3\partial_x + y^3\partial_y)$ form a basis for $D(\mathcal{A}, m)$ with exponents $(4, 4)$.

Such a "jump" of the exponents of a two-dimensional arrangement is a peculiar phenomenon in the case that the arrangement has a multiplicity. The two main themes of this section are to study: 1) under what conditions the exponents do/don't jump, and 2) how the exponents jump. In what follows, we will introduce several basic results.

For a line arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$, let $\alpha_i \in S = \boldsymbol{C}[x, y]$ be the defining polynomial of H_i. Since $D(\mathcal{A}, m)$ is a reflexive S-module, it is a free module (because it is two-dimensional). Let δ_1, δ_2 be a basis for the module and $m_i = m(H_i)$; then

$$\deg \delta_1 + \deg \delta_2 = \sum_{i=1}^{n} m_i. \tag{3.1}$$

It is known that the jumping phenomena like Example 3.12 do not occur under the following conditions:

Theorem 3.13. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$, $m : \mathcal{A} \longrightarrow \boldsymbol{Z}_{>0}$ be a two-dimensional multiarrangement, and the multiplicities satisfy $m_1 \geq m_2 \geq \cdots \geq m_n$. Put $|m| = \sum_{i=1}^{n} m_i$ the sum of the multiplicities.*

(1) *If $m_1 \geq |m|/2$, then* $\exp(\mathcal{A}, m) = (m_1, |m| - m_1)$.
(2) *If $n \geq (|m|/2) + 1$, then* $\exp(\mathcal{A}, m) = (m - n + 1, n - 1)$.
(3) *If $m_1 = m_2 = \cdots = m_n = 2$, then* $\exp(\mathcal{A}, m) = (n, n)$.
(4) *If $n = 3$ and $m_1 \leq m_2 + m_3$, then*

$$\exp(\mathcal{A}, m) = \begin{cases} (k, k), & \textit{when } |m| = 2k, \\ (k, k+1), & \textit{when } |m| = 2k+1. \end{cases}$$

In regard to (1) and (2), we can explicitly construct a basis. (3) was verified by Wakefield-Yuzvinsky [84]. As for (4), Wakamiko [83] constructed an explicit basis by using the special values of the Schur functions. The assertion of Theorem 3.13 (1) is, roughly speaking, if there is a line with extremely high multiplicity (specifically, the multiplicity is greater than or equal to half of the sum of the multiplicities), then we can explicitly construct a basis, and determine the exponents. We may naturally regard such a multiplicity to be exceptional.

Definition 3.14. Let $(\mathcal{A}, m)$ be a two-dimensional multiarrangement. When the inequality

$$m(H) \leq \frac{1}{2} \sum_{K \in \mathcal{A}} m(K)$$

holds for any $H \in \mathcal{A}$, we say that the multiplicity m is **balanced**.

The sum of the two exponents of a two-dimensional multiarrangement is the sum of the multiplicities because of (3.1). Therefore, determining the exponents is equivalent to determining the difference between them. Put

$$\Delta(\mathcal{A}, m) = |d_1 - d_2| \tag{3.2}$$

the difference of $\exp(\mathcal{A}, m) = (d_1, d_2)$. In the unbalanced case, it turns out that $\Delta(\mathcal{A}, m)$ is determined only by the combinatorial information because of Theorem 3.13. However, in the balanced case, the jumping phenomena like Example 3.12 can occur, hence Δ may not be determined only by the combinatorial information. More precisely, let $\mathcal{M}_n = \{(a_1, \ldots, a_n) \in (\mathbf{P}^1)^n \mid a_i \neq a_j \ (i \neq j)\}$ be the configuration space of n points on $\mathbf{P}^1$. Fix a multiplicity $(m_1, \ldots, m_n)$. For a point $(a_1, \ldots, a_n) \in \mathcal{M}_n$, consider the multiarrangement $(\mathcal{A}, m)$ defined by $Q(\mathcal{A}, m) = \prod_{i=1}^n (y - a_i x)^{m_i}$. Then we can define the function $\Delta : \mathcal{M}_n \longrightarrow \boldsymbol{Z}_{\geq 0}, (a_1, \ldots, a_n) \longmapsto \Delta(\mathcal{A}, m)$. The most basic results concerning the function Δ are as follows:

Theorem 3.15. *Let $m : \{1, \ldots, n\} \longrightarrow \boldsymbol{Z}_{>0}$ be a balanced multiplicity. Then*

(i) $\Delta : \mathcal{M}_n \longrightarrow \boldsymbol{Z}_{\geq 0}$ *is upper semi-continuous. (In other words, $\{\mathcal{A} \in \mathcal{M}_n \mid \Delta(\mathcal{A}, m) < k\} \subset \mathcal{M}_n$ is a Zariski open set for any $k \in \boldsymbol{R}$.)*
(ii) ([84]) *There exists a non-empty Zariski open set $U \subset \mathcal{M}_n$ such that $\Delta(\mathcal{A}, m) \leq 1$ for any $\mathcal{A} \in U$.*
(iii) ([1]) *For any $\mathcal{A} \in \mathcal{M}_n$, $\Delta(\mathcal{A}, m) \leq n - 2$.*

Therefore, for a generic arrangement $\mathcal{A}$, $\Delta(\mathcal{A}, m) = 0$ or 1 according to whether $|m|$ is even or odd. Moreover, in general, it is known that $\Delta(\mathcal{A}, m)$ is not greater than the number of lines minus two. However, for example, it is not known that for what kind of $m : \{1, \ldots, n\} \to \boldsymbol{Z}_{>0}$, there exists $\mathcal{A} \in \mathcal{M}_n$ which attains the upper bound $\Delta(\mathcal{A}, m) = n - 2$.

3.6. **The primitive derivation and the freeness of Coxeter multiarrangements.** Coxeter arrangements are the first class of arrangements verified to be free ([59]). Later, the freeness of the Coxeter multiarrangements with constant multiplicities were also proved ([79]). From the topological viewpoints, Deligne ([27]) proved that the higher homotopy groups of the complexified complements of the Coxeter arrangements vanish. The class of the Coxeter arrangements have been studied more deeply than any other classes so far ([62]). In this section, we will introduce several results about the freeness of Coxeter (multi)arrangements.

Let V be a real vector space of dimension ℓ, W a finite group generated by reflections of V, acting irreducibly on V. Then the set $\mathcal{A}$ of all reflecting hyperplanes is called the **Coxeter arrangement**. Moreover, then the W-invariant inner product on V is uniquely determined (up to a non-zero constant multiple). Let $S = S(V^*)$ be the symmetric algebra of V^*. The isomorphism $I : V^* \xrightarrow{\simeq} V$ obtained from the inner product I is extended to the isomorphism (as S-modules) $I : \Omega_V \longrightarrow \mathrm{Der}_V$ between $\Omega_V = S \otimes V^*$ and $\mathrm{Der}_V = S \otimes V$.

It is known that the invariant subring S^W is expressed as a polynomial ring $\boldsymbol{R}[P_1, \dots, P_\ell]$ (Chevalley [22]), where $P_1, \dots, P_\ell$ are algebraically independent homogeneous polynomials which are called basic invariants. The tuple $(e_1, \dots, e_\ell)$ of $e_i = \deg P_i - 1$ is called the **exponents** of W. Order them in such a way that $e_1 \le e_2 \le \cdots \le e_\ell$. Then $e_{\ell-1} < e_\ell$ holds. This inequality will play an important role when we define the primitive derivation later. The product $\Delta = \prod_{H \in \mathcal{A}} \alpha_H$ of defining polynomials α_H of hyperplanes H is a generator of the set $S^{-W} = \{f \in S \mid w(f) = \det(w) f, w \in W\} = S^W \cdot \Delta$ of antiinvariants as an S^W-module. Fix coordinates $x_1, \dots, x_\ell$; then it is classically known that

$$\Delta = \det \left(\frac{\partial P_i}{\partial x_j} \right)_{ij} \tag{3.3}$$

holds true up to a constant multiple. By using the equality (3.3) and Saito's criterion, we can explicitly construct a free basis for $D(\mathcal{A})$ of the Coxeter arrangement.

Theorem 3.16 (Saito [59]). *With the notation above, for the Coxeter arrangement $\mathcal{A}$, the module of W-invariant vector fields decomposes as $\mathrm{Der}_V^W = S^W \cdot I(dP_1) \oplus \cdots \oplus S^W \cdot I(dP_\ell)$. Moreover, $\mathrm{Der}_V^W \otimes_{S^W} S = D(\mathcal{A})$ holds. In particular, $\mathcal{A}$ is a free arrangement with exponents $\exp(\mathcal{A}) = (e_1, \dots, e_\ell)$.*

We regard basic invariants $(P_1, \dots, P_\ell)$ as coordinates of the quotient space $V/W = \operatorname{Spec} S^W$, and $(\partial/\partial P_1), \dots, (\partial/\partial P_\ell)$ as vector fields on the quotient space. Since the degrees satisfy $\deg P_\ell > \deg P_i, (\forall i \neq \ell)$, $D := (\partial/\partial P_\ell)$ is independent of the choice of basic invariants (up to a non-zero constant multiple). We call this vector field D the **primitive derivation**. The primitive derivation is characterized by $DP_i = \delta_{i\ell}$. By using a coordinate system of V, the primitive derivation D is a W-invariant rational vector field which has an expression

$$D = \frac{1}{\Delta} \cdot \det \begin{pmatrix} (\partial P_1/\partial x_1) & \cdots & (\partial P_{\ell-1}/\partial x_1) & (\partial/\partial x_1) \\ (\partial P_1/\partial x_2) & \cdots & (\partial P_{\ell-1}/\partial x_2) & (\partial/\partial x_2) \\ \vdots & \ddots & \vdots & \vdots \\ (\partial P_1/\partial x_\ell) & \cdots & (\partial P_{\ell-1}/\partial x_\ell) & (\partial/\partial x_\ell) \end{pmatrix}. \tag{3.4}$$

Let ∇ denote the Levi-Civita connection defined by the metric I. (This can be expressed as $\sum_i f_i(\partial/\partial x_i) = \sum_i \delta(f_i)(\partial/\partial x_i))$ by the coordinates $x_1, \dots, x_\ell$.) In the theory of Kyoji Saito's primitive derivation, for the fixed primitive form, we can identify the relative de Rham cohomology of a family of semi-universal deformations of isolated hypersurface singularities with the vector fields of the base space. From this identification, geometric structures such as the Gauss-Manin connection of a relative de Rham cohomology and the Hodge filtration bring about the flat structure (the Frobenius structure) to the vector fields in the base space. When the singularities are of the types ADE, the parameter space of the semi-universal deformation can be identified with the quotient space V/W of the corresponding Weyl group, and the flat structure can be described from the viewpoints of the invariant theory of the Coxeter groups. The following Theorem 3.18 forms a foundation of invariant-theoretic construction of the flat structure. It plays an important role also in the freeness of the Coxeter multiarrangement.

Theorem 3.17 ([61, 62]). *The covariant derivative ∇_D defined by the primitive derivation induces an isomorphism*[6] $\nabla_D : D(\mathcal{A})^W \xrightarrow{\simeq} \mathrm{Der}_{V/W}$.

From this result, the inverse map ∇_D^{-1} acts on $D(\mathcal{A})^W$ and has a filtration

$$\cdots \subset \nabla_D^{-2} D(\mathcal{A})^W \subset \nabla_D^{-1} D(\mathcal{A})^W \subset D(\mathcal{A})^W. \tag{3.5}$$

This filtration plays an important role in studying the Coxeter multiarrangements. The inverse map ∇_D^{-1} has an action which raises the contact order of vector fields to each hyperplane.

Theorem 3.18 ([69, 79, 80, 85, 10]). *Let $\mathcal{A}$ be a Coxeter arrangement with* $\exp(\mathcal{A}) = (e_1, \ldots, e_\ell)$*, and $h = e_\ell + 1$ the Coxeter number. Let $m : \mathcal{A} \longrightarrow \{0,1\}$ be a $\{0,1\}$-valued multiplicity. Then*

(1)

$$\begin{aligned} &D(\mathcal{A}, 2k+m) \simeq D(\mathcal{A}, m)[-kh] \quad (k \in \boldsymbol{Z}_{\geq 0}), \\ &D(\mathcal{A}, 2k-m) \simeq \left(D(\mathcal{A}, m)^\vee\right)[-kh] \simeq \Omega^1(\mathcal{A}, m)[-kh] \quad (k \in \boldsymbol{Z}_{>0}), \end{aligned} \tag{3.6}$$

where, for a graded module $M = \bigoplus_{n \in \boldsymbol{Z}} M_n$, $M[d]$ is a graded module obtained by shifting the degree by d ($M[d]_n := M_{d+n}$).

(2) ($m \equiv 1$) *The multiarrangement $(\mathcal{A}, 2k+1)$ is free with* $\exp(\mathcal{A}, 2k+1) = (e_1 + kh, \ldots, e_\ell + kh)$ *when $k \in \boldsymbol{Z}_{\geq 0}$.*

(3) ($m \equiv 0$) *The multiarrangement $(\mathcal{A}, 2k)$ is free with* $\exp(\mathcal{A}, 2k) = (kh, kh, \ldots, kh)$ *when $k \in \boldsymbol{Z}_{\geq 0}$.*

3.7. **The freeness of the generalized Catalan/Shi arrangements.** In 1996, Edelman-Reiner [33] discovered a family of free arrangements associated with the root system of the type A when they were studying the condition for the discriminant arrangement of hyperplanes defined by a two-dimensional zonotope to be free. They conjectured the freeness of the generalized Catalan arrangements and the generalized Shi arrangements associated with root systems in the form of generalizing the discovery ([33, Conjecture 3.3]). Fix a positive root system $\Phi^+ \subset \Phi$ of a root system $\Phi \subset (\boldsymbol{R}^\ell)^*$. Put $H_{\alpha,p} = \{v \in \boldsymbol{R}^\ell \mid \alpha(v) = p\}$ for $\alpha \in \Phi^+$ and an integer $p \in \boldsymbol{Z}$. Define an affine arrangement of hyperplanes $\mathcal{A}_\Phi^{[a,b]}$ for integers $a \leq b$ by

$$\mathcal{A}_\Phi^{[a,b]} := \{H_{\alpha,p} \mid \alpha \in \Phi^+, a \leq p \leq b\}. \tag{3.7}$$

This can be regarded as a finite subarrangement of the arrangement consisting of reflecting hyperplanes of an affine Weyl group. Among the cones of these arrangements, Edelman and Reiner conjectured that $c\mathcal{A}_\Phi^{[-k,k]}$ (the cone over the Catalan arrangement) and $c\mathcal{A}_\Phi^{[1-k,k]}$ (the cone over the Shi arrangement) are free. The conjecture was settled in 2004 by combining results in [79] and [86].

Theorem 3.19 (Yoshinaga [86]). *Let Φ be a root system, $e_1, \ldots, e_\ell$ its exponents, and h the Coxeter number.*

1. *The cone over the generalized Catalan arrangement $c\mathcal{A}_\Phi^{[-k,k]}$ is free with exponents* $\exp(c\mathcal{A}_\Phi^{[-k,k]}) = (1, e_1 + kh, \ldots, e_\ell + kh)$.
2. *The cone over the generalized Shi arrangement $c\mathcal{A}_\Phi^{[1-k,k]}$ is free with exponents* $\exp(c\mathcal{A}_\Phi^{[1-k,k]}) = (1, kh, \ldots, kh)$.

[6] This map is not a homomorphism as S-modules. Define the subring T of S to be $T = \{f \in S^W \mid Df = 0\}$; then ∇_D induces an isomorphism as T-modules.

The proof follows from an induction on the rank of the root system. First, when the rank is 2, it is sufficient to verify the case of the types A_2, B_2, G_2 due to the classification of root systems, hence we check them separately. Next, when the rank is greater than or equal to 3, we apply Theorem 3.9 to the restriction onto the hyperplane H_0 at infinity. The condition (i) in Theorem 3.9 "the freeness of the restriction" can be proved in Theorem 3.18 (2), (3), and the condition (ii) "locally free along H_0" is derived from the induction hypothesis. In more detail, for $X \subset H_0$, $(c\mathcal{A}_\Phi^{[a,b]})_X$ is the direct sum of several arrangements $\mathcal{A}_{\Phi'}^{[a,b]}$ for root systems Φ' of lower ranks. Therefore, $(c\mathcal{A}_\Phi^{[a,b]})_X$ turns out to be free from this theorem for root systems of lower ranks. As bases for the modules of logarithmic vector fields of the Coxeter arrangements are constructed (Theorem 3.16), it is expected that basis for the modules of logarithmic vector fields of the generalized Catalan arrangements and the generalized Shi arrangements are also constructed explicitly in some way. Research is now in progress for individual cases (Suyama-Terao [74], Suyama [73], Gao-Pei-Terao [39]). In these cases, the Bernoulli polynomials essentially appear in the constructions of bases.

Applying Theorem 3.4 to the above freeness, we may explicitly describe the characteristic polynomials.

Corollary 3.20. (1) *The characteristic polynomial of the generalized Catalan arrangement $\mathcal{A}_\Phi^{[-k,k]}$ is $\chi(A_\Phi^{[-k,k]}, t) = \prod_{i=1}^{\ell}(t - kh - e_i)$.*

(2) *The characteristic polynomial of the generalized Shi arrangement $\mathcal{A}_\Phi^{[1-k,k]}$ is $\chi(\mathcal{A}_\Phi^{[1-k,k]}, t) = (t - kh)^\ell$.*

There are various studies with respect to combinatorial properties of arrangements of hyperplanes associated with such root systems. It is known that the characteristic polynomial $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$ factors into linear polynomials only in the above two cases. Even when the characteristic polynomial does not factor, the following remarkable property is conjectured:

Conjecture 3.21 ([54])**.** *Let integers a, b satisfy $0 \le a < b$. Then the real part of any root $t \in \boldsymbol{C}$ of the characteristic polynomial $\chi(\mathcal{A}_\Phi^{[-a,b]}, t) = 0$ satisfies $\Re(t) = (a + b + 1)h/2$.*

This conjecture asserts that the zeros of the characteristic polynomial lie on the line satisfying that the real parts are $(a + b + 1)h/2$, as a conclusion, also asserts the following non-trivial relation holds:

$$\chi(\mathcal{A}_\Phi^{[-a,b]}, (a + b + 1)h - t) = (-1)^\ell \chi(\mathcal{A}_\Phi^{[-a,b]}, t).$$

Conjecture 3.21 verified for the root systems of the types A, B, C, D, and the several other cases of special parameters by using the classification of root systems at this writing ([17]). It is expected that we can explain the conjecture by algebraic properties of logarithmic vector fields, however it is still open.

In the proof of the Edelman-Reiner conjecture, the multifreeness of the restriction onto the hyperplane H_0 at infinity plays an important role. Theorem 3.18 shows that there are many non-constant multiplicities $m : \mathcal{A} \longrightarrow \{n, n + 1\}$ which make $D(\mathcal{A}, m)$ free. In order to answer the question asking whether there exists a free arrangement whose restriction is equal to the $(\mathcal{A}, m)$ or not, we have to study the existence of a series of the free arrangements interpolating between the generalized Catalan and Shi arrangements. We do not know how to answer the question even when the multiplicity is $m : \mathcal{A} \longrightarrow \{0, 1\}$ (this corresponds to free arrangements interpolating between the empty arrangement and the Coxeter arrangement). However, for example, we know the following beautiful relation between a combinatorial structure of a root system and the exponents: Let $\Phi^+ = \Phi_1^+ \sqcup \Phi_2^+ \sqcup \cdots \sqcup \Phi_p^+$ be a partition by heights[7] of positive roots. Then the dual partition[8] $m_\ell \geq \cdots \geq m_1$ of a partition $|\Phi_1^+| \geq |\Phi_2^+| \geq \cdots \geq |\Phi_p^+|$ of a natural number $|\Phi^+|$ coincides with the exponents (Arnold Shapiro, R. Steinberg [72, §9], B. Kostant [46]). The following conjectures which naturally extend the above relation were brought up:

Conjecture 3.22. *We say that $\Psi \subset \Phi^+$ is the* ***height subset*** *if $\alpha \in \Psi, \beta \in \Phi^+, \mathrm{ht}(\beta) < \mathrm{ht}(\alpha)$ implies $\beta \in \Psi$. If $\Psi \subset \Phi^+$ is a height subset, then $\mathcal{A}_\Psi = \{H_{\alpha,0} \mid \alpha \in \Psi\}$ is a free arrangement and its exponents are given by the dual partition of height distribution of Ψ.*[9]

Conjecture 3.23. *Let $(n_1, n_2, \ldots, n_q)$ be the dual partition of the height distribution of a height subset Ψ.*

(1) $c\left(\mathcal{A}_\Phi^{[1-k,k]} \cup \{H_{\alpha,-k} \mid \alpha \in \Psi\}\right)$ *is a free arrangement and its ℓ exponents are $(kh + n_1, \ldots, kh + n_q, kh, \ldots, kh)$.*

(2) $c\left(\mathcal{A}_\Phi^{[1-k,k]} \setminus \{H_{\alpha,k} \mid \alpha \in \Psi\}\right)$ *is a free arrangement and its ℓ exponents are $(kh - n_1, \ldots, kh - n_q, kh, \ldots, kh)$.*[10]

Recently in [4], Abe-Terao introduced the special basis for $D_0\left(c\left(\mathcal{A}_\Phi^{[1-k,k]}\right)\right)$ which is called SRB (simple-root basis), the behavior of the freeness in the case of adding one hyperplane to (or deleting one hyperplane from) the generalized Shi arrangement was completely understood by the following theorem:

Theorem 3.24 ([4]). (1) $c\left(\mathcal{A}_\Phi^{[1-k,k]} \cup \{H_{\alpha,-k}\}\right)$ *is a free arrangement if and only if $\alpha \in \Phi^+$ is a simple root.*

(2) $c\left(\mathcal{A}_\Phi^{[1-k,k]} \setminus \{H_{\alpha,k}\}\right)$ *is a free arrangement if and only if $\alpha \in \Phi^+$ is a simple root.*

4. Topologies of complex complements

There are a large number of topological researches of arrangements of hyperplanes. Thus we focus the two basic themes of the "combinatorial determinativeness" and the "minimality" in this chapter.

4.1. **The Orlik-Solomon algebra.** In studies of arrangements of hyperplanes, we often formulate problems in the form of "Is it combinatorially determined"? The problems originate from the description, due to Orlik-Solomon, of the cohomology ring of the complement. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement over $\boldsymbol{C}$. Fix a letter e_i for each hyperplane H_i. Let $E = \bigoplus_{i=1}^n \boldsymbol{Z} e_i$ be the free additive group generated by the e_i's, and $\bigwedge E$ the exterior algebra over $\boldsymbol{Z}$. Define the linear

[7] When we express a positive root as a linear combination of simple roots, the sum of coefficient is called the **height**. For example, a simple root is of height 1.

[8] In the dual partition of a partition $(m_1, m_2, \ldots, m_p)_\geq$ of a positive integer N, a positive integer k $(1 \leq k \leq p)$ appears exactly $m_k - m_{k+1}$ times, where we agree that $m_{p+1} = 0$.

[9] This conjecture has been recently proved in [3] (added in translation).

[10] This conjecture has been recently proved in [5] (added in translation).

map $\partial : \bigwedge E \longrightarrow \bigwedge E$ by $\partial(e_{i_1}e_{i_2}\dots e_{i_k}) = \sum_{p=1}^{k}(-1)^{p-1}e_{i_1}\dots\widehat{e_{i_p}}\dots e_{i_k}$. Call the subset $\{H_{i_1},\dots,H_{i_k}\} \subset \mathcal{A}$ dependent when $\operatorname{codim}(H_{i_1}\cap\dots\cap H_{i_k}) < k$. Let $I_{\mathcal{A}}$ be the ideal of $\bigwedge E$ generated by $\{\partial(e_{i_1}e_{i_2}\dots e_{i_k}) \mid H_{i_1},\dots,H_{i_k}$ are dependent$\}$. Then,

Theorem 4.1 ([50])**.** $(\bigwedge E)/I_{\mathcal{A}} \simeq H^*(M(\mathcal{A}), \boldsymbol{Z})$.

The left hand side $(\bigwedge E)/I_{\mathcal{A}}$ is called the **Orlik-Solomon algebra** of $\mathcal{A}$. This corresponds to the cohomology by

$$e_i \longmapsto \frac{1}{2\pi\sqrt{-1}} \cdot \frac{d\alpha_{H_i}}{\alpha_{H_i}}.$$

4.2. **Oriented matroid and the Salvetti complex.** By definition, the structure of the Orlik-Solomon algebra uses only combinatorial information of "intersections" of an arrangement. In other words, the cohomology ring of the complement $M(\mathcal{A})$ is determined only by the intersection poset $L(\mathcal{A})$.

When an arrangement of hyperplanes is defined over the real number field $\boldsymbol{R}$, we can consider a stronger "combinatorial structure", which is called the oriented matroid. The oriented matroid is a combinatorial object with more information than $L(\mathcal{A})$. It is known that we can completely restore the topological type of $M(\mathcal{A})$ out of its oriented matroid. Here, we briefly describe the construction of the Salvetti complex which plays an important role in the topological study of $M(\mathcal{A})$. First, let $\{-1, 0, +1\}$ be the set of signs, and define the partial order $0 \succeq -1$, $0 \succeq +1$ (± 1 are incomparable). Then it induces the natural partial order $(a_i)_{i=1}^n \preceq (b_i)_{i=1}^n \iff a_i \preceq b_i, \forall i$ on sign vectors $\boldsymbol{a} = (a_i)_{i=1}^n, \boldsymbol{b} = (b_i)_{i=1}^n \in \{-1, 0, +1\}^n$. Define the composition $\boldsymbol{a} \circ \boldsymbol{b} \in \{-1, 0, +1\}^n$ of sign vectors by

$$(\boldsymbol{a} \circ \boldsymbol{b})_i = \begin{cases} b_i & (b_i \neq 0), \\ a_i & (b_i = 0). \end{cases}$$

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in $\boldsymbol{R}^\ell$ over the real number field $\boldsymbol{R}$. Fix a defining polynomial $\alpha_H \in (\boldsymbol{R}^\ell)^*$ for each hyperplane $H \in \mathcal{A}$. Define the map $\sigma_{\mathcal{A}}$ as

$$\sigma_{\mathcal{A}} : \boldsymbol{R}^\ell \longrightarrow \{-1, 0, +1\}^n,\ \boldsymbol{x} \longmapsto (\operatorname{sgn}\alpha_{H_1}(\boldsymbol{x}), \dots, \operatorname{sgn}\alpha_{H_n}(\boldsymbol{x})).$$

The image $\sigma_{\mathcal{A}}(\boldsymbol{R}^\ell) \subset \{-1, 0, +1\}^n$ of the map $\sigma_{\mathcal{A}}$ is called (the set of covectors of) the **oriented matroid** of $\mathcal{A}$. For a sign vector $a \in \sigma_{\mathcal{A}}(\boldsymbol{R}^\ell)$, $\sigma_{\mathcal{A}}^{-1}(a) \subset \boldsymbol{R}^\ell$ is defined by a bunch of equalities and inequalities of linear functions. Hence it is contractible, and $\boldsymbol{R}^\ell = \bigsqcup_{a \in \sigma_{\mathcal{A}}(\boldsymbol{R}^\ell)} \sigma_{\mathcal{A}}^{-1}(a)$ is a stratification of $\boldsymbol{R}^\ell$ defined by $\mathcal{A}$.

We can regard $\sigma_{\mathcal{A}}(\boldsymbol{R}^\ell)$ as a combinatorial interpretation of the stratification of $\boldsymbol{R}^\ell$ defined by $\mathcal{A}$. The minimal elements with respect to the order $\preceq$ can be characterized by sign vectors with all its entries being non-zero, namely, being chambers. Express the set of the minimal elements by $\mathcal{T} = \sigma_{\mathcal{A}}(\boldsymbol{R}^\ell) \cap \{-1, +1\}^n$. The partial order $\preceq$ corresponds to the adjacency relationship determined by the topology on $\boldsymbol{R}^\ell$. We can define a partial order $\leq$ on $\mathcal{S}(\mathcal{A}) = \{(X, T) \in \sigma_{\mathcal{A}}(\boldsymbol{R}^\ell) \times \mathcal{T} \mid X \succeq T\}$ as follows:

$$(X', T') \leq (X, T) \iff X' \preceq X \text{ and } T' = T \circ X'.$$

Then, there exists a regular cell complex $\Delta_{Sal}(\mathcal{A})$ with partially ordered set $(\mathcal{S}(\mathcal{A}), \leq)$ as its face lattice. This cell complex is called the **Salvetti complex**. It is known that the Salvetti complex $\Delta_{Sal}(\mathcal{A})$ is homotopy equivalent to the complexified complement $M(\mathcal{A}_{\boldsymbol{C}})$ ([63]). More strongly, we can restore the topology type out of $\Delta_{Sal}(\mathcal{A})$ ([19]). The Salvetti complexes have been applied to calculation of homologies of the Artin groups or the Coxeter groups.

4.3. **Combinatorial structure and topologies.** Identifying -1 with $+1$ in a sign vector, we obtain the map $|\sigma_{\mathcal{A}}| : \boldsymbol{R}^\ell \longrightarrow \{0,1\}^n$. Moreover, we have the following diagram by using the absolute value function $|\cdot| : \pm 1 \longmapsto 1$:

$$\begin{array}{cccc} |\cdot|: & \{0,\pm 1\}^n & \longrightarrow & \{0,1\}^n \\ & \cup & & \cup \\ & \sigma_{\mathcal{A}}(\boldsymbol{R}^\ell) & \longrightarrow & |\sigma_{\mathcal{A}}|(\boldsymbol{R}^\ell). \end{array}$$

As we saw above, $\sigma_{\mathcal{A}}(\boldsymbol{R}^\ell)$ has enough information to restore the topological type of the complement $M(\mathcal{A})$. On the other hand, $|\sigma_{\mathcal{A}}|(\boldsymbol{R}^\ell)$ and the intersection poset $L(\mathcal{A})$ essentially have the same amount of information, thus we can restore the cohomology ring with coefficients in $\boldsymbol{Z}$ out of $|\sigma_{\mathcal{A}}|(\boldsymbol{R}^\ell)$ by Orlik-Solomon's result. For a rank 1 local system $\mathcal{L}$ on $M(\mathcal{A})$ which is close to the trivial local system, [11] it is known that we can restore the cohomology with coefficients in $\mathcal{L}$ from $|\sigma_{\mathcal{A}}|(\boldsymbol{R}^\ell)$ ([35]). However, $|\sigma_{\mathcal{A}}|(\boldsymbol{R}^\ell)$ seems to have only much less information than $\sigma_{\mathcal{A}}(\boldsymbol{R}^\ell)$. Indeed, Rybnikov constructed a pair $\mathcal{A}_1, \mathcal{A}_2$ of plane arrangements in $\boldsymbol{C}^3$ whose intersection posets are isomorphic but the fundamental groups of their complements are not isomorphic as groups ([57, 16]). Also, for an arrangement defined over the real number field, it is conjectured that there exists an example that the intersection posets are isomorphic but the fundamental groups are not, and many candidates for them are found. However no one has ever verified that the fundamental groups are not isomorphic (other than Rybnikov) at this writing. Another open problem is whether the (co)homologies $H^*(M(\mathcal{A}), \mathcal{L})$ with coefficients in local systems or Betti numbers of Milnor fibers of $\mathcal{A}$ are determined only by information of $L(\mathcal{A})$ or not. As we saw in this section, how much combinatorial structure is necessary to restore various topological invariants is a central problem in the study of arrangements of hyperplanes.

4.4. **The minimality.** We say that a finite ℓ-dimensional CW-complex X is a minimal CW-complex if

$$\text{(the number of } k\text{-dimensional cells)} = b_k(X),$$

for $k = 0, 1, \ldots, \ell$. Recall the process for calculating the homology by making use of the chain complex from a CW-complex. Then we find that the inequality $\geq$ always holds. In other words, the number of k-dimensional cells is bounded from below by $b_k(X)$. The above equality holds when a CW-complex X has as few cells as possible in CW-complexes of the same homotopy type. Around 2000, Dimca-Papadima [29] and Randell [55] established that $M(\mathcal{A})$ and a minimal CW-complex are homotopy equivalent. The proof uses the Lefschetz type theorem [40] for affine varieties (the complements of hypersurfaces). The minimality has been observed for various cases since the 1970s ([41, 37, 53]).

[11] A local system of rank 1 one-to-one corresponds to a group homomorphism $\rho : \pi_1(M(\mathcal{A})) \longrightarrow \boldsymbol{C}^*$. Precise meaning of "local system $\mathcal{L}$ on $M(\mathcal{A})$ which is close to the trivial local system" is that there exists an open neighborhood U of the trivial local system in $\mathrm{Hom}(\pi_1(M(\mathcal{A})), \boldsymbol{C}^*)$ such that local system cohomology is combinatorially determined if $\mathcal{L} \in U$.

Since the complement of a general hypersurface does not have minimality,[12] the minimality can be said to be a characteristic property for arrangements of hyperplanes. It is expected that the minimality has many applications for topologies of arrangements of hyperplanes. When we try to apply it, then we face a bottleneck of how to attach the cells of a minimal cell complex.

In the proof by using the Lefschetz theorem, there is no information with respect to how to attach the cells. Currently, for an arrangement defined over the real number field, description of the minimality including how to attach the cells is addressed through two approaches. One is by Yoshinaga [88]: Analyzing the Morse theoretic proof of the Lefschetz theorem, he corresponded the critical points of a Morse function with a certain real domain by employing the real structure. In this way, he described the homotopy types of the ℓ-dimensional cells which are attachable to the hyperplane section $M(\mathcal{A})\cap F$. The other is by Salvetti-Settepanella [64]: Defining a discrete Morse function on the Salvetti complex, they combinatorially described the gradient flow deforming the complex from the Salvetti complex to the minimal CW-complex. For the former, there are applications with respect to 1) the vanishing of cohomologies with coefficients in local systems of dimension two and 2) homogeneous representations for fundamental groups ([90, 91]). For the latter, Delucchi generalized the result of [64] to oriented matroids, and verified that the Salvetti complex of an oriented matroid has also the minimality ([28]).

5. Hypergeometric integrals associated with an arrangement

In this chapter, for simplicity, let $\mathcal{A}$ be a real arrangement of rank ℓ and $\mathcal{A}_{\boldsymbol{C}}$ its complexification. Let $\mathcal{L}$ be a local system of rank 1 on $M(\mathcal{A}_{\boldsymbol{C}})$. As mentioned in Chapter 1, Aomoto's hypergeometric integrals are expressed as the twisted de Rham pairing of the local system cohomology and the local system homology. Thus studies on the local system (co)homology are essential for us to understand the hypergeometric functions. Especially, vanishing conditions, dimensions and basis descriptions of the local system (co)homology are, among others, very important. It goes without saying that the knowledge of topology of complex complements is indispensable for the study. For example, as an application of the minimality of $M(\mathcal{A}_{\boldsymbol{C}})$ in the previous chapter, the following inequality in "Theory of Hypergeometric Functions" [14, Proposition 2.1, Remark 2.3] by Aomoto-Kita follows immediately ([24]):

$$\dim H_k(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}) \leq b_k(M(\mathcal{A}_{\boldsymbol{C}})).$$

5.1. The vanishing theorem of local system (co)homologies.

Next, we will explain the refinement of the vanishing theorem, which is obtained as an application of the minimal cell decomposition including how to attach the cells. First, we state the vanishing and a basis description of the following local system (co)homology:

Theorem 5.1 ([13, 45, 30]). *When $\mathcal{L}$ satisfies suitable genericity conditions, then*

$$H^i(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}) \simeq H^{lf}_{2\ell-i}(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}) = 0 \qquad (i \neq \ell)$$

and the bounded chambers $\mathrm{b}\mathcal{C}(\mathcal{A})$ *provide a basis for*

$$H^\ell(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}) \simeq H^{lf}_\ell(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}) \simeq H_\ell(M(\mathcal{A}_{\boldsymbol{C}}),\mathcal{L}).$$

[12] For example, the first Betti number of $\boldsymbol{C}^2 \setminus \{y^2 = x^3\}$ is $b_1 = 1$, however the fundamental group is the braid group on three strings, so is not isomorphic to $\boldsymbol{Z}$.

(Here H_k^{lf} is a locally finite homology.)

The above result without a basis description is simply called the "the vanishing theorem", which holds true under various conditions ([44, 35, 66, 23, 25]). It is known that, when $\ell = 2$, the resonance conditions by Cohen-Dimca-Orlik [25] are necessary and sufficient for the vanishing theorem with a basis description.

Theorem 5.2 (Yoshinaga [91])**.** *Let $\mathcal{A}$ be a line arrangement in $\boldsymbol{R}^2$ which has greater than or equal to three intersection points on the line H_∞ at infinity. Then the following are equivalent:*

(1) *The monodromies around the line H_∞ at infinity and the intersection points on H_∞ is non-trivial.*
(2) $H^0(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L}) = H^1(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L}) = 0$ *and the bounded chambers* $\mathbf{b}\mathcal{C}(\mathcal{A})$ *form a basis for* $H^2(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L}) \simeq H_2^{lf}(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L})$.
(3) $H_2^{lf}(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L})$ *is generated by the bounded chambers* $\mathbf{b}\mathcal{C}(\mathcal{A})$.

5.2. **Bases for local system (co)homologies.** When the vanishing theorem (Theorem 5.1) holds true, the dimension of the unique non-trivial cohomology $H^\ell(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L})$ is equal to the absolute value of the Euler characteristic of $M(\mathcal{A}_{\boldsymbol{C}})$. We will introduce a construction of a basis for the cohomology in accordance with [38]. First, we impose a total order $H_1 \prec H_2 \prec \cdots \prec H_n$ on $\mathcal{A}_{\boldsymbol{C}}$.

Definition 5.3 (Björner-Ziegler [18], [43])**.** A minimal linearly dependent subset of $\mathcal{A}_{\boldsymbol{C}}$ is called a **circuit**. When $\mathcal{B}$ is a circuit, a set of the form $\mathcal{B} \setminus \{\min \mathcal{B}\}$ ($\min \mathcal{B}$ is the minimum element in $\mathcal{B}$ with respect to the given total order) is called a **broken circuit**. Finally, we say that $\mathcal{C}$ is an **nbc** if $\mathcal{C}$ contains no broken circuit. We refer to an **nbc** such that $|\mathcal{C}| = \ell$ as an **nbc frame**.

The set of all **nbc** provides a basis for the Orlik-Solomon algebra (4.1) as follows:

Theorem 5.4 (Björner-Ziegler [18], [43])**.** *The set*

$$\left\{ e_{i_1} \dots e_{i_k} + I_{\mathcal{A}} \in \left(\bigwedge E\right)/I_{\mathcal{A}} \mid (H_{i_1}, \dots, H_{i_k})_\prec \text{ is an } \mathbf{nbc} \right\}$$

provides a basis for the Orlik-Solomon algebra (as a vector space). (Here the notation $(H_{i_1}, \dots, H_{i_k})_\prec$ implies that the sequence in the parentheses satisfies $H_{i_1} \prec \cdots \prec H_{i_k}$.)

Definition 5.5 (Ziegler [94])**.** When $\mathcal{C} = (H_{i_1}, \dots, H_{i_\ell})_\prec$ is an nbc frame, then we say that $\mathcal{C}$ is a $\boldsymbol{\beta}$**nbc frame** if, for an arbitrary $H \in \mathcal{C}$, there exists some $H' \in \mathcal{A}_{\boldsymbol{C}}$ such that $H' \prec H$ and $(\mathcal{C} \setminus \{H\}) \cup \{H'\}$ is linearly independent.

Let $\beta\mathbf{nbc}(\mathcal{A}_{\boldsymbol{C}})$ be the set of all β**nbc** frames.

Theorem 5.6 ([38])**.** *Suppose that a local system has a suitable genericity, and the vanishing theorem (Theorem 5.1) holds. Define*

$$X_p := \bigcap_{k=p}^{\ell} H_{i_k} \quad (1 \le p \le \ell), \qquad \xi(B) := (X_1 < \cdots < X_\ell),$$

$$\omega_\lambda(X) := \sum_{X \subseteq H \in \mathcal{A}_{\boldsymbol{C}}} \lambda_H (d\alpha_H/\alpha_H), \qquad \zeta(B) := \bigwedge_{p=1}^{\ell} \omega_\lambda(X_p),$$

*for each β**nbc** frame $B = (H_{i_1}, \dots, H_{i_\ell})_\prec$. Then $\{\zeta(B) \mid B \in \beta\mathbf{nbc}(\mathcal{A}_{\boldsymbol{C}})\}$ provides a basis for the local system cohomology $H^\ell(M(\mathcal{A}_{\boldsymbol{C}}), \mathcal{L})$ (via the twisted de Rham* *correspondence). Here we assume that the monodromy of $\mathcal{L}$ around H is equal to* $\exp(2\pi\sqrt{-1}\lambda_H)$.

Schechtman and Varchenko [65], [82] (by using flags) constructed the dual basis of the basis in Theorem 5.6 to describe solutions of the Knizhnik-Zamolodchikov equations.

The above β**nbc** is naturally indexed by the set of bounded chambers $\mathbf{b}\mathcal{C}(\mathcal{A})$. Therefore, under the assumption of Theorem 5.1, we can construct a period matrix whose rows and columns are both indexed by $\mathbf{b}\mathcal{C}(\mathcal{A})$ due to the twisted de Rham pairing

$$H^\ell(M(\mathcal{A}),\mathcal{L}) \times H_\ell(M(\mathcal{A}),\mathcal{L}^\vee) \longrightarrow \boldsymbol{C}.$$

Varchenko [81] conjectured a formula for the determinant of the period matrix. The conjecture was affirmatively settled in [30] by adopting a β**nbc** basis for the local system cohomology. The proof deeply relies on the fact that β**nbc** bases behave well for the triples (Definition 2.3).

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