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GEODESIC FLOWS ON NEGATIVELY CURVED MANIFOLDS AND THE SEMI-CLASSICAL ZETA FUNCTION

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ABSTRACT. In this article, we report some recent advances in the study of spectral properties of transfer operators for geodesic flows on negatively curved manifolds. We first review related studies, explaining important concepts and introduce basic definitions. We then discuss recent results on spectral properties of the (generator of) transfer operators and also related analytic properties of dynamical zeta functions.

1. INTRODUCTION

The *geodesic flow* is a flow that describes the motion of a free particle on a Riemann manifold $(M, \|\cdot\|)$. It is known that the dynamical properties of the geodesic flow depends much on the curvature of the manifold M. If the (sectional) curvature of M is negative everywhere, the flow is unstable in the sense that the orbits depend on their initial conditions sensitively. Indeed the geodesic flows on negatively curved manifolds is known as types of hyperbolic flows and exhibit diverse and complex behavior of the orbits, which is called "Chaos". On the other hand, the unstable property of the flow takes effect as "diffusion" on the orbits and, as a consequence, the statistical properties of the orbits become stable and observable. Statistical properties of the orbits of dynamical systems have been studied extensively in the field of ergodic theory and the geodesic flows on negatively curved manifolds have been one of the prominent examples since the work [1] of Hadamard in the late nineteenth century.

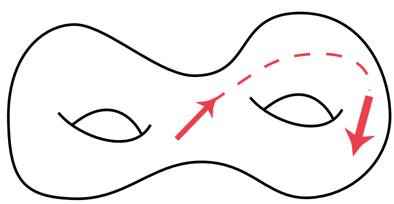

FIGURE 1. The geodesic flow on a surface.

The natural action of a flow on functions on the phase space and its variants are called *transfer operators* and are useful when we study the statistical properties of dynamical systems. For geodesic flows on negatively curved manifolds (or more

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general hyperbolic flows), recent studies have revealed that, if we consider the one-parameter group of transfer operators on some appropriate function spaces, the action is strongly continuous and, rather surprisingly, the generator has *discrete spectrum.* If we call the eigenvalues of the Laplacian on M "quantum mechanical spectrum of M", we may call the eigenvalues in such discrete spectrum "classical mechanical spectrum of M" in contrast. The main theorem (Theorem 5.1) presented in this article states that the discrete spectrum of the generator of transfer operators has "band structure". This is obtained in the author's joint work with Frédéric Faure at the Fourier Institute (Grenoble, France).

It is usually not possible to define the trace of transfer operators in the standard manner. But the Atiyah-Bott(-Guillemin) trace, which is defined as the integration of the Schwartz kernel of the operator on the diagonal, is well-defined for transfer operators under a mild condition on the flow and given as a weighted sum on the periodic orbits. Hence, having in mind the expression of the trace as the sum of eigenvalues, we are lead to the idea that the spectral properties of transfer operators dominate asymptotic distribution of periods of periodic orbits. This is a classical idea behind the dynamical zeta functions and enables us to deduce a result (Theorem 6.6) on the "semi-classical" (or "Gutzwiller-Voros") zeta function from the "band structure" of the spectrum.

In what follows, we first overview the ergodic properties of geodesic flows on negatively curved manifolds and then present some of the more recent results. The author attempts to present not only the results but the ideas behind the proofs and the difficulties that prevent further developments. But this seems beyond his ability and we are afraid that the readers may find some parts of the explanation not clear. We ask the readers to skip such parts.

2. Geodesic flows on negatively curved manifolds

2.1. **Geodesic flows.** The geodesic flow of a Riemann manifold $(M, \|\cdot\|)$ describes the motion of a free particle on M. Suppose that a particle is now at a point $x \in M$ on a Riemann manifold and has velocity $v \in T_xM$ as illustrated in Figure 1. Then the particle will move along the geodesic that is tangent to $v \in T_xM$ and proceeds $t \times \|v\|$ in length by time t. Since the norm $\|v\|$ of the initial velocity affects only the speed of time evolution, we will suppose that $\|v\| = 1$. Hence the phase space of the geodesic flow is the set of pairs of a point $x \in M$ and a unit tangent vector $v \in T_xM$ at x, that is, the unit tangent bundle $T_1M = \{v \in TM \mid \|v\| = 1\}$. The geodesic flow of M is the flow on T_1M:

$$f^t : T_1M \to T_1M. \tag{2.1}$$

We can present the geodesic flow as a Hamiltonian flow. We consider a Hamiltonian function $H(x, \xi) = \|\xi\|^2/2$ on the cotangent bundle T^*M, which corresponds to the kinetic energy of the particle. We consider local coordinates $x = (x_1, x_2, \dots, x_m)$ with $m = \dim M$ on M and suppose that the pair (x, ξ) with the dual coordinates $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ is a local chart on the cotangent bundle T^*M. Then the Hamiltonian flow for the Hamilton function $H(x, \xi)$ is the flow generated by the ordinary differential equations

$$\dot{x}_i = \frac{\partial H}{\partial \xi_i}, \qquad \dot{\xi}_i = -\frac{\partial H}{\partial x_i}, \qquad i = 1, 2, \dots, m$$

in such local coordinates. If we restrict such flow to the unit cotangent bundle T_1^*M, given by the equation $H \equiv 1/2$, we obtain a flow

$$f^t : T_1^*M \to T_1^*M. \tag{2.2}$$

This flow is equivalent to the geodesic flow in the sense that (2.1) and (2.2) are the same if we identify the unit tangent bundle T_1M and the unit cotangent bundle T_1^*M in the standard manner by using the Riemann metric.

2.2. **Curvature and exponential instability.** Dynamical properties of the geodesic flow depends much on the curvature of the manifold. If the sectional curvature of the manifold is negative everywhere, the geodesic flow has exponential instability, that is, the distances between nearby orbits grow exponentially fast. The next definition abstracts a geometric property of the geodesic flows on a negatively curved manifold, which leads to exponential instability.

Definition 2.1 (Anosov flow)**.** A flow $f^t : N \to N$ on a closed manifold N is called an *Anosov flow* if there exists a Df^t-invariant continuous decomposition of the tangent bundle TN,

$$TN = \langle V \rangle \oplus E_s \oplus E_u \tag{2.3}$$

such that

(1) $\langle V \rangle$ is the one-dimensional subbundle spanned by the generating vector field V of the flow. (Especially the vector field V is non-singular.
(2) The action of the differential Df^t on E_s (resp. E_u) is exponentially contracting (resp. expanding), that is, there exist some constants $\chi_0 > 0$ and $C \geq 1$ such that, the following holds for $t \geq 0$:

$$\|Df^t(v)\| \leq C \exp(-\chi_0 t)\|v\| \qquad (\forall v \in E_s),$$
$$\text{resp. } \|Df^t(v)\| \geq C^{-1} \exp(+\chi_0 t)\|v\| \quad (\forall v \in E_u).$$

Lemma 2.2 (Anosov [2])**.** *The geodesic flow on a closed negatively curved manifold is an Anosov flow.*

The subbundles E_s and E_u are unique for an Anosov flow f^t and called the *stable and unstable subbundle.* From the stable manifold theorem, the subbundle E_s (resp. E_u) is integrable and the integral manifold $W^s(p)$ (resp. $W^s(p)$) passing through $p \in N$ is called the *stable manifold* (resp. *unstable manifold*) of p. The stable and unstable manifolds are characterized as

$$W^s(p) = \{q \in N \mid d(f^t(q), f^t(p)) \to 0 \ (t \to +\infty)\},$$
$$W^u(p) = \{q \in N \mid d(f^t(q), f^t(p)) \to 0 \ (t \to -\infty)\}.$$

The foliation that the stable (resp. unstable) manifolds $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) constitute is called the *stable* (resp. *unstable*) *foliation.*

Remark 2.3. Beware that the stable and unstable subbundles are not smooth, that is, the subspaces $E_s(p)$ and $E_u(p)$ do not depend on p smoothly and the dependence is only Hölder continuous in general. This is the case even if we assume that the flow is real-analytic. We cannot expect more smoothness than Hölder continuous if we do not put strong conditions on symmetry of the system[1].

[1]See [3, 4]. The Hölder exponent of $E_s(p)$ and $E_u(p)$ depend on some dynamical exponents of the flow. See [5, Appendix A] and the references therein.

2.3. **Contact Flow.** The geodesic flow on a negatively curved manifold has an important geometric property other than it is an Anosov flow: It preserves a differential one-form called *contact form* on the phase space.

Definition 2.4 (Contact flow)**.** Suppose that a closed manifold N is of odd dimension and $\dim N = 2d+1$ for some integer $d \geq 1$.

(1) A differential one-form α on N is said to be a *contact form* if it satisfies the condition

$$\alpha \wedge (d\alpha)^d(p) \neq 0 \quad (\forall p \in N) \tag{2.4}$$

called the "complete non-integrability condition[2]". The $(2d+1)$-form $\alpha \wedge (d\alpha)^d$ is a volume form and called the Liouville volume.

(2) A flow $f^t : N \to N$ is said to be a *contact flow* if it preserves a contact form α on N. (Consequently a contact flow f^t preserves the Liouville volume $\alpha \wedge (d\alpha)^d$.)

On the cotangent bundle T^*M, we can define a differential one-form η, called the canonical one-form, by

$$\eta(v) = \xi(D\pi(v)) \quad \text{for } v \in T_{(x,\xi)}(T^*M),$$

where $\pi : T^*M \to M$ is the projection. For the geodesic flow, regarded as a flow on the unit cotangent bundle T_1^*M as in (2.2), we have the following.

Lemma 2.5. *The canonical one-form η restricted to the unit cotangent bundle T_1^*M, $\alpha := \eta|_{T_1^*M}$, is a contact form. The geodesic flow $f^t : T_1^*M \to T_1^*M$ is a contact flow that preserves this contact form α.*

From Lemma 2.2 and Lemma 2.5, the geodesic flow on a negatively curved manifold is an Anosov flow and a contact flow simultaneously, that is, a *contact Anosov flow.*

Remark 2.6. For a geodesic flow, the generating vector field V satisfies $\alpha(V) \equiv 1$ for the contact form α given in Lemma 2.5. For a general contact Anosov flow, $\alpha(V)$ is a non-zero constant[3]. Below we may and do suppose that $\alpha(V) \equiv 1$ by multiplying α by a constant.

For a contact Anosov flow, it is not difficult to see the following fact if we note that the flow f^t preserves the contact form α while the vectors in E_s and E_u are exponentially contracted and expended, respectively, and also that $d\alpha$ restricted to $\ker \alpha = E_s \oplus E_u$ is a symplectic form preserved by the flow.

Lemma 2.7. *For a contact Anosov flow $f^t : (N, \alpha) \circlearrowleft$, we have*

$$\ker \alpha = E_s \oplus E_u, \quad \dim E_s = \dim E_u = \dim \ker \alpha/2 = (\dim N - 1)/2 = d.$$

We are going to explain that the stable and unstable foliation for a contact Anosov flow satisfies a "non-integrability" condition. For facility of explanation, we suppose $\dim N = 3$, $\dim E_s = \dim E_u = 1$, and consider (in a flow box) coordinates. We draw a piecewise smooth curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ in N, as illustrated in Figure 2, so that:

[2]This implies that the restriction of $d\alpha$ to $\ker \alpha$ is non-degenerate (as a bilinear form). This is opposite to the situation where $\ker \alpha$ is integrable (i.e. the restriction of $d\alpha$ to $\ker \alpha$ vanishes).

[3]Since $\alpha(V)$ is invariant with respect to the flow, it is a constant from Theorem 3.1 and non-zero from Lemma 2.7.

- each of the curves γ_1 and γ_3 is contained in an unstable manifold, while each of the curves γ_2 and γ_4 is contained in a stable manifold, and
- the projection of γ along the flow is a closed curve that bounds a region R.

Then, the end points of the curve γ are on an orbit of the flow and the time T to move one to another by the flow is given as the integral

$$T = \int_R d\alpha.$$

This does not vanish provided that the region R has positive area. Hence, though the distributions E^s and E^u are integrable, the sum $E^s \oplus E^u$ are far from integrable. In other words, the stable foliation $\mathcal{F}^s$ and the unstable foliation $\mathcal{F}^u$ are "jointly" non-integrable.

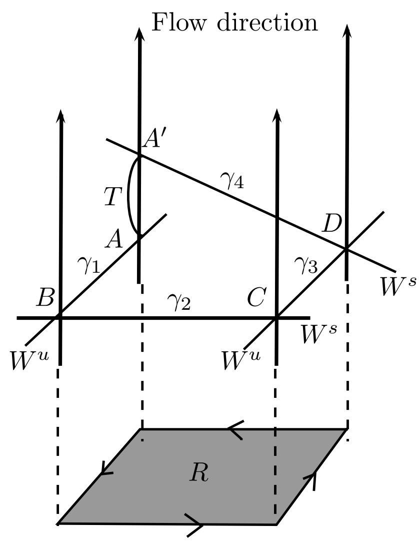

FIGURE 2. Non-integrability of the stable and unstable foliation.

3. ERGODIC PROPERTIES AND DECAY OF CORRELATION

Ergodic theory studies statistical properties of the orbits of dynamical systems. Though its scopes and applications are diverse, we restrict ourselves here to the case of geodesic flows on a negatively curved manifolds. For simplicity, we write $N = T_1^*M$ and denote the geodesic flow as $f^t : N \to N$. The contact form on $N = T_1^*M$ (given as the restriction of the canonical one-form) is written α. Let μ denote the Liouville measure normalized so that $\mu(N) = 1$.

3.1. **Ergodicity.** We regard a function $\psi : N \to \mathbb{R}$ as an observable. Its time evolution is given by the family

$$\psi \circ f^t(x) = \psi(f^t(x)) \qquad (-\infty < t < +\infty). \tag{3.1}$$

A standpoint of ergodic theory is that we regard (3.1) as a one-parameter family of random variable on the probability space (N, μ) and apply probability theory to it. To this end, we first have to check that the system is ergodic.

Theorem 3.1 (Hopf [6], Anosov [2]). *If an Anosov flow $f^t : N \to N$ on a closed manifold N preserves a smooth volume μ on N such that $\mu(N) = 1$, it is ergodic with respect to μ, that is, for any L^1 function $\psi \in L^1(N, \mu)$, the time average*

$$\overline{\psi}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(f^t(x))dt$$

exists and coincides with the space average $\int \psi \, d\mu$ at μ-almost every point $x \in N$. In particular, the geodesic flow on a negatively curved manifold is ergodic with respect to the Liouville measure.

The idea of the proof is not difficult and given by the following argument, called the Hopf argument. (But see the remark below.)

(1) From Birkhoff ergodic theorem [7] the time average $\overline{\psi}(x)$ exists and coincides with the limit for $T \to -\infty$ at μ-a.e. $x \in N$.
(2) The time average $\overline{\psi}(x)$ takes a constant value on each of the stable manifolds because the points on a stable manifold become close to each other as T gets large. Since $\overline{\psi}(x)$ coincides with the limit for $T \to -\infty$, it also takes a constant value on each of the unstable manifolds (if we ignore the sets of the measure zero with respect to μ).
(3) The time average $\overline{\psi}(x)$ also takes a constant value for points on each orbit.
(4) Any two points in N are connected by a piecewise smooth curve that is obtained by connecting curves each of which is contained in a stable manifold, an unstable manifold or an orbit. Therefore, by (2) and (3), the time average $\overline{\psi}(x)$ takes a constant value at almost every point $x \in N$.

Remark 3.2. The explanation above is simplistic actually. To justify the argument above, we need a basic property of the stable (unstable) foliation, that is, *absolute continuity of the homonomy map.* Hopf was able to prove Theorem 3.1 in the case of geodesic flows on surfaces because the homonomy map of the stable (unstable) foliation is C^1 in such case. But, for the higher dimensional case, it remained open until the work of Anosov [2] in which the homonomy map is proved to be absolutely continuous (though it is only Hölder continuous in general). See, for example, [7, Ch.3].

3.2. **Mixing and decay of correlation.** Once we established ergodicity, we next consider the correlations between the "random variables" (3.1). If the correlation between $\psi \circ f^t$ and $\psi \circ f^s$ decay fast as $|t - s|$ gets large, we may regard that the random variables (3.1) are almost independent of each other and expect that some limit laws, such as the central limit theorems, hold true. If the correlation decays only slowly, we observe deterministic properties of the dynamical system in such slow decay.

For functions $\varphi, \psi \in L^2(N, \mu)$ and $t \geq 0$, we define[4]

$$\begin{aligned}\mathrm{Cor}(\varphi, \psi; t) &= \int \varphi \cdot (\psi \circ f^t) d\mu - \int \varphi d\mu \cdot \int \psi d\mu \\ &= \int (\varphi \circ f^{-t}) \cdot \psi d\mu - \int \varphi d\mu \cdot \int \psi d\mu. \qquad (3.2)\end{aligned}$$

[4]The second equality follows from the fact that μ is f^t invariant.

Then the correlation between $\psi \circ f^t$ and $\psi \circ f^s$ with $t > s$ is given by

$$\mathrm{Cor}(\psi, \psi; t-s) = \int (\psi \circ f^t) \cdot (\psi \circ f^s) d\mu - \left(\int \psi d\mu \right)^2 .$$

We consider how fast the correlation $\mathrm{Cor}(\varphi, \psi; t)$ decays as $t \to \infty$. The next theorem due to Sinai is fundamental.

Theorem 3.3 (Sinai [8, 9])**.** *The geodesic flow $f^t : N \to N$ on a negatively curved manifold is mixing. That is, for any L^2 functions $\varphi, \psi \in L^2(N, \mu)$, we have*

$$\lim_{t\to\infty} \mathrm{Cor}(\varphi, \psi; t) = 0. \tag{3.3}$$

Here we used the word "mixing" because, if we let ψ and φ be the indicator functions of Borel measurable subsets A and B, respectively, and suppose $\mu(A) > 0$, the condition (3.3) implies

$$\lim_{t\to\infty} \frac{\mu(A \cap f^t(B))}{\mu(A)} = \mu(B), \tag{3.4}$$

that is, the image $f^t(B)$ becomes homogeneous in N as $t \to \infty$.

In Theorem 3.3, we have to use the fact that the geodesic flow is not only an Anosov flow but also a contact flow (or some other general condition on joint non-integrability of the stable and unstable foliations). Indeed the next example shows that, if we only assume that the flow is an Anosov flow preserving a smooth volume, the conclusion of the theorem is not true in general.

Example 3.4. From the Arnold's cat map

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

we construct the mapping torus $X_f = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]/(x,1) \sim (f(x), 0)$ and consider the suspension flow (with a constant roof function)

$$f^t : X_f \to X_f, \quad f^t(x,s) = (x, s+t).$$

This is obviously an Anosov flow preserving the volume, but (3.4) does not hold for $A = B = \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1/2]$.

As we noted in the last paragraph of Section 2, the situation in the case of geodesic flows (or contact Anosov flows) is opposite to that in the example above.[5]

3.3. **Exponential decay of correlation.** Though the mixing property given in Theorem 3.3 is fundamental, for the purpose of applying the argument in probability theory to the random variables (3.1), we need some more quantitative estimates on the rate of decay in (3.3). Below we discuss such quantitative estimates. Note that, if we only assume that the functions φ and ψ are L^2 functions, it is not possible to get an appropriate quantitative estimate on $\mathrm{Cor}(\varphi, \psi; t)$, as is known even for much simpler cases of discrete dynamical systems. Below we assume that φ and ψ are C^∞ functions for simplicity.

Even under such additional assumptions on functions φ and ψ, the quantitative estimate for the decay rate in (3.3) was a difficult problem. (This is in contrast

[5]Mixing property holds true under a much more general assumption than contact Anosov flows. For Anosov flows preserving a smooth volume, they are not mixing only if it is conjugated to the suspension flow of an Anosov diffeomorphism with a constant roof function [10].

to the case of discrete hyperbolic dynamical systems where the exponential decay rate was obtained rather easily.) In the case of geodesic flow on surfaces with negative constant curvature, Ratner [11] and Moore [12] proved exponential decay of correlations using a method from representation theory[6]. But there had not been much progress for more general cases, until a breakthrough was made by Chernov and Dolgopyat in the late 1990's. Chernov [13] proved that the decay rate is bounded by a stretched exponential rate $\mathcal{O}(e^{-c\sqrt{t}})$ by using a method called Markov approximation. Then Dolgopyat [14] was succeeded to prove that the decay rate is actually exponential right after the work of Chernov.

Theorem 3.5 (Dolgopyat, Liverani [5])**.** *The geodesic flow on a negatively curved manifold exhibits exponential decay of correlation. That is, there exists some constant $\epsilon > 0$ such that, for any C^∞ functions φ and ψ, we have*

$$|\mathrm{Cor}(\varphi, \psi; t)| \leq Ce^{-\epsilon t} \quad (\forall t \geq 0),$$

where C is a constant depending on ψ and φ.

Remark 3.6. Dolgopyat proved the theorem above for the case of geodesic flows on surfaces (but under weaker assumptions on differentiability of ψ and φ). Liverani [5] extended the result to general contact Anosov flows, by using a functional analytic method, invented by him, together with the argument by Dolgopyat.

Theorem 3.5 is one of the landmarks of ergodic theory in 1990's. The argument used in the proof is called Dolgopyat argument and extended to many directions later. The basic idea of the argument is not very difficult as we sketch below. But, because the (un)stable foliation is not smooth, one need to overcome some technical difficulties. This is indeed a great achievement obtained only by the outstanding ability of Dolgopyat (and Chernov, Liverani) in analysis.

The idea behind the proof of Theorem 3.5 (to the understanding of the author) is sketched roughly as follows.

(1) We regard the definition (3.2) of $\mathrm{Cor}(\varphi, \psi; t)$ as the value that the distribution $\varphi \circ f^{-t}$ takes against a smooth function ψ. From this viewpoint, the problem is the decay of $\varphi \circ f^{-t}$ as a distribution.
(2) Since flow f^t is a translation in the flow direction (at least if we look them in the flow box coordinates), it virtually preserves the Fourier components of the functions in the flow direction. Hence we decompose functions on N with respect to the frequency in the flow diction and consider the action of the flow on each of the components. (We focus on the high frequency components because the low frequency components can be treated in the same manner as the case of discrete dynamical systems.)
(3) Since the flow f^t is exponentially expanding in on the unstable manifolds, we may suppose that the function $\varphi \circ f^{-t}$ for large $t > 0$ is virtually constant on each of the unstable manifolds.
(4) We apply the operation of "averaging the function $\varphi \circ f^{-t}$ along the stable manifolds". For instance, if the function $\varphi \circ f^{-t}$ takes values a_1, a_2 at two nearby points p_1, p_2 on a stable manifold $W^s(p)$, we replace those values by the average $(a_1 + a_2)/2$. This operation changes the value of $\mathrm{Cor}(\varphi, \psi; t)$,

[6]In this case, the unit tangent bundle T_1M of M is identified with a quotient space of $SL(2, \mathbb{R})$ with respect to a discrete subgroup and the geodesic flow may be regarded as a part of the action of $SL(2, \mathbb{R})$ on it. From this fact, we can make use of the theory on the representations of $SL(2, \mathbb{R})$.

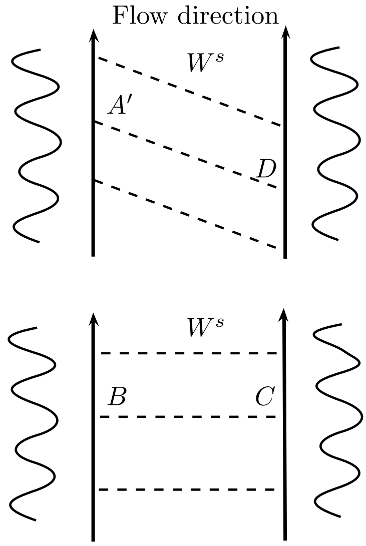

FIGURE 3. Averaging along stable manifolds and non-integrability.

but the difference (as a distribution) will decay exponentially fast by the time evolution afterward because the flow is exponentially contracting along the stable manifolds. Hence such differences will not affect the proof of exponential decay of correlation. (We ignore such differences.)

(5) At each of certain time intervals, we apply the operation of averaging as above and show that the operation reduce $\varphi \circ f^{-t}$ at a small but fixed rate. For this, we use the joint non-integrability of the stable and unstable foliation, explained in Section 3. Consider the situation illustrated in Figure 2. If the averaging does not take effect at the points B and C on the front side (unfortunately), we can find, by virtue of the joint non-integrability, another pair of points A' and D at which the averaging takes effect. (See Figure 3 and compare it with Figure 2.) Hence the function $\varphi \circ f^{-t}$ decays at a certain rate at each time interval by the operation of averaging.

As we are showing a quantitative estimate, the explanation above is of course not sufficient. But we stress that the operation of "averaging along the stable manifolds" yields the exponential decay of correlation. A motivation of the author's study explained in the next section is to put this argument into a framework of functional analysis.

4. Spectral gap

4.1. **Spectral gap and exponential decay of correlation.** We consider the transfer operator $\mathcal{L}^t : C^\infty(N) \to C^\infty(N)$ defined by

$$\mathcal{L}^t u(z) = u \circ f^{-t}(z).$$

Then we may write the correlation $\mathrm{Cor}(\varphi, \psi; t)$ as

$$\mathrm{Cor}(\varphi, \psi; t) = \langle \mathcal{L}^t(\varphi - \Pi_0(\varphi)), \psi - \Pi_0(\psi) \rangle,$$

where Π_0 is the rank one projection operator that carries a function to a constant function taking the value of its average. (The operator Π_0 commutes with $\mathcal{L}^t$.) Here the bracket $\langle \cdot, \cdot \rangle$ on the right hand side denotes the pairing of a distribution on the left with a function on the right.

We will take a Hilbert space $\mathcal{H}$ of distributions such that $C^\infty(N) \subset \mathcal{H} \subset C^\infty(N)'$ and that the action of the transfer operators $\mathcal{L}^t$ on it is bounded. The essential spectral radius of the bounded operator $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$ is by definition the infimum of the spectral radius of its perturbations by finite rank operators (or, equivalently, by compact operators):

$$\rho_{ess}(\mathcal{L}^t|_{\mathcal{H}}) = \inf\{\rho((\mathcal{L}^t - K)|_{\mathcal{H}}) \mid K : \mathcal{H} \to \mathcal{H} \text{ is of finite rank.}\}$$

From the perturbation theory of operators, we see that the spectral set of the operator $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$ on the outside of the disk of radius $\rho_{ess}(\mathcal{L}^t|_{\mathcal{H}})$ consists of discrete eigenvalues with finite multiplicity.

If the essential spectral radius $\rho_{ess}(\mathcal{L}^t|_{\mathcal{H}})$ is smaller than the spectral radius, which is 1, we say that the transfer operator $\mathcal{L}^t$ on $\mathcal{H}$ is *quasi-compact* or has *spectral gap*. If such condition is fulfilled, then we can show, by a simple argument[7] using the mixing property in Theorem 3.3, that the leading eigenvalue 1 is simple and is the only eigenvalue on the unit circle and that the other part of the spectral set is contained in a disk $\{|z| < 1 - \eta\}$ with $\eta > 0$. In particular, the exponential decay of correlation in Theorem 3.5 follows immediately.

4.2. **Quasi-compactness of transfer operators.** It was noted (as a footnote) in [5] that the exponential decay of correlation would be deduced from quasi-compactness of the transfer operator on some function space. This is achieved in the author's results presented below. Note that the Hilbert space $\mathcal{H}$ in the statement below is a generalized Sobolev space with a weight function $\mathcal{W}^r$ on T^*N adapted to the hyperbolic decomposition (2.3) and called an *anisotropic Sobolev space.*

Theorem 4.1 (Tsujii [15, 16])**.** *There exists a Hilbert space $C^\infty(N) \subset \mathcal{H} \subset C^\infty(N)'$ such that the transfer operator $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$ acting on $\mathcal{H}$ is bounded at least for sufficiently large $t > 0$ and the essential spectral radius is* $\exp(-\chi_u t/2)$*, where*

$$\chi_u = \lim_{t\to\infty} \frac{1}{t} \log \left\| \det(Df^t|_{E_u}) \right\|_\infty^{1/t} > 0.$$

As illustrated in Figure 4, the spectrum of $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$ on $|z| > \exp(-\chi_u t/2)$ consists of discrete eigenvalues of finite multiplicity. There is only one simple eigenvalue 1 *on the unit circle and the rest of the spectral set is contained in the interior of the unit disk.*

As we have explained, Theorem 4.1 implies exponential decay of correlation in Theorem 3.5. Moreover, since we now have a concrete bound on the essential spectral radius, we can give the following asymptotic estimate on the correlation.

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[7]Suppose that $e^{i\lambda t}$ is an eigenvalue of $\mathcal{L}^t$ with absolute value 1 and that $u \in \mathcal{H}$ belongs to the corresponding eigenspace S. Since the limit $\lim_{T\to\infty} T^{-1} \int_0^T e^{-i\lambda t}\mathcal{L}^t dt$ is the projection to S and since $C^\infty(N)$ in dense in $\mathcal{H}$, we can find $\varphi \in C^\infty(N)$ such that $\lim_{T\to\infty} T^{-1} \int_0^T e^{-\lambda t}\mathcal{L}^t \varphi dt = u$. The left hand side converges weakly in L^2 for some subsequence of T (because of boundedness in L^2). Hence $u \in L^2$. From the mixing property, this implies $u \equiv 1$ and $\lambda = 0$.

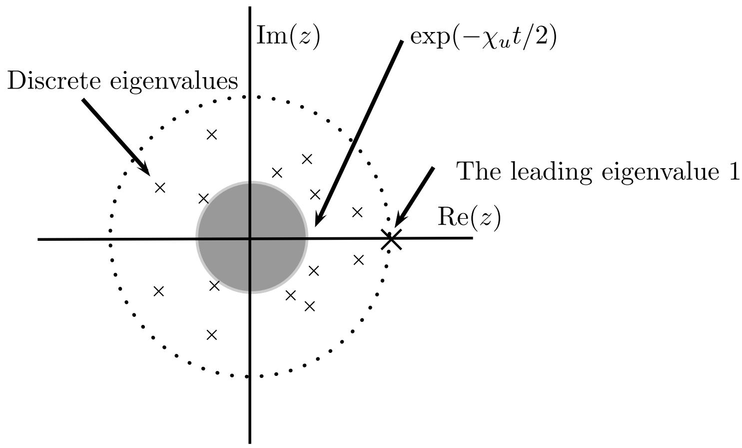

FIGURE 4. Spectrum of $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$.

Corollary 4.2. *For any $\epsilon > 0$, there exists complex numbers χ_k, $1 \le k \le \ell$, such that, for sufficiently large $t \ge 0$, the discrete eigenvalues of $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$ on the region $|z| \ge \exp(-(\chi_u - \epsilon)t/2)$ are $\exp(\chi_k t)$ for $1 \le k \le \ell$. Let m_k be the maximum of the size of the Jordan blocks of $\mathcal{L}^t$ for the eigenvalue $\exp(\chi_k t)$. Then, for any C^∞ functions $\varphi, \psi \in C^\infty(N)$, we have*

$$\mathrm{Cor}(\varphi, \psi; t) = \sum_{k=1}^{\ell} \sum_{m=0}^{m_k-1} C_{k,m}(\varphi, \psi) t^m e^{\chi_k t} + \mathcal{O}\big(\exp(-(\chi_u - \epsilon)t/2)\big),$$

where $C_{k,m}(\varphi, \psi)$ is a constant depending on φ and ψ.

Remark 4.3. If we take a different Hilbert (or Banach) space in Theorem 4.1, it may be possible to get better bound on the essential spectral radius than that in the theorem. Indeed it is desirable that a better bound is given as a quantity such as topological pressure for some potential. However, note that, since the bound given in Theorem 4.1 is optimal in the case of geodesic flow on a negative constant curvature, it is not possible to give a bound like $\exp(-\chi_u t)$ in general.[8]

Below we explain the idea in the proof of Theorem 4.1. The following lengthy explanation is for the readers who are familiar with dynamical systems theory and interested in technical matters. The readers can skip it if otherwise.

(1) We analyze the transfer operators $\mathcal{L}^t$ regarding them as Fourier integral operators. That is, we decompose each function u on N into "wave packets" and consider how the wave packets are transferred by $\mathcal{L}^t$. Here a "wave packet" is a function which is localized in the space N and also whose Fourier transform (on local coordinates) is localized. For instance, if we define a function $\phi_{x,\xi} : \mathbb{R}^d \to \mathbb{C}$ for a point $(x, \xi) \in \mathbb{R}^{2d} = T^*\mathbb{R}^d$ and $\hbar > 0$ by

$$\phi_{x,\xi}(y) = \exp(i\xi \cdot (y - x) - |y - x|^2/\hbar)$$

[8]This is in contrast to the situation for Anosov diffeomorphism in [17, 19], where we can take Hilbert spaces on which the action of the transfer operator has arbitrarily small essential spectral radius.

in local coordinates, it is a wave packet localized in the $\hbar^{1/2}$-neighborhood of the point x and its Fourier transform is localized around the $\hbar^{-1/2}$-neighborhood of ξ ($\hbar > 0$ is a parameter that indicates the size of the wave packet). Intuitively, it is clear that the transfer operator $\mathcal{L}^t$ carries a wave packet $\phi_{x,\xi}$ for $(x,\xi) \in T^*N$ to wave packets for points around $(x',\xi') = (Df^{-t})^*(x,\xi)$. So, for our analysis of $\mathcal{L}^t$, it is important to consider the dynamics of the map

$$(Df^{-t})^* : T^*N \to T^*N, \tag{4.1}$$

which is called the *canonical map associated to* $\mathcal{L}^t$ regarded as a Fourier integral operator.

(2) When we study a dynamical system by topological methods, one of the first things to do is to identify its non-wandering set. For the dynamics of the canonical map $(Df^{-t})^*$, the non-wandering set is the one-dimensional subbundle $\langle\alpha\rangle$ of the cotangent bundle T^*N spanned by the contact form α. To see this, we first note that, through the one-to-one correspondence $(x,s) \in X \times \mathbb{R} \mapsto s \cdot \alpha(x) \in T^*N$ between $X \times \mathbb{R}$ and $\langle\alpha\rangle$, we can identify the flow $(Df^{-t})^*$ on $\langle\alpha\rangle$ with $f^t \times \mathrm{Id}$ on $X \times \mathbb{R}$. So $\langle\alpha\rangle$ is contained in the non-wandering set. On the other hand, since the flow $(Df^{-t})^*$ is hyperbolic in the normal directions to $\langle\alpha\rangle$, the points on the outside of $\langle\alpha\rangle$ are not recurrent.

(3) Since the points on the outside of the non-wandering set $\langle\alpha\rangle$ are not recurrent, we can assign some weight to each of the corresponding wave packets so that the action of the transfer operator $\mathcal{L}^t$ on such wave packets looks strongly dissipative. More concretely, for each positive number $r > 0$, we take a smooth function $\mathcal{W}^r : T^*N \to \mathbb{R}_+$ so that, if $(x,\xi) \in T^*N$ is not close to $\langle\alpha\rangle$, we have

$$\mathcal{W}^r((Df^{-t})^*(x,\xi)) \le e^{-r\chi_0 t} \cdot \mathcal{W}^r(x,\xi).$$

Then we introduce a norm which counts the wave packet corresponding to a point $(x,\xi) \in T^*N$ with the weight $\mathcal{W}^r(x,\xi)$. (Indeed the Hilbert space $\mathcal{H}$ in Theorem 4.1 is defined as the generalized Sobolev space with weight $\mathcal{W}^r(\cdot)$.) The function $\mathcal{W}^r(\cdot)$ is constructed as follows. We consider the dual of the decomposition (2.3), $T^*N = \langle\alpha\rangle \oplus E_s^* \oplus E_u^*$ where E_s^* (resp. E_u^*) is the subbundle on which the action of $(Df^{-t})^*$ is expanding (resp. contracting)[9]. Then the point in T^*N is expressed as $s\alpha(x) + \xi_s + \xi_u$, $x \in N$, $s \in \mathbb{R}$, $(\xi_s,\xi_u) \in E_s^*(x) \oplus E_u^*(x)$). We define

$$\mathcal{W}^r(s\alpha(x) + \xi_s + \xi_u) = \Big\langle \langle s\rangle^{-1/2} \cdot \|(\xi_s,\xi_u)\| \Big\rangle^{\mathrm{ord}([(\xi_s,\xi_u)])}, \tag{4.2}$$

where we set $\langle s\rangle = \sqrt{1+s^2} \ge 1$ and $\mathrm{ord} : \mathbb{P}(E_s^* \oplus E_u^*) \to [-r,r]$ is a function on the projective bundle $\mathbb{P}(E_s^* \oplus E_u^*)$ such that $\mathrm{ord}([\xi_s,\xi_u]) = -r$ (resp. $\mathrm{ord}([\xi_s,\xi_u]) = +r$) if $\|\xi_s\| \ge 2\|\xi_u\|$ (resp. $\|\xi_u\| \ge 2\|\xi_s\|$).

(4) The norm introduced above allows us to focus on the action of $\mathcal{L}^t$ on the wave packets corresponding to the points in (a small neighborhood of) $\langle\alpha\rangle$. If we let $\mathcal{L}^t$ act on the wave packet corresponding to a point $(x,\xi) = s \cdot \alpha(x) \in \langle\alpha\rangle$ and decompose the image into wave packets, we will get a set of wave packets corresponding to points near the point $(Df^{-t})^*(x,\xi) \in T^*N$, but the image will spread along the direction E_s^*. This operation is unitary if we consider it with respect to the usual L^2 norm. But it is contracting with respect to the norm introduced above because the wave packets are counted with the weight function $\mathcal{W}^r(x,\xi)$ which takes small

[9] The notation E_s^* for the expanding direction may be misleading. For the latter argument, we would like to take E_s^* as the dual space of E_s.

values at the points far from $\langle\alpha\rangle$ in the direction of E_s^*. Indeed we can see that the norm is contracted by the rate $|\det(Df^t|_{E_u})|^{-1/2} \lesssim \exp(-\chi_u t/2)$. This rate appears as the bound on the essential spectral radius in Theorem 4.1. But notice that this is not the end of the argument. (In fact, we have not used the non-integrability of α!)

(5) The last important problem is that, if several wave packets, which are placed mutually in the stable direction, are transferred by $\mathcal{L}^t$ with large $t > 0$, the image will come very close to each other and hence the norm may increase by the interference between them. Indeed, if we did not have the joint non-integrability condition of the stable and unstable foliation, the effect of the interference might compensate the the contraction given in (4) above. (This in fact happens in the case of Example 3.4.) The complete non-integrability of the contact form α implies that such an increase of the norm by interference will not occur. In fact, since the non-integrability condition implies that the direction of α is "twisted" along stable manifolds and hence there is some difference of the frequency vectors corresponding to the images of the wave packets, then the images are virtually orthogonal to each other. With this, we conclude that the operator norm of the transfer operator $\mathcal{L}^t$ on $\mathcal{H}$ is bounded by the rate $\exp(-\chi_u t/2)$ if we ignore its action on the low frequency part of the functions, which is compact.

We have to say that the explanation above is (again) not very precise and ignores some technical details. But we would like to emphasize that the mechanism of cancellation by "averaging along the stable manifolds", explained after Theorem 3.5 in Subsection 3.3, is replaced by the orthogonality between the image of the wave packets explained in (5) above.

Remark 4.4. For the parallel results for discrete hyperbolic dynamical systems, such as Anosov diffeomorphisms, we refer to [17, 18, 19, 20, 21].

5. Semiclassical analysis of transfer operators

We continue discussing the spectrum of transfer operators for the geodesic flow $f^t : N = T_1^*M \circlearrowleft$ on a closed Riemann manifold M. In the following, we present the results of the author's joint work with F. Faure [22].

Generalizing the definition of the transfer operator in the last section slightly, we consider the transfer operators of the form

$$\mathcal{L}^t u(x) = (g^t \cdot u)(f^{-t}(x)), \tag{5.1}$$

where $g^t(x)$ is a one-parameter family of functions on N satisfying the cocycle condition

$$g^{t+s}(x) = g^t(f^s(x)) \cdot g^s(x),$$

which is necessary for the family $\mathcal{L}^t$ to be a one-parameter group.

The next theorem shows that the generator of a one-parameter group of transfer operators $\mathcal{L}^t$ as above has discrete spectrum and, further, the discrete spectrum has a "band structure" parallel to the imaginary axis. The Hilbert space $\widetilde{\mathcal{H}}^r$ in the statement is essentially same as $\mathcal{H}$ in Theorem 4.1, but modified slightly so that $\mathcal{L}^t$ for small $t \geq 0$ is bounded on $\widetilde{\mathcal{H}}^r$.

Theorem 5.1 (Faure-Tsujii [22]). *For any $r > 0$, there exists a Hilbert space $C^\infty(N) \subset \widetilde{\mathcal{H}}^r \subset C^\infty(N)'$ such that $\mathcal{L}^t : \widetilde{\mathcal{H}}^r \to \widetilde{\mathcal{H}}^r$ for $t \geq 0$ is a strongly continuous one-parameter family of bounded operators and the spectrum of its generator A in* *the region* $\mathrm{Re}(s) > -r\chi_0$ *consists of discrete eigenvalues with finite multiplicity, where* $\chi_0 > 0$ *is the constant in Definition* 2.1 *of Anosov flow. For any* $\epsilon > 0$, *the discrete spectrum of* A *is contained in the* ϵ*-neighborhood of the union of zonal regions*

$$B = \bigcup_{k=0}^{\infty} B_k, \qquad B_k = \{z \in \mathbb{C} \mid \gamma_k^- \le \mathrm{Re}\,(z) \le \gamma_k^+\},$$

but for finitely many exceptions (depending on ϵ*), where*

$$\gamma_k^- := \lim_{t\to\infty} \frac{1}{t} \log \min_{x\in N} \left(|g^t(x)| \cdot (\det |(Df^t|_{E_u})(x)|)^{-1/2} \cdot \|Df^t|_{E_u}\|_{\max}^{-k} \right)$$
$$\le \gamma_k^+ := \lim_{t\to\infty} \frac{1}{t} \log \max_{x\in N} \left(|g^t(x)| \cdot \det |(Df^t|_{E_u})(x)|^{-1/2} \cdot \|Df^t|_{E_u}\|_{\min}^{-k} \right)$$

(See Figure 5. $\|L\|_{\max}$ *and* $\|L\|_{\min}$ *denotes the maximum and minimum singular values of a linear map* L*.)*

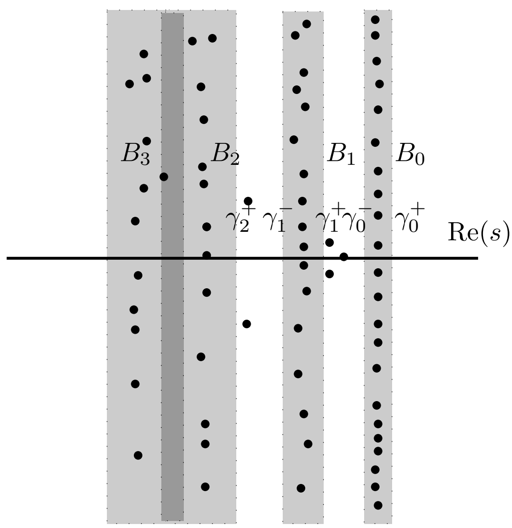

FIGURE 5. Spectrum of the generator of $\mathcal{L}^t$.

Remark 5.2. The discrete eigenvalues of the generator A given in Theorem 5.1 above are intrinsic to the one-parameter (semi-)group $\mathcal{L}^t$ and does not depend on the function space (under a very mild condition). In particular, if we let $r > 0$ be larger in Theorem 5.1, we observe discrete spectrum of the generator A in a larger region, but those in the region for the original r will not change.

In general, the zonal regions B_k, $k = 0, 1, 2, \ldots$, may overlap each other especially for large $k \ge 0$. In some cases, the region B may coincide with the half plane $\mathrm{Re}(s) \le \gamma_0^+$ and the "band structure" is vacuous in such cases. But we observe several bands B_k disjoint from the others in the following cases:

(a) If the (sectional) curvature of the manifold M is close to constant and if the cocycle $g^t(x)$ is also, the bounds γ_k^+ and γ_k^- are close to each other and each zonal region B_k is narrow. In this case, the zonal regions B_k with small k will be disjoint from the others.

(b) If the cocycle g^t is close to $g_0^t(x) = (\det |(Df^t|_{E_u})(x)|)^{1/2}$, the bounds γ_0^+ and γ_0^- are close to 0. So B_0 is a small neighborhood of the imaginary axis and disjoint from the other zonal regions B_k with $k > 0$ because $\gamma_1^+ \le -\chi_0 < 0$.

Theorem 5.3 (Faure-Tsujii [22])**.** *(Continued from Theorem* 5.1.*) If the zonal region B_0 is disjoint from the other zonal regions B_k, $k = 1, 2, \dots$, there are infinitely many eigenvalues of the generator A in B_0. Further, we have that*

(1) (An analogy of the Weyl law) *For any $\delta > 0$, there is a constant $C > 1$ such that, if $|\nu|$ is sufficiently large, we have*

$$C^{-1}\nu^d \le \frac{\#\{\text{eigenvalues of } A \text{ in } [r_0^- - \epsilon, r_0^+ + \epsilon] \times i[\nu, \nu + |\nu|^\delta]\}}{|\nu|^\delta} \le C\nu^d.$$

(2) (Concentration of eigenvalues along a line parallel to the imaginary axis) *In the limit* $\mathrm{Im}(s) \to \pm\infty$, *most of the eigenvalues in the region B_0 are contained in a small neighborhood of the line*

$$\mathrm{Re}(s) = \bar{\gamma}_0 := (1/t) \int \log(|g^t| \cdot |\det(df^t|_{E_u})|^{-1/2}) d\mu \in [\gamma_0^-, \gamma_0^+].$$

(Note that $\bar{\gamma}_0$ does not depend on $t > 0$.) More precisely, for any $0 < \epsilon' < \epsilon$, the ratio

$$\frac{\#\{\text{eigenvalues of } A \text{ in } ([\gamma_0^- - \epsilon, \gamma_0^+ + \epsilon] \setminus [\bar{\gamma}_0 - \epsilon', \bar{\gamma}_0 + \epsilon']) \times i[\nu - 1, \nu + 1]\}}{|\nu|^d}$$

tends to 0 *as* $\nu \to \pm\infty$.

Remark 5.4. In the claims of Theorem 5.1 and Theorem 5.3, the following points are not very satisfactory, but technical difficulties prevent us from improving them.

(1) We expect that the bounds for the sides of the zonal regions B_k will be improved. In the definition of $\gamma_k^\pm$, we take the maximum (or minimum) over the points $x \in N$, but it will be much better if we can replace it by some average on $x \in N$ and $\gamma_k^\pm$ is given as a quantity such as topological pressure for some potential.
(2) In Theorem 5.3, we assumed that B_0 is separated from other B_k with $k \ge 1$. But this assumption will not be necessary.
(3) It will be possible to prove the analogue (1) of the Weyl law for $\delta = 0$.
(4) In the proof of Theorem 5.1, we construct the spectral projection operator Π_k for the spectral set in B_k from the corresponding projection operator on local charts. But we expect that the spectral projection operators Π_k can be constructed directly by a more global and geometric manner.

The proof of Theorem 5.1 and Theorem 5.3 is obtained as an extension of the argument in that of Theorem 4.1, but we use a much more refined argument using the method of semi-classical analysis. (Below we suppose $g^t \equiv 1$ for simplicity for a while.) As explained in the last section, in the proof of Theorem 4.1, the dynamical properties of the flow of the canonical maps, $(Df^{-t})^* : T^*N \to T^*N$, around the non-wandering set $\langle\alpha\rangle \subset T^*N$ played the key roles. In particular, we used the non-integrability of the contact form in the last step. The point of the proofs of Theorem 5.1 and Theorem 5.3 is to interpret the non-integrability (2.4) of the contact form α as the fact that[10] the non-wandering set $\langle\alpha\rangle \subset T^*N$ is a symplectic submanifold of T^*N (except for the zero section). From this, we see that the tangent space $T_q(T^*N)$ at $q \in \langle\alpha\rangle$ is decomposed as the direct sum of the tangent space $T_q\langle\alpha\rangle$ of $\langle\alpha\rangle$ and its symplectic orthogonal $(T_q\langle\alpha\rangle)^\perp$:

$$T_qN = T_q\langle\alpha\rangle \oplus (T_q\langle\alpha\rangle)^\perp. \tag{5.2}$$

Since $(Df^{-t})^*$ preserves the symplectic structure on T^*N, it preserves also the decomposition (5.2). This implies that the transfer operator $\mathcal{L}^t$ is decomposed (micro-)locally as[11]

$$\mathcal{L}^t = \mathcal{L}^t_{\parallel} \otimes \mathcal{L}^t_{\perp},$$

where the operators $\mathcal{L}^t_{\parallel}$ and $\mathcal{L}^t_{\perp}$ describe the action of $\mathcal{L}^t$ on wave packets in the direction along $T_q\langle\alpha\rangle$ and those along $(T_q\langle\alpha\rangle)^\perp$, respectively (locally around $q \in \langle\alpha\rangle$). They are unitary operators if we view them in the L^2 norm. But the canonical map corresponding to the latter $\mathcal{L}^t_{\perp}$ is the restriction of $(Df^{-t})^*$ to the subspace $(T_q\langle\alpha\rangle)^\perp$ normal to $\langle\alpha\rangle$ and it is therefore dissipating if we view it with respect to the weight function $\mathcal{W}^r(\cdot)$. By a little more precise consideration, we see that $\mathcal{L}^t_{\perp}$ is equivalent to the L^2 normalized transfer operator

$$L^t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad L^t u(x) = |\det A^t|^{-1/2} \cdot u((A^t)^{-1}x)$$

associated to an expanding map A^t corresponding to $Df^t|_{E_u}$. Since we consider the weight $\mathcal{W}^r(\cdot)$, we focus on the action of L^t on the wave packets that correspond to a neighborhood of the origin $(0,0)$ in $T^*\mathbb{R}^d$. Roughly, this implies that we consider the action of L^t restricted on a smooth function supported on a neighborhood of the origin. If we consider the Taylor expansion of functions at the origin 0, we see that the principal part of the action of L^t is that on the low order polynomials. Since the transfer operator L^t preserves the space of homogeneous polynomials of each order, we may restrict L^t to each of those subspaces of homogeneous polynomials. But the action of L^t is obviously more contracting on the spaces of homogeneous polynomials of higher order. This is the origin of the "band structure" given in Theorem 5.1. Indeed the action of L^t on the space of homogeneous polynomial of order k is related to the spectral restriction of $\mathcal{L}^t$ to the k-th zonal region B_k.

6. The semi-classical (Gutzwiller-Voros) zeta function

In this section, we give a result on the distribution of zeros of the semi-classical (Gutzwiller-Voros) zeta function. We first explain about Atiyah-Bott trace formula and the dynamical zeta function. Then we will discuss the semi-classical zeta function.

6.1. Atiyah-Bott trace formula.

The Atiyah-Bott trace formula for the transfer operator $\mathcal{L}^t$ defined in (5.1) reads

$$\mathrm{Tr}^\flat \mathcal{L}^t := \int K(x,x;t)dx = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \frac{|\gamma| \cdot g^{m|\gamma|}(p_\gamma)}{|\det(1-D^m_\gamma)|} \cdot \delta(t - m|\gamma|), \tag{6.1}$$

[10]This idea is due to F. Faure, from whom the author learn much about semi-classical analysis and related subjects.

[11]This decomposition is realized exactly for the linear setting; see [23, §4]. In the non-linear setting, the decomposition is realized only locally and approximately.

where Γ is the set of prime periodic orbit, $|\gamma|$ denotes the prime period of $\gamma \in \Gamma$, D_γ denotes the derivative of Poincaré map along $\gamma \in \Gamma$ and p_γ is a point on γ. The left equality := is the definition of the Atiyah-Bott trace (or the flat trace), which is defined as the integration of the Schwartz kernel of $\mathcal{L}^t$,

$$K(x, y; t) = g^t(y) \cdot \delta_0(f^{-t}(x) - y), \quad (\delta_0(\cdot) \text{ is the Dirac delta function})$$

on the diagonal set $\Delta = \{(x, x) \in N \times N \mid x \in N\}$. The formula above just implies that this integral is well defined and equals the quantity on the right hand side. If the Atiyah-Bott trace was expressed as the sum of the eigenvalues of the operator $\mathcal{L}^t$, the formula (6.1) would lead to an interesting asymptotic formula of the (weighted) distribution of periods of the periodic orbits in terms of the spectrum of the (generator of) transfer operators. The difficulty in realizing this idea is that the Atiyah-Bott trace is not the trace in the usual sense and, to resolve it, we have to do with various technical problems. Below we will not deal with such technical problems, but we content ourselves with some formal computations (which is amusing) and state a few related results.

On the right hand side of the formula (6.1), the sum is taken over the periodic orbits, but with the weight $|\gamma| \cdot g^{m|\gamma|}(p_\gamma) \cdot |\det(1 - D_\gamma^m)|^{-1}$. Below we explain that we can get a formula (6.4) which counts the periodic orbits without weight by considering a few vector-valued transfer operators. This argument is due to Ruelle [24]. To begin with, we consider a vector bundle $\pi : W \to N$ and a one-parameter group of vector bundle maps $F^t : W \to W$ which project onto the flow f^t, that is, the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{F^t} & W \\ \pi \big\downarrow & & \pi \big\downarrow \\ N & \xrightarrow{f^t} & N. \end{array}$$

We define a vector-valued transfer operator acting on the space of sections $\Gamma(W)$ of W by

$$\mathcal{L}^t : \Gamma(W) \to \Gamma(W), \qquad \mathcal{L}^t u(x) = F^t(u(f^{-t}(x))).$$

The Atiyah-Bott trace formula (6.1) is extended to such a transfer operator as

$$\text{(6.2)} \qquad \mathrm{Tr}^\flat \mathcal{L}^t := \int \mathrm{Tr}\, K(x, x; t) dx = \sum_{\gamma \in \Gamma} \sum_{m=1}^{\infty} \frac{|\gamma| \cdot \mathrm{Tr}\,(F^t|_{\pi^{-1}(p_\gamma)})}{|\det(1 - D_\gamma^m)|} \cdot \delta(t - m|\gamma|).$$

As special cases, we consider the vector bundle W_k for $k = 0, 1, 2, \dots, 2d$ ($d = (\dim N - 1)/2$) defined by

$$W_k = \{\eta \mid \eta \text{ is a differential } k\text{-form on } N \text{ satisfying } \eta(V) = 0\},$$

where V is the generating vector field of the flow f^t. Then we let the vector bundle map $F^t : W_k \to W_k$ be the natural push-forward action of the flow f^t. For the corresponding vector-valued transfer operator $\mathcal{L}_k^t : \Gamma(W_k) \to \Gamma(W_k)$, the formula (6.2) reads

$$\text{(6.3)} \qquad \mathrm{Tr}^\flat \mathcal{L}_k^t = \sum_{\gamma \in \Gamma} \sum_{m=1}^{\infty} \frac{|\gamma| \cdot \mathrm{Tr}\,((D_\gamma^m)^{\wedge k})}{|\det(1 - D_\gamma^m)|} \cdot \delta(t - m|\gamma|).$$

Noting that[12]

$$|\det(1-D_\gamma^m)| = (-1)^d \cdot \sum_{k=0}^{2d} (-1)^k \cdot \mathrm{Tr}\,((D_\gamma^m)^{\wedge k}),$$

we obtain the formula

$$\sum_{k=0}^{2d} (-1)^{d-k} \cdot \frac{1}{t} \cdot \mathrm{Tr}^\flat \mathcal{L}_k^t = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \frac{1}{m} \cdot \delta(t - m|\gamma|). \tag{6.4}$$

Note that we still have the factor $1/m$ on the right hand side. But this is not a problem because our argument is not so precise. In fact, if we integrate both sides over the interval $[T, T+1]$, the right hand side equals

$$\sum_{m=1}^{\infty} \frac{1}{m} \cdot \sharp\{\gamma \in \Gamma \mid |\gamma| \in [T/m, (T+1)/m]\}$$

but the contribution of the terms for $m \geq 2$ are exponentially smaller than that of the term for $m = 1$ in the limit $T \to \infty$ (and negligible).

Now we examine each term on the left hand side of (6.4). The spectral radius of the transfer operator $\mathcal{L}_k^t$ is $\exp(tP_k)$ where P_k is the topological pressure

$$P_k := P_{top}(\{f^t\}; \{g_k^t\}) = \lim_{T\to\infty} \frac{1}{T} \log \left(\int_0^T \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \exp(g_k^{m|\gamma|}(p_\gamma)) \cdot \delta(t - m|\gamma|) \right)$$

in which p_γ is a point on the periodic orbit γ and the function $g_k^t(x)$ is defined by

$$g_k^t(x) = \log \|(Df^t(x))^{\wedge k}\| - \log |\det(Df^t|_{E_u})(x)|.$$

The topological pressure P_k takes the maximum value $h := h_{top}(\{f^t\})$ when $k = \dim E_u = d$ and takes smaller values for other $k < d$.

If the transfer operators $\mathcal{L}_d^t$ in the case $k = \dim E_u = d$ have spectral gap (in the sense that we explained in Section 4), that is, if the spectral set except for the leading eigenvalue $\exp(ht)$ is contained in the disk $\{|z| < \exp((h-\epsilon)t)\}$ for some $\epsilon > 0$, we expect the asymptotic estimate[13]

$$\#\{\gamma \in \Gamma \mid |\gamma| < T\} = \int_{+0}^{T} \frac{e^{ht}}{t} dt + \mathcal{O}\left(e^{(h-\epsilon)T}\right). \tag{6.5}$$

Further, if we have more information on the spectral properties of the transfer operators $\mathcal{L}_k^t$, it will be possible[14] to get a more precise estimate on the error term on the right hand side similar to that in Corollary 4.2.

For the expected formula (6.5), there is a famous result by Parry and Pollicott, called "Prime Orbit Theorem", which holds under a much weaker assumption.

[12] We assume that E_u is orientable. This is true when the manifold M is orientable. If E_u is not orientable, we have to consider the tensor product of W_k with the orientation line bundle of E_u; see [25].

[13] The lower bound of the integration denoted by $+0$ is actually a fixed number T_0. This is the same in a few expressions below.

[14] In fact, there exists such an estimate by Huber [26] in the case of geodesic flows on surfaces with negative constant curvature.

Theorem 6.1 (Parry-Pollicott [10]). *If an Axiom A flow $f^t : N \to N$ is topologically weak mixing, we have*

$$\#\{\gamma \in \Gamma \mid |\gamma| < T\} \sim \int_{+0}^{T} \frac{e^{ht}}{t} dt \quad \left(\sim \frac{e^{hT}}{hT} \right),$$

where $\sim$ implies that the ratio between the both sides converges to 1 as $T \to +\infty$.

The assumption above holds for the geodesic flows on negatively curved manifolds. The estimate on the error term has been studied since the theorem above is obtained and the following is one of the most recent results.

Theorem 6.2 (Giulietti-Liverani-Pollicott [27]). *If the geodesic flow of a negatively curved manifold M (or more generally a contact Anosov flow) $f^t : N \to N$ satisfies the following condition (1/3-bunching condition) for some $t > 0$,*

$$\|Df^t(x)|_{E_u}\|_{\min} > \|Df^t(x)|_{E_u}\|_{\max}^{1/3}, \tag{6.6}$$

then the asymptotic formula (6.5) holds for some $\epsilon > 0$. Especially, if the manifold M satisfies the 1/9-pinching condition (that is, the values of the sectional curvature are contained in $[-9\sigma, -\sigma]$ for some $\sigma > 0$), the assumption holds true.

Remark 6.3. For the case of geodesic flows on surfaces, the dimension of E_u is 1 and hence the assumption (6.6) holds trivially. The result above for such a case has been proved by Pollicott and Sharp [28]. The assumption (6.6) is used to ensure that the holonomy mapping of the stable and unstable foliation is at least (2/3)-Hölder continuous. Recently, Stoyanov [29] announced a decisive result which claims that the asymptotic estimate (6.5) holds for any contact Anosov flows.

6.2. **Dynamical zeta function.** The dynamical zeta function is an object that is closely related to the Atiyah-Bott trace formula (6.1). For non-singular flows, it is introduced by S. Smale [30] as a a generalization of Selberg zeta function in terms of dynamical systems. This is a function of a complex variable $s \in \mathbb{C}$ defined formally as

$$Z(s) = \prod_{n=1}^{\infty} \prod_{\gamma \in \Gamma} (1 - e^{-(s+n)|\gamma|}) = \prod_{n=1}^{\infty} \exp\left(-\sum_{\gamma \in \Gamma} \sum_{m=1}^{\infty} \frac{1}{n} e^{-(s+n)\cdot m|\gamma|} \right). \tag{6.7}$$

The infinite sum (and product) on the right hand side converges on the region $\mathrm{Re}(s) > h = h_{top}(\{f^t\})$ and hence the definition above gives a holomorphic function without zeros on such a region. Using (6.4), we can write the right hand side as

$$Z(s) = \prod_{n=1}^{\infty} \prod_{k=0}^{2d} \exp\left((-1)^{d-k} \cdot \int_{+0}^{\infty} \frac{e^{-(s+n)t}}{t} \cdot \mathrm{Tr}_k^{\flat} \mathcal{L}_k^t \right), \tag{6.8}$$

In Section 5, we saw that the generator of the transfer operators $\mathcal{L}_k^t$ has discrete spectrum. From this fact and the relation

$$\frac{d}{ds}\left(\int_{+0}^{\infty} \frac{e^{-st}}{t} \cdot e^{\lambda t} dt \right) = \int_{+0}^{\infty} e^{-(s-\lambda)t} dt = (s-\lambda)^{-1}$$

we expect that $Z(s)$ extends to a meromorphic function on the complex plane and the eigenvalues of the generators of $\mathcal{L}_k^t$ will appear as zeros and poles of the extension depending on the parity of $d - k$.

For analytic extension, we have the following result, which holds for general Anosov flows.

Theorem 6.4 (Giulietti-Liverani-Pollicott [27]). *For a general C^∞ Anosov flow $f^t : N \to N$, the dynamical zeta function $Z(s)$ extends to a meromorphic function on the complex plane $\mathbb{C}$.*

Remark 6.5. There are studies [31, 32] on special values of the dynamical zeta functions $Z(s)$ (mainly at $s = 0$).

6.3. **The semi-classical (Gutzwiller-Voros) zeta function.** Below we discuss the semi-classical zeta function $Z_{sc}(s)$, which is another generalization of the Selberg zeta function. For the geodesic flows on a negatively curved manifold (or more generally for a contact Anosov flow), we define

$$Z_{sc}(s) = \exp\left(-\sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-sm|\gamma|}}{\sqrt{\det(1-D_\gamma^m)}}\right). \tag{6.9}$$

This zeta function is introduced by physicists in the field of (the semi-classical theory of) quantum chaos and defined on purpose of expressing the quantum spectrum (the eigenvalues of the Laplacian on M) approximately in terms of periodic orbits of the classical limit [33]. Frankly the author does not know to what extent the related argument is justified on the basis of mathematical rigor[15]. However, for the case of surfaces of negative constant curvature ($\equiv -1$) the semi-classical zeta function $Z_{sc}(s)$ (as well as the dynamical zeta function $Z(s)$) coincides[16] with the Selberg zeta function up to shift of the variable s by $1/2$.

Similarly to the case of the dynamical zeta function $Z(s)$, the sum on the right hand side of (6.9) converges absolutely on the region where $\mathrm{Re}(s)$ is sufficiently large and hence the semi-classical zeta function is initially defined as a holomorphic function without zeros on such region. In order to investigate its meromorphic extension and derive information on the zeros and poles of the extension, we express $Z_{sc}(s)$ using the Atiyah-Bott trace of some transfer operators, similarly to the case of the dynamical zeta function $Z(s)$ considered in the last subsection. First we consider the k-th exterior product $\hat{W}_k = (E_s^*)^{\wedge k}$ $(k = 0, 1, \dots, d)$ of the dual E_s^* of the stable subbundle[17] E_s and define the vector bundle map $F_k^t : \hat{W}_k \to \hat{W}_k$ by

$$\hat{F}_k^t(w) = |\det Df^t|_{E_s}|^{1/2} \cdot ((Df^{-t})^*)^{\wedge k}(w). \tag{6.10}$$

The corresponding transfer operator $\hat{\mathcal{L}}_k^t : \Gamma(\hat{W}_k) \to \Gamma(\hat{W}_k)$ is

$$\hat{\mathcal{L}}_k^t u(x) = F_k^t(u(f^{-t}(x))) = |\det Df^t|_{E_s}|^{1/2} \cdot ((Df^{-t})^*)^{\wedge k}(u(f^{-t}(x))). \tag{6.11}$$

[15] The Gutzwiller trace formula is for finite time and for the semi-classical limit $\hbar \to 0$. But, in the related argument in physics, it seems that the limit $t \to \infty$ is considered and the order of limits are exchanged.

[16] We can check this by computation using the fact that $D_\gamma = \begin{pmatrix} \exp(|\gamma|) & 0 \\ 0 & \exp(-|\gamma|) \end{pmatrix}$. In the definition (6.7) of the dynamical zeta function $Z(s)$, we do not find a specific reason why we have to take the product with respect to n (besides the reason that this is an analogue of the Selberg zeta function). Interestingly, in the case of the semi-classical zeta function, the product with respect to n appears naturally (in the case of surface with negative constant curvature). The author personally thinks that the semi-classical zeta function $Z_{sc}(s)$ is a more natural generalization of the Selberg zeta function compared with the dynamical zeta function $Z(s)$ introduced by Smale.

[17] Since the stable subbundle E_s is not smooth, we need to provide some technical argument to treat the non-smoothness. In the paper [23, §2], we avoid this difficulty by considering the natural extension of the flow f^t to the d-dimensional Grassmann bundle over N.

The Atiyah-Bott trace formula for this one-parameter group of transfer operators reads

$$\mathrm{Tr}^\flat\, \hat{\mathcal{L}}_k^t = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty} \frac{|\gamma|\cdot|\det D_\gamma^s|^{m/2}\cdot \mathrm{Tr}\,((D_\gamma^s)^{-m})^{\wedge k})}{|\det(\mathrm{Id}-D_\gamma^{-m})|}\cdot\delta(t-m\cdot|\gamma|),$$

where D_γ^s is the restriction of D_γ to E_s. Since the contact Anosov flow f^t preserves the symplectic form $d\alpha$ on $\ker\alpha = E_s\oplus E_u$, we have

$$\sqrt{|\det(\mathrm{Id}-D_\gamma^{-m})|} = |\det(D_\gamma^s)|^{m/2}\cdot\left|\det\left(\mathrm{Id}-(D_\gamma^s)^{-m}\right)\right|$$

and hence

$$\sum_{k=0}^{d}(-1)^{d-k}\,\mathrm{Tr}^\flat\,\hat{\mathcal{L}}_k^t = \sum_{\gamma\in\Gamma}\sum_{m=1}^{\infty}\frac{|\gamma|}{\sqrt{|\det(\mathrm{Id}-D_\gamma^{-m})|}}\cdot\delta(t-m\cdot|\gamma|).$$

We therefore express the semi-classical zeta function as

$$Z_{sc}(s) = \exp\left(-\int_{+0}^{\infty}\frac{e^{-st}}{t}\sum_{k=0}^{d}(-1)^{d-k}\mathrm{Tr}^\flat\,\hat{\mathcal{L}}_k^t dt\right). \tag{6.12}$$

From this expression (6.12), we see that the semi-classical zeta function $Z_{sc}(s)$ has meromorphic extension to the complex plane $\mathbb{C}$ and the (discrete) eigenvalues of the generator of $\hat{\mathcal{L}}_k^t$ appear as its zeros and poles depending on the parity of $d-k$. Since the real parts of the eigenvalues for the generator of $\hat{\mathcal{L}}_k^t$ $(k=0,1,\dots,d)$ will be largest in the case $k=d$, where the transfer operator $\hat{\mathcal{L}}_d^t$ is scalar-valued and obtained by setting

$$g^t(x) = g_0^t(x) = (\det|(Df^t|_{E_s})(x)|)^{-1/2} = (\det|(Df^t|_{E_u})(x)|)^{1/2} \tag{6.13}$$

in the definition (5.1). As we remarked after Theorem 5.1, we have $r_1^+ < r_0^- = r_0^+$ in this case and the rightmost zonal region B_0 coincides with the imaginary axis $\mathrm{Re}(s)=0$ and is isolated from the other zonal regions B_k, $k\geq 1$. From this fact, the zeros of the semi-classical zeta function $Z_{sc}(s)$ concentrate along the imaginary axis. In fact, we have the following.

Theorem 6.6 (Faure-Tsujii [23])**.** *The semi-classical zeta function $Z(s)$ extends to a meromorphic function on the complex plane $\mathbb{C}$. The zeros of the extension are contained in the region*

$$R = R_0\cup R_1,\quad R_0=\{|\mathrm{Re}(s)|<\epsilon\},\quad R_1=\{\mathrm{Re}(s)<-\chi_0+\epsilon\}$$

for arbitrarily small $\epsilon>0$ but for finitely many exceptions depending on ϵ. (See Figure 6.) The poles are contained in the region R_1 again with finitely many exceptions. Further, there exist infinitely many zeros in R_0 and the imaginary part of them satisfies the distribution law given in Theorem 5.1(1).

The conclusion of Theorem 6.6 is expected from the motivation in the definition of the semi-classical zeta function and also may be regarded as a generalization of the classical result of Selberg (See [34] for instance) in terms of dynamical systems.

Remark 6.7. For the dynamical zeta function $Z(s)$ and more general dynamically defined zeta functions, we will be able to get a description of its zeros and poles corresponding to the band structure described in Theorem 5.1 and Theorem 5.3. We conjecture that, in the region $\mathrm{Re}(s)>C$ with any C, the zeros and poles will concentrate to several lines parallel to the imaginary axis in the limit $\mathrm{Im}(s)\to\pm\infty$.

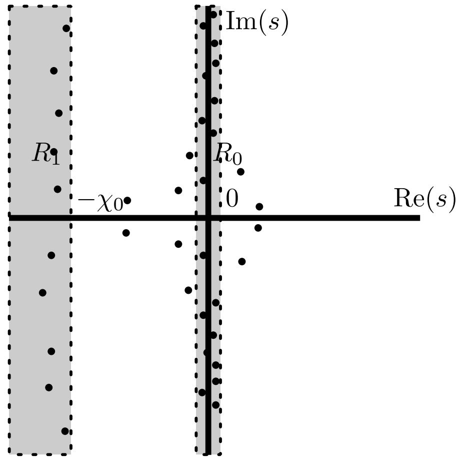

FIGURE 6. The zeros of the semi-classical zeta function $Z_{sc}(s)$.

Remark 6.8. It is actually more natural to consider the zeros (and poles) as the eigenvalues for the generator of the action of the transfer operators on the cohomology space along the stable foliation $\mathcal{F}^s$ (or the unstable foliation $\mathcal{F}^u$). (See [35, 36] for instance.) Let us ignore the problems related to non-smoothness of the subbundles E_s^* and the foliation $\mathcal{F}^s$. Let d_s be the exterior derivative operator along the stable foliation and let Λ^k be the space of sections of the vector bundle $(E_s^*)^k$. Then, the following diagram commutes[18],

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Lambda^0 & \xrightarrow{d_s} & \Lambda^1 & \xrightarrow{d_s} & \cdots & \xrightarrow{d_s} & \Lambda^d & \longrightarrow & 0 \\ & & \downarrow \hat{\mathcal{L}}_0^t & & \downarrow \hat{\mathcal{L}}_1^t & & & & \downarrow \hat{\mathcal{L}}_d^t & & \\ 0 & \longrightarrow & \Lambda^0 & \xrightarrow{d_s} & \Lambda^1 & \xrightarrow{d_s} & \cdots & \xrightarrow{d_s} & \Lambda^d & \longrightarrow & 0 \end{array} \tag{6.14}$$

Therefore we expect that good part of the eigenvalues for the generator of $\hat{\mathcal{L}}_k^t$ will be cancelled by those for $\hat{\mathcal{L}}_{k\pm 1}^t$ and only the eigenvalues for the action on the cohomology space will survive as the zeros and poles on the semi-classical zeta function. Though this idea has not been justified to date, again by the difficulties caused by the non-smoothness of the stable foliation, it seems possible and interesting to find a way to realize this idea and derive more information on the analytic property of the semi-classical zeta function[19].

The spectrum of the transfer operators and the analytic properties of dynamically defined zeta function has been studied extensively in the case of constant curvature

[18] The multiplication by the cocycle (6.13) is a bit problematic in showing this. But, by considering appropriate metric on the stable leafs, we may regard it locally constant along the leafs and hence the diagram (6.14) commutes. See [35, p. 503].

[19] To the best of the author's knowledge, among the dynamically defined zeta functions, only the semi-classical zeta function allows the reduction to the action on the space of cohomology as explained above. For the case of discrete dynamical systems, only the simplest dynamical zeta function, called Artin-Mazur zeta function, is reduced to the action on the space of cohomology and, for Anosov diffeomorphisms, we can see that it is not only meromorphic but also rational and their zeros and poles are distributed symmetrically as a consequence of Poincaré duality.

by using the Selberg trace formula and representation theory. But, for the case of variable curvature, the study was started only recently and much is left to be studied at present. Its relation to the semi-classical theory of quantum chaos is a fascinating subject, with various heuristic arguments and numerical computations. But it seems necessary to do further fundamental research for a while before we really tackle such problems.

Remark 6.9. As explained in the text, our argument is closely related to the semi-classical analysis. Recently there appeared a few papers [38, 39, 40, 41] on contact Anosov flows and dynamical zeta functions authored by specialists of semi-classical analysis such as Datchev, Dyatlov, Nonnenmacher and Zworski.

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