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STOCHASTIC ANALYSIS AND RANDOM SCHRÖDINGER OPERATORS

NAOMASA UEKI

1. Introduction

The Donsker-Varadhan theory on large deviations [12] and the Malliavin calculus [43] are the main theories in the stochastic analysis founded in the latter twentieth century. Among them the theory on large deviations has been related with the research on the random Schrödinger operators since the theory was founded. Indeed the theory on large deviations is well known to be applied to prove mathematically the Lifschitz asymptotic behavior immediately after the theory was founded [13], [46]. On the other hand, the Malliavin calculus had not been related with the research on the random Schrödinger operators for a long time and had been developed mainly for the stochastic differential equations of Itô type among random differential equations. In this article, we describe parts of the research on the random Schrödinger operators: in Section 2, we describe the development of the relating research after the mathematical proof of the Lifschitz asymptotic behavior and in Section 3 we describe the recent research relating also with the Malliavin calculus. The subject in Section 2 is the behavior of the integrated density of states and those in Section 3 are the mathematical proof of the Anderson localization by the random magnetic field and the Wegner estimate needed in the proof.

2. Wiener functional integrations and the asymptotic behavior of the integrated density of states

2.1. **In the Poisson model.** Let μ be the Poisson random measure on $\mathbb{R}^d$: μ is a random variable such that its value is in the Borel measures on $\mathbb{R}^d$ and, for any disjoint Borel sets $A_1, \ldots, A_n$ of $\mathbb{R}^d$, the random variables $\mu(A_1), \ldots, \mu(A_n)$ are mutually independent and the distribution of each $\mu(A_j)$ is the Poisson distribution with the mean $|A_j|$, where $|A_j|$ is the d-dimensional Lebesgue measure of A_j. Let h be a positive constant and let u be a positive continuous function on $\mathbb{R}^d$ such that $u(x) = o(|x|^{-d-\epsilon})$ as $|x| \to \infty$ with some $\epsilon \in (0, \infty)$. Then we can define a self-adjoint operator on $L^2(\mathbb{R}^d)$ of the form

$$H_\mu = -h\Delta + V_\mu, \qquad V_\mu(x) = \int u(x-y)\mu(dy).$$

This is given as the Hamiltonian of one electron in the electric field given by the scalar potential V_μ in the non-relativistic quantum mechanics and is called the Schrödinger operator. V_μ is a model of the random media where the single site

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potential u is located at each sample point of the Poisson random measure, and this is a random field which is stationary and ergodic with respect to the shift of the space variable. For such a Schrödinger operator with a stationary and ergodic potential, we can define a deterministic monotone increasing function $N(\lambda; H_\mu)$ satisfying

$$\frac{1}{(2R)^d}\#\{\text{the eigenvalues of } H_{\mu,(-R,R)^d} \leq \lambda\} \overset{R\to\infty}{\longrightarrow} N(\lambda; H_\mu)$$

with the probability one and is called the integrated density of states of H_μ, where $H_{\mu,(-R,R)^d}$ is the restriction of H_μ to $(-R,R)^d$ by the Dirichlet boundary condition. This function increases only on the spectral set of H_μ and the gradient reflects the density of the spectrum. Thus this function represents the distribution of the spectrum. Here we remark that both the spectrum of H_μ and the integrated density of states $N(\lambda; H_\mu)$ are proven to be independent of the sample value of μ by the ergodicity. The research of the asymptotic behavior of this function at the infimum of the spectrum is one of the subjects developed earlier among the subjects on the random Schrödinger operators. In particular, the following result is called the Lifschitz behavior and is well known with the history that the physicist, Lifschits [42], pointed out the existence of the behavior and Nakao [46] showed that the behavior was essentially proved rigorously in Donsker and Varadhan [13]:

Theorem 1 ([13, 42, 46]). *In the above setting, we further assume that $u(x) = o(|x|^{-d-2})$ as $|x| \to \infty$. Then we have*

$$\lim_{\lambda\downarrow 0} \lambda^{d/2} \log N(\lambda; H_\mu) = -h^{d/2}\lambda_1(-\Delta_{B(1)})^{d/2},$$

where $-\Delta_{B(1)}$ is the Dirichlet Laplacian on the d-dimensional unit ball $B(1) = \{x \in \mathbb{R}^d : |x| \leq 1\}$ and $\lambda_1(-\Delta_{B(1)})$ is the least eigenvalue of $-\Delta_{B(1)}$.

The character of this theorem is that the decay of $N(\lambda; H_\mu)$ as $\lambda \downarrow 0$ is very rapid and this fact becomes the character in the case that the operator is random. Indeed if we replace V_μ by a non-random and periodic function, then the order of the decay of the integrated density of states defined similarly at the infimum of the spectrum is known to be some power and is bigger than that of this theorem. This means that the spectrum in the disordered media is much thinner than that in the ordered media.

The proof was given by the fact that Donsker and Varadhan's theory on the large deviations determined the leading term as $t \to \infty$ of the Laplace transform

$$\int_{\mathbb{R}} e^{-t\lambda} dN(\lambda; H_\mu) = \mathbb{E}[\exp(-tH_\mu)(0,0)] \tag{2.1}$$

of the integrated density of states represented by the expectation of the diagonal part of the integral kernel $\exp(-tH_\mu)(x,y)$ of the heat semigroup generated by the Schrödinger operator and represented by the Feynman-Kac formula as

$$\begin{aligned}&\exp(-tH_\mu)(x,y)\\&\quad= E_{0,x}^{2ht,y}\Big[\exp\Big(-\frac{1}{2h}\int_0^{2ht} V_\mu(w(s))ds\Big)\Big]\exp\Big(-\frac{|x-y|^2}{2ht}\Big)\frac{1}{(2h\pi t)^{d/2}},\end{aligned}$$

where $E_{0,x}^{2ht,y}$ is the expectation with respect to the d-dimensional Brownian motion w conditioned that $w(0) = x$ and $w(2ht) = y$. This asymptotic behavior is the same as the case where u is replaced by a hard potential which is ∞ on $B(1)$ and is 0 on $\mathbb{R}^2 \setminus B(1)$, and it is also well known that (2.1) is represented in terms of the d-dimensional volume of the Wiener sausage obtained by fattening the trajectory of the Brownian motion $\bigcup_{0\le s\le 2ht}(w(s)+B(1))$, whose behavior was the main object of [13].

The case where the decay of u is slower than that in the above theorem is rather easy and Pastur [51] showed the following immediately after the above theorem.

Theorem 2 ([51]). *In the above setting, we further assume that $u(x) = C_0|x|^{-\alpha}(1+o(1))$ as $|x|\to\infty$ with some $\alpha\in(d,d+2)$. Then we have*

$$\lim_{\lambda\downarrow 0}\lambda^{\kappa}\log N(\lambda;H_\mu) = -C_{d,\alpha}C_0^{\kappa},$$

where $\kappa = d/(\alpha-d)$,

$$C_{d,\alpha} = \frac{\alpha-d}{\alpha^{\kappa+1}d}\Big(|S^{d-1}|\Gamma\Big(\frac{\alpha-d}{\alpha}\Big)\Big)^{\kappa+1},$$

and $|S^{d-1}|$ is the $d-1$-dimensional volume of the $d-1$-dimensional unit surface $S^{d-1}=\{x\in\mathbb{R}^d : |x|=1\}$.

The proof is also shown by determining the leading term of (2.1) as $t\to\infty$, which is easily obtained without applying the theory on the large deviations since the strong effect of u weakens the effect of the Brownian motion and the estimate is reduced to that of the moment generating function of $V_\mu(0)$. Indeed the leading term is independent of h, which is the character of the case that the classical effect only appears on the leading term and the quantum effect does not appear in the term. The leading term in Theorem 2 is the same as that of the classical integrated density of states defined by

$$N_c(\lambda : H_\mu) = \mathbb{E}[|\{(x,p)\in(-R,R)^d\times\mathbb{R}^d : H_{\mu,c}(x,p)\le\lambda\}|](4\pi\sqrt{h}R)^{-d}, \tag{2.2}$$

where $|\cdot|$ is the $2d$-dimensional volume, $H_{\mu,c}(x,p) = h|p|^2 + V_\mu(x)$ is the classical Hamiltonian, R is a positive number chosen arbitrarily: the right hand side of (2.2) is independent of the choice of R (cf. [41]). On the other hand, in Theorem 1, the quantum effect appears in the leading term and the effect of the potential is rather abstract. For the critical case between these two theorems, Ôkura [48] showed that both the quantum effect and the potential appear in the leading term in detail as follows.

Theorem 3 ([48]). *In the above setting, we further assume that*

$$u(x) = C_0|x|^{-d-2}(1+o(1))$$

as $|x|\to\infty$ with some $\alpha\in(d,d+2)$. Then we have

$$\lim_{\lambda\downarrow 0}\lambda^{d/2}\log N(\lambda;H_\mu) = \frac{-2d^{d/2}}{(d+2)^{1+d/2}}C(h,C_0)^{1+d/2},$$

where $C(h,C_0)$ is

$$\inf\Big\{h\int_{\mathbb{R}^d}|\nabla\psi(x)|^2dx + \int_{\mathbb{R}^d}dy\Big(1-\exp\Big(-C_0\int_{\mathbb{R}^d}\frac{\psi(x)^2dx}{|x-y|^{d+2}}\Big)\Big) : \psi\in C_0^\infty(\mathbb{R}^d),$$
$$\int\psi(x)^2 = 1\Big\}.$$

2.2. **In the Poisson model under the uniform magnetic field.** The above theorems are extended to the following 2-dimensional Schrödinger operator

$$H_\mu^B = \Big(i\partial_1 - \frac{Bx_2}{2}\Big)^2 + \Big(i\partial_2 + \frac{Bx_1}{2}\Big)^2 + V_\mu$$

with the uniform magnetic field $B > 0$:

Theorem 4. (i) (*Classical asymptotic behavior I: the case where u decays with a finite power* [5]) *In the above setting, we further assume that $u(x) = C_0|x|^{-\alpha}(1 + o(1))$ as $|x| \to \infty$ with some $\alpha \in (2, \infty)$. Then we have*

$$\lim_{\lambda\downarrow 0} \lambda^\kappa \log N(\lambda + B; H_\mu^B) = -C_{2,\alpha} C_0^\kappa,$$

where $\kappa = 2/(\alpha - 2)$, $C_{2,\alpha}$ is the constant $C_{d,\alpha}$ in Theorem 2 with $d = 2$.

(ii) (*Classical asymptotic behavior II: the case where u decays exponentially* [30]) *In the above setting, we further assume that $u(x) = \exp(-|x|^\alpha(1 + o(1))/C_0)$ as $|x| \to \infty$ with some $\alpha \in (0, 2)$. Then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-2/\alpha} \log N(\lambda + B; H_\mu^B) = -\pi C_0^{2/\alpha}.$$

(iii) (*Quantum asymptotic behavior* [15, 30]) *In the above setting, we further assume that $\overline{\lim}_{|x|\to\infty} |x|^{-2} \log u(x) = -\infty$. Then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-1} \log N(\lambda + B; H_\mu^B) = -\frac{2\pi}{B}.$$

(iv) (*Critical asymptotic behavior* [16, 30]) *In the above setting, we further assume that $u(x) = \exp(-|x|^2(1 + o(1))/C_0)$ as $|x| \to \infty$. Then we have*

$$\lim_{\lambda\downarrow 0}(\log(1/\lambda))^{-1} \log N(\lambda + B; H_\mu^B) = -\pi\Big(\frac{2}{B} + C_0\Big).$$

As for the proof, (i) was first proven by Broderix-Hundertmark-Kirsch-Leschke [5]. Erdös [15] next proved (iii), where u is replaced by a function with compact support. Hupfer, Leschke, and Warzel [30] next proved (iii) in a general case, (ii) and the lower estimate in (iv). Erdös [16] finally completed the proof of (iv). The necessary and sufficient condition for the appearance of the quantum effect in this setting is that the single site potential decays as a Gaussian kernel or decays faster than a Gaussian kernel as the above theorem indicates.

Moreover, the extension to the 3-dimensional Schrödinger operator,

$$\mathbb{H}_\mu^B = \Big(i\partial_1 - \frac{Bx_2}{2}\Big)^2 + \Big(i\partial_2 + \frac{Bx_1}{2}\Big)^2 - \partial_3^2 + V_\mu,$$

with a uniform magnetic field $B > 0$ has been treated below.

Theorem 5. (i) (*Classical asymptotic behavior*) *In the above setting, we further assume the existence of $C_0 \in (0, \infty)$, $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0, \infty)^2$ satisfying*

$$\frac{2}{\alpha_\perp} + \frac{3}{\alpha_3} > 1 > \frac{2}{\alpha_\perp} + \frac{1}{\alpha_3} \tag{2.3}$$

and $\tilde{p} \in [1, \infty]$ such that

$$u(x) = \frac{C_0}{\|x\|_{\tilde{p}}^{\boldsymbol{\alpha}}}(1 + o(1)) \tag{2.4}$$

as $|x| \to \infty$. *Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{\kappa(\boldsymbol{\alpha})} \log N(\lambda + B : \mathbb{H}_\mu^B) = -C(\boldsymbol{\alpha}, \tilde{p}) C_0^{\kappa(\boldsymbol{\alpha})}, \tag{2.5}$$

where, denoting $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *as* $(x_\perp, x_3) \in \mathbb{R}^2 \times \mathbb{R}$*,*

$$\|\boldsymbol{x}\|_{\tilde{p}}^{\boldsymbol{\alpha}} := \begin{cases} \| |x_\perp|^{\alpha_\perp}, |x_3|^{\alpha_3} \|_{\tilde{p}} = (|x_\perp|^{\alpha_\perp \tilde{p}} + |x_3|^{\alpha_3 \tilde{p}})^{1/\tilde{p}} & (\textit{if } \tilde{p} \in [1, \infty)), \\ |x_\perp|^{\alpha_\perp} \vee |x_3|^{\alpha_3} & (\textit{if } \tilde{p} = \infty), \end{cases}$$

$\kappa(\boldsymbol{\alpha}) = (2/\alpha_\perp + 1/\alpha_3)(1 - 2/\alpha_\perp - 1/\alpha_3)^{-1}$, *and*

$$C(\boldsymbol{\alpha}, \tilde{p}) = \begin{cases} \frac{1}{\kappa(\boldsymbol{\alpha})} \left\{ \frac{4\pi}{\alpha_\perp \alpha_3 \tilde{p}} B\left(\frac{2}{\tilde{p}\alpha_\perp}, \frac{1}{\tilde{p}\alpha_3} \right) \Gamma\left(1 - \frac{2}{\alpha_\perp} - \frac{1}{\alpha_3} \right) \right\}^{(1-2/\alpha_\perp - 1/\alpha_3)^{-1}} & (\textit{if } \tilde{p} \in [1, \infty)), \\ \frac{1}{\kappa(\boldsymbol{\alpha})} \left\{ 2\pi \left(\frac{2}{\alpha_\perp} + \frac{1}{\alpha_3} \right) \Gamma\left(1 - \frac{2}{\alpha_\perp} - \frac{1}{\alpha_3} \right) \right\}^{(1-2/\alpha_\perp - 1/\alpha_3)^{-1}} & (\textit{if } \tilde{p} = \infty). \end{cases}$$

(ii) (*Quantum asymptotic behavior, Critical asymptotic behavior*) *In the above setting, we further assume the existence of* $C_0 \in (0, \infty)$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0, \infty)^2$ *satisfying*

$$\frac{2}{\alpha_\perp} + \frac{3}{\alpha_3} \le 1 \tag{2.6}$$

and $\tilde{p} \in [1, \infty]$ *such that* (2.4) *holds as* $|x| \to \infty$. *Then we have*

$$\lim_{\lambda \downarrow 0} \frac{\log(-\log N(\lambda + B : \mathbb{H}_\mu^B))}{\log(1/\lambda)} = \frac{2/\alpha_\perp}{1 - 2/\alpha_\perp - 1/\alpha_3} + \frac{1}{2}. \tag{2.7}$$

As for the proof, (i) was proven by Hundertmark-Kirsch-Warzel [29], a special case of (ii) with $\alpha_\perp = \alpha_3 = 0$ was proven by Warzel [66] and a review article by Kirsch-Metzger [36] reported that the general case of (ii) was proven by Kirsch, Leschke, and Warzel. To treat 3-dimensional uniform magnetic field, we should distinguish the perpendicular direction and the parallel direction with the magnetic fields as above. A similar problem is treated by Kirsch and Warzel [37] in the case that the scalar potential is anisotropic without magnetic fields. Here we note that V_μ is not defined if $2/\alpha_\perp + 1/\alpha_3 \ge 1$.

The other related works are reported in the review article by Kirsch and Metzger [36].

2.3. In an intermediate model between the Poisson model and a periodic model.

In this article, we discuss developments after the publication of the review by Kirsch-Metzger [36]: we focus on topics the author of this article related.

Fukushima [21] extended Theorem 1 to the following model:

$$H_\xi = -h\Delta + V_\xi, \qquad V_\xi(x) = \sum_{q \in \mathbb{Z}^d} u(x - q - \xi_q), \tag{2.8}$$

where h is a positive constant, u is a non-zero, non-negative and continuous function with a compact support, and $\xi = (\xi_q)_{q \in \mathbb{Z}^d}$ is a system of $\mathbb{R}^d$-valued independent random variables whose common distribution is

$$\mathbb{P}_\theta(\xi_q \in dx) = \exp(-|x|^\theta) dx / Z(d, \theta) \tag{2.9}$$

with some $\theta \in (0, \infty)$. In this equation, $Z(d, \theta)$ is the normalizing constant. The result is the following.

Theorem 6 ([21]). *In the above situation with $d \geq 2$,*

$$\log N(\lambda; H_\xi) \asymp \begin{cases} -\lambda^{-1-\theta/2}\left(\log \dfrac{1}{\lambda}\right)^{-\theta/2} & (\textit{in the case of } d = 2), \\ -\lambda^{-d/2-\theta/d} & (\textit{in the case of } d \geq 3). \end{cases} \tag{2.10}$$

In this equation, "$f(\lambda) \asymp g(\lambda)$ "means" $0 < \underline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) \leq \overline{\lim}_{\lambda\downarrow 0} f(\lambda)/g(\lambda) < \infty$".

The significance of this theorem is that it reveals how the behavior in Theorem 1 transitions as the potential changes from the completely random one to an ordered one. For this argument, it was better to replace $|x|^\theta$ by $(1+|x|)^\theta$ in (2.9). This replacement does not change the proof essentially and under this replacement, the centers of the potentials $\{q + \xi_q\}_{q\in\mathbb{Z}^d}$ converges weakly to the lattice $\mathbb{Z}^d$ as $\theta \to \infty$. On the other hand, Appendix A in Fukushima [21] shows that $\sum_{q\in\mathbb{Z}^d} \delta_{q+\xi_q}$ converges weakly to the Poisson random measure as $\theta \downarrow 0$. Therefore the model in this subsection can be regarded as a model describing the transition from the completely random model given by the Poisson random measure to a completely ordered model. If we take the limit as $\theta \to 0$ in (2.10) formally, then the result is consistent with the result in Theorem 1. On the other hand, if we take the limit as $\theta \to \infty$, then the right hand side of (2.10) diverges to $-\infty$, which reflects that $N(\lambda; H_\xi)$ remains to be zero on a neighborhood of $\lambda = 0$ since the infimum of the spectrum of the Schrödinger operator with a positive and periodic potential is strictly positive. Fukushima started this research from the view point of investigating the lifetime of the Brownian motion under a random environment by (2.1). From this view point, Donsker and Varadhan's research [13] on the Wiener sausage was one of the first significant results. However, the Donsker-Varadhan theory on large deviations has not be applied to the model in this subsection. Instead of the theory, Fukushima applied Sznitman's method [60] of enlargement of obstacles to obtain the above result. Though this method is involved, it has been used widely: this method has been used to refine the result obtained by Donsker and Varadhan's theory or to develop the estimates of the lifetime of the Brownian motion under an random environment [59], [60].

We concentrate our concern only on the asymptotic behavior of the integrated density of states and discuss an extension of the above theorem. The author made an attempt to generalize Theorems 1–3 and obtained the following results in collaboration with Fukushima.

Theorem 7 ([22]). *In* (2.8)*, we assume that u is a positive valued continuous function on $\mathbb{R}^d$ satisfying $u(x) = C_0|x|^{-\alpha}(1+o(1))$ as $|x| \to \infty$ with some $C_0 \in (0,\infty)$ and $\alpha \in (d, \infty)$.*

(i) (*Classical asymptotic behavior*) *If $d < \alpha < d+2$, then*

$$\lim_{\lambda\downarrow 0} \lambda^\kappa \log N(\lambda; H_\xi) = \frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}}\left\{\int_{\mathbb{R}^d} dq \inf_{y\in\mathbb{R}^d}\left(\frac{C_0}{|q+y|^\alpha} + |y|^\theta\right)\right\}^{\kappa+1}, \tag{2.11}$$

where $\kappa = (d+\theta)/(\alpha - d)$.

(ii) (*Quantum asymptotic behavior*) *If $d = 1$ and $\alpha > 3$, then*

$$\lim_{\lambda\downarrow 0} \lambda^{(1+\theta)/2} \log N(\lambda; H_\xi) = \frac{-\pi^{1+\theta} h^{(1+\theta)/2}}{(1+\theta)2^\theta}. \tag{2.12}$$

If $d \geq 2$ *and* $\alpha > d+2$*, then*

$$\log N(\lambda; H_\xi) \asymp \begin{cases} -\lambda^{-1-\theta/2}\left(\log \dfrac{1}{\lambda}\right)^{-\theta/2} & (\textit{in the case of } d=2), \\ -\lambda^{-(d+\mu\theta)/2} & (\textit{in the case of } d \geq 3), \end{cases} \tag{2.13}$$

where $\mu = 2(\alpha-2)/(d(\alpha-d))$.

(iii) (*Critical asymptotic behavior*) *If* $\alpha = d+2$*, then*

$$\varliminf_{\lambda \downarrow 0} \lambda^{(d+\theta)/2} \log N(\lambda; H_\xi) \geq \frac{-2(d+\theta)^{(d+\theta)/2}}{(d+2+\theta)^{(d+2+\theta)/2}} K_0(h, C_0)^{(d+2+\theta)/2}, \tag{2.14}$$

where

$$\begin{aligned} &K_0(h, C_0) \\ &= \inf \Big\{ h\|\nabla\psi\|_2^2 + \int_{\mathbb{R}^d} dq \inf_{y \notin supp(\psi)-q} \Big(\int_{\mathbb{R}^d} \frac{dx C_0 \psi(x)^2}{|x-q-y|^{d+2}} + |y|^\theta \Big) \\ &\quad : \psi \in W^{2,1}(\mathbb{R}^d), \|\psi\|_2 = 1 \Big\} \end{aligned} \tag{2.15}$$

and $W^{2,1}(\mathbb{R}^d) = \{\psi \in L^2(\mathbb{R}^d) : \nabla\psi \in L^2(\mathbb{R}^d)\}$.

The formal limits of these results as $\theta \to 0$ also consist of the results of Theorems 1–3 as in Theorem 6 and the limits as $\theta \to \infty$ diverge. The latter fact consists of the fact that the infimum of the spectrum of a Schrödinger operator is strictly positive. As for the transition as the variation of the speed of the decaying of the potential, only the classical effect appears in the leading term if the potential decays slowly as $\alpha < d+2$ and the quantum effect appears if the potential decays faster as $\alpha > d+2$, as in the Poisson model. In $d=1$, the leading order varies continuously at $\alpha = d+2$ and does not vary for $\alpha > d+2$ as in the Poisson model. However, in $d=2$, the logarithmic term appears discontinuously in the leading order at $\alpha = d+2$, though the leading order still does not vary for $\alpha > d+2$. In $d \geq 3$, the leading order varies for $\alpha > d+2$, though the order still varies continuously. These differences occur since the states having holes in their supports contribute to the leading term. Such states do not contribute to the leading term in the Poisson model since their kinetic energies are rather large. However, not only the energies of the states but also the costs for the centers of the single site potentials to move from the sites of the lattice contribute to the asymptotic behavior of $N(\lambda; H_\xi)$ as $\lambda \downarrow 0$. Since the costs are small in the Poisson model, only the states with small energies contribute to the leading term. Moreover, the leading term is independent of α if $\alpha > d+2$, since only the kinetic energies of the states contribute to the leading term. However, in the model in this subsection, the costs for the potentials are rather large. The optimal combination of the potentials and the states seems to be the one where the locations of the centers of the single potentials are near the site of the lattice and the states have holes around the centers. For such combinations, the potential energies and the cost for the movements of the centers reduce. Moreover if d is big, then the kinetic energies do not so increase. This phenomenon contributes to the results in the case of $d \geq 2$ and $\alpha > d+2$. As for the analysis of the phenomenon that the energies of the Dirichlet Laplacian do not also increase, even if many small holes are drilled in the domain, interesting facts are known. For this aspect, please refer to Ozawa's review [50] for example. Kac [33] first analyzed this phenomenon by the stochastic analysis using the Wiener sausage. Rauch and Taylor [53, 54] next analyzed this phenomenon by relating it to the scattering length. Moreover, in [53, 54], this phenomenon is introduced in the crushed ice problem: the problem appearing when we cool juices by crushed ice. For this aspect, we can also refer to Simon's textbook [56]. For the proof of Theorem 7, we refer to the results obtained for this phenomenon and we also use the method of Dirichlet-Neumann bracketing. This method dominates the integrated density of states from above and below by the averaged number of eigenvalues of the Schrödinger operators restricted to the d-dimensional cube by the Dirichlet or the Neumann boundary conditions, and estimates the averaged numbers by functional analytic methods [6], [34], [35], [52]. This method can be applied widely, though the obtained results are not so sharp. Indeed (ii) in the above theorem is not so sharp. For this case, a little more detailed estimate is obtained in [23] by referring to Sznitman's method of the enlargement of the obstacles. However, the results for the case of $\alpha = d+2$ are still unsatisfactory. Since the decay of $N(\lambda; H_\xi)$ becomes slower as α increases, the result in (ii) of the above theorem gives an upper estimate for $\alpha = d+2$. Therefore we see that the quantum effect appears if $d = 1$, $\alpha = 3$, and h is big. We can also see that the quantum effect appears if $d \geq 3$, $\alpha = d+2$, and h is big by sharpening the estimate in (ii). However, we have no such results for $d = 2$.

2.4. **In the intermediate model of the last subsection under the uniform magnetic field.** The author next extends Theorem 4 to the following.

Theorem 8 ([63, 64])**.** *In the 2-dimensional Schrödinger operator $H_\xi^B = (i\partial_1 - Bx_2/2)^2 + (i\partial_2 + Bx_1/2)^2 + V_\xi$ with the uniform magnetic field $B > 0$, let V_ξ be the function obtained by replacing the function u by a positive continuous function on $\mathbb{R}^2$ in* (2.8)*. For this operator, the following hold:*

(i) (*Classical asymptotic behavior I: the case where u decays with a finite power*) *We further assume that $u(x) = C_0|x|^{-\alpha}(1+o(1))$ as $|x| \to \infty$ with some $\alpha \in (2, \infty)$. Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^\kappa \log N(\lambda + B; H_\xi^B) = \frac{-\kappa^\kappa}{(\kappa+1)^{\kappa+1}} \left\{ \int_{\mathbb{R}^2} dq \inf_{y \in \mathbb{R}^2} \left(\frac{C_0}{|q+y|^\alpha} + |y|^\theta \right) \right\}^{\kappa+1},$$

where $\kappa = (2+\theta)/(\alpha-2)$. This leading term is the same as (2.11)*.*

(ii) (*Classical asymptotic behavior II: the case where u decays exponentially*) *We further assume that $u(x) = \exp(-|x|^\alpha(1+o(1))/C_0)$ as $|x| \to \infty$ with some $\alpha \in (0,2)$. Then we have*

$$\lim_{\lambda \downarrow 0} \left(\log \frac{1}{\lambda} \right)^{-(2+\theta)/\alpha} \log N(\lambda + B; H_\xi^B) = -\frac{2\pi C_0^{(2+\theta)/\alpha}}{(\theta+1)(\theta+2)}.$$

(iii) (*Quantum asymptotic behavior*) *We further assume that*

$$\overline{\lim}_{|x| \to \infty} |x|^{-2} \log u(x) = -\infty.$$

Then we have

$$\underline{\lim}_{\lambda \downarrow 0} \left(\log \frac{1}{\lambda} \right)^{-(1+\theta/2)} \log N(\lambda + B; H_\xi^B) \geq \frac{-2^{2+\theta/2}\pi}{(\theta+1)(\theta+2)B^{1+\theta/2}}. \tag{2.16}$$

Moreover, if $\theta > 4$, then there exists a finite positive constant K such that

$$\overline{\lim}_{\lambda \downarrow 0} \left(\log \frac{1}{\lambda} \right)^{-(1+(\theta-4)/6)} \log N(\lambda + B; H_\xi^B) \leq \frac{-K}{B^{1+(\theta-4)/6}}. \tag{2.17}$$

(iv) (*Critical asymptotic behavior*) *We further assume that* $u(x) = \exp(-|x|^2(1+o(1))/C_0)$ *as* $|x| \to \infty$. *Then we have*

$$\varliminf_{\lambda\downarrow 0}\Big(\log\frac{1}{\lambda}\Big)^{-(1+\theta/2)}\log N(\lambda+B;H_\xi^B) \geq \frac{-2\pi}{(\theta+1)(\theta+2)}\Big(\frac{2}{B}+C_0\Big)^{1+\theta/2}.$$

For this model, we do not obtain sufficient upper estimates including the quantum effect and we do not obtain conditions for the appearance of the quantum effect except for the case of $\theta > 4$. The author conjectures that an upper estimate like (2.16) exists and a condition for the appearance of the quantum effect is still that u decays as the Gaussian kernel or u decays faster than the Gaussian kernel. To solve this problem, we should extend (2.17) to a general θ and improve the estimate. For this, we should improve the lower estimate of the infimum of the spectrum of the Schrödinger operator with a uniform magnetic field on a domain with holes with the Dirichlet boundary condition. To prove Theorem 4, we do not need the information of the effect of the holes and we have only to show that the infimum of the spectrum attains its minimum under the given area of the domain when the domain becomes the disk. This necessary fact was proven in Erdös [14], where the key of the proof was a result on the isoperimetric problem. This problem shows that the maximum of the area of the domain under the given length of the boundary is attained when the domain is the disk. Now we should estimate the effect caused by opening holes in the domain and we should apply effectively the estimate to that of the spectrum of the Schrödinger operator with a uniform magnetic field. Equation (2.17) is based on a lower estimate of the infimum of the spectrum obtained by applying the improvement by Osserman [49] of an inequality for the isoperimetric problem as in Erdös [14].

We extend also Theorem 5 to the following.

Theorem 9 ([63, 64])**.** *In the 3-dimensional Schrödinger operator* $\mathbb{H}_\xi^B = (i\partial_1 - Bx_2/2)^2 + (i\partial_2 + Bx_1/2)^2 - \partial_3^2 + V_\xi$ *with the uniform magnetic field* $B > 0$, *let* u *be a positive continuous function and* $\xi = (\xi_q)_{q\in\mathbb{Z}^3}$ *is a system of* $\mathbb{R}^3$*-valued independent random variables whose common distribution is*

$$P_{\boldsymbol{\theta}}(\xi_q \in dx) = \exp(-\|x\|_p^{\boldsymbol{\theta}})dx/Z(\boldsymbol{\theta},p) \tag{2.18}$$

with some $\boldsymbol{\theta} = (\theta_\perp,\theta_3) \in (0,\infty)^2$ *and* $p \in [1,\infty]$, *where* $Z(\boldsymbol{\theta},p)$ *is the normalizing constant. For this operator, the following hold:*

(i) (*Classical asymptotic behavior*) *We further assume* (2.4) *as* $|x| \to \infty$ *with some* $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp,\alpha_3) \in (0,\infty)^2$ *satisfying* (2.3). *Then we have*

$$\lim_{\lambda\downarrow 0}\lambda^{\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}\log N(\lambda+B;\mathbb{H}_\xi^B) = \frac{-\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})^{\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}}{(1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta}))^{1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}}C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0)^{1+\kappa(\boldsymbol{\alpha},\boldsymbol{\theta})}, \tag{2.19}$$

where

$$\kappa(\boldsymbol{\alpha},\boldsymbol{\theta}) = \frac{(\theta_\perp/\alpha_\perp)\vee(\theta_3/\alpha_3) + 2/\alpha_\perp + 1/\alpha_3}{1 - 2/\alpha_\perp - 1/\alpha_3}, \tag{2.20}$$

$$\begin{aligned} &C(\boldsymbol{\alpha},\boldsymbol{\theta},C_0) \\ &= \int_{\mathbb{R}^3} d\boldsymbol{q} \inf_{\boldsymbol{y}=(y_\perp,y_3)\in\mathbb{R}^3}\Big(\frac{C_0}{\|\boldsymbol{q}+\boldsymbol{y}\|_{\tilde{p}}^{\boldsymbol{\alpha}}} + \Big\| 1_{\frac{\theta_\perp}{\alpha_\perp}\geq\frac{\theta_3}{\alpha_3}}|y_\perp|^{\theta_\perp}, 1_{\frac{\theta_\perp}{\alpha_\perp}\leq\frac{\theta_3}{\alpha_3}}|y_3|^{\theta_3}\Big\|_p\Big). \end{aligned} \tag{2.21}$$

(ii) (*Quantum and critical asymptotic behavior*) *We further assume* (2.4) *as* $|x| \to \infty$ *with some* $C_0 \in (0,\infty)$, $\tilde{p} \in [1,\infty]$ *and* $\boldsymbol{\alpha} = (\alpha_\perp, \alpha_3) \in (0,\infty)^2$ *satisfying* (2.6). *Then we have*

$$\varliminf_{\lambda\downarrow 0} \lambda^{\mu_1(\alpha_\perp,\boldsymbol{\theta})} \log N(\lambda + B; \mathbb{H}_\xi^B) > -\infty, \tag{2.22}$$

$$\varlimsup_{\lambda\downarrow 0} \lambda^{\mu_2(\boldsymbol{\alpha},\boldsymbol{\theta})} \log N(\lambda + B; \mathbb{H}_\xi^B) < 0\ , \tag{2.23}$$

where

$$\mu_1(\alpha_\perp, \boldsymbol{\theta}) = \frac{3}{\alpha_\perp - 2} + \frac{1}{2} + \frac{3\theta_\perp}{2(\alpha_\perp - 2)} \vee \frac{\theta_3}{2}, \tag{2.24}$$

$$\mu_2(\boldsymbol{\alpha}, \boldsymbol{\theta}) = \frac{2/\alpha_\perp}{1 - 1/\alpha_3 - 2/\alpha_\perp} + \frac{1}{2} + \frac{\theta_\perp/\alpha_\perp}{1 - 1/\alpha_3 - 2/\alpha_\perp} \wedge \frac{\theta_3}{2}. \tag{2.25}$$

Estimates appearing the quantum effect are still insufficient and we do not even know the leading order. However, since the upper estimate (2.23) is effective, we know that the condition for the appearance of the quantum effect is (2.6) as in the Poisson model. The estimate (2.23) is proven by reducing the asymptotic problem to that of the integrated density of states of a 1-dimensional Schrödinger operator $-\partial_3^2 + V_\xi(x_\perp, x_3)$ without magnetic fields by the inequality

$$\begin{aligned}
&\mathrm{Tr}[\exp(-t(H_\xi^B + \varepsilon|x|^2) - B)] \\
\le &\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\Big(\Big(i\nabla_\perp + \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\Big)^2 - B\Big)\Big) \exp(-t(-\partial_{x_3}^2 + V^\xi + \varepsilon|x|^2)) \\
&\times \exp\Big(-\frac{t}{2}\Big(\Big(i\nabla_\perp + \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\Big)^2 - B\Big)\Big)\Big]
\end{aligned}$$

using the Golden-Thompson inequality as in Warzel [66].

3. Malliavin calculus and Wegner type estimate

The Malliavin calculus succeeded to show the absolute continuity of the probability distribution of the solution of a stochastic differential equation (3.8) of Itô type only by the calculus on the probability space without using the theories on differential equations usually used [43], and has been extended also to stochastic partial differential equations (see [47] for example) and to stochastic differential equations satisfied by stochastic processes of jump type (see [11] for example). On the other hand, the Wegner type estimate is an important estimate for random Schrödinger operators. This estimate also shows the smoothness of the probability distributions of eigenvalues by the integration by parts on the probability space.

However, the Wegner type estimate has not been discussed as a related topic to the Malliavin calculus. The reason seems to be that the derivative of the eigenvalues of random operators with respect to the random elements cannot be estimated effectively except for the first derivative. This situation is different with the solutions of the stochastic differential equations. Indeed, in the Malliavin calculus, the smoothness of the probability distributions of the solutions of the stochastic differential equations is discussed by differentiating the solutions many times and the necessary condition for this argument is summarized to the non-degeneracy in the Malliavin sense. On the other hand, for the Wegner type estimate, the integration by parts has been used only one time and the necessary condition for this argument to deduce the differentiability of the probability distributions of the eigenvalues is the monotonicity of the eigenvalues with respect to the random elements. However, the monotonicity is too restrictive for some cases. For example, the Schrödinger operators with random magnetic fields do not have monotonicity. Therefore the Wegner-type estimate for these operators has been very restrictive until recently (see [27], [28], [38], [39], [62]). Indeed, in the works [27], [28], [39], [62] on the Wegner type estimate for random magnetic fields, some conditions are posed on the vector potentials. However, the gauge invariance implies that the spectrum depends only on the magnetic fields and is independent of the choice of the scalar potentials. Thus the conditions should be posed only on the magnetic fields.

Recently Erdös and Hasler [17], [18], [19] succeeded to obtain such a theory. They succeeded to reduce the estimates of the derivatives of the eigenvalues to those of the derivatives of the resolvent operators and by this reduction, they gave a Wegner type estimate for the Schrödinger operators with a positive random magnetic field by 2 times the integration by parts on the probability space. For this, they showed non-degeneracy corresponding to the non-degeneracy in the Malliavin sense by dominating the quantity corresponding to the Malliavin covariance from below by a positive quadratic form of the magnetic field. The same argument has been used for the stochastic differential equations of Itô type. The case where the magnetic field is positive corresponds to the case where the generator is elliptic in the stochastic differential equations of Itô type. However, the Malliavin calculus has treated the case where the generator satisfies only Hölmander's condition, which is more general than the ellipticity, since the calculus was created. Therefore we may obtain a Wegner type estimate without the positivity of the magnetic field by applying the techniques developed for the Malliavin calculus. This idea is applied to a fundamental Gaussian random magnetic field in [65].

In this section, we discuss this work and its background.

3.1. **Setting.** Let $\omega = (\omega(h) = \int h(x)\omega(dx))_{h\in L^2(\mathbb{R}^2)}$ be a 2-dimensional white noise, which is a Gaussian random field on $L^2(\mathbb{R}^2)$ such that, for any $h_1, \dots, h_n \in L^2(\mathbb{R}^2)$, $(\omega(h_1), \dots, \omega(h_n))$ obeys the n-dimensional normal distribution with the mean vector $\mathbf{0}$ and the covariance matrix $((h_i, h_j)_{L^2})_{1\le i,j\le n}$. As for the fundamental tools for this random field, we can refer to [47] .

As the magnetic field, we take the random field

$$B^\omega(x) = B + \widetilde{B^\omega}(x), \qquad \widetilde{B^\omega}(x) = \int_{\mathbb{R}^2} \sigma(x-y)\omega(dy), \tag{3.1}$$

where $B \in \mathbb{R}$,

$$\sigma(x) = (\overline{\sigma}^2 - |x|^2)_+^\nu, \tag{3.2}$$

$\overline{\sigma} \in (0,\infty)$, $\nu \in (3/2, \infty)$, and for any $a \in \mathbb{R}$, $a_+ = \max\{a, 0\}$. One character of σ is that its Fourier transform is written as

$$\widehat{\sigma}(\xi)\Big(:= \int_{\mathbb{R}^2} \exp(-2\pi i\xi\cdot x)\sigma(x)dx\Big) = \frac{\overline{\sigma}^{\nu+1}\Gamma(\nu+1)}{\pi^\nu|\xi|^{\nu+1}} J_{\nu+1}(2\pi\overline{\sigma}|\xi|), \tag{3.3}$$

where

$$J_{\nu+1}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (t/2)^{2m+\nu+1}}{m!\Gamma(m+\nu+2)} \tag{3.4}$$

is the Bessel function of the order $\nu+1$. Under the assumption $\nu > 3/2$, the sample path $B^\omega(x)$ is locally square integrable with its derivatives of order less than or equal to 2. Thus the sample path is C^1 in x by the Sobolev embedding theorem.

Now, for any $L \geq 1$ and ω, let H_L^ω be the self-adjoint operator on the Hilbert space $L^2((-L,L)^2)$ of square integrable functions on the square $(-L,L)^2$ obtained by restricting the operator

$$H^\omega := \sum_{\iota=1}^{2} \Big(i\frac{\partial}{\partial x_\iota} + A_\iota^\omega(x)\Big)^2 \tag{3.5}$$

by the Dirichlet boundary condition, where A^ω is a vector potential: a vector field satisfying $\nabla \times A^\omega = \partial_2 A_1^\omega - \partial_1 A_2^\omega = B^\omega$.

3.2. **Main results: Wegner type estimate.** Our main result is the following.

Theorem 10 (Wegner type estimate for a Gaussian random magnetic field, [65])**.** *Under the setting in the last subsection, there exist finite positive constants C_0, C_1, C_2 such that*

$$\mathbb{E}[\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(H_L^\omega)]] \leq C_0 R^2 \eta L^{C_1} \tag{3.6}$$

for any $R \in [1,\infty)$, $L \geq \sqrt{R} \vee C_2$ and $E, \eta > 0$ satisfying $E+\eta \leq R$.

This is an extension of the estimate obtained by Wegner [67] for the Anderson model, which is the self-adjoint operator

$$(A^X\varphi)(\boldsymbol{n}) = \sum_{\boldsymbol{m}=(m_1,\dots,m_d)\in\mathbb{Z}^d:\max_i |m_i-n_i|=1} \varphi(\boldsymbol{m}) + X_{\boldsymbol{n}}\varphi(\boldsymbol{n})$$

on $\ell^2(\mathbb{Z}^d) = \{\varphi : \mathbb{Z}^d \to \mathbb{C}, \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\varphi(\boldsymbol{n})|^2 < \infty\}$ defined from the system of real random variables $X = \{X_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}$. If we denote the restriction of this operator to $\ell^2(\{-n,-n+1,\dots,n\}) = \{\varphi \in \ell^2(\mathbb{Z}^d) : \mathrm{supp}\,\varphi \subset \{-n,-n+1,\dots,n\}\}$ by A_L^X, then the estimate obtained by Wegner [67] is

$$\mathbb{E}[\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(A_L^X)]] \leq C\eta L^d \tag{3.7}$$

when $X = \{X_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d}$ is a system of independently and identically distributed random variables whose common distribution has a bounded and smooth density function. One crucial difference between this estimate and our estimate (3.6) is that the power of L is d. From this estimate, we can obtain an upper bound of an important quantity called the density of states and defined by

$$\lim_{L\uparrow\infty,\eta\downarrow 0} \mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(A_L^X)]/(\eta L^d) = n(E).$$

To obtain the upper bound was the motivation of Wegner [67]. However, C_1 in (3.6) is known only as a very big number, and improving the estimate for the random magnetic field to an estimate like (3.7) is an issue that remains for the future. On the other hand, by applying Chebyshev's inequality to (3.7), we obtain an estimate of the probability of the existence of the eigenvalues in a small energy interval:

$$\mathbb{P}(\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(A_L^X)] \geq 1) \leq C\eta L^d.$$

Minami [44] developed this estimate to an estimate of the probability of the event that 2 or more eigenvalues exist in a small energy interval,

$$\mathbb{P}(\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(A_L^X)] \geq 2) \leq \frac{1}{2}(C\eta L^d)^2,$$

to obtain results on the level statistics, and this developed estimate is called a Minami type estimate. Moreover, Combes, Germinet, and Klein [7] developed these estimates to several estimates including an estimate of the probability of the event that n or more eigenvalues exist in a small energy interval,

$$\mathbb{P}(\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(A_L^X)] \geq n) \leq \frac{1}{n!}(C\eta L^d)^n,$$

and these estimates are applied to obtain results on the multiplicity of each eigenvalue and on the conductance. Germinet and Klopp [26] further improve the results. The development in this direction has been mainly considered on discrete models. However, extensions to continuous models are also attempted as in [8].

On the other hand, even the estimate, as in Theorem 10, is effective for the proof of the Anderson localization by the multi-scale analysis founded by Fröhlich and Spencer [20] as we can obtain the results in the next subsection from Theorem 10.

3.3. **Application to the proof of the Anderson localization.** By Theorem 10 in the last subsection, we obtain the following.

Corollary 11. *For the self-adjoint operator H^ω on the Hilbert space $L^2(\mathbb{R}^2)$ of square integrable functions on the whole space $\mathbb{R}^2$ of the form of* (3.5)*, the Anderson localization on a low energy interval occurs as follows:*

(i) *there exists a finite positive constant ε_0 such that the interval $[0, \varepsilon_0]$ is included in the pure point spectrum of H^ω;*

(ii) *the corresponding eigenfunctions decay exponentially;*

(iii) *for any positive number p, any interval I included in the interval $[0, \varepsilon_0]$ and any compact set K in $\mathbb{R}^2$, we have*

$$\mathbb{E}\left[\sup_t \left\| |x|^p e^{-itH^\omega} 1_I(H^\omega) 1_K \right\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}\right] < \infty,$$

where $\|\cdot\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$ is the operator norm of the bounded linear operators on $L^2(\mathbb{R}^2)$.

By the Anderson localization, we mean the phenomenon that the electron in random media as semiconductors made by doping impurities is bounded spacially by the randomness as its energy closes to the boundary of the admitted energy intervals. The existence of this phenomenon was pointed out in the work [3] in 1958 by Anderson, a researcher on the condensed matter physics. In the progress of the research on this phenomenon, the Anderson transition has also attracted attention. The Anderson transition is the phenomenon that the state of the electron in random media changes from bounded states to unbounded states as the energy varies from a neighborhood of the boundary of the energy interval to inside. The energy changing the property of the states is called the mobility edge. The research on these phenomena has been one of the main subjects in condensed matter physics. For the related contribution, the Nobel Prize in physics in 1977 was awarded jointly to Anderson, van Vleck, and Mott. The relating mathematical problem has also been studied actively. The Anderson localization is mathematically formulated as (i), (ii), (iii) in Corollary 11 above, for example. The Anderson transition is formulated as follows: the spectrum becomes a pure point outside mobility edges and becomes continuous inside pairs of adjacent mobility edges. However, the mathematical proof is far from the completion. One exception is the 1-dimensional case and it is known that the localization occurs for all energies in all interested

1-dimensional models. For the 1-dimensional case, it is crucial that we can apply the techniques for the ordinary differential equations, which is insufficient for the multi-dimensional case. The present achievements for the multi-dimensional case are the proofs of the Anderson localizations in the fundamental Anderson models, their continuous analogues are called the alloy type models, and the related models, and we have almost no results on the Anderson transition including the existence of the continuous spectrum (see [6], [40], [52]). For random magnetic fields, even the proof of the Anderson localization was given only in restrictive cases (see [17], [27], [28], [39], [62]). The case of the random magnetic fields may be imagined to be more complicated than the case of the Anderson model as follows: in the Anderson model, we have only to consider the random geographical features of mountains and valleys made by the random scalar potentials. However, in the random magnetic field, the rotating effects by the vector potentials vary randomly. The existence of the Anderson localization in random magnetic fields is not so established as in the models related to Anderson models.

The multi-scale analysis is a method of an induction combining estimates of the resolvent operators on small cubes to obtain similar estimates on larger cubes by the Wegner type estimate. This method is the present main tool to prove the Anderson localization in the multi-dimensional cases. Recently Bourgain and König [4] extended this method to an alloy type model where the random variables obey a discrete distribution, and their theory was developed by Germinet and Klein [25]. Thus we can also expect the development for this direction. As another method for the proof of the Anderson localization in discrete models, Aizenman and Milchanov [2] found a method called the fractional moment method, which estimates the fractional moment of the integral kernel of the resolvent operator on the whole space. Their method was extended to continuous models by Aizenman, Elgart, Naboko, Schenker and Stolz [1]. However, this method is too new and has not been extended to the random magnetic fields.

Corollary 11 is obtained by applying also Germinet and Klein's theory on the bootstrap multi-scale analysis [24], which is the most powerful method among all of the multi-scale analyses. Their theory gives a series of necessary conditions for the statements of the Anderson localization to hold. These conditions are easily checked by standard arguments for Schrödinger operators except for the Wegner type estimate and the initial estimate for the first step of the induction of the multi-scale analysis. The initial estimate is usually proven by using the exponential decay of the integrated density of states at the infimum of the spectrum as is discussed in Section 2. The exponential decay in random magnetic fields is proven by Nakamura [45] and the author [61], which are sufficient for the proof of Corollary 11.

Corollary 11 (ii) is called the exponential localization and this is proven only by a simpler multi-scale analysis without using the bootstrap multi-scale analysis [24]. Thus the importance of the bootstrap multi-scale analysis [24] is that this method proves Corollary 11 (iii). This phenomenon is called the strong dynamical localization. The strong dynamical localization states that the dynamical localization discussed in [9] and [10] holds in a stronger sense. The dynamical localization discussed in [9] and [10] states that the quantity inside of the expectation in (iii) with $p = 1$ is finite with the probability one. This statement is stronger than Corollary 11 (i).

3.4. **Malliavin calculus.** Before discussing the proof of Theorem 10, we summarize the related issues in the Malliavin calculus.

Let $x(t,w)$ be the solution of the stochastic differential equation

$$x(t) = \int_0^t a(x(s))ds + \int_0^t b(x(s))dw(s) \tag{3.8}$$

of Itô type, then this is a functional on the classical Wiener space $W := \{w : [0,t] \to \mathbb{R} : \text{continuous}, w(0) = 0\}$. However, its functional derivatives are well defined only in the direction of the elements of the space $H = \{h : [0,t] \to \mathbb{R} : \text{absolutely continuous}, h' \in L^2, h(0) = 0\}$, which is called as the Cameron-Martin space. The reasons are that $x(t,w)$ is defined only with probability one, where the probability is the Wiener measure, the probability distribution of the Brownian motion, and that the Wiener measure is absolutely continuous under the parallel transport only in the direction of the Cameron-Martin space. Thus we follow Definition 2.6 in [57]: a functional $\mathcal{F}(\omega)$ on W is said to be H-differentiable at $\omega_0 \in W$ if there exists $D\mathcal{F}(\omega_0) \in H$ such that

$$\lim_{\varepsilon \to 0} \{\mathcal{F}(\omega_0 + \varepsilon\Phi) - \mathcal{F}(\omega_0)\}/\varepsilon = (D\mathcal{F}(\omega_0), \Phi)_H$$

for any $\Phi \in H$. Here we note that H becomes a Hilbert space with $(h,k)_H = \int_0^1 h'(s)k'(s)ds$ as its inner product. Then we can show that $x(t,w)$ is H-differentiable for any times at almost all w under the condition that a and b are smooth and their all derivatives are bounded. By a formal integration by parts on W, for any $n \in \{0,1,2,\ldots\}$, there exist $N(n) \in \mathbb{N}$ and $F_n(t,\cdot,w) \in \bigcup_{p\in[1,\infty)} L^p(W)$ such that

$$\partial_x^n \frac{P(x(t,w) \in dx)}{dx} = E[\|Dx(t,w)\|_H^{-N(n)} F_n(t,w,x)]$$

(see [32] V-(9.6), (9.7)). This calculation is justified if

$$E[\|Dx(t,w)\|_H^{-n}] < \infty \text{ for any } n \in \mathbb{N}. \tag{3.9}$$

This is the statement of the non-degeneracy in the Malliavin sense and the quantity $\|Dx(t,w)\|_H^2$ in this it is called the Malliavin covariance. The reason for the name "covariance" is that the covariance becomes $\det((Dx^j(t,w), Dx^k(t,w))_H)_{1\le j,k\le d}$ for the solution $(x^j(t,w))_{1\le j\le d}$ of a stochastic differential equation on $\mathbb{R}^d$.

Accordingly the important step for the application of the Malliavin calculus is the proof of the non-degeneracy in the Malliavin sense. The proof is usually given by applying estimates of stopping times of the Brownian motion. Stopping times are useful tools for subjects related to stochastic processes including Brownian motions. However, applying them to problems on random fields as ours is not straightforward. However, another important step of the proof of the non-degeneracy is to reduce the estimate of the Malliavin covariance to that of a positive quadratic form of the Brownian motion. Such methods are useful also for subjects on random fields. Thus we refer to this step to consider our problem. Another subject where quadratic forms of the Brownian motion play an important role is the asymptotic problem of a stochastic oscillatory integral

$$E[\exp(\sqrt{-1}\xi F(w))]$$

as $\xi \to \infty$. This problem is related closely with the smoothness of the distribution of $F(w)$. As for this subject and the related fields, refer to the review by Ikeda [31].

3.5. **Erdös and Hasler's Wegner type estimate.** In this subsection we discuss the Wegner type estimate given in Erdös and Hasler [17]. Their random magnetic field is represented as

$$\mathcal{B}^{\boldsymbol{\omega}}(x) = \sum_{k\in\mathbb{Z}_+} c_k \sum_{a\in\mathbb{Z}^2/2^k} \omega_a^k u(2^k(x-a)) \tag{3.10}$$

and satisfies

$$C_1 \le \mathcal{B}^{\boldsymbol{\omega}}(x) \le C_2, \tag{3.11}$$

where $\boldsymbol{\omega} = \{\omega_a^k\}_{a\in\mathbb{Z}^2/2^k, k\in\mathbb{Z}_+}$ is a system of independent real random variables such that the distribution of ω_a^k is independent of a, u is a positive smooth function with a compact support on $\mathbb{R}^2$, $\{c_k\}_{k\in\mathbb{Z}_+}$ is a series of positive numbers, and C_1, C_2 are finite positive constants.

For this, they showed

$$\sum_{k,a}(\partial_{\omega_a^k}\lambda_\ell(\mathcal{H}_L^{\boldsymbol{\omega}}))^2 \ge C_3/L^{C_4} > 0 \tag{3.12}$$

and, under this estimate, they gave a Wegner type estimate for $\mathcal{H}_L^{\boldsymbol{\omega}}$. In this sentence, $\mathcal{H}_L^{\boldsymbol{\omega}}$ is the Schrödinger operator H_L^ω where the magnetic field $B^\omega(x)$ is replaced by $\mathcal{B}^{\boldsymbol{\omega}}(x)$ in (3.10), and $\lambda_\ell(\mathcal{H}_L^{\boldsymbol{\omega}})$ is its ℓth smallest eigenvalue. To show (3.12), they used a technical method based on the special structure of the magnetic field in (3.10). To extend this to Gaussian random fields, we use other methods discussed below. The estimate (3.12) corresponds to the non-degeneracy in the Malliavin sense in the framework of the stochastic differential equations of Itô type. The monotonicity used before their work corresponds to the similar estimate where $(\partial_{\omega_a^k}\lambda_\ell(\mathcal{H}_L^{\boldsymbol{\omega}}))^2$ is replaced by $\partial_{\omega_a^k}\lambda_\ell(\mathcal{H}_L^{\boldsymbol{\omega}})$ in (3.12). However, their theory implies that the monotonicity is not necessary for the Wegner type estimate and it is sufficient that the eigenvalues always move sensitively in some direction on the probability space. We will show this in this subsection:

Proof of a Wegner type estimate under (3.12)**.** We first introduce the function $t(h) := (h+1)(5R)^3(5R+h+1)^{-3}$ to cutoff a high energy part. Since the first derivative of this function is dominated from below by a positive constant on the interval $[-1, 2R-1]$, we have

$$\mathrm{Tr}[\chi_{[E-\eta,E+\eta]}(\mathcal{H}_L^{\boldsymbol{\omega}})] \le \mathrm{Tr}[\chi_{[t(E)-\eta,t(E)+\eta]}(t(\mathcal{H}_L^{\boldsymbol{\omega}}))] \tag{3.13}$$

if $[E-\eta, E+\eta] \subset [-1, 2R-1]$. Since we obtain

$$\sum_{k,a}(\partial_{\omega_a^k}\lambda_\ell(t(\mathcal{H}_L^{\boldsymbol{\omega}})))^2 \ge C_5/L^{C_4} > 0$$

from (3.12), the right hand side of (3.13) is dominated from above by

$$L^{C_4}\sum_{k,a,\ell}\chi_{[t(E)-\eta,t(E)+\eta]}(\lambda_\ell(t(\mathcal{H}_L^{\boldsymbol{\omega}})))(\partial_{\omega_a^k}\lambda_\ell(t(\mathcal{H}_L^{\boldsymbol{\omega}})))^2. \tag{3.14}$$

Here we introduce

$$F(t) = \int_{-\infty}^t \chi_{[t(E)-\eta,t(E)+\eta]}(s), \qquad G(t) = \int_{-\infty}^t F(s)ds.$$

Then (3.14) is rewritten as

$$L^{C_4}\sum_{k,a,\ell}\{\partial^2_{\omega^k_a}G(\lambda_\ell(t(\mathcal{H}^{\boldsymbol{\omega}}_L)))-F(\lambda_\ell(t(\mathcal{H}^{\boldsymbol{\omega}}_L)))\partial^2_{\omega^k_a}\lambda_\ell(t(\mathcal{H}^{\boldsymbol{\omega}}_L))\}. \tag{3.15}$$

Now the key point in Erdös and Hasler [17] is that they dominate (3.15) from above by

$$L^{C_4}\sum_{k,a}\{\partial^2_{\omega^k_a}\mathrm{Tr}[G(t(\mathcal{H}^{\boldsymbol{\omega}}_L))]-\mathrm{Tr}[F(t(\mathcal{H}^{\boldsymbol{\omega}}_L))\partial^2_{\omega^k_a}t(\mathcal{H}^{\boldsymbol{\omega}}_L)]\} \tag{3.16}$$

(See [17] Lemma 5.2), since the second derivatives of eigenvalues, which are too complicated, are replaced by derivatives of resolvent operators and so on, which are simpler, in (3.16). Then, by a simple estimate, we have

$$|\mathrm{Tr}[F(t(\mathcal{H}^{\boldsymbol{\omega}}_L))\partial^2_{\omega^k_a}t(\mathcal{H}^{\boldsymbol{\omega}}_L)])|\le C_6\eta L^{C_7}. \tag{3.17}$$

Moreover, by repeating the integration by parts on the probability space for two times, we have

$$\mathbb{E}[\partial^2_{\omega^k_a}\mathrm{Tr}[G(t(\mathcal{H}^{\boldsymbol{\omega}}_L))]]=\mathbb{E}[f_{a,k}(\boldsymbol{\omega})\mathrm{Tr}[G(t(\mathcal{H}^{\boldsymbol{\omega}}_L))]]\le C_8\eta L^{C_9} \tag{3.18}$$

and we can obtain a Wegner type estimate, where $f_{a,k}(\boldsymbol{\omega})$ is a suitable function of $\boldsymbol{\omega}$.

3.6. **Lower estimate of the gradients of the eigenvalues in terms of a quadratic form of a magnetic field.** In this subsection we give an estimate corresponding to (3.12) used for the proof of the Wegner type estimate in the setting in Subsection 3.1.

We use the vector potential

$$A^\omega_L(x):=\begin{pmatrix}\partial_2\\ -\partial_1\end{pmatrix}(-\Delta_L)^{-1}B^\omega(x) \tag{3.19}$$

only on $(-L,L)^2$, where Δ_L is the Dirichlet Laplacian on $(-L,L)^2$. Its eigenvalues and eigenfunctions are

$$E_{\boldsymbol{n},L}=\Big(\frac{\pi|\boldsymbol{n}|}{2L}\Big)^2 \text{ and } \Phi_{\boldsymbol{n},L}(x)=\frac{1}{L}\prod_{\iota=1}^{2}\sin\Big(\frac{n_\iota\pi}{2}\Big(\frac{x_\iota}{L}+1\Big)\Big)$$

for $\boldsymbol{n}=(n_1,n_2)\in\mathbb{N}^2$. By using these, we can write this as

$$(-\Delta_L)^{-1}B^\omega(x)=\sum_{\boldsymbol{n}\in\mathbb{N}^2}\frac{\Phi_{\boldsymbol{n},L}(x)}{E_{\boldsymbol{n},L}}\int_{(-L,L)^2}\Phi_{\boldsymbol{n},L}(y)B^\omega(y)dy. \tag{3.20}$$

Since $\varepsilon\mapsto\lambda_\ell(H^{\omega+\varepsilon\Phi}_L)$ is analytic for any $\Phi\in L^2(\mathbb{R}^2)$ by the perturbation theory (see [55], §XII.2), $\lambda_\ell(H^\omega_L)$ is H-differentiable for any times everywhere in the sense of Subsection 3.4. Here we note that the Wiener space W is the space $\mathcal{D}'(\mathbb{R}^2)$ of the distributions on $\mathbb{R}^2$ and the Cameron-Martin space H is $L^2(\mathbb{R}^2)$ in the setting of Subsection 3.1. Then the left hand side of (3.12) corresponds to $\|D\lambda_\ell(H^\omega_L)\|^2_H$. In the following we denote the H-derivative $(D\mathcal{F},\Phi)_H$ by $D_\Phi\mathcal{F}$ for any $\Phi\in H$ and any functional $\mathcal{F}$ on W. Then by the Feynman-Hellmann theorem ([58]), we have

$$D_\Phi\lambda_\ell(H^\omega_L)=(j^\omega_\ell,D_\Phi A^\omega_L)_H=(\nabla\times j^\omega_\ell,(-\Delta_L)^{-1}\sigma*\Phi)_{L^2}.$$

In this equation j_ℓ^ω is an $\mathbb{R}^2$-valued function called the current of the normalized eigenfunction ψ_ℓ^ω of the Schrödinger operator H_L^ω with the eigenvalue $\lambda_\ell(H_L^\omega)$, it is written as

$$j_\ell^\omega(x) = 2\mathrm{Re}\,\overline{\psi_\ell^\omega(x)} \cdot (i\nabla + A_L^\omega(x))\psi_\ell^\omega(x), \tag{3.21}$$

and its properties are investigated in Erdös and Hasler [17] to show (3.12).

We will use the following modification of Lemma 6.2 in [17].

Lemma 12 (cf. [17] Lemma 6.2)**.**

$$\|j_\ell^\omega\|_{L^2(\mathbb{R}^2)}^2 \geq \frac{1}{4L^2}\mathcal{F}(B^\omega), \tag{3.22}$$

where

$$\mathcal{F}(B^\omega) = \int_0^{cL^{-11}R(\omega)^{-2}} \Big| \int_{x\in\mathbb{R}^2:|x-x_{\omega,L}|\leq r} B^\omega(x)dx\Big|^2 \frac{dr}{2\pi r}, \tag{3.23}$$

$x_{\omega,L}$ is a point in $(-L,L)^2 \cap cL^{-11}R(\boldsymbol{\omega})^{-2}\mathbb{Z}^2$ determined by $\{B^\omega(x)\}_{x\in(-L,L)^2}$, c is a finite positive constant, and $R(\omega) = \|B^\omega\|_{W^{2,2}((-L,L)^2)}^2 + R$. In the last equality, $\|\cdot\|_{W^{2,2}((-L,L)^2)}$ is a norm of the Sobolev space $W^{2,2}((-L,L)^2)$ of the functions on $(-L,L)^2$ whose derivatives of order less than or equal to 2 are all square integrable, and is defined by

$$\|\varphi\|_{W^{2,2}((-L,L)^2)} = \Big(\sum_{j=0}^{2} \|\nabla^j\varphi\|_{L^2((-L,L)^2)^{2^j}}\Big)^{1/2}$$

for $\varphi \in C^\infty((-L,L)^2)$, for example.

For the proof of this lemma, we take $x_{\omega,L}$ so that $|\psi_\ell^\omega| \geq 1/(2L)$ on $\{x \in \mathbb{R}^2 : |x - x_{\omega,L}| \leq cL^{-11}R(\omega)^{-2}\}$. This is possible by $\max|\psi_\ell^\omega| \geq 1/L$ and estimates on the continuity of ψ_ℓ^ω on a neighborhood of the point attains $\max|\psi_\ell^\omega|$. Then we can show

$$\|j_\ell^\omega\|_{L^2(D(x_{\omega,L},cL^{-11}R(\omega)^{-2}))}^2 \geq \frac{1}{4L^2}\|A_\ell^\omega - \nabla\mathrm{arg}\psi_\ell^\omega\|_{L^2(D(x_\omega,cL^{-11}R(\omega)^{-2}))}^2$$

by (3.21). Moreover, by using the Stokes theorem, we can dominate the right hand side from below by $\mathcal{F}(B^\omega)$.

On the other hand, since $\nabla\cdot j_\ell^\omega = 0$, we have

$$\|\nabla \times j_\ell^\omega\|_H^2 = \|\nabla j_\ell^\omega\|_H^2 \geq \Big(\frac{\pi}{L}\Big)^2 \|j_\ell^\omega\|_{L^2((-L,L)^2)}^2. \tag{3.24}$$

In the last inequality, we use the fact that the least eigenvalue of $-\Delta_L$ is $(\pi/L)^2$.

To obtain an analogue of (3.12), Φ may be taken from an orthonormal basis of H and the basis may be taken as $\{\Phi_{\boldsymbol{n},L}\}_{\boldsymbol{n}\in\mathbb{N}^2}$ referring to (3.19) and (3.20). However, in

$$D_{\Phi_{\boldsymbol{n},L}}\lambda_\ell(H_L^\omega) = \widehat{\sigma}\Big(\frac{\boldsymbol{n}}{2L}\Big)\Big(\frac{L}{\pi|\boldsymbol{n}|}\Big)^2(\nabla\times j_\ell^\omega, \Phi_{\boldsymbol{n},L})_H,$$

one difficulty is that $\widehat{\sigma}(\boldsymbol{n}/(2L))$ may be 0. Now we use (3.3) and results on the behavior of the Bessel functions. Useful results for our purpose are that the positive zero points $\{j_{\nu+1,s}\}_{s\in\mathbb{N}}$ of $J_{\nu+1}$ are all simple, $j_{\nu+1,s} \sim (s+\nu/2+1/4)\pi$ as $s\to\infty$, and

$$\sqrt{t}J_{\nu+1}(t) \overset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\cos\Big(t - \frac{2\nu+3}{4}\pi\Big), \frac{J_{\nu+1}(t)}{t^{\nu+1}} \overset{t\to 0}{\sim} \frac{1}{2^{\nu+1}\Gamma(\nu+2)}.$$

Then to avoid the zero points of the Bessel function, we modify $\boldsymbol{n}$ in $\Phi_{\boldsymbol{n},L}(x)$ as

$$(\boldsymbol{n};\varepsilon) := \begin{cases} \boldsymbol{n} + \dfrac{\varepsilon\bar{j}L}{8\pi\overline{\sigma}}\dfrac{\boldsymbol{n}}{|\boldsymbol{n}|} & \Big(\text{if } |\boldsymbol{n}| \in \dfrac{L}{\pi\overline{\sigma}}\Big[j_{\nu+1,s}, j_{\nu+1,s} + \dfrac{\bar{j}}{8}\Big) \text{ for some } s \in \mathbb{N}\Big), \\ \boldsymbol{n} - \dfrac{\varepsilon\bar{j}L}{8\pi\overline{\sigma}}\dfrac{\boldsymbol{n}}{|\boldsymbol{n}|} & \Big(\text{if } |\boldsymbol{n}| \in \dfrac{L}{\pi\overline{\sigma}}\Big(j_{\nu+1,s} - \dfrac{\bar{j}}{8}, j_{\nu+1,s}\Big) \text{ for some } s \in \mathbb{N}\Big), \\ \boldsymbol{n} & (\text{otherwise}) \end{cases}$$

by taking $\varepsilon \in (0,1)$, where $\bar{j} := \inf_{s\in\mathbb{N}}(j_{\nu+1,s+1} - j_{\nu+1,s}) \wedge j_{\nu+1,1} > 0$. Moreover we should reduce the summation in $\boldsymbol{n}$ to a finite sum. For this, we use (3.24),

$$\|\nabla \times j_\ell^\omega\|_{L^2} \le CL^9 R(\omega)^2, \qquad \text{and} \qquad \|\nabla(\nabla \times j_\ell^\omega)\|_{L^2} \le CL^{22}R(\omega)^6.$$

Then we have

$$\sum_{|\boldsymbol{n}|^2 \le N(\omega)} (D_{\Phi_{(\boldsymbol{n};\varepsilon(\omega)),L}}\lambda_\ell(H_L^\omega))^2 \ge C_1 \frac{\varepsilon(\omega)^2 L^{2\nu-1}}{N(\omega)^{\nu+5/2}}\mathcal{F}(B^\omega), \tag{3.25}$$

where

$$N(\omega) = C_2 L^{30} R(\omega)^6 \mathcal{F}(\omega)^{-1} \text{ and } \varepsilon(\omega) = C_3 \mathcal{F}(\omega)^2 L^{-58} R(\omega)^{-11}.$$

In [17], since we have $\mathcal{F}(\mathcal{B}^{\boldsymbol{\omega}}) \ge C_2/L^{C_3}$ by the assumption (3.11), we reach (3.12). However, in our case, since $\mathcal{F}(\mathcal{B}^{\boldsymbol{\omega}})$ still may attain 0, we should use the results in the next subsection.

3.7. **Proof of the non-degeneracy.** The main part of the quantity given in (3.23) is the quadratic form

$$X(t) := \int_0^t \Big| \int_{|x|\le r} B^\omega(x)dx \Big|^2 \frac{dr}{2\pi r} \tag{3.26}$$

of the magnetic field. The quadratic form in (3.26) is positive definite and its rank is infinity. Therefore we can show the following estimate.

Theorem 13 ([65], Lemma 4.2)**.** *For any $\overline{R} \in (0,\infty)$, there exists $c \in (0,\infty)$ such that*

$$\mathbb{E}[\exp(-sX(R))] \le \exp(-cRs^{1/(2\nu+5)})$$

for any $s \in [1,\infty)$ and $R \in (0,\overline{R}]$ satisfying $Rs^{1/(2\nu+5)} \ge 1$.

By this theorem and the arguments in the Tauberian theory, we can obtain the following estimate.

Corollary 14 ([65], Corollary of Lemma 4.1)**.** *For any $p, \overline{R} \in (0,\infty)$, there exists $c \in (0,\infty)$ such that*

$$\mathbb{E}[X(R)^{-p}] \le cR^{-p(2\nu+5)}$$

for any $R \in (0,\overline{R}]$.

This estimate and (3.25) play the role of (3.12).

For the proof of Theorem 13, the key point is that the rank of $X(R)$ as a quadratic form is infinite. To use this, we take a sequence $\{R_j\}_j$ such that $R = R_0 > R_1 > R_2 > \cdots > R_n \downarrow 0$ and use the property that the part

$$B(R_{j-1}) \setminus B(R_j) \ni x \mapsto \int_{B(R_j+\overline{\sigma})^c} \sigma(x-y)\omega(dy)$$

of the increment $X(R_{j-1}) - X(R_j)$ is independent of $X(R_j)$. Then we obtain

$$\mathbb{E}[\exp(-sX(R))] \le \prod_{j=1}^{\infty} \mathbb{E}[\exp(-sX(R_{j-1}, R_j))],$$

where

$$X(R_{j-1}, R_j) := \int_{R_j}^{R_{j-1}} \frac{dr}{2\pi r} \Big(\int_{B(r)\setminus B(R_j)} dx \int_{B(R_j+\overline{\sigma})^c} \sigma(x-y)\omega(dy) \Big)^2.$$

In the setting of Subsection 3.1, we can show

$$c_1(R_{j-1} - R_j)^{2\nu+5} \Big(\frac{R_j}{R_{j-1}} \Big)^3 \le \mathbb{E}[X(R_{j-1}, R_j)] \le c_2(R_{j-1} - R_j)^{2\nu+5} \Big(\frac{R_{j-1}}{R_j} \Big)^2.$$

Since ω is a Gaussian random field, we can estimate $\mathbb{E}[\exp(-sX(R_{j-1}, R_j))]$. Then by taking $\{R_j\}_j$ appropriately, we can complete the proof of Theorem 13.

3.8. **Proof of the main theorem.** To prove Theorem 10 in the setting of Subsection 3.1, we have only to extend the arguments in (3.13)–(3.18) of Subsection 3.5. For the step corresponding to (3.18), we should dominate quantities depending on ω from above by H-differentiable functionals of ω so that the integrations by parts proceed well. Moreover, to apply Corollary 14, we classify the cases according to the values of $x_{\boldsymbol{\omega},L} \in (-L, L)^2 \cap cL^{-11}R(\omega)^{-2}\mathbb{Z}^2$ and $R(\omega)$. For details, refer to [65].

References

1. M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker and G. Stolz, *Moment analysis for localization in random Schrödinger operators*, Invent. Math., **163** (2006), 343–413. MR2207021
2. M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys., **157** (1993), 245–278. MR1244867
3. P. W. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev., **109** (1958), 1492–1505.
4. J. Bourgain and C. E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math., **161** (2005), 389–426. MR2180453
5. K. Broderix, D. Hundertmark, W. Kirsch and L. Leschke, *The fate of Lifshits tails in magnetic fields*, J. Statist. Phys., **80** (1995), 1–22. MR1340551
6. R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators*, Probab. Appl., Birkhäuser Boston Inc., (1990). MR1102675
7. J. -M. Combes, F. Germinet and A. Klein, *Generalized eigenvalue-counting estimates for the Anderson model*, J. Stat. Phys., **135** (2009), 201–216. MR2505733
8. J. -M. Combes, F. Germinet and A. Klein, *Poisson statistics for eigenvalues of continuum random Schrödinger operators*, Anal. PDE **3** (2010), 49–80. MR2663411
9. R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, *What is localization?* Phys. Rev. Lett., **75** (1995), 117–119.
10. R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, *Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one pertubations, and localization*, J. Anal. Math., **69** (1996), 153–200.
11. G. Di Nunno, B. Øksendal and F. Proske, *Malliavin calculus for Lévy processes with applications to finance*, Springer (2009). MR2460554
12. M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time. I, II, III, IV*, Communications on Pure and Applied Mathematics, **28** (1975), 1–47, **28** (1975), 279–301, **29** (1976), 389–461, **36** (1983), 183–212.
13. M. D. Donsker and S. R. S. Varadhan, *Asymptotics for the Wiener sausage*, Comm. Pure Appl. Math., **28** (1975), 525–565. MR0397901
14. L. Erdös, *Rayleigh-type isoperimetric inequality with a homogeneous magnetic field*, Calc. Var. Partial Differential Equations, **4** (1996), 283–292. MR1386738

15. L. Erdös, *Lifschitz tail in a magnetic field: the nonclassical regime*, Probab. Theory Related Fields, **112** (1998), 321–371. MR1660914
16. L. Erdös, *Lifschitz tail in a magnetic field: coexistence of classical and quantum behavior in the borderline case*, Probab. Theory Related Fields, **121** (2001), 219–236. MR1865486
17. L. Erdös and D. Hasler, *Wegner estimate and Anderson localization for random magnetic fields*, Comm. Math. Phys., **309** (2012), 507–542. MR2864802
18. L. Erdös and D. Hasler, *Wegner estimate for random magnetic Laplacian on* $\mathbb{Z}^2$, Annales Henri Poincaré, **13** (2012), 1719–1731. MR2994757
19. L. Erdös and D. Hasler, *Anderson localization at band edges for random magnetic fields*, J. Stat. Phys, **146** (2012), 900–923. MR2902447
20. J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys., **88** (1983), 151–184. MR696803
21. R. Fukushima, *Brownian survival and Lifshitz tail in perturbed lattice disorder*, J. Funct. Anal., **256** (2009), 2867–2893. MR2502426
22. R. Fukushima and N. Ueki, *Classical and quantum behavior of the integrated density of states for a randomly perturbed lattice*, Annales Henri Poincaré, **11** (2010), 1053–1083. MR2737491
23. R. Fukushima and N. Ueki, *Moment asymptotics for the parabolic Anderson problem with a perturbed lattice potential*, J. Funct. Anal., **260** (2011), 724–744. MR2737395
24. F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media*, Comm. Math. Phys., **222** (2001), 415–448. MR1859605
25. F. Germinet and A. Klein, *A comprehensive proof of localization for continuous Anderson models with singular random potentials*, J. Eur. Math. Soc., **15** (2013) 53–143. MR2998830
26. F. Germinet and F. Klopp, *Enhanced Wegner and Minami estimates and eigenvalue statistics of random Anderson models at spectral edges*, Annales Henri Poincaré, **14** (2013), 1263–1285. MR3070753
27. F. Ghribi, P. D. Hislop and F. Klopp, *Localization for Schrödinger operators with random vector potentials*, In: Adventures in Mathematical Physics, Contemp. Math., **447** (2007), 123–138. MR2423576
28. P. D. Hislop and F. Klopp, *The integrated density of states for some random operators with nonsign definite potentials*, J. Funct. Anal., **195** (2002), 12–47. MR1934351
29. D. Hundertmark, W. Kirsch and S. Warzel, *Classical magnetic Lifshits tails in three space dimensions: impurity potentials with slow anisotropic decay*, Markov Process. Related Fields, **9** (2003), 651–660. MR2072247
30. T. Hupfer, H. Leschke and S. Warzel, *Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields*, J. Statist. Phys., **97** (1999), 725–750. MR1736101
31. N. Ikeda, *Van Vleck's formula and Jacobi fields in connection with Wiener integrals*, (Japanese), Sugaku, **52** (2000), 225–244. MR1784771
32. N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland Math. Library, 24, North-Holland/Kodan-sha, (1989). MR637061
33. M. Kac, *Probabilistic methods in some problems of scattering theory*, Rocky Mountain J. Math., **4** (1974), 511–537. MR0510113
34. W. Kirsch and F. Martinelli, *On the density of states of Schrödinger operators with a random potential*, J. Phys. A, **15** (1982), 2139–2156. MR665829
35. W. Kirsch and F. Martinelli, *On the spectrum of Schrödinger operators with a random potential*, Comm. Math. Phys., **85** (1982), 329–350. MR678150
36. W. Kirsch and B. Metzger, *The integrated density of states for random Schrödinger operators*, Proc. Sympos. Pure Math., Amer. Math. Soc., **76** (2007), 649–696. MR2307751
37. W. Kirsch and S. Warzel, *Lifshits tails caused by anisotropic decay: the emergence of a quantum-classical regime*, Mathematical Physics, Analysis and Geometry, **8** (2005), 257–285. MR2177468
38. F. Klopp, *Localization for some continuous random Schrödinger operators*, Comm. Math. Phys., **167** (1995), 553–569. MR1316760
39. F. Klopp, S. Nakamura, F. Nakano and Y. Nomura, *Anderson localization for 2D discrete Schrödinger operators with random magnetic fields*, Ann. Henri Poincaré, **4**, (2003), 795–811. MR2015430
40. S. Kotani, *Problems on random potentials*, (Japanese), Sugaku, **38** (1986), 53–61, 193–201. MR876441

41. H. Leschke, P. Müller and S. Warzel, *A survey of rigorous results on random Schrödinger operators for amorphous solids*, Markov Process. Related Fields, **9** (2003), 729–760. MR2072253
42. I. M. Lifshitz, *Energy spectrum structure and quantum states of disordered condensed systems*, Soviet Physics Uspekhi, **7** (1965), 549–573. MR0181368
43. P. Malliavin, *Stochastic calculus of variation and hypoelliptic operators*, In: Proceedings of the International Symposium on Stochastic Differential Equations, RIMS, Kyoto, 1976, (ed. K. Itô), Wiley, (1978), 195–263. MR536013
44. N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight binding model*, Commun. Math. Phys., **177**, (1996) 709–725. MR1385082
45. S. Nakamura, *A remark on the Dirichlet-Neumann decoupling and the integrated density of states*, J. Funct. Anal., **179**, (2001), 136–152. MR1807255
46. S. Nakao, *On the spectral distribution of the Schrödinger operator with random potential*, Japan. J. Math. (N.S.) **3** (1977), 111–139. MR0651925
47. D. Nualart, *The Malliavin calculus and related topics*, Probability and its Applications (New York), Springer-Verlag, New York, 1995. MR1344217
48. H. Ôkura, *An asymptotic property of a certain Brownian motion expectation for large time*, Proc. Japan Acad. Ser. A Math. Sci., **57** (1981), 155–159 MR618081
49. R. Osserman, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly, **86** (1979), 1–29. MR519520
50. S. Ozawa, *Spectra of random media* (Japanese), Sugaku, **44** (1992), 306–319. MR1197556
51. L. A. Pastur, *The behavior of certain Wiener integrals as $t \to \infty$ and the density of states of Schrödinger equations with random potential*, Teoret. Mat. Fiz., **32** (1977), 88–95. MR0449356
52. L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Grundlehren Math. Wiss., 297, Springer, (1992). MR1223779
53. J. Rauch, *The mathematical theory of crushed ice*, In: Partial Differential Equations and Related Topics, Lecture Notes in Math., **446** (1975), 370–379. MR0427863
54. J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal., **18** (1975), 27–59. MR0377303
55. M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, (1978). MR0493421
56. B. Simon, *Functional integration and quantum physics*, 2nd ed., AMS Chelsea Publishing, Providence, RI, (2005). MR2105995
57. I. Shigekawa, *Stochastic analysis*, Translations of Mathematical Monographs, 224. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2004. xii+182 pp. MR2060917
58. P. Stollmann, *Caught by disorder, Bound states in random media*, Progr. Math. Phys., 20, Birkhäuser, (2001). MR1935594
59. A. -S. Sznitman, *Lifschitz tail and Wiener sausage, I, II*, J. Funct. Anal., **94** (1990), 223–246, 247–272. MR1081644
60. A. -S. Sznitman, *Brownian motion, obstacles and random media*, Springer Monogr. Math., Springer, (1998) MR1717054
61. N. Ueki, *Simple examples of Lifschitz tails in Gaussian random magnetic fields*, Ann. Henri Poincaré, **1** (2000), 473–498. MR1777309
62. N. Ueki, *Wegner estimates and localization for random magnetic fields*, Osaka J. Math., **45** (2008), 565–608. MR2468583
63. N. Ueki, *Classical behavior of the integrated density of states for the uniform magnetic field and a randomly perturbed lattice*, Markov Processes and Related Fields, **17** (2011), 347-368. MR2895556
64. N. Ueki, *Quantum behavior of the integrated density of states for the uniform magnetic field and a randomly perturbed lattice*, In: Spectra of Random Operators and Related Topics, (ed. N. Minami), RIMS Kokyuroku Bessatsu, B**27** (2011), 149–169. MR2885261
65. N. Ueki, *Wegner estimate for Gaussian random magnetic fields*, J. Funct. Anal., **263** (2012) 2024-2050. MR2956933
66. S. Warzel, *Lifshits tails in magnetic fields*, PhD Thesis, Universität Erlangen-Nürnberg, (2001).

67. F. Wegner, *Bounds on the density of states in disordered systems*, Z. Phys. B, **44** (1981), 9–15. MR639135

Translated by THE AUTHOR

Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

Email address: ueki@math.h.kyoto-u.ac.jp