

**AN ESTIMATE OF THE RATE OF CONVERGENCE
 TO THE NORMAL LAW**

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ABSTRACT. The paper contains an estimate of the rate of convergence to the normal law for nonidentically distributed random variables.

A generalization of a V. M. Zolotarev result for the case of nonidentically distributed random variables is obtained in [1]. This paper continues the line of research initiated in [1] on applications of pseudomoments to the rate of convergence to the normal law. However below we use other pseudomoments different from those used in [1].

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of independent random variables such that $M \xi_i = 0$ and $\text{Var } \xi_i = \sigma_i^2$. Let $F_i(x)$ and $f_i(t)$ be the distribution function and characteristic function of the random variable ξ_i , $i \geq 1$, respectively. Denote by $\Phi_n(x)$ the distribution function of the random variable $(\xi_1 + \xi_2 + \dots + \xi_n)/B_n$ and put $B_n^2 = \sigma_1^2 + \dots + \sigma_n^2$, $\bar{\sigma}_i = \min(1; \sigma_i)$, and $\bar{B}_n^2 = \bar{\sigma}_1^2 + \dots + \bar{\sigma}_n^2$. Further let $\Phi(x)$ be the standard normal distribution function, $\rho_n = \sup_x |\Phi_n(x) - \Phi(x)|$, and

$$\begin{aligned} \lambda_{ni}^{(1)} &= \int_{|x| \leq B_n} \max(1, |x|^3) \left| d \left(F_i(x) - \Phi \left(\frac{x}{\sigma_i} \right) \right) \right|, \\ \lambda_{ni}^{(2)} &= \int_{|x| \geq B_n} x^2 \left| d \left(F_i(x) - \Phi \left(\frac{x}{\sigma_i} \right) \right) \right|, \\ \Lambda_n^{(1)} &= \frac{1}{B_n^2} \sum_{i=1}^n \lambda_{ni}^{(1)}, \quad \Lambda_n^{(2)} = \frac{1}{B_n^2} \sum_{i=1}^n \lambda_{ni}^{(2)}, \quad b_n = \min(\bar{\sigma}_1, \dots, \bar{\sigma}_n). \end{aligned}$$

Theorem. For all $n \geq 1$,

$$(1) \quad \rho_n \leq C \left(\frac{1}{B_n} \Lambda_n^{(1)} + \Lambda_n^{(2)} \right) b_n^{-3},$$

where C is a universal constant.

Proof. Let n be fixed and $c \in (0; 1)$ be a certain constant; the exact value of c will be chosen below. Assume that $\Lambda_n^{(2)} \leq c$ and note that estimate (1) is obvious otherwise.

For $n = 1$ we have

$$\begin{aligned} \rho_1 &= \sup_x |F_1(x\sigma) - \Phi(x)| = \sup_x |F_1(x) - \Phi(x/\sigma_1)| = \sup_x \left| \int_{-\infty}^{+\infty} d(F_1(x) - \Phi(x/\sigma_1)) \right| \\ &\leq \int_{-\infty}^{+\infty} |d(F_1(x) - \Phi(x/\sigma_1))| \leq \lambda_{n1}^{(1)} + \frac{1}{\sigma_1^2} \lambda_{n1}^{(2)}, \end{aligned}$$

and inequality (1) follows. Now let $n \geq 2$.

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Put $\phi_n(t) = \prod_{i=1}^n f_i(t/B_n)$. We apply the following inequality:

$$(2) \quad \rho_n \leq \min \left\{ \max \left(\int_0^X \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t}, \frac{6\sqrt{2}}{X} \right); X > 0 \right\}$$

(see [2, p. 377]) with $X = TB_n$, $T = c^{5/2}\Lambda_n^{-1}$, and $\Lambda_n = \max(\Lambda_n^{(1)}; \Lambda_n^{(2)})$. Let $T_1 = \sqrt{c}$, $T_2 = \min(T_1, T)$. Using the inequalities

$$\left| \phi_n(t) - e^{-t^2/2} \right| \leq \sum_{i=1}^n \left| \omega_i \left(\frac{t}{B_n} \right) \right| |\psi_{ni}(t)|,$$

where $\omega_i(t) = f_i(t) - \exp\{-t^2\sigma_i^2/2\}$, $\psi_{ni}(t) = \prod_{k=1}^{i-1} \exp\{-t^2\sigma_k^2/2B_n^2\} \prod_{k=i+1}^n f_k(t/B_n)$, and

$$|\omega_i(t)| \leq \min \left(\frac{|t|^3}{6} \lambda_{ni}^{(1)} + 2t^2 \lambda_{ni}^{(2)}; \lambda_{ni}^{(1)} + \frac{1}{B_n^2} \lambda_{ni}^{(2)} \right),$$

we get

$$\begin{aligned} |f_k(t)| &\leq \left| \exp\{-t^2\sigma_k^2/2\} + \omega_k(t) \right| \leq \exp\{-t^2\sigma_k^2/2\} + \frac{|t|^3}{6} \lambda_{nk}^{(1)} + 2t^2 \lambda_{nk}^{(2)} \\ &\leq \exp\{-t^2\bar{\sigma}_k^2/2\} + t^2 \left(\frac{c^{5/2}}{6} + 2c \right) \leq \exp\{-c_1 t^2 \bar{\sigma}_k^2\} \end{aligned}$$

for $|t| \leq T_2$, while

$$|\psi_{ni}(t)| \leq e \cdot \exp \left\{ -t^2 \frac{\bar{B}_n^2}{B_n^2} c_1 \right\}$$

for $|t| \leq T_2 B_n$, where $c_1 = (1 - e^{-c/2})/c - 13c/6$ and c is chosen such that $c_1 > 0$. Then (see inequality (12) in [1])

$$(3) \quad \left| \phi_n(t) - e^{-t^2/2} \right| \leq e \left(\frac{|t|^3}{6} \Lambda_n^{(1)} + 2t^2 \Lambda_n^{(2)} \right) \exp \left\{ -t^2 \frac{\bar{B}_n^2}{B_n^2} c_1 \right\}.$$

We represent the integral in (2) as follows:

$$(4) \quad \begin{aligned} I &= \int_0^{TB_n} \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t} \\ &= \int_0^{T_2 B_n} \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t} + \int_{T_2 B_n}^{TB_n} \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t} = I_1 + I_2. \end{aligned}$$

Inequality (3) implies

$$(5) \quad I_1 = \int_0^{T_2 B_n} \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t} \leq C^{(1)} \left(\frac{1}{B_n} \Lambda_n^{(1)} + \Lambda_n^{(2)} \right).$$

The integral I_2 is nonzero if $T_2 = T_1$, in which case $\Lambda_n^{(1)} \leq c^2$. Thus

$$\begin{aligned} \left| f_i \left(\frac{t}{B_n} \right) \right| &\leq \exp \left\{ -t^2 \sigma_i^2 / (2B_n^2) \right\} + \left| \omega_i \left(\frac{t}{B_n} \right) \right| \\ &\leq \exp \left\{ \left(e^{-c/2} - 1 \right) \bar{\sigma}_i^2 + \lambda_{ni}^{(1)} + \frac{1}{B_n^2} \lambda_{ni}^{(2)} \right\}, \\ \prod_{\substack{k=1 \\ k \neq i, j}}^n \left(\exp \left\{ -t^2 \sigma_1^2 / (2B_n^2) \right\} + \left| \omega_k \left(\frac{t}{B_n} \right) \right| \right) &\leq e^{2+c} \exp \left\{ - \left(\frac{1 - e^{-c/2}}{c} - c \right) c \bar{B}_n^2 \right\} \end{aligned}$$

for $T_2B_n \leq |t| \leq TB_n$. Moreover by inequality (13) in [1],

$$\begin{aligned} \left| \phi_n(t) - e^{-t^2/2} \right| &\leq \sum_{i=1}^n \left| \omega_i \left(\frac{t}{B_n} \right) \right| \left| \sum_{\substack{j=1 \\ j \neq i}}^n e^{-t^2 \sigma_j^2 / (2B_n^2)} \prod_{\substack{k=1 \\ k \neq i, j}}^n \left(e^{-t^2 \sigma_k^2 / (2B_n^2)} + \left| \omega_k \left(\frac{t}{B_n} \right) \right| \right) \right| \\ &\quad + \prod_{k=1}^n \left| \omega_k \left(\frac{t}{B_n} \right) \right| \\ &\leq \sum_{i=1}^n \left| \omega_i \left(\frac{t}{B_n} \right) \right| e^{-t^2 b_n^2 / (2B_n^2)} (n-1) e^{2+c} \exp \{ -c_3 c \bar{B}_n^2 \} \\ &\quad + \prod_{k=1}^n \left(\lambda_{nk}^{(1)} + \frac{1}{B_n^2} \lambda_{nk}^{(2)} \right), \end{aligned}$$

where $c_3 = (1 - e^{-c/2})/c - c > 0$.

Thus

$$\begin{aligned} I_2 &= \int_{T_2B_n}^{TB_n} \left| \phi_n(t) - e^{-t^2/2} \right| \frac{dt}{t} \\ &\leq (n-1) e^{2+c} \exp \{ -c_3 c \bar{B}_n^2 \} \sum_{i=1}^n \left(\lambda_{ni}^{(1)} + \frac{1}{B_n^2} \lambda_{ni}^{(2)} \right) \int_{T_2B_n}^{TB_n} e^{-t^2 b_n^2 / (2B_n^2)} \frac{dt}{t} \\ (6) \quad &+ \prod_{k=1}^n \left(\lambda_{nk}^{(1)} + \frac{1}{B_n^2} \lambda_{nk}^{(2)} \right) \int_{T_2B_n}^{TB_n} \frac{dt}{t} \\ &\leq C^{(3)} \left(\frac{\Lambda_n^{(1)}}{\bar{B}_n} + \Lambda_n^{(2)} \right) b_n^{-3} \end{aligned}$$

and (1) follows from (2) and (4)–(6). \square

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