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THE COMPLETE CONVERGENCE IN THE STRONG LAW OF LARGE NUMBERS FOR DOUBLE SUMS INDEXED BY A SECTOR WITH FUNCTION BOUNDARIES

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Dedicated to M. I. Yadrenko on his 70th birthday.

Abstract. We find necessary and sufficient conditions for the complete convergence in the strong law of large numbers for double sums of independent identically distributed random variables indexed by a sector with function boundaries.

1. Introduction

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to 0 if

$$\sum_{n=1}^{\infty} \mathsf{P}(|U_n| \geq \varepsilon) < \infty \qquad \text{for all } \varepsilon > 0.$$

This kind of convergence was introduced in 1947 by Hsu and Robbins [5]. They also proved that the condition

$$\mathsf{E}\, X = 0, \qquad \mathsf{E}\, X^2 < \infty \tag{1}$$

implies the complete convergence of S_n/n to 0, where

$$S_n = X_1 + \cdots + X_n, \quad n \geq 2, \qquad S_1 = X_1,$$

and $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables. The random variable X involved in condition (1) is a copy of X_1. Erdös in 1949 [1] proved the converse, so that the complete convergence of S_n/n to 0 is equivalent to condition (1).

It is natural to introduce the notion of complete convergence for sequences depending on two parameters. Namely, a sequence of random variables $\{U(m,n); m \geq 1, n \geq 1\}$ is said to converge completely to 0 if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathsf{P}(|U(m,n)| \geq \varepsilon) < \infty \qquad \text{for all } \varepsilon > 0.$$

Smythe [7] was the first to consider the complete convergence for double sums (in fact, for d-multiple sums, $d \geq 1$). He proved that the condition

$$\mathsf{E}\, X = 0, \qquad \mathsf{E}\, X^2 \log^+ |X| < \infty \tag{2}$$

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is equivalent to the complete convergence of $S(m,n)/mn$ to 0, where

$$S(m,n)=\sum_{i=1}^{m}\sum_{j=1}^{n}X(i,j),$$

$\{X(i,j); i\ge 1, j\ge 1\}$ are independent identically distributed random variables, X is a copy of $X(1,1)$, and $\log^+ z=\log(1+z)$ for $z\ge 0$.

Gut [3] considered the complete convergence of $S(m,n)/mn$ on a sector. Namely, let $\theta>1$ and $A=\{(m,n)\colon m\le n\le \theta n\}$. The complete convergence of a sequence of random variables $\{U(m,n); m\ge 1, n\ge 1\}$ on the set A means that

$$\sum_A \mathsf{P}(|U(m,n)|\ge\varepsilon)<\infty \qquad \text{for all } \varepsilon>0,$$

where $\sum_A$ stays for the summation over all $(m,n)\in A$. Gut [3] proved that the complete convergence of $S(m,n)/mn$ to 0 on the sector A is equivalent to condition (1) whatever $\theta>1$ is. In fact, Gut considered the case of the set $A=\{(m,n)\colon \theta^{-1}m\le n\le\theta n\}$. However the result for this set easily follows from the case mentioned above.

In this paper we consider the case of a more general set A. More precisely, let

$$f\colon \mathbf{R}\to\mathbf{R}$$

be a function such that

$$f(x)\ge x \qquad \text{for all } x\ge 1. \tag{3}$$

Note that $f(x)$ can be infinite for some x (even for all x). Define the set A by

$$A=\{(m,n)\colon m\le n\le f(m)\}. \tag{4}$$

We study the complete convergence of $S(m,n)/mn$ to 0 on the set A defined by (4). The condition for the complete convergence depends in this case on the behavior of the sequence

$$\tau_k=\operatorname{card}\{(i,j)\colon ij=k, (i,j)\in A\}, \qquad k\ge 1, \tag{5}$$

where i and j are positive integers. The conditions for the complete convergence on A are given by

$$\mathsf{E}\,X=0, \tag{6}$$

$$\sum_{k=1}^{\infty}k\tau_k\,\mathsf{P}(|X|\ge\delta k)<\infty \quad \text{for all } \delta>0 \tag{7}$$

(see Theorem 1 below). One can easily get the above results by Smythe and Gut from Theorem 1. Indeed we choose $f(x)=\infty$ and $f(x)=\theta x$ for all x in the Smythe and Gut cases, respectively. Then we transform the series on the left-hand side of (7) and prove that (7) is equivalent to

$$\sum_{k=j}^{\infty}\mathsf{P}(\delta j\le|X|<\delta(j+1))\sum_{k=1}^{j}k\tau_k<\infty \quad \text{for all } \delta>0.$$

Now we use the asymptotic behavior of the sequence $\{\tau_k\}$:

$$\tau_1+\cdots+\tau_k\asymp k\log k$$

in the Smythe case, and

$$\tau_1+\cdots+\tau_k\asymp k$$

in the Gut case, and obtain their results from (7). This completes the proof.

Another example is presented by the function $f(x) = x \log^+ x$ in which case

$$\tau_1 + \cdots + \tau_k \asymp k \log \log k$$

and (7) is equivalent to $\mathsf{E}\,|X| \log^+ \log^+ |X| < \infty$.

All three results above are special cases of the following proposition.

Proposition. *Let f be such that*

$$(\star) \qquad \lim_{x\to\infty} \frac{f(x)}{x} \quad \textit{exists.}$$

The complete convergence of $S(m,n)/mn$ to 0 on the set A defined by (4) is equivalent to the following conditions:

$$\mathsf{E}\,X = 0, \qquad \sum_{k=1}^{\infty} k\tau_k\, \mathsf{P}(|X| \ge k) < \infty.$$

In the above proposition, we were able to weaken condition (7) to the condition that the series on the left-hand side of (7) converges only for $\delta = 1$. This is possible in view of the nice behavior of the function

$$V(x) = \sum_{k \le x} k\tau_k$$

under condition ($\star$). We will study the function V elsewhere. Note that the condition of the proposition can be written in terms of the function $V(x)$ as follows:

$$\mathsf{E}\,V(|X|) < \infty.$$

2. Main result and proof

Let $\{X(i,j); i \ge 1, j \ge 1\}$ be independent identically distributed random variables with the distribution function F; $\{S(m,n); m \ge 1, n \ge 1\}$ be their double sums; X be a random variable with the distribution function F; $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that (3) holds; the set A be defined by (4); the sequence $\{\tau_k, k \ge 1\}$ be defined by (5); and $T_k = \tau_1 + \cdots + \tau_k$ for $k \ge 2$ and $T_1 = \tau_1$ and $T_0 = 0$.

Theorem 1. *The complete convergence of $S(m,n)/mn$ to 0 on the set A is equivalent to conditions (6)–(7).*

Proof. We prove that conditions (6)–(7) imply the complete convergence, that is, the condition

$$(8) \qquad \sum_{A} \mathsf{P}(|S(m,n)| \ge \varepsilon mn) < \infty \quad \text{for all } \varepsilon > 0.$$

The converse can be proved by using the same method as in [6]. First we consider the case of a symmetric distribution function F. Note that condition (7) with $\delta = 1$ implies that

$$\sum_{k=1}^{\infty} p_k \sum_{i=1}^{k} i\tau_i < \infty,$$

where

$$p_k = \mathsf{P}(k \le |X| < k+1).$$

For any k, there exists a number m such that $m^2 \le k < (m+1)^2$. Therefore,

$$\sum_{i=1}^{k} i\tau_i \ge \sum_{i=1}^{m^2} i\tau_i \ge \sum_{i=1}^{m} i^2 \ge \frac{m^3}{6} \ge \frac{k^{3/2}}{48},$$

whence $\sum_{k=1}^{\infty} p_k k^{3/2} < \infty$ and

$$\mathsf{E}\,|X|^{3/2} < \infty. \tag{9}$$

Let $\frac{8}{9} < v < 1$. Consider truncated random variables $X'(i,j) = X(i,j)I(|X(i,j)| < mn)$ for $(i,j) \le (m,n)$ and their sums $S'(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} X'(i,j)$. Now we introduce three random events:

$$\begin{aligned} E_1(m,n) &= \left\{\omega\colon \max_{(i,j)\le(m,n)} |X(i,j)| \ge \frac{\varepsilon}{3} mn\right\}, \\ E_2(m,n) &= \left\{\omega\colon |S'(m,n)| \ge \frac{\varepsilon}{3} mn\right\}, \\ E_3(m,n) &= \big\{\omega\colon \text{ there are at least three pairs } (i,j) \\ &\qquad \text{such that } (i,j) \le (m,n) \text{ and } |X(i,j)| \ge (mn)^v\big\}. \end{aligned}$$

It is clear that

$$\{\omega\colon |S(m,n)| \ge \varepsilon mn\} \subseteq E_1(m,n) \cup E_2(m,n) \cup E_3(m,n) \tag{10}$$

and for $\delta = \varepsilon/3$

$$\begin{aligned} \mathsf{P}(E_1(m,n)) &\le mn\,\mathsf{P}(|X| \ge \delta mn), \\ \mathsf{P}(E_3(m,n)) &\le \big[mn\,\mathsf{P}\left(|X| \ge (mn)^v\right)\big]^3 \le \left[\frac{\mathsf{E}\,|X|^{3/2}}{(mn)^w}\right]^3, \end{aligned}$$

where $w = \frac{3v}{2} - 1$. The above estimate for $\mathsf{P}(E_1(m,n))$ and condition (7) imply that

$$\sum_A \mathsf{P}(E_l(m,n)) < \infty \tag{11}$$

for $l = 1$. Since $v > \frac{8}{9}$, we have $w > \frac{1}{3}$, whence the above estimate for $\mathsf{P}(E_3(m,n))$ and (9) imply (11) for $l = 3$.

To prove (11) for $l = 2$ we apply the Fuk–Nagaev inequality [2]. The absolute constant involved in the Fuk–Nagaev inequality is denoted by C_{F-N} (see Lemma 3.2 in [4] for an estimate of C_{F-N}):

$$\begin{aligned} \mathsf{P}(|S'(m,n)| \ge \delta mn) &\le mn\,\mathsf{P}\left(|X'| \ge \frac{\delta}{2} mn\right) + C_{F-N}\, mn \left[\frac{1}{\delta^2 mn}\int_{|t|<(mn)^v} t^2\,dF\right]^2 \\ &\le mn\,\mathsf{P}\left(|X'| \ge \frac{\delta}{2} mn\right) + C_{F-N}\left(\mathsf{E}\,|X|^{3/2}\right)^2 \frac{1}{\delta^2 (mn)^{2-v}} \end{aligned}$$

(this is the only place in the proof where we use the assumption that F is symmetric). Since $v < 1$, condition (7) implies relation (11) for $l = 2$ as well. Relation (11) for $l = 1, 2, 3$ and (10) imply (8) in the case of a symmetric distribution F.

Now we turn to the case of a general distribution F. Consider the symmetric versions of random variables, namely $X^s(m,n) = X(m,n) - Y(m,n)$, where $\{Y(m,n); m \ge 1, n \ge 1\}$ is an independent copy of $\{X(m,n); m \ge 1, n \ge 1\}$. Condition (7) holds for X^s as well, and the above proof shows that

$$\sum_A \mathsf{P}(|S^s(m,n)| \ge \varepsilon mn) < \infty \quad \text{for all } \varepsilon > 0,$$

where $S^s(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n} X^s(i,j)$. Using the symmetrization inequality we prove that

$$\sum_A \mathsf{P}(|S(m,n) - \mu(m,n)| \ge \varepsilon mn) < \infty \quad \text{for all } \varepsilon > 0,$$

where $\mu(m,n)$ is a median of $S(m,n)$. Now we use (9) to obtain $\mu(m,n) = o(mn)$ and therefore (8) holds in the case of a general distribution function F. □

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