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THE EXPONENTIAL INTEGRABILITY OF QUASI-ADDITIVE FUNCTIONALS OF GAUSSIAN VECTORS

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Abstract. We study the exponential integrability of quasi-additive functionals of Gaussian random vectors.

1. Definitions and auxiliary results

Let $(V, \mathcal{F})$ be a *measurable vector space*. This means that V is a real vector space where the addition of its elements and the multiplication of its elements by real numbers agree with the σ-field $\mathcal{F}$ of subsets of the space V. Let (Ω, F, P) be a probability space.

A mapping X from Ω to V measurable with respect to the σ-fields F and $\mathcal{F}$ is called a *V-valued random vector*. The independence of a family of V-valued random vectors is defined in a standard way (see [1]). A V-valued random vector Y is called a *copy* of a V-valued random vector X if X and Y have the same distribution.

Definition 1. A V-valued random vector X is called *Gaussian* if, for each pair (X_1, X_2) of two independent copies of the random vector X, the pair (Z_1, Z_2), where

$$Z_1 = 2^{-1/2}(X_1 + X_2), \qquad Z_2 = 2^{-1/2}(X_1 - X_2),$$

is also a pair of independent copies of X.

If $V = \mathbf{R}$ is the set of real numbers and $\mathcal{F} = B(\mathbf{R})$ is the σ-field of Borel sets, then the zero mean Gaussian random variables (and only they) satisfy Definition 1 (see, for example, [1]). This implies that for Banach spaces as well as for some general topological vector spaces V Definition 1 is equivalent to the definition of a zero mean Gaussian random vector expressed in terms of the characteristic functional (see, for example, [2]).

We consider the problem on the integrability of quasi-additive functionals of V-valued Gaussian random vectors.

Definition 2. Let $(V, \mathcal{F})$ be a measurable vector space. A real-valued functional $g(\cdot) = (g(x), x \in V)$ is called a *measurable quasi-additive functional* (in other words, a *measurable C-quasi-additive functional*) if

1) $g(x) \geq 0$, $x \in V$;
2) there exists $C \geq 1$ such that $g(x + y) \leq C(g(x) + g(y))$ for $x, y \in V$;
3) the mapping $x \mapsto g(x)$ is measurable with respect to the σ-fields $\mathcal{F}$ and $B(\mathbf{R})$.

If the following additional condition holds:

4) $g(x) = g(-x)$, $x \in V$,

then the functional $g(\cdot)$ is called a *C-measurable quasi-additive symmetric functional*.

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Remark 1. The constant C involved in condition 2) is not unique. If condition 2) holds for some constant C, then it also holds for all $C' > C$. The minimum constant $C_{\min}$ for which condition 2) holds can be defined as the infimum of the set of numbers C satisfying that condition. This minimum constant $C_{\min}$ can be viewed as a kind of characteristic of the functional g. However the exact value of $C_{\min}$ does not play any role in the further discussion.

Remark 2. One can consider C-quasi-additive functionals for $0 < C < 1$, too. However

$$0 \le \sup_{x \in V} g(x) \le \frac{Cg(0)}{1-C} < \infty$$

for $C \in (0,1)$, since $g(x) = g(x+0) \le C(g(x) + g(0))$, that is, the functional $g(\cdot)$ is uniformly bounded on V. This is a trivial case, thus we consider the case of $C \ge 1$ in condition 2).

A measurable quasi-additive functional $g(\cdot)$ is called *measurable semiadditive* if $C = 1$. Measurable seminorms give a well-known example of semiadditive symmetric functionals. Measurable seminorms satisfy conditions 1)–4) and an extra (homogeneity) condition: $g(\lambda x) = |\lambda| g(x)$ for all $x \in V$ and $\lambda \in \mathbf{R}$. Measurable quasi-norms are a more general example of semiadditive functionals for which $g(\lambda x) \le g(x)$ for all $x \in V$ and $\lambda \in [-1, 1]$.

Simple examples of *measurable* C-quasi-additive functionals for $C > 1$ are given by convex measurable functionals $(g(x), x \in V)$ such that $g(x) > 0$, $x \ne 0$, and

$$K = \sup_{x \ne 0} (g(2x)/g(x)) \in (2, \infty).$$

In this case, $C = K/2$. In particular, $g(\cdot) = q^r(\cdot)$ is a C-quasi-additive functional if $q(\cdot)$ is a measurable quasi-norm and $r > 1$. In this case, $C = 2^{r-1} > 1$.

It is proved in [3, 4, 5] that for various infinite-dimensional vector spaces V the following holds: *if $g(\cdot)$ is a measurable seminorm and X is a Gaussian random vector, then there exists $a_0 > 0$ such that $\mathsf{E}\exp\{ag^2(X)\} < \infty$ for all $a \in [0, a_0)$.* The Fernique [4] proof of this result is based on Definition 1. A generalization of Fernique's method allows one to prove an analogous result for semiadditive functionals (see [6]).

The aim of this paper is to prove that there are numbers b such that

$$\mathsf{E}\exp\left\{ag^b(X)\right\} < \infty$$

for every measurable C-quasi-additive functional g of a Gaussian random vector X and to show how b depends on the constant C.

2. Main results

The first result concerns a general C-quasi-additive functional.

Theorem 1. *Let $(V, \mathcal{F})$ be a measurable vector space, X be a V-valued Gaussian random vector, $C \ge 1$, and let $g(\cdot)$ be a measurable C-quasi-additive functional on V. Then for any $\varepsilon > 0$ there is a number $a_0 > 0$ (depending, generally speaking, on X, g, and ε) such that*

$$\mathsf{E}\exp\left\{ag^b(X)\right\} < \infty \tag{1}$$

for all $a \in [0, a_0)$, where

$$b = b(\varepsilon,\ C) = \frac{2}{1 + (1+\varepsilon)\log_2 C}.$$

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Note that $b = 2$ if $C = 1$. Therefore Theorem 1 generalizes the result on the exponential integrability of measurable semiadditive functionals.

Inequality (1) holds for all $b \in [0, b(C))$ where

$$b(C) = \frac{2}{1 + \log_2 C}$$

and $C > 1$. Note however that inequality (1) may hold with $b = b(C)$ for some measurable C-quasi-additive functionals.

Example 1. Let

$$g(x) = p^r(x), \qquad x \in V,$$

where $p(\cdot)$ is a measurable semiadditive functional and $r > 1$. Then $g(\cdot)$ is a measurable 2^{r-1}-quasi-additive functional, that is, $C = 2^{r-1}$. Due to the theorem on the exponential integrability of semiadditive functionals, inequality (1) holds for $b = 2/r$. It remains to note that

$$b = \frac{2}{r} = \frac{2}{1 + \log_2 2^{r-1}} = \frac{2}{1 + \log_2 C}.$$

Example 1 shows that not only the constant b in inequality (1) can attain the value $b(C)$ but also that b does not exceed $b(C)$ in the class of C-quasi-additive functionals. A natural question arises whether inequality (1) may hold for $b = b(C)$ in the class of quasi-additive functionals. The answer is positive under certain additional restrictions on the functional $g(\cdot)$.

Theorem 2. *Let $(V, \mathcal{F})$ be a measurable vector space, X be a V-valued Gaussian random vector, $C \in [1, 2)$, and let $g(\cdot)$ be a measurable convex C-quasi-additive functional on V. Then there is a number $a_0 > 0$ (depending, generally speaking, on X and g) such that*

$$\mathsf{E} \exp\left\{a g^b(X)\right\} < \infty$$

for all $a \in [0, a_0)$, where

$$b = \frac{2}{1 + \log_2 C}.$$

An answer to the question on whether Theorem 2 holds for $C \geq 2$ requires an additional investigation.

3. Proofs of the theorems

We need several auxiliary results in order to prove Theorem 1.

Let $X_1, \ldots, X_n$ be independent copies of a V-valued Gaussian random vector X, where $n = 2^l$ and $l \geq 1$. We introduce random vectors $S_1, \ldots, S_n$ as follows:

$$(S_1, \ldots, S_n)^\top = U_n (X_1, \ldots, X_n)^\top, \tag{2}$$

where "$\top$" denotes the transposition,

$$U_n = n^{-1/2} H_n$$

are real $n \times n$ matrices, and $H_n = (h_{kj})_{k,j=1}^n$ are defined by

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_n = \begin{pmatrix} H_{\frac{n}{2}} & H_{\frac{n}{2}} \\ H_{\frac{n}{2}} & -H_{\frac{n}{2}} \end{pmatrix}, \qquad n = 2^l, \ l > 1.$$

Therefore

$$S_k = 2^{-l/2} \sum_{j=1}^{2^l} h_{kj} X_j$$

for all $k = 1, \ldots, 2^l$, where $h_{kj} = \pm 1$.

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Lemma 1 ([6]). *The random vectors $S_1, \dots, S_n$ are independent copies of the Gaussian random vector X.*

Lemma 2. *Let $C \geq 1$ and let $(g(x), x \in V)$ be a measurable C-quasi-additive functional. Then*

$$g(2^n x) \leq (2C)^n g(x), \tag{3}$$

$$g(x_1 + \dots + x_n) \leq Cg(x_1) + \dots + C^{n-2} g(x_{n-2}) + C^{n-1} g(x_{n-1}) + C^{n-1} g(x_n), \tag{4}$$

$$g(x_1 + \dots + x_n) \geq \frac{g(x_1)}{C} - C^{n-2} \sum_{k=2}^{n} g(-x_k) \tag{5}$$

for all positive integers n and all $x, x_1, \dots, x_n \in V$.

Proof of Lemma 2. Inequalities (3) and (4) can be proved by induction. Inequality (5) follows from (4). □

Lemma 3. *Let X be a V-valued Gaussian random vector, $C \geq 1$, and let $(g(x), x \in V)$ be a measurable C-quasi-additive functional on V. Then*

$$(\mathsf{P}\{g(X) \leq s\})^{4^m - 1} \mathsf{P}\{g(X) > t\} \leq \left(\mathsf{P}\left\{g(X) > \frac{t - (4^m - 1)C^{4^m - 1} s}{C(2C)^m}\right\}\right)^{4^m} \tag{6}$$

for all positive integers m and all $t > 0$ and $s > 0$.

Proof of Lemma 3. Let $X_1, \dots, X_{4^m}$ be independent copies of X. In view of Lemma 1, the random vectors

$$S_k = 2^{-m} \sum_{j=1}^{4^m} h_{kj} X_j, \qquad k = 1, \dots, 4^m,$$

are independent copies of X.

Since $g(\cdot)$ is a C-quasi-additive symmetric functional, Lemma 2 implies

$$\begin{aligned}(2C)^m g(S_k) &\geq g(2^m S_k) = g\left(\sum_{j=1}^{4^m} h_{kj} X_j\right) \\ &\geq \frac{1}{C} g(h_{k1} X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(-h_{kj} X_j) = \frac{1}{C} g(X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(X_j)\end{aligned}$$

for all $k = 1, \dots, 4^m$. This yields

$$\begin{aligned}(\mathsf{P}\{g(X) \leq s\})^{4^m - 1} \mathsf{P}\{g(X) > t\} &= \mathsf{P}\{g(X_1) > t, g(X_2) \leq s, \dots, g(X_{4^m}) \leq s\} \\ &\leq \mathsf{P}\left\{\frac{1}{C} g(X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(X_j) > \frac{t - (4^m - 1)C^{4^m - 1} s}{C}\right\} \\ &\leq \mathsf{P}\left\{\bigcap_{k=1}^{4^m} \left\{g(S_k) > \frac{t - (4^m - 1)C^{4^m - 1} s}{C(2C)^m}\right\}\right\} \\ &= \left(\mathsf{P}\left\{g(X) > \frac{t - (4^m - 1)C^{4^m - 1} s}{C(2C)^m}\right\}\right)^{4^m}\end{aligned}$$

for all $t > 0$ and $s > 0$. □

Proof of Theorem 1. First we consider the case of a C-quasi-additive symmetric functional $g(\cdot)$. Consider a positive number $s = t_0$ such that

$$c = \mathsf{P}\{g(X) \leq t_0\} > 1/2.$$

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For a given positive integer m let the sequence of positive numbers $(t_n, n \ge 1)$ be defined as follows:

$$t_n = ((\gamma+1)\beta^n - \gamma)t_0, \qquad n \ge 1,$$

where

$$\gamma = \frac{\alpha}{\beta - 1}, \qquad \alpha = (4^m - 1)C^{4^m - 1}, \qquad \beta = C(2C)^m \ge 2.$$

The sequence $(t_n, n \ge 1)$ is such that

$$\frac{t_n - \alpha t_0}{\beta} = t_{n-1}, \qquad n \ge 1,$$

and $t_n \to \infty$, $n \to \infty$.

Let z_n, $n \ge 0$, be real numbers such that

$$z_0 = \frac{1-c}{c}, \qquad c z_n = \mathsf{P}\left\{g(X) > t_n\right\}, \qquad n \ge 1.$$

Note that $z_0 < 1$. It follows from (6) that $z_n \le (z_{n-1})^{4^m}$, $n \ge 1$. Thus

$$\mathsf{P}\left\{g(X) > t_n\right\} \le c(z_0)^{4^{mn}}, \qquad n \ge 1,$$

whence

$$\begin{aligned}
\mathsf{E}\exp\left\{a g^b(X)\right\} &\le c\left[\exp\left\{a t_0^b\right\} + \sum_{n=1}^{\infty} z_0^{4^{m(n-1)}} \exp\left\{a(t_0(1+\gamma))^b \beta^{bn}\right\}\right] \\
&= c\left[\exp\left\{a t_0^b\right\} + \sum_{n=1}^{\infty} z_0^{4^{m(n-1)}} \exp\left\{a(t_0(1+\gamma))^b 2^{nb\log_2\beta}\right\}\right] \\
&= c\left[\exp\left\{a t_0^b\right\} + \sum_{n=1}^{\infty} z_0^{4^{m(n-1)}} \exp\left\{a(t_0(1+\gamma))^b 4^{mn}\right\}\right] \\
&= c\left[\exp\left\{a t_0^b\right\} + \sum_{n=1}^{\infty} \exp\left\{4^{mn}[a(t_0(1+\gamma))^b - (\left|\ln z_0\right|/4^m)]\right\}\right]
\end{aligned}$$

for

$$b = b\left(\frac{1}{m}, C\right) = \frac{2}{1 + (1 + \frac{1}{m})\log_2 C}.$$

This implies that the series converges and

$$\mathsf{E}\exp\left\{a g^b(X)\right\} < \infty$$

for $a < \left|\ln z_0\right| / [4^m a(t_0(1+\gamma))^b]$. This completes the proof of the theorem for C-quasi-additive symmetric functionals $g(\cdot)$, since m is arbitrary.

Now let $g(\cdot)$ be a measurable C-quasi-additive functional on V. Then

$$\widetilde{g}(x) = \max\{g(x), g(-x)\}, \qquad x \in V,$$

is a measurable C-quasi-additive symmetric functional on V. Thus there is a number $a_0 > 0$ such that

$$\mathsf{E}\exp\left\{a g^b(X)\right\} \le \mathsf{E}\exp\left\{a \widetilde{g}^b(x)\right\} < \infty$$

for all $a \in [0, a_0)$. □

Remark 3. The proof of Theorem 1 is simpler for $C = 1$. In this case, Theorem 1 follows from Lemma 3 for $m = 1$ (see [6]).

The following result is needed for the proof of Theorem 2.

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Lemma 4. *Let X be a V-valued Gaussian random vector, $C \in [1,2)$, and let $g(\cdot)$ be a measurable convex C-quasi-additive functional on V. If*

$$\mathsf{E}\exp\{a_1 g(X)\} < \infty \tag{7}$$

for some $a_1 > 0$, then there is $a > 0$ such that

$$\mathsf{E}\exp\left\{a g^b(X)\right\} < \infty,$$

where

$$b = \frac{2}{1 + \log_2 C}.$$

Proof of Lemma 4. It follows from Lemma 1 that

$$\mathsf{P}\{g(X) \ge t2^s\} = \mathsf{P}\left\{g\left(\frac{X_1 + \cdots + X_{4^n}}{2^n}\right) \ge t2^s\right\}$$

for all positive integers n and all $s, t > 0$ where $X_1, \ldots, X_{4^n}$ are independent copies of the random vector X.

Since $g(\cdot)$ is a convex C-quasi-additive functional,

$$\begin{aligned}\mathsf{P}\left\{g\left(\frac{X_1 + \cdots + X_{4^n}}{2^n}\right) \ge t2^s\right\} &= \mathsf{P}\left\{g\left(\frac{2^n(X_1 + \cdots + X_{4^n})}{4^n}\right) \ge t2^s\right\} \\ &\le \mathsf{P}\left\{g(2^n X_1) + \cdots + g(2^n X_{4^n}) \ge t2^{s+2n}\right\} \\ &\le \mathsf{P}\left\{g(X_1) + \cdots + g(X_{4^n}) \ge \frac{t2^{s+n}}{C^n}\right\}.\end{aligned}$$

Condition (7) and the Markov inequality imply that

$$\begin{aligned}\mathsf{P}\{g(X) \ge t2^s\} &\le \mathsf{P}\left\{g(X_1) + \cdots + g(X_{4^n}) \ge \frac{t2^{s+n}}{C^n}\right\} \\ &\le \exp\left\{-a_1 t\frac{2^{s+n}}{C^n}\right\}\mathsf{E}\exp\{a_1(g(X_1) + \cdots + g(X_{4^n}))\} \\ &= \exp\left\{-a_1 t\frac{2^{s+n}}{C^n}\right\}\left(\mathsf{E}\exp\{a_1 g(X)\}\right)^{4^n} = \exp\left\{-a_1 t\frac{2^{s+n}}{C^n} + a_2 4^n\right\}\end{aligned}$$

for all positive integers n and all $s, t > 0$ where $a_2 = \ln \mathsf{E}\exp\{a_1 g(X)\} \in (0, \infty)$. In the case of $s = n(1 + \log_2 C) = 2n/b$ one has

$$\mathsf{P}\left\{g(X) \ge t4^{n/b}\right\} \le \exp\{-(a_1 t - a_2)4^n\}$$

for all positive integers n and all $t > 0$. This implies that

$$\mathsf{E}\exp\left\{a g^b(X)\right\} \le \exp\left\{4at^b\right\} + \sum_{n=1}^{\infty} \exp\left\{-[a_1 t - 4at^b - a_2]4^n\right\}.$$

The series converges and

$$\mathsf{E}\exp\left\{a g^b(X)\right\} < \infty$$

if $t > a_2/a_1$ and $0 \le a < (a_1 t - a_2)/4t^b$. □

Proof of Theorem 2. Theorem 2 for $C = 1$ follows from Theorem 1 immediately. Condition (7) holds by Theorem 1 if $C \in (1, 2)$, in which case one needs to apply Lemma 4. □

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