THE EXPONENTIAL INTEGRABILITY OF QUASI-ADDITIVE FUNCTIONALS OF GAUSSIAN VECTORS

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Abstract. We study the exponential integrability of quasi-additive functionals of Gaussian random vectors.

1. Definitions and auxiliary results

Let \((V, \mathcal{F})\) be a measurable vector space. This means that \(V\) is a real vector space where the addition of its elements and the multiplication of its elements by real numbers agree with the \(\sigma\)-field \(\mathcal{F}\) of subsets of the space \(V\). Let \((\Omega, F, \mathbb{P})\) be a probability space.

A mapping \(X\) from \(\Omega\) to \(V\) measurable with respect to the \(\sigma\)-fields \(F\) and \(\mathcal{F}\) is called a \(V\)-valued random vector. The independence of a family of \(V\)-valued random vectors is defined in a standard way (see [1]). A \(V\)-valued random vector \(Y\) is called a copy of a \(V\)-valued random vector \(X\) if \(X\) and \(Y\) have the same distribution.

Definition 1. A \(V\)-valued random vector \(X\) is called Gaussian if, for each pair \((X_1, X_2)\) of two independent copies of the random vector \(X\), the pair \((Z_1, Z_2)\), where

\[
Z_1 = 2^{-1/2}(X_1 + X_2), \quad Z_2 = 2^{-1/2}(X_1 - X_2),
\]

is also a pair of independent copies of \(X\).

If \(V = \mathbb{R}\) is the set of real numbers and \(\mathcal{F} = B(\mathbb{R})\) is the \(\sigma\)-field of Borel sets, then the zero mean Gaussian random variables (and only they) satisfy Definition 1 (see, for example, [1]). This implies that for Banach spaces as well as for some general topological vector spaces \(V\) Definition 1 is equivalent to the definition of a zero mean Gaussian random vector expressed in terms of the characteristic functional (see, for example, [2]).

We consider the problem on the integrability of quasi-additive functionals of \(V\)-valued Gaussian random vectors.

Definition 2. Let \((V, \mathcal{F})\) be a measurable vector space. A real-valued functional \(g(\cdot) = (g(x), x \in V)\) is called a measurable quasi-additive functional (in other words, a measurable \(C\)-quasi-additive functional) if

1) \(g(x) \geq 0, x \in V\);
2) there exists \(C \geq 1\) such that \(g(x + y) \leq C(g(x) + g(y))\) for \(x, y \in V\);
3) the mapping \(x \mapsto g(x)\) is measurable with respect to the \(\sigma\)-fields \(\mathcal{F}\) and \(B(\mathbb{R})\).

If the following additional condition holds:
4) \(g(x) = g(-x), x \in V\),

then the functional \(g(\cdot)\) is called a \(C\)-measurable quasi-additive symmetric functional.
Remark 1. The constant $C$ involved in condition 2) is not unique. If condition 2) holds for some constant $C$, then it also holds for all $C'>C$. The minimum constant $C_{\min}$ for which condition 2) holds can be defined as the infimum of the set of numbers $C$ satisfying that condition. This minimum constant $C_{\min}$ can be viewed as a kind of characteristic of the functional $g$. However the exact value of $C_{\min}$ does not play any role in the further discussion.

Remark 2. One can consider $C$-quasi-additive functionals for $0<C<1$, too. However

$$0 \leq \sup_{x \in V} g(x) \leq \frac{Cg(0)}{1-C} < \infty$$

for $C \in (0,1)$, since $g(x) = g(x+0) \leq C(g(x) + g(0))$, that is, the functional $g(\cdot)$ is uniformly bounded on $V$. This is a trivial case, thus we consider the case of $C \geq 1$ in condition 2).

A measurable quasi-additive functional $g(\cdot)$ is called *measurable semiadditive* if $C=1$. Measurable seminorms give a well-known example of semiadditive symmetric functionals. Measurable seminorms satisfy conditions 1)–4) and an extra (homogeneity) condition: $g(\lambda x) = |\lambda|g(x)$ for all $x \in V$ and $\lambda \in \mathbb{R}$. Measurable quasi-norms are a more general example of semiadditive functionals for which $g(\lambda x) \leq g(x)$ for all $x \in V$ and $\lambda \in [-1,1]$.

Simple examples of measurable $C$-quasi-additive functionals for $C > 1$ are given by convex measurable functionals $(g(x), x \in V)$ such that $g(x) > 0$, $x \neq 0$, and

$$K = \sup_{x \neq 0} (g(2x)/g(x)) \in (2, \infty).$$

In this case, $C = K/2$. In particular, $g(\cdot) = q^r(\cdot)$ is a $C$-quasi-additive functional if $q(\cdot)$ is a measurable quasi-norm and $r > 1$. In this case, $C = 2^{r^{-1}} > 1$.

It is proved in [3, 4] that for various infinite-dimensional vector spaces $V$ the following holds: if $g(\cdot)$ is a measurable seminorm and $X$ is a Gaussian random vector, then there exists $a_0 > 0$ such that $\mathbb{E}\exp\{ag^2(X)\} < \infty$ for all $a \in [0, a_0)$. The Fernique [4] proof of this result is based on Definition 1. A generalization of Fernique’s method allows one to prove an analogous result for semiadditive functionals (see [5]).

The aim of this paper is to prove that there are numbers $b$ such that

$$\mathbb{E}\exp\{ag^b(X)\} < \infty$$

for every measurable $C$-quasi-additive functional $g$ of a Gaussian random vector $X$ and to show how $b$ depends on the constant $C$.

2. Main results

The first result concerns a general $C$-quasi-additive functional.

**Theorem 1.** Let $(V, \mathcal{F})$ be a measurable vector space, $X$ be a $V$-valued Gaussian random vector, $C \geq 1$, and let $g(\cdot)$ be a measurable $C$-quasi-additive functional on $V$. Then for any $\varepsilon > 0$ there is a number $a_0 > 0$ (depending, generally speaking, on $X$, $g$, and $\varepsilon$) such that

$$\mathbb{E}\exp\{ag^b(X)\} < \infty$$

for all $a \in [0, a_0)$, where

$$b = b(\varepsilon, C) = \frac{2}{1 + (1+\varepsilon) \log_2 C}.$$
Note that $b = 2$ if $C = 1$. Therefore Theorem 1 generalizes the result on the exponential integrability of measurable semiadditive functionals.

Inequality (1) holds for all $b \in [0, b(C))$ where

$$b(C) = \frac{2}{1 + \log_2 C}$$

and $C > 1$. Note however that inequality (1) may hold with $b = b(C)$ for some measurable $C$-quasi-additive functionals.

**Example 1.** Let

$$g(x) = p^r(x), \quad x \in V,$$

where $p(\cdot)$ is a measurable semiadditive functional and $r > 1$. Then $g(\cdot)$ is a measurable $2^{r-1}$-quasi-additive functional, that is, $C = 2^{r-1}$. Due to the theorem on the exponential integrability of semiadditive functionals, inequality (1) holds for $b = 2/r$. It remains to note that

$$b = \frac{2}{r} = \frac{2}{1 + \log_2 2^{r-1}} = \frac{2}{1 + \log_2 C}.$$ 

Example 1 shows that not only the constant $b$ in inequality (1) can attain the value $b(C)$ but also that $b$ does not exceed $b(C)$ in the class of $C$-quasi-additive functionals. A natural question arises whether inequality (1) may hold for $b = b(C)$ in the class of quasi-additive functionals. The answer is positive under certain additional restrictions on the functional $g(\cdot)$.

**Theorem 2.** Let $(V, \mathcal{F})$ be a measurable vector space, $X$ be a $V$-valued Gaussian random vector, $C \in [1, 2)$, and let $g(\cdot)$ be a measurable convex $C$-quasi-additive functional on $V$. Then there is a number $a_0 > 0$ (depending, generally speaking, on $X$ and $g$) such that

$$E \exp \{a g^b(X)\} < \infty$$

for all $a \in [0, a_0)$, where

$$b = \frac{2}{1 + \log_2 C}.$$ 

An answer to the question on whether Theorem 2 holds for $C \geq 2$ requires an additional investigation.

3. **Proofs of the theorems**

We need several auxiliary results in order to prove Theorem 1.

Let $X_1, \ldots, X_n$ be independent copies of a $V$-valued Gaussian random vector $X$, where $n = 2^l$ and $l \geq 1$. We introduce random vectors $S_1, \ldots, S_n$ as follows:

$$S_k = \sum_{j=1}^{2^l} h_{kj} X_j$$

for all $k = 1, \ldots, 2^l$, where $h_{kj} = \pm 1$. 

Therefore

$$S_k = 2^{-l/2} \sum_{j=1}^{2^l} h_{kj} X_j$$

and $C > 1$. Note however that inequality (1) may hold with $b = b(C)$ for some measurable $C$-quasi-additive functionals.
Lemma 1 (14). The random vectors $S_1, \ldots, S_n$ are independent copies of the Gaussian random vector $X$.

Lemma 2. Let $C \geq 1$ and let $(g(x), x \in V)$ be a measurable $C$-quasi-additive functional. Then

\begin{align*}
(3) \quad g(2^n x) & \leq (2C)^n g(x), \\
(4) \quad g(x_1 + \cdots + x_n) & \leq C g(x_1) + \cdots + C^{n-2} g(x_{n-2}) + C^{n-1} g(x_{n-1}) + C^{n-1} g(x_n), \\
(5) \quad g(x_1 + \cdots + x_n) & \geq \frac{g(x_1)}{C} - C^{n-2} \sum_{k=2}^n g(-x_k)
\end{align*}

for all positive integers $n$ and all $x, x_1, \ldots, x_n \in V$.

Proof of Lemma 2. Inequalities (3) and (4) can be proved by induction. Inequality (5) follows from (4).

Lemma 3. Let $X$ be a $V$-valued Gaussian random vector, $C \geq 1$, and let $(g(x), x \in V)$ be a measurable $C$-quasi-additive functional on $V$. Then

\begin{align*}
(6) \quad (P \{ g(X) \leq s \})^{4^m - 1} P \{ g(X) > t \} & \leq \left( P \left\{ g(X) > \frac{t - (4^m - 1)C^{4^m - 1}s}{C(2C)^m} \right\} \right)^{4^m}
\end{align*}

for all positive integers $m$ and all $t > 0$ and $s > 0$.

Proof of Lemma 3. Let $X_1, \ldots, X_{4^m}$ be independent copies of $X$. In view of Lemma 1, the random vectors

\[ S_k = 2^{-m} \sum_{j=1}^{4^m} h_{kj} X_j, \quad k = 1, \ldots, 4^m, \]

are independent copies of $X$.

Since $g(\cdot)$ is a $C$-quasi-additive symmetric functional, Lemma 2 implies

\[ (2C)^m g(S_k) \geq g(2^m S_k) = g\left( \sum_{j=1}^{4^m} h_{kj} X_j \right) \]

\[ \geq \frac{1}{C} g(h_{k1} X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(-h_{kj} X_j) = \frac{1}{C} g(X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(X_j) \]

for all $k = 1, \ldots, 4^m$. This yields

\[ (P \{ g(X) \leq s \})^{4^m - 1} P \{ g(X) > t \} = P \{ g(X_1) > t, g(X_2) \leq s, \ldots, g(X_{4^m}) \leq s \} \]

\[ \leq P \left\{ \frac{1}{C} g(X_1) - C^{4^m - 2} \sum_{j=2}^{4^m} g(X_j) > \frac{t - (4^m - 1)C^{4^m - 1}s}{C} \right\} \]

\[ \leq P \left\{ \bigcap_{k=1}^{4^m} \left\{ g(S_k) > \frac{t - (4^m - 1)C^{4^m - 1}s}{C(2C)^m} \right\} \right\} \]

\[ = \left( P \left\{ g(X) > \frac{t - (4^m - 1)C^{4^m - 1}s}{C(2C)^m} \right\} \right)^{4^m} \]

for all $t > 0$ and $s > 0$. \hfill \Box

Proof of Theorem 1. First we consider the case of a $C$-quasi-additive symmetric functional $g(\cdot)$. Consider a positive number $s = t_0$ such that

\[ c = P \{ g(X) \leq t_0 \} > 1/2. \]
For a given positive integer $m$ let the sequence of positive numbers $(t_n, n \geq 1)$ be defined as follows:

$$t_n = ((\gamma + 1)\beta^n - \gamma)t_0, \quad n \geq 1,$$

where

$$\alpha = \frac{\alpha}{\beta - 1}, \quad \beta = (4^m - 1)C^m, \quad \gamma = C(2C)^m \geq 2.$$

The sequence $(t_n, n \geq 1)$ is such that

$$t_n < t_{n+1}; \quad t_n \to 1, \quad n \to \infty.$$

Let $z_n, n \geq 0$, be real numbers such that

$$z_0 = \frac{1 - c}{c}, \quad cz_n = P \{g(X) > t_n\}, \quad n \geq 1.$$

Note that $z_0 < 1$. It follows from (6) that $z_n \leq (z_{n-1})^{4^m}, n \geq 1$. Thus

$$P \{g(X) > t_n\} \leq c(z_0)^{4^m n}, \quad n \geq 1,$$

whence

$$\mathbb{E} \exp \{ag^b(X)\} \leq c \left[ \exp \left\{ a \left( \sum_{n=1}^{\infty} z_0^{4^m n - 1} \exp \left\{ a(t_0(1 + \gamma))^b \beta^n \right\} \right) \right]$$

for

$$b = b \left( \frac{1}{m}, C \right) = \frac{2}{1 + \frac{1}{m} \log_2 C}.$$

This implies that the series converges and

$$\mathbb{E} \exp \{ag^b(X)\} < \infty$$

for $a < |\ln z_0| / [4^m a(t_0(1 + \gamma))^b]$. This completes the proof of the theorem for $C$-quasi-additive symmetric functionals $g(\cdot)$, since $m$ is arbitrary.

Now let $g(\cdot)$ be a measurable $C$-quasi-additive functional on $V$. Then

$$\bar{g}(x) = \max\{g(x), g(-x)\}, \quad x \in V,$$

is a measurable $C$-quasi-additive symmetric functional on $V$. Thus there is a number $a_0 > 0$ such that

$$\mathbb{E} \exp \{ag^b(X)\} \leq \mathbb{E} \exp \{ag^b(x)\} < \infty$$

for all $a \in [0, a_0).$

**Remark 3.** The proof of Theorem 1 is simpler for $C = 1$. In this case, Theorem 1 follows from Lemma 3 for $m = 1$ (see [1]).

The following result is needed for the proof of Theorem 2.
**Lemma 4.** Let $X$ be a $V$-valued Gaussian random vector, $C \in [1, 2)$, and let $g(\cdot)$ be a measurable convex $C$-quasi-additive functional on $V$. If

$$E \exp\{a_1 g(X)\} < \infty$$

for some $a_1 > 0$, then there is $a > 0$ such that

$$E \exp\{a g^b(X)\} < \infty,$$

where

$$b = \frac{2}{1 + \log_2 C}.$$

**Proof of Lemma 4.** It follows from Lemma 1 that

$$P\{g(X) \geq t^{2s}\} = P\left\{g\left(\frac{X_1 + \cdots + X_{4^n}}{2^n}\right) \geq t^{2s}\right\}$$

for all positive integers $n$ and all $s, t > 0$ where $X_1, \ldots, X_{4^n}$ are independent copies of the random vector $X$.

Since $g(\cdot)$ is a convex $C$-quasi-additive functional,

$$P\left\{g\left(\frac{X_1 + \cdots + X_{4^n}}{2^n}\right) \geq t^{2s}\right\} = P\left\{g\left(\frac{2^n(X_1 + \cdots + X_{4^n})}{4^n}\right) \geq t^{2s}\right\}$$

$$\leq P\left\{g(2^n X_1) + \cdots + g(2^n X_{4^n}) \geq t^{2s+2n}\right\}$$

$$\leq P\left\{g(X_1) + \cdots + g(X_{4^n}) \geq \frac{t^{2s+n}}{C^n}\right\}.$$

Condition (7) and the Markov inequality imply that

$$P\{g(X) \geq t^{2s}\} \leq E \exp\left\{-a_1 t \frac{2^{s+n}}{C^n} g(X_1) + \cdots + g(X_{4^n})\right\}$$

$$= \exp\left\{-a_1 t \frac{2^{s+n}}{C^n} \left(E \exp\{a_1 g(X_1) + \cdots + g(X_{4^n})\}\right)^{4^n} \right\} = \exp\left\{-a_1 t \frac{2^{s+n}}{C^n} + a_2 4^n\right\}$$

for all positive integers $n$ and all $s, t > 0$ where $a_2 = \ln E \exp\{a_1 g(X)\} \in (0, \infty)$. In the case of $s = n(1 + \log_2 C) = 2n/b$ one has

$$P\{g(X) \geq t^{4n/b}\} \leq \exp\{-a_1 t - a_2\} 4^n$$

for all positive integers $n$ and all $t > 0$. This implies that

$$E \exp\{a g^b(X)\} \leq \exp\left\{4a t^{b}\right\} + \sum_{n=1}^{\infty} \exp\left\{-a_1 t - 4a t^{b} - a_2 4^n\right\}.$$

The series converges and

$$E \exp\{a g^b(X)\} < \infty$$

if $t > a_2/a_1$ and $0 < a < (a_1 t - a_2)/4t^b$. \[\square\]

**Proof of Theorem 2.** Theorem 2 for $C = 1$ follows from Theorem 1 immediately. Condition (7) holds by Theorem 1 if $C \in (1, 2)$, in which case one needs to apply Lemma 4. \[\square\]
Bibliography


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