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THE ERDÖS–RÉNYI LAW FOR RENEWAL PROCESSES

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Abstract. The Erdös–Rényi law and strong law of large numbers are proved for renewal processes constructed from nonidentically distributed random variables.

1. Introduction

Renewal processes form a class of stochastic processes important from both theoretical and applied points of view. Renewal processes are quite often used in models describing devices with renewable parts. Another application of renewal processes is the finance and actuarial mathematics where renewal processes serve as a model for the total number of claims occurred according to an insurance portfolio. Poisson processes are one of the important examples of renewal processes.

An interest to renewal processes arose quite long ago. Many results concerning the renewal processes (say, the law of large numbers, central limit theorem, etc.) are included in textbooks for advanced students (see, for example, Borovkov [1], Gnedenko [2], Feller [3]).

The main goal of this paper is to study the almost sure asymptotic behavior of increments of renewal processes.

Let $X_1, X_2, \dots$ be a sequence of nondegenerate positive independent identically distributed random variables, and put $S_n = X_1 + \cdots + X_n$. The renewal process is defined as follows:

$$N(t) = \max\{n \colon S_n \le t\}, \qquad t \ge 0.$$

The problem is to find necessary and/or sufficient conditions for

$$\text{(L)} \qquad \limsup_{t\to\infty} b_t^{-1} \max_{0\le u\le t-a_t} \{N(u+a_t) - N(u)\} = 1 \quad \text{a.s.}$$

("a.s." is the abbreviation for "almost surely") or for an analogue of relation (L) with lim instead of lim sup. The functions $a_t \le t$ and b_t are nonrandom and nondecreasing. One can also use other functionals of the increments instead of max involved in (L). The maximum on the left-hand side of (L) becomes $N(t)$ for the case of $a_t = t$. Many various results on the almost sure behavior of $N(t)$ can be found in Gut [4], Gut, Klesov, and Steinebach [5], Frolov, Martikainen, and Steinebach [6], and in the papers cited therein.

The interest to increments of renewal processes is initiated by corresponding results on the limit behavior of increments of sums of independent random variables. The pioneering paper on this topic is Erdös and Rényi [7], where increments of a small (logarithmic) length are studied for sums. Nowadays several dozens of researches in this field are

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known in the literature. Limit theorems for increments of renewal processes are studied by Steinebach [8], Deheuvels and Steinebach [9], Bacro, Deheuvels, and Steinebach [10], Steinebach [11]. Both small ($a_t = O(\log t)$) and large ($a_t/\log t \to \infty$) increments are studied in those papers. The rate of convergence for both cases is found for an analogue of (L) where lim stands instead of lim sup.

Possible generalizations of the above results to the case of nonidentically distributed random variables X_i make sense from both theoretical and practical points of view. For example, in the case of a model of a device with renewable parts, nonidentically distributed random variables mean that parts of the device have different behavior in time, caused, for instance, by different qualities of parts purchased from different producers.

The current paper is devoted to the study of the almost sure limit behavior of short increments of renewal processes constructed from nonidentically distributed random variables. We prove the Erdös–Rényi law and strong law of large numbers for such processes.

2. Preliminary remarks

Let $m \geq 1$ and let $\beta_1, \beta_2, \ldots, \beta_m$ be positive numbers such that

$$\beta_1 + \beta_2 + \cdots + \beta_m = 1.$$

Further let $V_1(x), V_2(x), \ldots, V_m(x)$ be distribution functions of random variables with positive mean values. Assume that

$$H_i = \sup\left\{t \colon \varphi_i(t) = \int_{-\infty}^{+\infty} e^{tx} dV_i(x) < \infty\right\} > 0, \qquad i = 1, 2, \ldots, m.$$

For $t < H_i$ and $i = 1, 2, \ldots, m$ we put

$$m_i(t) = (\log \varphi_i(t))', \qquad f_i(t) = t m_i(t) - \log \varphi_i(t), \qquad A_i = \lim_{t \uparrow H_i} m_i(t),$$

$$\frac{1}{c_i} = \lim_{t \uparrow H_i} f_i(t).$$

Properties of the functions $m_i(t)$ and $f_i(t)$ are well known in the theory of large deviations (see, for example, Borovkov [1], Petrov [12]). The function $m_i(t)$ is the mathematical expectation of the Esher transformation of a random variable with the distribution function $V_i(x)$. Note that $m_i(0) = 0$ and $m_i(t) \nearrow$. The function $f_i(t)$ is related to the known deviation function (defined as the logarithm of the Chernoff function)

$$\zeta_i(z) = \sup\{zt - \log \varphi_i(t) \colon t \geq 0, \varphi_i(t) < \infty\}$$

and to the function $m_i(t)$ as follows:

$$\zeta_i(z) = f_i(m_i^{-1}(z)) \quad \text{for } z \in [0, A_i),$$

where $m_i^{-1}(z)$ is the inverse function to $m_i(t)$. Therefore $f_i(0) = 0$ and $f_i(t) \nearrow$.

Moreover, it is known that $c_i > 0$ in each of the following two cases:

1) $H_i = \infty$, $A_i < \infty$, and $q = V_i(A_i + 0) - V_i(A_i) > 0$. In this case $c_i = -1/\log q$;
2) $H_i < \infty$ and $A_i < \infty$. In this case $c_i = 1/(H_i A_i - \log \varphi_i(H_i)) > 0$

(see, for example, Deheuvels, Devroye, and Lynch [13]).

In all other cases $c_i = 0$. Note also that $A_i = \sup\{x \colon V_i(x) < 1\}$ provided the latter supremum is finite.

Put

$$H_0 = \min\{H_1, H_2, \ldots, H_m\}.$$

For $t < H_0$ and $i = 1, 2, \ldots, m$ let

$$\varphi(t) = \prod_{i=1}^m \varphi_i^{\beta_i}(t), \qquad m(t) = \sum_{i=1}^m \beta_i m_i(t), \qquad f(t) = \sum_{i=1}^m \beta_i f_i(t).$$

Note that

$$m(t) = (\log \varphi(t))', \qquad f(t) = tm(t) - \log \varphi(t).$$

Put

$$A = \lim_{t \uparrow H_0} m(t), \qquad \frac{1}{c_0} = \lim_{t \uparrow H_0} f(t). \tag{1}$$

Assume that $\alpha \in (0, A)$, $c > c_0$, and $t^* \in (0, H_0)$ are such that

$$f(t^*) = \frac{1}{c}, \tag{2}$$

$$m(t^*) = \alpha. \tag{3}$$

It is clear that $m(0) = 0$, $m(t) \nearrow$, $f(0) = 0$, and $f(t) \nearrow$. Therefore any one of the parameters $\alpha \in (0, A)$, $c > c_0$, or $t^* \in (0, H_0)$ uniquely determines the other two.

3. Results

Let $X_1, X_2, \dots$ be a sequence of positive independent random variables such that $\mathsf{E}\, X_i = \mu_i > 0$ and $\sigma_i^2 = \mathsf{E}(X_i - \mu_i)^2 > 0$ $(i = 1, 2, \dots)$.

Assume that the following conditions hold:

(A1) $\lim_{k\to\infty} \sup_n \bigl| k^{-1} \sum_{i=n+1}^{n+k} \mu_i - \mu \bigr| = 0$.

(A2) There exist positive constants H and $d_1, d_2, \dots$ such that

$$\left| \log \mathsf{E}\, e^{z(X_i - \mu_i)} \right| \le d_i$$

in the circle $|z| < H$.

(A3) $\limsup_{n\to\infty} \sup_k B_{nk}^{-1} \sum_{i=n+1}^{n+k} (d_i^2 + d_i) < \infty$ where $B_{nk} = \sum_{i=n+1}^{n+k} \sigma_i^2$.

(A4) There exist $\delta > 0$ and k_0 such that $B_{nk} > \delta k$ for all n and $k \ge k_0$.

We denote partial sums by $S_n = X_1 + X_2 + \dots + X_n$, and let

$$\begin{aligned} N(t) &= \max\{n \colon S_n \le t\}, \\ \Delta_T(b) &= \sup_{0 \le t \le T - b\log T} \{N(t + b\log T) - N(t)\}, \\ \delta_T(b) &= \inf_{0 \le t \le T - b\log T} \{N(t + b\log T) - N(t)\} \end{aligned}$$

for $b > 0$. The process $N(t)$ is an ordinary renewal process if the random variables $\{X_i\}$ are identically distributed.

We start with the strong law of large numbers for $N(t)$.

Theorem 1. *Assume that conditions (A1)–(A4) hold. Then*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

Now we consider increments of the process $N(t)$.

Theorem 2. *Assume that conditions (A1)–(A4) hold. Then*

1) *there exist a constant $C \ge 0$ and functions $b_i \colon (C, +\infty) \to (0, +\infty)$, $i = 1, 2$, such that*

$$\limsup_{T\to\infty} \frac{\Delta_T(b_1)}{\log T} \le c \le \liminf_{T\to\infty} \frac{\Delta_T(b_2)}{\log T} \quad \text{a.s.}$$

for all $c > C$, where $b_i = b_i(c)$, $i = 1, 2$.

2) *there exist a constant $C' \ge 0$ and functions $b'_i\colon (C', +\infty) \to (0, +\infty)$, $i = 1, 2$, such that*
$$\limsup_{T\to\infty} \frac{\delta_T(b'_1)}{\log T} \le c' \le \liminf_{T\to\infty} \frac{\delta_T(b'_2)}{\log T} \quad a.s.$$
for all $c' > C'$ where $b'_i = b'_i(c)$, $i = 1, 2$.

Simple expressions for the constant C and functions $b_i(c)$ in Theorem 1 are unknown unless we impose additional requirements on the structure of the sequence $\{X_i\}$. Also, without such requirements, one cannot claim that equalities hold in Theorem 2 instead of inequalities. Therefore we assume that there exists a finite set of distributions such that the portion of random variables with those distributions in the moving blocks $X_{n+1}, \dots, X_{n+k}$ approaches 1 as $k \to \infty$ uniformly with respect to n. A sequence of random variables whose distribution functions belong to a set of given m distribution functions is the simplest example where the above property is satisfied.

Let $m \ge 1$ and let $\beta_1, \beta_2, \dots, \beta_m$ be positive numbers such that
$$\beta_1 + \beta_2 + \dots + \beta_m = 1.$$
Put $F_k(x) = \mathsf{P}(X_k - \mu_k < x)$ and
$$I^i_{nk} = \{j\colon n \le j \le n+k, F_j = G_i\},$$
$$J_{nk} = \{j\colon n \le j \le n+k,\} \setminus \bigcup_{i=1}^m I^i_{nk},$$
$$N^i_{nk} = \# I^i_{nk},$$
where $G_1(x), G_2(x), \dots, G_m(x)$ are some distribution functions, and $\#D$ is the cardinality of a set D.

Theorem 3. *Assume that conditions (A1)–(A3) and*

(A5) $\lim_{k\to\infty} \sup_n \max_{1\le i\le m} \bigl| N^i_{nk}/k - \beta_i \bigr| = 0$,

(A6) $\lim_{k\to\infty} \sup_n B^{-1}_{nk} \sum_{i\in J_{nk}} d_i = 0$

hold.

Put $V_i(x) = 1 - G_i(-x)$ $(i = 1, 2, \dots, m)$, and let the constants A and c_0 be defined by (1)*, while constants $c > c_0$ and $\alpha \in (0, \min\{A, \mu\})$ satisfy* (2) *and* (3)*. Finally put $b = c(\mu - \alpha)$.*

Then
$$\lim_{T\to\infty} \frac{\Delta_T(b)}{\log T} = c \quad a.s.$$

Note that conditions (A1) and (A5) imply
$$\mu = \sum_{i=1}^m \beta_i \mu_{k(i)}, \quad \text{where } k(i) = \min\{k\colon F_k = G_i\}.$$
Below we prove that conditions (A2), (A3), (A5), and (A6) yield
$$\lim_{k\to\infty} \sup_n \left| \frac{B_{nk}}{k} - \sigma^2 \right| = 0, \quad \text{where } \sigma^2 = \sum_{i=1}^m \beta_i \int_{-\infty}^{+\infty} x^2 \, dG_i(x),$$
whence (A4) follows (see Lemma 3).

Theorem 4. *Assume that conditions (A1)–(A6) hold. Let*
$$V_i(x) = G_i(x), \qquad i = 1, 2, \dots, m,$$
let the constants A and c_0 be defined by relation (1)*, while constants $c > c_0$ and $\alpha \in (0, A)$ satisfy relations* (2) *and* (3)*, respectively. Put $b = c(\mu + \alpha)$.*

Then

$$\lim_{T\to\infty}\frac{\delta_T(b)}{\log T}=c \quad a.s.$$

It is worthwhile to mention that the constants A, c_0, and α in Theorems 3 and 4 are different since these constants are constructed in Theorems 3 and 4 from different distributions.

Due to duality, the study of the asymptotic behavior of $\Delta_T(b)$ and $\delta_T(b)$ is reduced to that of

$$m_N(c)=\min_{0\le n\le N-[c\log N]}(S_{n+[c\log N]}-S_n)$$

and

$$M_N(c)=\max_{0\le n\le N-[c\log N]}(S_{n+[c\log N]}-S_n),$$

respectively. Here $c>0$.

To conclude this section we state two results to be applied in the proofs of Theorems 3 and 4. Moreover these two results are of independent interest.

Theorem 5. *Assume that all the assumptions of Theorem 3 hold. Then*

$$\lim_{N\to\infty}\frac{m_N(c)}{c\log N}=\mu-\alpha \quad a.s. \tag{4}$$

Since $\max\{u,v\}=-\min\{-u,-v\}$, Theorem 5 is equivalent to the following result.

Theorem 6. *Assume that all the assumptions of Theorem 4 hold. Then*

$$\lim_{N\to\infty}\frac{M_N(c)}{c\log N}=\mu+\alpha \quad a.s. \tag{5}$$

It is worthwhile to mention that the constants α in Theorems 5 and 6 are essentially different, since these constants are constructed from different distributions.

4. Proofs

For $t\in(0,H)$ and all $i\ge 1$ let

$$\psi_i(t)=\mathsf{E}\,e^{t(X_i-\mu_i)},\qquad \mu_i(t)=(\log\psi_i(t))',\qquad g_i(t)=tm_i(t)-\log\psi_i(t).$$

We also put for all n and k

$$\Psi_{nk}(t)=\sum_{i=n+1}^{n+k}\psi_i(t),\qquad M_{nk}(t)=\sum_{i=n+1}^{n+k}\mu_i(t),\qquad G_{nk}(t)=\sum_{i=n+1}^{n+k}g_i(t).$$

It is clear that $G_{nk}(t)=tM_{nk}(t)-\log\Psi_{nk}(t)$.

Let $c>0$, $\alpha_{nk}=\alpha_{nk}(c)$, and $t^*_{nk}=t^*_{nk}(c)$ be defined by

$$G_{nk}(t^*_{nk})=\frac{k}{c}, \tag{6}$$

$$M_{nk}(t^*_{nk})=\alpha_{nk}B_{nk}. \tag{7}$$

To prove Theorems 1, 2, and 3 we need the following two results (Theorem 7 and Lemma 1) from [14].

Theorem 7. *Let $X_1,X_2,\dots$ be a sequence of nondegenerate independent random variables satisfying conditions (A2)–(A4). Then*

1) *there exists a constant $C_0\ge 0$ such that equations* (6) *and* (7) *have a unique solution for all $c>C_0$, all n, and all sufficiently large k.*

2) *Let*

$$U_N = \max_{0\le n\le N-K} \frac{t^*_{nK}}{\log K}\left(S_{n+K} - S_n - \sum_{i=n+1}^{n+K} \mu_i - \alpha_{nK} B_{nK}\right),$$
$$K = [c\log N].$$

Then

(8) $$\limsup_{N\to\infty} U_N = \frac{1}{2} \quad a.s.,$$

(9) $$\liminf_{N\to\infty} U_N = -\frac{1}{2} \quad a.s.$$

Lemma 1. *Assume that conditions (A2) and (A3) hold. Then there exists a positive constant* Δ *such that* $B_{nk} \le \Delta k$ *for all* n *and all* $k \ge K_0$.

Lemma 2. *Assume that all the assumptions of Theorem 3 hold. Then*

(10) $$\lim_{k\to\infty} \sup_n |t^*_{nk} - t^*| = 0.$$

Proof. We have

$$\frac{G_{nk}(t)}{k} = f(t) + \sum_{i=1}^{m}\left(\frac{N^i_{nk}}{k} - \beta_i\right) f_i(t) + g_{nk}(t), \quad \text{where } g_{nk}(t) = \sum_{i\in J_{nk}} g_i(t).$$

Denoting by γ_{ri} the rth order cumulant of the random variable X_i we get

$$g_{nk}(t) = \sum_{i\in J_{nk}} \sum_{r=1}^{\infty} \frac{(1-r)}{r!}\gamma_{ri} t^r.$$

Condition (A2) and the Cauchy inequality imply that for all i

$$|\gamma_{ri}| \le \frac{r!\, d_i}{H^r}.$$

Taking conditions (A6) and (A3) into account we obtain from Lemma 1 that

$$\left|\sum_{i\in J_{nk}} \gamma_{ri}\right| \le \frac{r!}{H^r}\sum_{i\in J_{nk}} d_i \le \varepsilon \frac{r!}{H^r} k$$

for all $\varepsilon > 0$ and for all sufficiently large k. Therefore $|g_{nk}(t)| \le \varepsilon A_1 k$ for $|t| < H' < H$ and for all sufficiently large k. Similarly $|\sum_{i=1}^m f_i(t)| \le A_2$ for $|t| < H' < H$. Thus we get by (A5) that $G_{nk}(t)/k \to f(t)$ as $k\to\infty$ uniformly with respect to n. Since $f(0) = 0$ and $f(t)$ is continuous, strictly increases, and is bounded from above for $0 < t < H'$, we conclude that

$$\lim_{k\to\infty} \sup_{t\in(0,H')} \left|\frac{G_{nk}(t)}{k} - f(t)\right| = 0$$

uniformly with respect to n. The latter relation together with (6) and (2) implies (10). □

Lemma 3. *Assume that all the assumptions of Theorem 3 hold. Then*

(11) $$\lim_{k\to\infty} \sup_n \left|\frac{B_{nk}}{k} - \sigma^2\right| = 0,$$

where

$$\sigma^2 = \sum_{i=1}^m \beta_i \tilde\sigma_i^2, \qquad \tilde\sigma_i^2 = \int_{-\infty}^{+\infty} x^2\, dG_i(x), \qquad i = 1, 2, \dots, m.$$

Proof. Since

$$\frac{B_{nk}}{k} = \sigma^2 + \sum_{i=1}^{m} \left(\frac{N_{nk}^i}{k} - \beta_i \right) \tilde{\sigma}_i^2 + \sum_{i \in J_{nk}} \sigma_i^2$$

and $\sigma_i^2 \le 2d_i/H^2$, we derive relation (11) from conditions (A5) and (A6) and Lemma 1. □

Lemma 4. *Assume that all the assumptions of Theorem 3 hold. Then*

$$\lim_{k\to\infty} \sup_n \left| \alpha_{nk} - \frac{\alpha}{\sigma^2} \right| = 0, \tag{12}$$

where σ^2 is defined in Lemma 3.

Proof. We have

$$\frac{M_{nk}(t)}{k} = m(t) + \sum_{i=1}^{m} \left(\frac{N_{nk}^i}{k} - \beta_i \right) m_i(t) + m_{nk}(t), \quad \text{where } m_{nk}(t) = \sum_{i \in J_{nk}} \mu_i(t).$$

Following the same lines as in the proof of (10) we obtain

$$|m_{nk}(t)| = \left| \sum_{i \in J_{nk}} \sum_{r=1}^{\infty} \frac{r}{r!} \gamma_{ri} t^{r-1} \right| \le \varepsilon A_3 k$$

for all $\varepsilon > 0$, $|t| < H' < H$, and all sufficiently large k. It is clear that $|\sum_{i=1}^m m_i(t)| \le A_4$ for all $|t| < H' < H$ and all sufficiently large k. Thus by (A5) $M_{nk}(t)/k \to m(t)$ as $k \to \infty$ uniformly with respect to n. Since $m(t)$ is a continuous and strictly increasing function, bounded for $|t| < H' < H$, we conclude that

$$\lim_{k\to\infty} \sup_{t \in (0,H')} \left| \frac{M_{nk}(t)}{k} - m(t) \right| = 0$$

uniformly with respect to n. Then

$$\lim_{k\to\infty} \sup_n \left| \frac{M_{nk}(t_{nk}^*)}{k} - m(t^*) \right| = 0 \tag{13}$$

in view of (10).

Applying (3) and (7) we obtain

$$\alpha_{nk} - \frac{\alpha}{\sigma^2} = \frac{k}{B_{nk}} \left(\frac{M_{nk}(t_{nk}^*)}{k} - m(t^*) \right) + \frac{\alpha}{\sigma^2} \left(\frac{k\sigma^2}{B_{nk}} - 1 \right),$$

whence (12) follows by (11) and (13). □

Proof of Theorem 6. Let $D = \{\omega \in \Omega\colon \text{either (8) or (9) does not hold}\}$, where Ω is the space of elementary events. It is clear that $\mathsf{P}(D) = 0$.

Choose $\omega \in \Omega \setminus D$.

Let $\{N_j = N_j(\omega)\}$ be a sequence of natural numbers such that $N_j \to \infty$ as $j \to \infty$ and

$$L = \limsup_{j\to\infty} \frac{U_{N_j}}{K_j}$$

where $K_j = [c \log N_j]$.

It is clear that $L \in [-0.5, 0.5]$ by (8) and (9).

First we consider the case of $L > 0$. Then U_{N_j} attains its maximum for positive values of $S_{n+K_j} - S_n - \sum_{i=n+1}^{n+K_j} \mu_i - \alpha_{nK_j} B_{nK_j}$, provided j is sufficiently large.

Putting $M_{N_j} = M_{N_j}(c)$ we obtain

$$L - \varepsilon \le U_{N_j} \le (1+\varepsilon)\frac{t^* K_j}{\log K_j}\left(\frac{M_{N_j}}{K_j} - (1-\varepsilon)(\mu+\alpha)\right)$$

for all $\varepsilon > 0$ and all sufficiently large j. When proving the latter relation we used (A1) and (10)–(12). Therefore

$$\liminf_{j\to\infty} \frac{M_{N_j}}{K_j} \ge \mu + \alpha.$$

On the other hand

$$L + \varepsilon \ge U_{N_j} \ge (1-\varepsilon)\frac{t^* K_j}{\log K_j}\left(\frac{M_{N_j}}{K_j} - (1+\varepsilon)(\mu+\alpha)\right)$$

for all $\varepsilon > 0$ and all sufficiently large j. Here we used (10)–(12) again. Thus

$$\limsup_{j\to\infty} \frac{M_{N_j}}{K_j} \le \mu + \alpha,$$

whence

$$\lim_{j\to\infty} \frac{M_{N_j}}{K_j} = \mu + \alpha. \tag{14}$$

Relation (14) is proved similarly for the case of $L < 0$.

Finally if $L = 0$, then $U_{N_j} \to 0$ as $j \to \infty$. In this case we split the sequence $\{N_j\}$ into two subsequences $\{N_j'\}$ and $\{N_j''\}$ such that $U_{N_j'} \ge 0$ and $U_{N_j''} < 0$ for all j.

Now we consider these two sequences separately and apply the above method to each of them. This completes the proof for the case of $L = 0$.

Therefore relation (14) holds whatever L is. Theorem 6 is proved. □

Proof of Theorem 1. Choose $\varepsilon > 0$. For simplicity let $N = t\mu^{-1} + t\varepsilon$ be an integer. We have

$$\{N(t) \ge N\} = \{S_N \le (\mu - \varepsilon_1)N\},$$

where $\varepsilon_1 = \mu(1-(1+\mu\varepsilon)^{-1}) > 0$. Since the series $\sum \sigma_i^2/i^2$ converges, the strong law of large numbers holds for the sequence $\{X_i - \mu_i\}$. Taking condition (A1) into account, we conclude that $t^{-1}N(t) \le \mu^{-1} + \varepsilon$ with probability one for sufficiently large t.

Put $N = t\mu^{-1} - t\varepsilon$ for $\varepsilon > 0$. Again we assume, for simplicity, that the number N is an integer. Then

$$\{N(t) \ge N\} = \{S_N \le (\mu + \varepsilon_1)N\},$$

where $\varepsilon_1 = \mu(1-(1-\mu\varepsilon)^{-1}) > 0$. We again use the strong law of large numbers and condition (A5) to conclude that $t^{-1}N(t) \ge \mu^{-1} - \varepsilon$ with probability one for sufficiently large t. This completes the proof of Theorem 1. □

Lemma 5. *For any positive numbers b and c, there are constants d' and d'' such that*

$$\mathsf{P}\big(\Delta_T(b) \ge c\log T \ \text{i.o.}\big) \le \mathsf{P}\big(m_N(c) \le b\log N + d' \ \text{i.o.}\big),$$
$$\mathsf{P}(\Delta_T(b) \le c\log T \ \text{i.o.}) \le \mathsf{P}\big(m_N(c) \ge b\log N + d'' \ \text{i.o.}\big)$$

(the abbreviation i.o. stands for "infinitely often").

Proof. Without loss of generality we assume that $c\log T$ is an integer.

Note that $\Delta_T(b)$ is an integer-valued random variable. It attains the maximum at the point $t = S_n - b\log T$ for some n such that $S_n \le T$. Thus

$$\Delta_T(b) = N(S_n) - N(S_n - b\log T)$$

for some $n \le N(T)$.

Note that $S_n - S_{n-c\log T} \le b\log T$ for some $n \le N(T)$ if $\Delta_T(b) \ge c\log T$. Put $N = [\delta' T]$ for some $\delta' > \mu^{-1}$. Then Theorem 1 implies $N(T) \le N$ for $T \ge T_0$. It is clear that $\log N = \log T + \log \delta' + o(1)$. Thus $S_n - S_{n-c\log N} \le b\log N + d'$ for some $n \le N$, whence $m_N(c) \le b\log N + d'$. This proves the first inequality.

Further $S_n - S_{n-c\log T} \ge b\log T$ for all $n \le N(T)$ if $\Delta_T(b) \le c\log T$. Put $N = [\delta'' T]$ for some $\delta'' < \mu^{-1}$. Then Theorem 1 implies $N(T) \ge N$ for $N \ge N_0$. Moreover $\log N = \log T + \log \delta'' + o(1)$. Thus $S_n - S_{n-c\log N} \ge b\log N + d''$ for all $n \le N$, whence $m_N(c) \ge b\log N + d''$. Lemma 5 is proved. □

Lemma 6. *For all positive numbers b and c, there are constants d' and d'' such that*

$$\mathsf{P}\big(\delta_T(b) \ge c\log T \ \textit{i.o.}\big) \le \mathsf{P}\big(M_N(c) \le b\log N + d' \ \textit{i.o.}\big),$$
$$\mathsf{P}\big(\delta_T(b) \le c\log T \ \textit{i.o.}\big) \le \mathsf{P}\big(M_N(c) \ge b\log N + d'' \ \textit{i.o.}\big).$$

The proof of Lemma 6 is similar to that of Lemma 5.

Proof of Theorem 3. Note that $b = b(c)$ is strictly increasing with respect to c by (2) and (3).

Choose $\varepsilon > 0$. It follows from Lemma 5 and Theorem 5 that

$$\mathsf{P}\big(\Delta_T(b) \ge (c+\varepsilon)\log T \text{ i.o.}\big) \le \mathsf{P}\big(m_N(c+\varepsilon) \le b\log N + d' \text{ i.o.}\big) = 0.$$

This implies that

$$\limsup_{T\to\infty} \frac{\Delta_T(b)}{\log T} \le c \quad \text{a.s.}$$

On the other hand, Lemma 5 and Theorem 5 imply

$$\mathsf{P}\big(\Delta_T(b) \le (c-\varepsilon)\log T \text{ i.o.}\big) \le \mathsf{P}\big(m_N(c-\varepsilon) \ge b\log N + d'' \text{ i.o.}\big) = 0.$$

Therefore

$$\liminf_{T\to\infty} \frac{\Delta_T(b)}{\log T} \ge c \quad \text{a.s.}$$

and Theorem 3 is proved. □

The proof of Theorem 4 is analogous; however, Theorem 6 and Lemma 6 must be used in its proof instead of Theorem 5 and Lemma 5, respectively.

We use the same method to prove Theorem 2 as that in the proof of Theorem 3. The following two auxiliary results are useful for the proof.

Lemma 7. *Assume that all the assumptions of Theorem 2 hold. Then there exists a constant $C \ge 0$ such that for all $c > C$ there are sequences $\{\alpha_{nk}\}$ and $\{t^*_{nk}\}$satisfying conditions* (6) *and* (7). *Moreover there are $\alpha_i = \alpha_i(c) > 0$ and $t_i = t_i(c) > 0$ $(i = 1, 2)$ such that $\alpha_1 \le \alpha_{nk} \le \alpha_2$ and $t_1 \le t^*_{nk} \le t_2$ for all n and all $k \ge K_1$. In addition, $\alpha_i(c) \searrow$ and $\alpha_i(c) \to 0$ as $c \to \infty$ $(i = 1, 2)$.*

Lemma 7 is proved in [14].

Lemma 8. *Suppose all the assumptions of Theorem 2 hold. Then*

1) *there exist a constant $C \ge 0$ and functions $A_i \colon (C, +\infty) \to (0, +\infty)$, $i = 1, 2$, such that for all $c > C$ and all $\varepsilon > 0$*

$$(A_1 - \varepsilon)K\log K \le m_N(c) \le (A_2 + \varepsilon)K\log K$$

with probability one, where N is sufficiently large. Here $A_1 = A_1(c)$, $A_2 = A_2(c)$, and $K = [c\log N]$. Moreover $A_i(c) \searrow$ and $A_i(c) \to \mu$ as $c \to \infty$, $i = 1, 2$.

2) *There exist a constant* $C' \geq 0$ *and functions* $A_i' \colon (C', +\infty) \to (0, +\infty)$, $i = 1, 2$, *such that for all* $c > C$ *and all* $\varepsilon > 0$

$$(A_1' - \varepsilon)K \log K \leq M_N(c) \leq (A_2' + \varepsilon)K \log K$$

with probability one where N *is sufficiently large. Here* $A_1' = A_1'(c)$ *and* $A_2' = A_2'(c)$. *Moreover* $A_i'(c) \searrow$ *and* $A_i'(c) \to \mu$ *as* $c \to \infty$, $i = 1, 2$.

Lemma 8 follows from Theorem 7 and Lemma 7.

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