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ERGODICITY AND STABILITY OF NONSTATIONARY QUEUEING SYSTEMS

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Abstract. We study stability and ergodicity of a special class of nonhomogeneous birth-death processes and consider applications of estimates for queue-length process for $M_t/M_t/S$ and $M_t/M_t/S/S$ queues.

1. Introduction

Nonstationary (nonhomogeneous in time) Markov chains have been studied over the last 20 years; see, for example, [11, 14, 17, 3]. Nonstationary Markov chains appear, in particular, when describing queueing processes. Most of the research on those chains is devoted to various problems of their approximations (see [4, 19, 20] and references therein).

Important are the problems on the rate of convergence to a limit regime and on stability of nonstationary queueing systems. The first study on these problems was initiated by B. V. Gnedenko; see [8, 9, 10] and [15, 16].

The main tool of the present paper is the method used for the first time in [8] and developed in [25, 26, 27, 28, 29, 1, 12]. The method is based on estimates using the log-norm of a linear operator function and on special transforms of the reduced operator function.

We apply this method to a class of Markov queueing systems with a special form of nonstationarity, which, however, is quite general for many applications.

2. Main notation

In what follows we consider a nonhomogeneous Markov process $X(t)$, $t \geq 0$ (a queue-length process or a total number of customers arrived to a system), that is, a birth-death process with the phase space $E = \{0, 1, \dots, N\}$, $N \leq \infty$. It is assumed that the birth intensities $\lambda_n(t)$, $t \geq 0$, and death intensities $\mu_n(t)$, $t \geq 0$, $n \in E$, depend on time. Namely

$$(1)\qquad \mathsf{P}(X(t+h) = j \,/\, X(t) = i) = \begin{cases} \lambda_i(t) \cdot h + o(h) & \text{if } j = i+1, \\ \mu_i(t) \cdot h + o(h) & \text{if } j = i-1, \\ 1 - (\lambda_i(t) + \mu_i(t)) \cdot h + o(h) & \text{if } j = i, \\ o(h) & \text{if } |i-j| > 1. \end{cases}$$

Here $h > 0$ and all symbols $o(h)$ are such that $o(h) = o(t,h) \to 0$ as $h \to 0$, $t \geq 0$, uniformly with respect to $i \in E$. Let

$$(2)\qquad p_{ij}(s,t) = \mathsf{P}(X(t) = j \mid X(s) = i), \qquad i, j \in E,\ 0 \leq s \leq t,$$

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and $p_i(t) = \mathsf{P}(X(t) = i)$, $i \in E$, $t \geq 0$, be transition probabilities and probabilities of states, respectively. Denote by $\mathbf{p}(t) = (p_0(t), \dots, p_N(t))^T$, $t \geq 0$, and by $A(t) = (a_{ij}(t))$, $t \geq 0$, the vector column of probabilities of states, and the matrix of intensities related to (1). Note that

$$a_{ij}(t) = \begin{cases} \lambda_{i-1}(t) & \text{if } j = i-1, \\ \mu_{i+1}(t) & \text{if } j = i+1, \\ -(\lambda_i(t) + \mu_i(t)) & \text{if } j = i, \\ 0 & \text{if } |i-j| > 1. \end{cases} \tag{3}$$

Then the behavior of the process is described by the forward Kolmogorov system

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}, \qquad \mathbf{p} = \mathbf{p}(t), \qquad t \geq 0. \tag{4}$$

The Cauchy operator $U(t,s)$, $0 \leq s \leq t$, of equation (4) is determined by the matrix

$$U^T(t,s) = P(s,t) := (p_{ij}(s,t))_{i,j=0}^N.$$

The norms of vectors and matrices used below are, as a rule, the ℓ_1-norms denoted by $\|\cdot\|$, that is, $\|\mathbf{x}\| = \sum_{i \in E} |x_i|$ and

$$\|B\| = \sup_{j \in E} \sum_{i \in E} |b_{ij}|$$

for $\mathbf{x} = (x_0, \dots, x_N)^T$ and $B = (b_{ij})_{i,j=0}^N$, respectively.

The set of all stochastic vectors is denoted by Ω:

$$\Omega = \left\{ \mathbf{x} = (x_0, \dots, x_N)^T : \mathbf{x} \geq 0, \|\mathbf{x}\| = 1 \right\}.$$

3. Stability

Since

$$p_0(t) = 1 - \sum_{i \geq 1} p_i(t), \qquad t \geq 0, \tag{5}$$

we obtain from (4) the system of differential equations:

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \qquad t \geq 0, \tag{6}$$

where

$$\begin{aligned} B(t) &= \{a_{ij}(t) - a_{i0}(t), i,j = 1, \dots, N\}, \\ \mathbf{z}(t) &= (p_1(t), \dots, p_N(t))^T, \\ \mathbf{f}(t) &= (a_{10}(t), \dots, a_{N0}(t))^T, \qquad t \geq 0. \end{aligned} \tag{7}$$

The solution of equation (6) is given by

$$\mathbf{z}(t) = V(t,s) \cdot \mathbf{z}(s) + \int_s^t V(t,\tau) \cdot \mathbf{f}(\tau)\, d\tau, \qquad 0 \leq s \leq t, \tag{8}$$

where $V(t,s)$ is the Cauchy operator of system (6).

When study the stability and ergodicity, we often apply the inequality

$$\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\| \leq \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\| \leq 2\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|, \qquad t \geq 0, \tag{9}$$

where

$$\mathbf{z}^{(i)} = \mathbf{z}^{(i)}(t) \quad \text{and} \quad \mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t), \qquad t \geq 0,\ i = 1,2,$$

is an arbitrary pair of solutions of systems (6) and (4), respectively.

The following transform:

$$D = \begin{pmatrix} d_0 & d_0 & d_0 & \dots \\ 0 & d_1 & d_1 & \dots \\ 0 & 0 & d_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{10}$$

where $d_i > 0$, $i = 0, 1, \dots, N-1$, is introduced in [27, 28] in order to reduce the matrix $B(t)$ of system (6) to a more convenient form. Note that the transform is determined by an $N \times N$ upper triangle matrix. We have

$$D^{-1} = \begin{pmatrix} d_0^{-1} & -d_1^{-1} & 0 & \ddots & \\ 0 & d_1^{-1} & -d_2^{-1} & 0 & \ddots \\ \ddots & 0 & \ddots\, d_2^{-1} & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Applying this transform to the matrix $B(t)$ in (7) that corresponds to a nonhomogeneous birth-death process we get the matrix $DB(t)D^{-1}$:

$$DB(t)D^{-1} = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & d_0 \cdot d_1^{-1} \cdot \mu_1 & 0 & & \dots \\ d_1 \cdot d_0^{-1} \cdot \lambda_1 & -(\lambda_1(t)+\mu_2(t)) & d_1 \cdot d_2^{-1} \cdot \mu_2 & 0 & \\ 0 & d_2 \cdot d_1^{-1} \cdot \lambda_2(t) & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ & \dots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{11}$$

Put $d_0 = 1$. For the case of a finite phase space we also put $d_N = 0$. Let $\delta_{i+1} = d_{i+1}/d_i$, $i = 0, \dots, N-1$. Then the log-norm is given by

$$\gamma(B(t))_{1D} = \gamma\left(DB(t)D^{-1}\right)_1 = -\inf_k \alpha_k(t) = -\underline{\alpha}(t), \qquad t \geq 0 \tag{12}$$

(see, for example, [5, 29]), where

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1} \cdot \lambda_{k+1}(t) - \delta_k^{-1} \cdot \mu_k(t), \qquad t \geq 0,\ k = 0, \dots, N-1. \tag{13}$$

Now we use the change $\mathbf{z}(t) = D^{-1}\mathbf{y}(t)$, $t \geq 0$, and transform system (6) to a system with the matrix $DB(t)D^{-1}$, $t \geq 0$. As in [26, 27] we put

$$\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|. \tag{14}$$

Then we get

$$\|\mathbf{z}\|_{1D} \geq \frac{g}{2}\|\mathbf{z}\| \tag{15}$$

by (10), where $g = \inf_k d_k$. If the weights δ_k, $k = 0, \dots, N-1$, in (13) are such that

$$\int_0^{+\infty} \underline{\alpha}(t)\, dt = +\infty,$$

then the birth-death process is weakly ergodic and

$$\left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\| \leq \frac{4}{g} \exp\left\{-\int_s^t \underline{\alpha}(u)\, du\right\} \left\|\mathbf{z}^{(1)}(s) - \mathbf{z}^{(2)}(s)\right\|_{1D}, \quad 0 \leq s \leq t. \tag{16}$$

Here

$$\left\|\mathbf{z}^{(1)}(s)-\mathbf{z}^{(2)}(s)\right\|_{1D} \le \sum_{i\ge 1} q_i \left|p_i^{(1)}(s)-p_i^{(2)}(s)\right|, \qquad t\ge 0, \tag{17}$$

according to notation (14), where $q_i=\sum_{m=0}^{i-1} d_m$, $i=1,\dots,N$.

Estimate (16) holds for ergodic birth-death processes. This is seen by putting

$$\mathbf{p}^{(2)}(t)=\boldsymbol{\pi}, \qquad t\ge 0.$$

We consider a class of nonhomogeneous birth-death processes satisfying the condition

$$\lambda_n(t)=\lambda_n\cdot a(t), \qquad \mu_n(t)=\mu_n\cdot b(t), \qquad t\ge 0,\ n\in E, \tag{18}$$

where

$$\mu_0=0, \qquad \lambda_n>0, \quad n=0,\dots,N-1, \qquad \mu_n>0, \quad n=1,\dots,N-1. \tag{19}$$

We assume that the following limits exist:

$$\lim_{n\to\infty}\lambda_n=\lambda>0, \qquad \lim_{n\to\infty}\mu_n=\mu>0 \tag{20}$$

if $N=\infty$. For the case of $N<\infty$ we put

$$\lambda=\lambda_{N-1}, \qquad \mu=\mu_N. \tag{21}$$

We also assume that the functions $a(t)\ge 0$, $t\ge 0$, and $b(t)\ge 0$, $t\ge 0$, are locally integrable on $[0,\infty)$ and possess the “uniform” mean values

$$\widetilde{a}=\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t a(\tau)\,d\tau, \qquad \widetilde{b}=\lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t b(\tau)\,d\tau. \tag{22}$$

Condition (22) holds, in particular, if the functions $a(t)$ and $b(t)$ are periodic (in general, with different periods) or asymptotically periodic.

It is easy to show that condition (22) implies that the functions $a(t)$ and $b(t)$ are integral bounded on $[0,\infty)$, that is,

$$\sup_{t\ge 0}\int_t^{t+1} a(\tau)\,d\tau=\mathsf{a}<\infty, \qquad \sup_{t\ge 0}\int_t^{t+1} b(\tau)\,d\tau=\mathsf{b}<\infty. \tag{23}$$

Note that the existence of ordinary means

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau)\,d\tau, \qquad \lim_{t\to\infty}\frac{1}{t}\int_0^t b(\tau)\,d\tau$$

does not imply the integral boundedness of $a(t)$ and $b(t)$.

In the current paper we study the stability of birth-death processes with respect to the perturbations of a more general form as compared to those in [25, 26, 27, 30, 12].

We assume that the perturbed Markov chain $\bar{X}(t)$, $t\ge 0$, is a process with bounded jumps, that is, its matrix of intensities

$$\bar{A}(t)=(\bar{a}_{ij}(t))_{i,j=0}^{N}$$

is such that $\bar{a}_{ij}(t)=0$ for all $t\ge 0$, $|i-j|>R$, and some positive integer R. Moreover we assume that

$$\bar{a}_{ij}(t)=\bar{a}_{ij}\bar{a}(t)$$

for $i>j$, while

$$\bar{a}_{ij}(t)=\bar{a}_{ij}\bar{b}(t)$$

for $i < j$, and

$$L = \sup_{i,j,\ i \neq j} (a_{ij}, \bar{a}_{ij}) < \infty, \tag{24}$$

$$\sup_{t \geq 0} \int_t^{t+1} \left(|\hat{a}(\tau)| + |\hat{b}(\tau)| \right) d\tau \leq \varepsilon, \tag{25}$$

where $\hat{a}(t) = a(t) - \bar{a}(t)$ and $\hat{b}(t) = b(t) - \bar{b}(t)$.

Put

$$f_k = \frac{\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu + \sqrt{(\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu)^2 + 4\Delta\lambda\mu \cdot (\lambda_{k-1}\mu_{k+1} - \lambda_k\mu_k)}}{2\Delta\lambda\mu}, \quad k = 0, \dots, N-1, \tag{26}$$

where $\Delta > 0$ and λ and μ are defined by (20) and (21), respectively.

In what follows we use the notation

$$\tilde{E} = \begin{cases} \{0, 1, \dots, N-1\} & \text{if } N < \infty, \\ E & \text{if } N = \infty. \end{cases}$$

Put

$$c_0 = \min \left(1, \inf_k f_k, \inf_k \frac{\mu_{k+1}}{\mu} \right). \tag{27}$$

It is proved in [12] that for every $0 < c \leq c_0$ there exists a positive sequence $\{\delta_k\}$ such that

$$\underline{\alpha}(t) \geq l(t) = c \cdot (\mu b(t) - \Delta\lambda a(t)), \qquad t \geq 0, \tag{28}$$

for some $\Delta > 1$ if $\inf_k f_k > 0$. Moreover $\liminf \delta_k > 1$ for $N = \infty$ and

$$0 < \mathsf{m}_1 \leq \frac{\mu_k}{\mu_{k+1} - c\mu} \leq \delta_k \leq \frac{\lambda_{k-1} + c\Delta\lambda}{\lambda_k} \leq \mathsf{m}_2 < \infty \tag{29}$$

under the latter condition.

Conditions (22) and (28) imply that there exists a constant K such that

$$\exp\left(-\int_s^t l(u)\, du \right) \leq K \exp(-l(t-s)) \tag{30}$$

for all $0 \leq s \leq t$, where

$$l = c\left(\mu\widetilde{b} - \Delta\lambda\widetilde{a}\right) - \varepsilon.$$

Let

$$\mu\widetilde{b} - \Delta\lambda\widetilde{a} > 0, \qquad \mathbf{p}(0) = \bar{\mathbf{p}}(0), \tag{31}$$

and the "perturbed" Markov chain $\bar{X}(t)$, $t \geq 0$, has bounded jumps and satisfies conditions (24) and (25).

We rewrite condition (6) as follows:

$$\begin{aligned} \frac{d\mathbf{z}}{dt} &= \bar{B}(t)\mathbf{z} + \bar{\mathbf{f}}(t) + (B(t) - \bar{B}(t))\mathbf{z} + \mathbf{f}(t) - \bar{\mathbf{f}}(t) \\ &= \bar{B}(t)\mathbf{z} + \bar{\mathbf{f}}(t) + \hat{B}(t)\mathbf{z} + \hat{\mathbf{f}}(t), \end{aligned} \tag{32}$$

where $\hat{B}(t) = B(t) - \bar{B}(t)$ and $\hat{\mathbf{f}}(t) = \mathbf{f}(t) - \bar{\mathbf{f}}(t)$, $t \geq 0$. Then

$$\begin{aligned} \mathbf{z}(t) &= \bar{V}(t,0) \cdot \mathbf{z}(0) + \int_0^t \bar{V}(t,\tau) \cdot \bar{\mathbf{f}}(\tau)\, d\tau + \int_0^t \bar{V}(t,\tau) \cdot [\hat{B}(\tau)\mathbf{z}(\tau) + \hat{\mathbf{f}}(\tau)]\, d\tau \\ &= \bar{\mathbf{z}}(t) + \int_0^t \bar{V}(t,\tau) \cdot [\hat{B}(\tau)\mathbf{z}(\tau) + \hat{\mathbf{f}}(\tau)]\, d\tau, \qquad t \geq 0, \end{aligned}$$

and we get

$$\|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\|_{1D} \le \int_0^t \|\bar V(t,\tau)\|_{1D}\cdot\|\hat B(\tau)\mathbf{z}(\tau)+\hat{\mathbf{f}}(\tau)\|_{1D}\,d\tau \\ \le \int_0^t \|\bar V(t,\tau)\|_{1D}\cdot\left\{\|\hat B(\tau)\|_{1D}\cdot\|\mathbf{z}(\tau)\|_{1D}+\|\hat{\mathbf{f}}(\tau)\|_{1D}\right\}\,d\tau. \tag{33}$$

According to (30) and (23) we have

$$\begin{aligned}\|\mathbf{z}(t)\|_{1D} &\le \|V(t,0)\|\cdot\|\mathbf{z}(0)\|_{1D}+\int_0^t \|V(t,\tau)\|_{1D}\cdot\|\mathbf{f}(\tau)\|_{1D}\,d\tau \\ &\le Ke^{-lt}\|\mathbf{z}(0)\|_{1D}+\lambda_0 e^{-lt}\int_0^t e^{l\tau}a(\tau)\,d\tau \\ &\le Ke^{-lt}\|\mathbf{z}(0)\|_{1D}+\lambda_0 e^{-lt}\left(\sum_{n=0}^{[t]-1}\int_n^{n+1} e^{l\tau}a(\tau)\,d\tau+\int_{[t]}^t e^{l\tau}a(\tau)\,d\tau\right) \\ &\le Ke^{-lt}\|\mathbf{z}(0)\|_{1D}+\lambda_0\mathsf{a}\left(1+\frac{e^l}{e^l-1}\right).\end{aligned} \tag{34}$$

Let $\hat a_{ij}(t)=a_{ij}(t)-\bar a_{ij}(t)$. Then

$$D\hat B D^{-1}=\begin{pmatrix} g_{11} & g_{12} & \dots \\ g_{21} & g_{22} & \dots \\ \dots & \dots & \dots\end{pmatrix},$$

where

$$\begin{aligned} g_{11} &= \hat a_{11}+\hat a_{21}+\dots-\hat a_{10}-\hat a_{20}-\dots, \\ g_{12} &= d_0 d_1^{-1}\big(\hat a_{12}+\hat a_{22}+\dots-\hat a_{11}-\hat a_{21}-\dots\big), \\ g_{21} &= d_1 d_0^{-1}\big(\hat a_{21}+\hat a_{31}+\dots-\hat a_{20}-\hat a_{30}-\dots\big), \\ g_{22} &= \big(\hat a_{22}+\hat a_{32}+\dots-\hat a_{21}-\hat a_{31}-\dots\big).\end{aligned}$$

Since the jumps are bounded, the entries of the above matrix are zero if the absolute value of the difference of indices is greater than R. Putting

$$\mathsf{m}_3=\big(\max(\mathsf{m}_2,\mathsf{m}_1^{-1})\big)^R, \tag{35}$$

we get

$$\left\|\hat B(t)\right\|_{1D}=\left\|D\hat B(t)D^{-1}\right\|_1\le 2LR\mathsf{m}_3\left(|\hat a(t)|+|\hat b(t)|\right), \tag{36}$$

$$\left\|\hat{\mathbf{f}}(t)\right\|_{1D}\le LR\mathsf{m}_3|\hat a(t)|, \tag{37}$$

and

$$\gamma\left(\bar B(t)\right)_{1D}\le\gamma(B(t))_{1D}+\left\|\hat B(t)\right\|\le -l(t)+2LR\mathsf{m}_3\left(|\hat a(t)|+|\hat b(t)|\right). \tag{38}$$

The last inequality implies that

$$\|\bar V(t,\tau)\|\le K\exp\left(-l(t-\tau)+\int_\tau^t 2LR\mathsf{m}_3(|\hat a(\tau)|+|\hat b(\tau)|)\,d\tau\right) \\ \le 2K\exp\{-(l-2LR\mathsf{m}_3\varepsilon)(t-\tau)\} \tag{39}$$

by (24) and (25). Thus

$$\begin{aligned}\|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\|_1 &\le \frac{4}{g}\|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\|_{1D}\\ &\le \frac{8}{g}KLR\mathsf{m}_3\int_0^t \exp\{-(l-2LR\mathsf{m}_3\varepsilon)(t-u)\}\left(|\hat{a}(u)|+|\hat{b}(u)|\right)\\ &\qquad\times\left(2K\|\mathbf{z}(0)\|_{1D}+2\lambda_0\mathsf{a}\left(1+\frac{e^l}{e^l-1}\right)+1\right)du\\ &\le \frac{16}{g}\varepsilon KLR\mathsf{m}_3\left(2K\|\mathbf{z}(0)\|_{1D}+2\lambda_0\mathsf{a}\left(1+\frac{e^l}{e^l-1}\right)+1\right)\\ &\quad\times\left(1+\frac{e^{(l-2LR\mathsf{m}_3\varepsilon)}}{e^{(l-2LR\mathsf{m}_3\varepsilon)}-1}\right).\end{aligned} \tag{40}$$

Therefore the following result holds.

Theorem 1. *Assume that condition* (31) *holds and the "perturbed" Markov chain* $\bar{X}(t)$, $t\ge 0$, *has bounded jumps and satisfies conditions* (24) *and* (25). *Then estimate* (40) *holds.*

Now we consider the case of a finite phase space. The equation for the "perturbed" chain, similar to (6), can be written as follows:

$$\frac{d\bar{\mathbf{z}}}{dt}=B(t)\bar{\mathbf{z}}+\mathbf{f}(t)+\left(\bar{B}(t)-B(t)\right)\bar{\mathbf{z}}+\bar{\mathbf{f}}(t)-\mathbf{f}(t)=B(t)\bar{\mathbf{z}}+\mathbf{f}(t)-\hat{B}(t)\bar{\mathbf{z}}-\hat{\mathbf{f}}(t). \tag{41}$$

Then

$$\|\bar{\mathbf{z}}(t)\|_1\le\|\bar{\mathbf{p}}(t)\|_1\le 1, \tag{42}$$

$$\left\|\hat{B}(t)\right\|_1\le 2L\left(|\hat{a}(t)|+|\hat{b}(t)|\right), \tag{43}$$

$$\left\|\hat{\mathbf{f}}(t)\right\|_1\le L|\hat{a}(t)|, \tag{44}$$

$$\|V(t,\tau)\|_1\le\|D\|\cdot\left\|D^{-1}\right\|\cdot\exp\left\{-\int_s^t l(u)\,du\right\}\le NG\cdot\frac{2}{g}\cdot Ke^{-l(t-s)}, \tag{45}$$

where $G=\max d_i$.

Thus for the ℓ_1-norm we get

$$\begin{aligned}\|\mathbf{z}(t)-\bar{\mathbf{z}}(t)\| &\le \int_0^t\|V(t,\tau)\|\cdot\left\|\hat{B}(\tau)\bar{\mathbf{z}}(\tau)+\hat{\mathbf{f}}(\tau)\right\|d\tau\\ &\le\int_0^t\|V(t,\tau)\|\cdot\left\{\|\hat{B}(\tau)\|\cdot\|\bar{\mathbf{z}}(\tau)\|+\|\hat{\mathbf{f}}(\tau)\|\right\}d\tau\\ &\le\frac{6GNKL}{g}\cdot\int_0^t e^{-l(t-\tau)}\left(|\hat{a}(\tau)|+|\hat{b}(\tau)|\right)d\tau\\ &\le\frac{6GNKL}{g}\cdot\varepsilon\left(1+\frac{e^l}{e^l-1}\right).\end{aligned} \tag{46}$$

Therefore the following result holds.

Theorem 2. *Let all the assumptions of Theorem 1 hold and* $N<\infty$. *If the intensities of the "perturbed" Markov chain* $\bar{X}(t)$, $t\ge 0$, *satisfy conditions* (24) *and* (25), *then*

$$\|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\|_1\le\frac{12GNKL\varepsilon}{g}\left(1+\frac{e^l}{e^l-1}\right). \tag{47}$$

4. Examples

First we consider a queueing system close to $M_t/M_t/S$. The corresponding references for the system $M_t/M_t/S$ can be found in [13, 21, 22, 2]. We assume that the number of customers in the queue (the queue length) is a birth-death process $X(t)$ whose intensities are locally integrable on $[0,\infty)$:

$$\lambda_n(t) = a(t), \qquad \mu_n(t) = b(t)\cdot \min(n,S). \tag{48}$$

The stability of these processes is studied in [29, 12]. Assume that the uniform means $\widetilde{a}$ and $\widetilde{b}$ exist. We also assume that the traffic intensity

$$\rho = \frac{\widetilde{a}}{S\widetilde{b}}$$

is such that $(1-1/S)^2 \le \rho < 1$. Then

$$c_0 = \min\left(\frac{1}{S}, \frac{\Delta-1}{\Delta}\right) = \frac{\Delta-1}{\Delta} = 1-\rho^{1/2},$$

$\delta_k = \rho^{-1/2} > 1$, and

$$l(t) = \left(1-\rho^{1/2}\right)\left(Sb(t)-\rho^{-1}a(t)\right). \tag{49}$$

Note that estimate (49) is quite precise, since the true value of the convergence parameter for a stationary queueing system $M/M/S$ is equal to

$$\dot{\alpha} = \left(\sqrt{a}-\sqrt{Sb}\right)^2$$

(see [6]).

Now we assume that a queueing system is close to $M_t/M_t/S$ in the sense that customers may arrive to the system and be served in groups. The number of customers in every group does not exceed R and the perturbations of arrival and service intensities of customers satisfy conditions (24) and (25).

Then the following estimate of the "stability" holds:

$$\begin{aligned} \|\mathbf{p}(t)-\bar{\mathbf{p}}(t)\|_1 &\le 16\varepsilon KLR\mathsf{m}_3\left(2K\|\mathbf{z}(0)\|_{1D} + 2\mathsf{a}\left(1+\frac{e^l}{e^l-1}\right)+1\right) \\ &\quad\times\left(1+\frac{e^{(l-2LR\mathsf{m}_3\varepsilon)}}{e^{(l-2LR\mathsf{m}_3\varepsilon)}-1}\right), \end{aligned} \tag{50}$$

where $\mathsf{m}_3 = \rho^{-R/2}$. If there are H customers in the system at the moment $t=0$, then

$$\|\mathbf{z}(0)\|_{1D} \le \sum_{k=0}^{H} \rho^{-k/2}.$$

It is worthwhile to mention that if the arrival and service intensities of customers are asymptotically periodic, then the corresponding process is "almost ergodic" in the sense that the limit

$$\tilde{p}_\infty = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{p}(\tau)\,d\tau \tag{51}$$

exists and is independent of the initial probability distribution of states.

Now we consider queueing systems close to $M_t/M_t/S/0$.

There is a number of papers devoted to the queueing system $M_t/M_t/S/0$ (the system with losses). Usually it is assumed that the intensities are constant (see references in [18] and follow up papers [7, 24, 23]). The number of customers $X(t)$ for this queue is a birth-death process with a finite phase space $E = \{0, 1, \dots, S\}$ and with intensities

$$\lambda_n(t) = a(t), \qquad \mu_n(t) = nb(t). \tag{52}$$

The first results and estimates for the general case are obtained in [27] (also see [29, 12]). Some of the results of [27] are obtained in a follow-up paper [20], where the non-Markovian case is also studied. We assume again that the uniform means $\widetilde{a}$ and $\widetilde{b}$ exist and moreover $\widetilde{b} > 0$. Putting $\delta_k = 1$ for all k we prove that $\alpha_k(t) \geq b(t)$ for all k and

$$l(t) = b(t). \tag{53}$$

Let a queueing system be close to $M_t/M_t/S/0$ in the sense that the total number of customers in the system does not exceed S and the perturbations of the arrival and service intensities satisfy conditions (24) and (25).

Then Theorem 1 implies the following estimate of the "stability":

$$\begin{aligned} &\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \\ &\quad \leq 16\varepsilon KLS \left(2KS + 2\mathsf{a}\left(1 + \frac{e^l}{e^l - 1}\right) + 1\right)\left(1 + \frac{e^{(l - 2LR\mathsf{m}_3\varepsilon)}}{e^{(l-2LR\mathsf{m}_3\varepsilon)} - 1}\right). \end{aligned} \tag{54}$$

Theorem 2 implies a sharper estimate, which is, naturally, of the same order:

$$\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq 12\varepsilon KLS\left(1 + \frac{e^l}{e^l - 1}\right). \tag{55}$$

In the case of asymptotically periodic arrival and service intensities, the corresponding process is "almost ergodic" in the sense that the limit

$$\tilde{p}_\infty = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{p}(\tau)\,d\tau \tag{56}$$

exists and is independent of the initial probability distribution of customers. Note that the limit above can be used as the "pseudostationary distribution".

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