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STRONG STABILITY IN RETRIAL QUEUES

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Abstract. In this paper we study the strong stability in retrial queues after perturbation of the retrial's parameter.

Our objective is to obtain the necessary and sufficient conditions to approximate the stationary characteristics of the $M/G/1/1$ retrial queue by the classical $M/G/1$ correspondent ones. After clarifying the approximation conditions, we obtain the stability inequalities with an exact computation of the constants.

1. Introduction

Queueing systems in which customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called retrial queues. Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication network, and computer systems [7].

Because of the complexity of retrial queueing models, analytic results are generally difficult to obtain. In contrast, there exists, when a practical study is performed in queueing theory, a common technique for substituting the real but complicated elements governing a queueing system by simpler ones, in some sense close to the real elements. The queueing model so constructed represents an idealization of the real queueing one, and hence the "stability" problem arises.

A queueing system is said to be stable if small perturbations in its parameters entail small perturbations in its characteristics [2, 11]. So, the margin between the correspondent characteristics of two stable queueing systems is obtained as function of the margin between their parameters.

Elaborated in the early 1980s [1, 11] the strong stability method (also called "method of operators") can be used to investigate the ergodicity and stability of the stationary and nonstationary characteristics of the imbedded Markov chains [2]. In contrast to other methods, we assume that the perturbations of the transition kernel are small with respect to some norms in the operator space. This stringent condition gives better stability estimates and enables us to find precise asymptotic expressions of the characteristics of the "perturbed" system [3].

In this work, we study the strong stability method in the retrial queueing systems. This paper is organized as follows: In Section 2, we describe the model of $M/G/1/1$ retrial queue and we pose the problem. Section 3 concerns the study of strong stability of the stationary distribution of the imbedded Markov chain an $M/G/1$ queueing system with infinite retrials, after perturbation of the retrial's parameter. So we clarify the

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approximation conditions of the $M/G/1$ retrial queue by the classical $M/G/1$ correspondent ones. In Section 4, we obtain the stability inequalities. Finally, in Section 5, we present our conclusion.

2. The $M/G/1/1$ retrial queue

We consider a single server queue which has Poisson arrivals (primary calls) with parameter λ; retrial times are independent, identically and exponentially distributed with rate θ, and service times are identically, independently distributed with distribution function F. The earliest paper giving analytic results for the $M/G/1/1$ retrial queue is due to Keilson et al. [12]. Using the method of supplementary variable, the ergodic solutions are obtained for the generating functions of the number of customers in the queue. Alexandrov [4] studied the same model and obtained similar results using residual service time as a supplementary variable rather than elapsed service time, as used in [12]. A few years later, Choo and Conolly [6] examined the $M/G/1/1$ retrial model, using the imbedded Markov chain method.

Consider the imbedded Markov chain defined by X_n: "the number of customers left behind in the system by the nth departure." We have the fundamental equation:

$$X_{n+1} = X_n - \delta_{X_n} + \Delta_{n+1},$$

where Δ_n is the number of customers arriving during the nth service time; δ_{X_n} is a Bernoulli variable and

$$\delta_{X_n} = \begin{cases} 1 & \text{if the } (n+1)\text{th served customer is from orbit,} \\ 0 & \text{otherwise.} \end{cases}$$

Its distribution is given by

$$\mathsf{P}(\delta_{X_n} = 1 \;/\; X_n = k) = \frac{k\theta}{\lambda + k\theta}, \qquad \mathsf{P}(\delta_{X_n} = 0 \;/\; X_n = k) = \frac{\lambda}{\lambda + k\theta}.$$

The random variable Δ_{n+1} has distribution

$$\mathsf{P}(\Delta_{n+1} = k) = P_k = \int \frac{(\lambda x)^k}{k!} e^{-\lambda x} dF(x).$$

The transition probabilities in one step (single step) are given by

$$\mathsf{P}(X_{n+1} = j \;/\; X_n = i) = P_{ij} = \frac{i\theta}{\lambda + i\theta} P_{j-i+1} + \frac{\lambda}{\lambda + i\theta} P_{j-i}.$$

Denote by P the transition operator of the $M/G/1/1$ retrial queue:

(1)

$$\begin{aligned} P_{ij} &= \mathsf{P}(X_{n+1} = j \;/\; X_n = i) \\ &= \begin{cases} P_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!}\, dF(x) & \text{if } i = 0, \\ \frac{i\theta}{\lambda + i\theta} \int_0^\infty e^{-\lambda x}\, dF(x) & \text{if } i = j+1, \\ \frac{i\theta}{\lambda + i\theta} \int_0^\infty e^{(-\lambda x)} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!}\, dF(x) + \frac{\lambda}{\lambda + i\theta} \int_0^\infty e^{(-\lambda x)} \frac{(\lambda x)^{j-i}}{(j-i)!}\, dF(x) & \text{if } 1 \le i \le j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consider also an $M/G/1$ system with infinite retrials (which is the classical $M/G/1$ system [7]) with Poisson arrivals with parameter λ and with the same general service time distribution function F. Let $\bar{P}$ be the transition operator for the corresponding

Markov chain $\bar{X}_n$, in the classical $M/G/1$ system. We have

(2)
$$\bar{P}_{ij} = \begin{cases} \bar{P}_j = \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^j}{j!}\, dF(x) & \text{if } i = 0, \\ \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^{j-i+1}}{(j-i+1)!}\, dF(x) & \text{if } 1 \le i \le j+1, \\ 0 & \text{otherwise.} \end{cases}$$

We propose to apply the strong stability criterion [11] for the $M/G/1$ queueing system with infinite retrials after perturbation of the retrial's parameter (i.e. after crossing to a finite rate of retrial).

3. Strong stability in $M/G/1$ queue with infinite retrials

3.1. **Preliminaries and notations.** Let $\mathcal{M} = \{\mu_j\}$ be the space of finite measures on $\mathbf{N}$, and $\eta = \{f(i)\}$ the space of bounded measurable function on $\mathbf{N}$. We associate with each transition kernel P the linear mapping

(3)
$$(\mu P)_k(j) = \sum_{i \ge 0} \mu_i P_{ik}(j),$$

(4)
$$Pf(k) = \sum_{i \ge 0} f(i) P_{ki}.$$

Introduce on $\mathcal{M}$ the class of norms of the form

(5)
$$\|\mu\|_v = \sum_{j \ge 0} v(j)|\mu_j|,$$

where v is an arbitrary measurable function (not necessarily finite) bounded below away from a positive constant, and $|\mu_j|$ is the variation of the measure μ.

This norm induces in the space η the norm

(6)
$$\|f\|_v = \sup_{k \ge 0} \frac{|f(k)|}{v(k)}.$$

Let us consider $\mathcal{B}$, the space of linear operators, with the norm

(7)
$$\|P\|_v = \sup_{k \ge 0} \frac{1}{v(k)} \sum_{j \ge 0} v(j)|P_{kj}|.$$

Definition. A Markov chain X with a transition kernel P and invariant measure π is said to be strongly v-stable with respect to the norm $\|\cdot\|_v$, if $\|P\|_v < \infty$, and each stochastic kernel Q on the space in some neighborhood $\{Q\colon \|Q-P\|_v < \varepsilon\}$ has a unique invariant measure $\nu = \nu(P)$ and $\|\nu - \pi\|_v \to 0$ as $\|Q - P\|_v \to 0$.

In the sequel we use the following results:

Theorem 1 ([1])**.** *The Markov chain X with the transition kernel P and invariant measure π is strongly v-stable with respect to the norm $\|\cdot\|_v$ if and only if there exist a measure α and a nonnegative measurable function h on $\mathbf{N}$ such that $\pi h > 0$, $\alpha \mathbb{1} = 1$, $\alpha h > 0$, and*

a) *The operator $T = P - h \circ \alpha$ is nonnegative.*
b) *There exists $\rho < 1$ such that $Tv(k) \le \rho v(k)$ for $k \in \mathbf{N}$.*
c) $\|P\|_v < \infty$.

Here $\mathbb{1}$ is the function identically equal to 1.

Theorem 2 ([10])**.** *Let X be a strongly v-stable Markov chain that satisfies the conditions of Theorem 1. If ν is the measure invariant of a kernel Q, then for the norm $\|Q-P\|_v$ sufficiently small, we have*

$$\nu = \pi[I - \Delta R_0(I-\Pi)]^{-1} = \pi + \sum_{t=1}^{\infty} \pi[\Delta R_0(I-\Pi)]^t,$$

where $\Delta = Q - P$, $R_0 = (I-T)^{-1}$ and $\Pi = \mathbb{1} \circ \pi$.

Corollary 1. *Under the conditions of Ttheorem 1,*

$$\nu = \pi + \pi\Delta R_0(I-\Pi) + o\left(\|\Delta\|_v^2\right) \tag{8}$$

as $\|\Delta\|_v \to 0$.

Corollary 2. *Under the conditions of Theorem 1, for*

$$\|\Delta\|_v < \frac{(1-\rho)}{c},$$

we have the estimate

$$\|\nu - \pi\|_v \le \|\Delta\|_v c \|\pi\|_v (1 - \rho - c\|\Delta\|_v)^{-1}, \tag{9}$$

where

$$c = m\|P\|_v^{m-1}(1 + \|\mathbb{1}\|_v \|\pi\|_v);$$
$$\|\pi\|_v \le (\alpha v)(1-\rho)^{-1}(\pi h) m \|P\|_v^{m-1}.$$

3.2. **Strong stability.** To apply Theorem 1 to the Markov chain $\bar{X}_n$, we choose the function $v(k) = \beta^k$, $\beta > 1$, $h_i = \mathbb{1}_{i=0}$ and

$$\alpha_j = \bar{P}_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} \, dF(x).$$

We have $\bar{\pi} h = \bar{\pi}_0 > 0$, $\alpha \mathbb{1} = 1$, $\alpha h = \alpha_0 = \bar{P}_0 > 0$.

a) If $i = 0$, then $T_{0j} = \bar{P}_{0j} - \bar{P}_{0j} = 0$. If $i > 1$, then $T_{ij} = \bar{P}_{ij} \ge 0$. Hence, T_{ij} is nonnegative.

b) According to equation (4),

$$Tv(k) = \sum_{j \ge 0} \beta^j T_{kj}$$

for $k = 0$ and $Tv(0) = 0$ for $k \neq 0$:

$$\begin{aligned} Tv(k) &= \sum_{j\ge 0} \beta^j \bar{P}_{kj} = \sum_{j \ge k-1} \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^{j-k+1}}{(j-k+1)!} \, dF(x) \\ &= \beta^{k-1} \int_0^\infty \exp(-\lambda x) \sum_{j \ge 0} \frac{(\lambda \beta x)^j}{j!} \, dF(x) \\ &= \beta^{k-1} \int_0^\infty \exp\left[(\lambda\beta - \lambda)x\right] dF(x). \end{aligned}$$

We set

$$\hat{f}(\lambda\beta - \lambda) = \int_0^\infty \exp[(\lambda\beta - \lambda)x] \, dF(x)$$

and obtain

$$Tv(k) = \beta^{k-1} \hat{f}(\lambda\beta - \lambda).$$

Then we prove that

$$\beta^{k-1}\hat{f}(\lambda\beta - \lambda) \le \rho\beta^k, \quad \text{i.e.} \quad \hat{f}(\lambda\beta - \lambda) \le \rho\beta,$$

or

$$Tv(k) = \beta^k \frac{\hat{f}(\lambda\beta - \lambda)}{\beta}.$$

To prove that

$$\rho = \frac{\hat{f}(\lambda\beta - \lambda)}{\beta} < 1$$

we use the following lemma.

Lemma 1 ([2]). *Suppose that geometric ergodicity and the Cramér conditions in the system $M/G/1$ hold: $\lambda\,\mathsf{E}(\xi) < 1$, where ξ is the service time and*

$$\mathsf{E}(\exp(a\xi)) = \int \exp(au)\, dF(u) < 1$$

for some $a > 0$. Then there exists $\beta > 1$ such that $\frac{\hat{f}(\lambda\beta-\lambda)}{\beta} < 1$.

c) $T = P - h \circ \alpha \Rightarrow P = T + h \circ \alpha$ and $\|P\|_v = \|T + h \circ \alpha\|_v \leq \|T\|_v + \|h\|_v \|\alpha\|_v$, or, according to equation (7),

$$\|T\|_v = \sup_{k \geq 0} [v(k)]^{-1} \sum_{j \geq 0} v(j) |T_{kj}| \leq \sup_{k \geq 0} [v(k)]^{-1} \rho v(k) = \rho < 1.$$

According to equations (5), (6), we have

$$\begin{aligned} \|\alpha\|_v &= \sum_{j \geq 0} v(j) |\alpha(j)| = \sum_{j \geq 0} \beta^j \int_0^\infty \exp(-\lambda x) \frac{(\lambda x)^j}{j!}\, dF(x) \\ &= \hat{f}(\lambda\beta - \lambda) < \beta < \beta_0 < \infty, \end{aligned}$$

and

$$\|h\|_v = \sup_{k \geq 0} 1/v(k) = 1.$$

Hence $\|P\|_v < \infty$.

Then we have the following result.

Theorem 3. *Suppose that the assumptions of Lemma 1 hold and*

$$\beta_0 = \sup\left(\beta\colon \hat{f}(\lambda\beta - \lambda) < \beta\right).$$

Then, for all β such that $1 < \beta < \beta_0$, the Markov chain $\bar{X}_n$ is strongly stable for the test function $v_n = \beta^n$.

The imbedded Markov chain $\bar{X}_n$ being strongly stable then, the characteristics of the $M/G/1/1$ retrial queue can be approximated by the corresponding ones of the classical $M/G/1$ queue.

4. Stability inequalities

To estimate the margin between the stationary distributions of Markov chains $\bar{X}_n$ and X_n, we first estimate the norm of the deviation of transition kernels.

Theorem 4. *Let P (resp. $\bar{P}$) be the transition operator of the imbedded Markov chain in the $M/G/1/1$ retrial queue (resp. in the classical $M/G/1$ system). Then, for all $1 < \beta < \beta_0$, we have*

$$\|P - \bar{P}\|_v \leq \frac{\lambda}{\lambda + \theta}(1 + \beta_0). \tag{10}$$

Proof. According to formula (7),

$$\|P-\bar{P}\|_v = \sup_{k\geq 0} \frac{1}{v(k)} \sum_{j\geq 0} v(j)|P_{kj}-\bar{P}_{kj}| = \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j\geq 0} \beta^j |P_{kj}-\bar{P}_{kj}|$$

for $k=0$ and $\|P-\bar{P}\|_v = 0$ for $k \geq 1$:

$$\begin{aligned} \|P-\bar{P}\|_v &= \sup_{k\geq 1} \frac{1}{\beta^k} \sum_{j\geq 0} \frac{\lambda}{\lambda+k\theta}\Big[\beta^{j-k+1}\int_0^\infty \exp(-\lambda x)\frac{(\lambda x)^j}{j!}\,dF(x) \\ &\qquad\qquad + \beta^{j+k}\int_0^\infty \exp(-\lambda x)\frac{(\lambda x)^j}{j!}\,dF(x)\Big] \\ &= \sup_{k\geq 1} \frac{\lambda}{\lambda+k\theta}(1+\beta)\frac{\hat{f}(\lambda\beta-\lambda)}{\beta} < \frac{\lambda}{\lambda+\theta}(1+\beta) \leq \frac{\lambda}{\lambda+\theta}(1+\beta_0). \qquad \square \end{aligned}$$

Lemma 2. *Let π (resp. $\bar{\pi}$) be the stationary distribution of the imbedded Markov chain in $M/G/1/1$ retrial queue (resp. of the classical $M/G/1$ system); then for all $1<\beta<\beta_0$, we have $\|\bar{\pi}\|_v \leq c_0$, where c_0 is given by*

$$c_0 = \frac{(1-\lambda m)(\beta-1)}{1-\rho}\rho \tag{11}$$

and $m = \mathsf{E}(\xi)$.

Proof. By definition,

$$\|\bar{\pi}\|_v = \sum_{j\geq 0} v(j)\bar{\pi}_j = \sum_{j\geq 0} \beta^j \bar{\pi}_j = \bar{\pi}(\beta).$$

Or

$$\bar{\pi}(\beta) = \frac{(\beta-1)(1-\lambda m)\hat{f}(\lambda\beta-\lambda)}{\beta-\hat{f}(\lambda\beta-\lambda)} \leq \frac{(1-\beta)(1-\lambda m)\rho\beta}{\beta-\rho\beta} = \frac{(1-\lambda m)(\beta-1)}{1-\rho}\rho. \quad \square$$

Theorem 5. *Let π (resp. $\bar{\pi}$) be the stationary distribution of the imbedded Markov chain in the $M/G/1$ retrial queue (resp. in the classical $M/G/1$ system); then for all $1<\beta<\beta_0$*

$$\|\pi-\bar{\pi}\|_v \leq c_0 c\|\Delta\|_v(1-\rho-\|\Delta\|_v c)^{-1}, \tag{12}$$

where $c = 1+\|\bar{\pi}\|_v$, $\Delta = P-\bar{P}$, and c_0 is defined in (11).

Proof. According to Theorem 2,

$$R_0 = (I-T)^{-1} = \sum_{t\geq 0} T^t,$$

$$\|R_0\|_v = \frac{1}{\|I-T\|_v}.$$

Since

$$\|T\|_v \leq \rho,$$

we have

$$\|R_0\|_v \leq \frac{1}{1-\rho}.$$

Hence

$$\|R_0(I-T)\|_v \leq \|R_0\|_v(1+\|\pi\|_v) \leq \frac{1+\|\pi\|_v}{1-\rho}.$$

Using Corollary 2,

$$\begin{aligned}\|\pi-\bar{\pi}\|_v &\le \|\pi\|_v\|\Delta\|_v\|R_0(I-T)\|_v\frac{1}{1-\|\Delta\|_v\|R_0(I-T)\|_v}\\ &\le \|\pi\|_v\|\Delta\|_v\frac{1+\|\pi\|_v}{1-\rho}\frac{1}{1-\|\Delta\|_v\frac{1+\|\pi\|_v}{1-\rho}}\\ &\le c_0c\|\Delta\|_v(1-\rho-\|\Delta\|_v c)^{-1}. \qquad \square\end{aligned}$$

5. Conclusion

In this work, we proved the applicability of the strong stability method in retrial queues. In fact, in practice, we often neglect the retrials. The problem is then to estimate the error due to this approximation.

The precision obtained allows us to confirm the efficiency of this method and its importance for practical problems.

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