Теорія Ймовір. та Матем. Статист.
Вип. 68, 2003

Theor. Probability and Math. Statist.
S 0094-9000(04)00598-8
Article electronically published on May 24, 2004

ON THE ORDER LAW OF THE ITERATED LOGARITHM

UDC 519.21

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Abstract. We study the classical laws of the iterated logarithm due to Kolmogorov and Hartman–Wintner for random variables assuming values in Banach lattices.

1. Introduction

Let (ξ_n) be a sequence of independent random variables such that $\mathsf{E}\,\xi_n = 0$ and $\mathsf{E}\,\xi_n^2 = \sigma^2$, let (b_n) be a sequence of real numbers, and let $V_n = \sum_{i=1}^n b_i^2$. We say that a sequence (ξ_n) obeys the law of the iterated logarithm if

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n b_i\xi_i}{\chi(V_n)} = \sigma \quad \text{a.s.},$$
$$\liminf_{n\to\infty} \frac{\sum_{i=1}^n b_i\xi_i}{\chi(V_n)} = -\sigma \quad \text{a.s.}, \tag{1}$$

where "a.s." is the abbreviation for "almost surely", $\chi(t) = (2tLL(t))^{1/2}$, and

$$L(t) = \max(1, \ln(t))$$

for $t > 0$.

A survey of results on the law of the iterated logarithm (1), its generalizations for Banach spaces, and relevant references can be found in [1]–[3].

Denote by B a Banach lattice equipped with a norm $\|\cdot\|$ and module $|\cdot|$. For a sequence (x_n) of elements of a σ-complete Banach lattice we define the upper and lower limits as follows:

$$\limsup_{n\to\infty} x_n = \inf_m \left(\sup_{n\geq m} x_n\right),$$
$$\liminf_{n\to\infty} x_n = \sup_m \left(\inf_{n\geq m} x_n\right).$$

In what follows we make use of notions and some results of the theory of Banach lattices that can be found in the books [4]–[6].

The mean quadratic deviation of a random element and some of its generalizations can be defined for a Banach lattice (see [7]).

Consider a random element X assuming values in B. The mean p-deviation, $1 < p < \infty$, of a random element X is defined by

$$\Delta_p X = \sup(x \in K_p),$$

where

$$K_p = \left(x = \mathsf{E}\,\zeta X\colon\ \mathsf{E}\,|\zeta|^{p'} \leq 1\right), \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

2000 *Mathematics Subject Classification.* Primary 60B12.

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The element $\Delta X = \Delta_2 X$ of a Banach lattice is called the mean quadratic deviation of a random element X. The mean quadratic deviation coincides with the usual pointwise mean quadratic deviation

$$\Delta X = \Big((\mathsf{E}\,|X(t)|^2)^{1/2}, t \in T \Big)$$

in the case of a Banach function lattice if $\mathsf{E}\,X = 0$.

Let (X_n) be a sequence of independent random elements in B such that $\mathsf{E}\,X_n = 0$ and

$$\Delta(X_n) = \Delta X, \qquad n = 1, 2, \dots, \tag{2}$$

and put $S_n = \sum_{i=1}^n b_i X_i$. The following relations are natural generalizations of the law of the iterated logarithm for the case of Banach lattices:

$$\begin{aligned} \limsup_{n\to\infty} \frac{S_n}{\chi(V_n)} &= \Delta X \quad \text{a.s.}, \\ \liminf_{n\to\infty} \frac{S_n}{\chi(V_n)} &= -\Delta X \quad \text{a.s.} \end{aligned} \tag{3}$$

Relations (3) are called the order law of the iterated logarithm in a Banach lattice B.

The order law of the iterated logarithm for identically distributed random variables was studied in the paper [8]. The current paper continues investigations originated in [8].

Remark 1. Condition (2) holds if $\mathsf{E}\,X_n = 0$, all X_n have the same covariance operator R, H_R is a Hilbert subspace of the space B associated to R [3], and the closed unit ball of the space H_R is order bounded in B (see [7]).

2. Kolmogorov's law of the iterated logarithm and its generalizations

The following assertion generalizes the classical Kolmogorov theorem ([1, Chapter X, §1, Theorem 1]) to the case of Banach lattices.

Theorem 1. *Let B be a separable Banach lattice that does not contain ℓ_∞^n uniformly. Let X_n, $n \ge 1$, be independent random elements in B such that $\mathsf{E}\,X_n = 0$, $V_n \to \infty$ as $n \to \infty$, and condition (2) hold. Assume that there is a positive element $u \in B$ and a sequence of positive numbers (M_n) such that*

$$M_n = o\left(\left(\frac{V_n}{LL(V_n)} \right)^{1/2} \right), \tag{4}$$

$$|b_n X_n| \le M_n \cdot u \quad a.s. \tag{5}$$

Then the order law of the iterated logarithm (3) holds.

Remark 2. Assertion (iv) of Proposition 3 in § 3 shows that Theorem 1 does not hold for Banach lattices that uniformly contain ℓ_∞^n.

To prove Theorem 1 we need the following auxiliary result.

Lemma 1. *Let (b_n) be a sequence of real numbers, $V_n \to \infty$ as $n \to \infty$, $1 \le r < \infty$, and let (ξ_n) be a sequence of independent random variables such that $\mathsf{E}\,\xi_n = 0$ and $\mathsf{E}\,\xi_n^2 = \sigma^2$. Assume that there exists a sequence $(\widehat{M_n})$ of positive real numbers such that*

$$\sup_{n\ge1} \widehat{M_n} b_n \left(\frac{V_n}{LL(V_n)} \right)^{-1/2} = H < \infty, \tag{6}$$

$$|\xi_n| \le \widehat{M_n} \quad a.s. \tag{7}$$

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Then

$$\left(\mathsf{E}\sup_{n\geq 1}\left|\frac{\sum_{k=1}^{n} b_k\xi_k}{\chi(V_n)}\right|^r\right)^{1/r} \leq C_r\cdot H, \tag{8}$$

where the constant $C_r < \infty$ does not depend on (ξ_n) (however, it may depend on the sequence (b_n)).

Proof of Lemma 1. It is known that inequalities (6) and (7) yield

$$\sup_{n\geq 1}\left|\frac{\sum_{k=1}^{n} b_k\xi_k}{\chi(V_n)}\right| < \infty \quad \text{a.s.}$$

(see [16]). This implies that the following two conditions are equivalent:

$$\mathsf{E}\sup_{n\geq 1}\left|\frac{\sum_{k=1}^{n} b_k\xi_k}{\chi(V_n)}\right|^r < \infty, \tag{9}$$

$$\mathsf{E}\sup_{n\geq 1}\left|\frac{b_n\xi_n}{\chi(V_n)}\right|^r < \infty \tag{10}$$

(see [3, p. 159]).

It is easy to check that

$$\left|\frac{b_n\xi_n}{\chi(V_n)}\right| \leq \frac{H}{\sqrt{2}} \quad \text{a.s.} \tag{11}$$

under the assumptions of the lemma, whence (10) follows. Therefore inequality (9) holds, too.

Let $(\Omega, A, \mathsf{P}) = (\prod_{n=1}^{\infty}\Omega_n, \prod_{n=1}^{\infty}A_n, \prod_{n=1}^{\infty}\mathsf{P}_n)$ be the product of probability spaces $(\Omega_n, A_n, \mathsf{P}_n)$, and let Σ be the family of all sequences (ξ_n) of independent random variables defined on Ω and such that, for every n, the random variable ξ_n depends only on $\omega_n \in \Omega_n$. Denote by Σ_K the space of sequences $(\xi_n) \in \Sigma$ satisfying the conditions

$$\mathsf{E}\,\xi_n = 0, \qquad \mathsf{E}\,\xi_1^2 = \mathsf{E}\,\xi_2^2 = \cdots = \mathsf{E}\,\xi_n^2 = \cdots,$$

$$\|(\xi_n)\|_K = \sup_{n\geq 1}\|\xi_n\|_\infty b_n\left(\frac{V_n}{LL(V_n)}\right)^{-1/2} < \infty,$$

where $\|\xi_n\|_\infty = \inf(\alpha\colon\ \mathsf{P}(\omega\in\Omega_n\colon |\xi_n(\omega)| > \alpha) = 0)$.

Note that Σ_K is a Banach space with respect to the norm $\|\cdot\|_K$ (Σ_K is the direct sum of Banach spaces that are equivalent to $L^\infty(\Omega_n)$).

On the space Σ_K we consider the new norm

$$\|(\xi_n)\|_{K,1} = \|(\xi_n)\|_K + \left(\mathsf{E}\sup_{n\geq 1}\left|\frac{\sum_{k=1}^{n} b_k\xi_k}{\chi(V_n)}\right|^r\right)^{1/r}.$$

Since conditions (9) and (10) are equivalent and estimate (11) holds, the following two conditions

$$\|(\xi_n)\|_K < \infty$$

and

$$\|(\xi_n)\|_{K,1} < \infty$$

are equivalent. Therefore $\|(\cdot)\|_K$ and $\|(\cdot)\|_{K,1}$ are Banach norms on the space Σ_K. It is clear that $\|(\cdot)\|_K \leq \|(\cdot)\|_{K,1}$, whence we immediately obtain the equivalence of these norms:

$$\|(\cdot)\|_K \leq \|(\cdot)\|_{K,1} \leq C\|(\cdot)\|_K. \tag{12}$$

If a sequence (ξ_n) satisfies conditions (6) and (7), then $\|((\xi_n))\|_K \leq H$. The latter estimate together with (12) implies (8). □

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Proof of Theorem 1. Since Banach spaces are order isomorphic to Banach ideal spaces, we restrict ourselves to the case where B is a separable Banach ideal space on a measurable space (T, Λ, μ). Moreover, one can assume that the Banach ideal space B is q-concave for some $q < \infty$ (see [4], [5]) and that $\mu(T) = 1$.

We prove the first equality in the law of the iterated logarithm (3); the second can be checked analogously. Let

$$X = (X(t), t \in T) \in B \quad \text{a.s.},$$
$$\Delta X = (\sigma(t), t \in T) \in B,$$
$$U_m = \left(U_m(t) = \sup_{n \geq m} \frac{S_n(t)}{\chi(V_n)}, t \in T \right),$$
$$U = U_1.$$

Following the method of [8], it is sufficient to show that

$$\operatorname*{o-lim}_{m\to\infty} U_m = \Delta X \quad \text{a.s.}$$

The latter equality follows from

$$\mu \left(t \in T \colon \lim_{m\to\infty} U_m(t) = \sigma(t) \right) = 1 \quad \text{a.s.} \tag{13}$$

and

$$\mathsf{E}\, \|U\|^q < \infty. \tag{14}$$

The Kolmogorov law of the iterated logarithm implies that

$$\lim_{m\to\infty} U_m(t) = \sigma(t) \quad \text{a.s.}$$

almost everywhere on T provided conditions (4) and (5) hold. Applying Fubini's theorem, we derive (13) from the latter equality.

To check (14) we use the following estimate:

$$(\mathsf{E}\, \|Y\|^q)^{1/q} \leq D_q \|\Delta_q Y\|, \tag{15}$$

where Y is a random element assuming values in a q-concave Banach lattice (see [7]). We obtain from (8) and (15) that

$$(\mathsf{E}\, \|U\|^q)^{1/q} \leq D_q \left\| \left(\mathsf{E} \sup_{n\geq 1} \left| \frac{S_n(t)}{\chi(V_n)} \right|^q \right)^{1/q} \right\| \leq C_q D_q H \|u\|,$$

whence (14) follows. □

A similar method allows one to prove the following result.

Proposition 1. *Let B be a separable and q-concave, $2 < q < \infty$, Banach lattice. Assume that (X_n) is a sequence of independent random elements in B that satisfy condition (2), $\mathsf{E}\, X_n = 0$, and $V_n \to \infty$ as $n \to \infty$. Let there exist a positive element $u \in B$ and a number $\delta > 0$ such that*

$$\Delta_q(X_n) \leq u,$$
$$b_n^2 = O\left(V_n (L(V_n))^{-(1+\delta)/(q/2-1)} \right).$$

Then the order law of the iterated logarithm (3) holds.

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Consider a sequence (ε_i) of independent symmetric Bernoulli random variables,

$$\mathsf{P}(\varepsilon_i = \pm 1) = \frac{1}{2}.$$

It is shown in [9] that

$$\limsup_{n\to\infty} \left| \frac{\sum_{i=1}^n \varepsilon_i b_i}{\chi(V_n)} \right| \le 1 \quad \text{a.s.} \tag{16}$$

if $V_n \to \infty$. We prove a similar result for Banach ideal spaces. Assume that a Banach ideal space B contains a unit element $I = (I(t) \equiv 1, t \in T)$, and a sequence $(x_n) = (x_n(t), t \in T)$ is such that

$$A_n(t) = \sum_{i=1}^n |x_i(t)|^2 \to \infty, \qquad n \to \infty, \tag{17}$$

almost everywhere on T, and

$$|x_1| \ge \delta I \tag{18}$$

for some $\delta > 0$.

Theorem 2. (i) *Let B be a separable Banach ideal space containing a unit element I, and assume that B does not contain ℓ_∞^n uniformly. Let a sequence (x_n) satisfy conditions (17) and (18). Then there exists a nonrandom positive element $I_x \in B$ such that*

$$\limsup_{n\to\infty} \left| \frac{\sum_{i=1}^n \varepsilon_i x_i}{\chi(A_n)} \right| = I_x \quad \textit{a.s.}$$

and $I_x \le I$.

(ii) *If additionally*

$$A_n^{-1}(t) \sum_{i=1}^n |x_i(t)|^2 J(t, G_i) \to 0, \qquad n \to \infty, \tag{19}$$

for all $d > 0$ and everywhere on T where

$$J(t,G) = \begin{cases} 1, & t \in G, \\ 0, & t \notin G, \end{cases} \qquad G_i = \Big(t \in T \colon |x_i(t)| > d\big(A_i(t)/LL(A_i(t))\big)^{1/2}\Big),$$

then $I_x = I$.

It is known that inequality (16) becomes an equality if (19) holds (see [10]).

The following auxiliary result is the main tool in the proof of Theorem 2. Put

$$V(n,\lambda) = \lambda + \sum_{i=1}^n b_i^2, \qquad s_n = \sum_{i=1}^n \varepsilon_i b_i,$$

$$\Theta = \sup_{n \ge 1} \frac{|s_n|}{\chi(V(n,\lambda))},$$

and let $\Gamma(t)$ be the gamma function.

Lemma 2. *If $\lambda > e$ and $1 \le r < \infty$, then*

$$\mathsf{E}\,\Theta^r \le (2\lambda)^{r/2} + \frac{\pi^2 r}{3} \left(\frac{\lambda}{\ln\ln\lambda} \right)^{r/2} \Gamma\left(\frac{r}{2} - 1 \right). \tag{20}$$

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Proof of Lemma 2. The well-known exponential inequality (see, for example, Chapter 3, §4 in [1])

$$\mathsf{P}\left(s_n > t\left(\sum_{i=1}^n b_i^2\right)^{1/2}\right) \le \exp\left(-\frac{t^2}{2}\right)$$

implies

$$\mathsf{P}\big(s_n > t\chi(V(n,\lambda))\big) \le (\ln V(n,\lambda))^{-t^2}. \tag{21}$$

Put $\Theta_n = (i \in N\colon \lambda^n \le V(i,\lambda) < \lambda^{n+1})$. Considering only nonempty terms in the sequence (Θ_n) we obtain the subsequence (Θ_{n_k}). Further let

$$s_n^* = \max_{1\le i\le n}(s_i),$$

$$\alpha_k = \min\big(i\colon i \in \Theta_{n_k}\big), \qquad \beta_k = \max\big(i\colon i\in\Theta_{n_k}\big).$$

Estimate (21) and Levy's inequality $\mathsf{P}(s_n^* > t) \le 2\,\mathsf{P}(s_n > t)$ together with

$$\lambda^{n_k} \le V(\alpha_k,\lambda) \le V(\beta_k,\lambda) \le \lambda^{n_k+1}$$

imply

$$\begin{aligned}\mathsf{P}(\Theta > t) &\le \sum_{k=1}^\infty \mathsf{P}\left(\sup_{i\in\Theta_{n_k}} \frac{|s_i|}{\chi(V(i,\lambda))} > t\right) \le \sum_{k=1}^\infty \mathsf{P}\left(\sup_{i\in\Theta_{n_k}} |s_i| > t\chi(V(\alpha_k,\lambda))\right)\\ &\le 2\sum_{k=1}^\infty \mathsf{P}\left(s_{\beta_k}^* > t\chi(V(\alpha_k,\lambda))\right) \le 4\sum_{k=1}^\infty \mathsf{P}(s_{\beta_k} > t\chi(V(\alpha_k,\lambda)))\\ &\le 4\sum_{k=1}^\infty (\ln V(\alpha_k,\lambda))^{-t^2/\lambda} \le 4(\ln\lambda)^{-t^2/\lambda}\sum_{k=1}^\infty (n_k)^{-t^2/\lambda}\\ &\le 4(\ln\lambda)^{-t^2/\lambda}\zeta\left(\frac{t^2}{\lambda}\right),\end{aligned} \tag{22}$$

where $\zeta(s)$ is the Riemann zeta function.

Further

$$\mathsf{E}\,\Theta^r = r\int_0^\infty t^{r-1}\,\mathsf{P}(\Theta > t)\,dt \le (2\lambda)^{r/2} + r\int_{\sqrt{2\lambda}}^\infty t^{r-1}\,\mathsf{P}(\Theta > t)\,dt \tag{23}$$

(see [11], Chapter V, §6). Now we substitute estimate (22) into the latter integral

$$\begin{aligned} r\int_{\sqrt{2\lambda}}^\infty t^{r-1}\,\mathsf{P}(\Theta>t)\,dt &\le 4r\int_{\sqrt{2\lambda}}^\infty t^{r-1}(\ln\lambda)^{-t^2/\lambda}\zeta\left(\frac{t^2}{\lambda}\right)\,dt\\ &\le 4r\zeta(2)\int_{\sqrt{2\lambda}}^\infty t^{r-1}(\ln\lambda)^{-t^2/\lambda}\,dt \le \frac{\pi^2}{3}r\left(\frac{\lambda}{\ln\ln\lambda}\right)^{r/2}\Gamma\left(\frac{r}{2}-1\right),\end{aligned}$$

whence (20) follows by (21). □

3. The law of the iterated logarithm for identically distributed random elements

This section contains some sharpening of the results of [8].

Let (X_n) be a sequence of independent copies of a random element X such that $\mathsf{E}\,X = 0$, and let $S_n = \sum_{i=1}^n X_i$. We say that a random element X satisfies the order law of the iterated logarithm if (3) holds with $b_n \equiv 1$ and $V_n \equiv n$.

It is known (see [8]) that a random element X satisfies the order law of the iterated logarithm in a q-concave Banach lattice, $1 \le q < \infty$, if

$$\Delta_\psi X \quad \text{exists,} \tag{24}$$

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where $\psi(t) = t^2$ for $1 \le q < 2$, and $\psi(t) = |t|^q \ln(1 + |t|)$ for $2 \le q < \infty$ (here $\Delta_\psi X$ is the mean ψ-deviation of the random element X (see [7])).

There exists a random element in the space ℓ_2 for which condition (24) cannot be weakened if $q = 2$ (see [8]). This is the case for $1 \le q < 2$ also, in view of the one-dimensional law of the iterated logarithm.

In contrast, condition (24) can be improved for the case of $q > 2$.

Proposition 2. *Let $2 < q < \infty$, and let B be a separable q-concave Banach lattice. Assume that X is a random element taking values in B and such that $\mathsf{E}\, X = 0$. If $\Delta_q X$ exists, then*

$$\mathsf{E}\left\|\sup_{n\ge1}\frac{|S_n|}{\chi(n)}\right\|^q \le C_q \|\Delta_q X\|^q$$

and X satisfies the order law of the iterated logarithm.

To prove Proposition 2 we use the argument of [8] completed by the following auxiliary result.

Lemma 3. *Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables. If $r > 1$, then*

$$\mathsf{E}\sup_{n\ge1}\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i\right|^r \le C_r\,\mathsf{E}\,|\xi|^r.$$

The last lemma is a particular case of Lemma 2 in [7].

Assume that there exists a random variable $\tau \in L_2(\Omega)$ and an element $u \in B$ such that

$$|X| \le \tau \cdot u. \tag{25}$$

It is known that condition (25) is sufficient for the existence of the mean square deviation ΔX of the random element X (see [7]).

Proposition 3. *Assume that B is a separable Banach lattice, X is a random element with values in B, and $\mathsf{E}\, X = 0$.*

(i) *If B does not contain ℓ_∞^n uniformly and there exists a random variable $\tau \in L_2(\Omega)$ and an element $u \in B$ such that inequality (25) holds, then the random element X satisfies the order law of the iterated logarithm, and moreover*

$$\left(\mathsf{E}\left\|\sup_{n\ge1}\frac{|S_n|}{\chi(n)}\right\|^p\right)^{1/p} \le C(p,B)\cdot\Delta(\tau)\cdot\|u\|$$

for $0 < p < 2$.

(ii) *If the assumptions of assertion (i) hold with $\tau \in L_\psi(\Omega)$ and $\psi(t) = |t^2|\ln(1+|t|)$, then*

$$\left(\mathsf{E}\left\|\sup_{n\ge1}\frac{|S_n|}{\chi(n)}\right\|^2\right)^{1/2} \le C(B)\cdot\Delta_\psi(\tau)\cdot\|u\|.$$

(iii) *If the assumptions of assertion (i) hold with $\tau \in L_p(\Omega)$ and $p > 2$, then*

$$\left(\mathsf{E}\left\|\sup_{n\ge1}\frac{|S_n|}{\chi(n)}\right\|^p\right)^{1/p} \le C(p,B)\cdot\Delta_p(\tau)\cdot\|u\|.$$

(iv) *If a Banach lattice B uniformly contains ℓ_∞^n, then there exists a random element X in B and an element $u \in B$ such that*

$$|X| \le u \quad a.s. \tag{26}$$

and the order law of the iterated logarithm does not hold for the random element X.

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Proof of Proposition 3. Assertions (i) and (ii) are proved in [8]. Assertion (iii) follows from the results of [8] and Lemma 2.

Proof of (iv). A counterexample to the central limit theorem for Banach spaces uniformly containing ℓ_∞^n is constructed in the paper [12]. A modification of the method of [12] allows one to construct an example for assertion (iv) (see also [13]).

If the space B uniformly contains ℓ_∞^n, then for all n and $\delta > 0$ there exists a sequence $(x_i)_1^n$ of pairwise disjoint elements in B such that

$$\max_{1\le i\le n} |a_i| \le \left\| \sum_{i=1}^n a_i x_i \right\| \le (1+\delta) \max_{1\le i\le n} |a_i|$$

for every family of numbers $(a_i)_1^n$ (see [4, p. 91]). This implies (see [14])) that there exists a sequence of positive elements (x_i) such that

$$\|x_i\| = (\ln\ln(i+7))^{-1}$$

and the series $\sum_{i=1}^\infty x_i$ converges unconditionally.

Put

$$u = \sum_{i=1}^\infty x_i, \qquad X = \sum_{i=1}^\infty \xi_i x_i,$$

where (ξ_i) is a sequence of independent random variables such that

$$\mathsf{P}(\xi_i = 1) = \mathsf{P}(\xi_i = -1) = \frac{1}{\ln(i+7)}, \qquad \mathsf{P}(\xi_i = 0) = 1 - \frac{2}{\ln(i+7)}.$$

It is obvious that

$$|X| \le \sum_{i=1}^\infty |x_i| = \sum_{i=1}^\infty x_i = u \quad \text{a.s.},$$

that is, condition (26) holds.

Let (X_k) and $(\xi_{ki})_{k=1}^\infty$ be sequences of independent copies of the random elements X and ξ_i, respectively,

$$X_k = \sum_{i=1}^\infty \xi_{ki} x_i.$$

Assume that the random element X satisfies the order law of the iterated logarithm. Then the inequality

$$\sup_{n\ge1} \left\| \frac{S_n}{\chi(n)} \right\| \le \left\| \sup_{n\ge1} \frac{|S_n|}{\chi(n)} \right\| \quad \text{a.s.}$$

implies

$$\sup_{n\ge1} \left\| \frac{S_n}{\chi(n)} \right\| < \infty \quad \text{a.s.} \tag{27}$$

Put

$$N_n = [\exp(n\ln n + \ln\ln n)],$$

$$W_{ni} = \bigcap_{k=1}^n \{\xi_{ki} = 1\}, \qquad W_n = \bigcup_{i=n}^{N_n} W_{ni}.$$

It is shown in [13] that

$$\liminf_{n\to\infty} \mathsf{P}(W_n) \ge 1 - \frac{1}{e}. \tag{28}$$

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Further we apply estimate (27) and Levy's inequality [15],

$$0 = \lim_{n\to\infty} \mathsf{P}\left(\sup_{m\ge 1} \frac{\|S_m\|}{\chi(m)} > \ln n\right) \ge \liminf_{n\to\infty} \mathsf{P}\left(\frac{\|S_n\|}{\chi(n)} > \ln n\right) \tag{29}$$
$$\ge \frac{1}{2} \liminf_{n\to\infty} \mathsf{P}\left(\max_{n\le i\le N_n} \left\|\sum_{k=1}^{n} \xi_{ki} x_i\right\| > \ln n\right).$$

If $\omega \in W_n$, then

$$\max_{n\le i\le N_n} \left\|\frac{\sum_{k=1}^{n} \xi_{ki} x_i}{\chi(n)}\right\| \ge \frac{n}{\chi(n)\ln\ln(N_n+7)} \sim \frac{n^{1/2}}{\sqrt{2\ln\ln n}\,\ln n} \ge \ln n$$

for sufficiently large n, whence we get that

$$\liminf_{n\to\infty} \mathsf{P}(W_n) = 0$$

by (29). The latter equality contradicts (28). □

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Received 1/SEP/2000

Translated by OLEG KLESOV

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