Теорія Ймовір. та Матем. Статист.
Вип. 68, 2003

Theor. Probability and Math. Statist.
S 0094-9000(04)00599-X
Article electronically published on May 25, 2004

ASYMPTOTIC BEHAVIOR OF INCREMENTS OF RANDOM FIELDS

UDC 519.21

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Abstract. Some results on the asymptotic behavior of increments of a d-dimensional random field are proved. Let N and $a_N \in \{1, 2, \dots\}$ be fixed and let $S_N^\star$ be the maximum increment of a d-dimensional random field of independent identically distributed random variables evaluated for d-dimensional rectangles $(i, j] = \{k : i < k \le j\}$ such that $|j| \le N$ and $|j - i| = a_N$. Denote also by S_N the maximum increment evaluated for rectangles such that $|j - i| \le a_N$.

We determine the asymptotic almost sure behavior of random variables S_N and $S_N^\star$. Steinebach (1983) proved a similar result for the case of rectangles belonging to the cube $(0, N^{1/d}]$ (of volume N) and under the condition that $a_N = O(N^\delta)$ as $N \to \infty$ for *all* $\delta \in (0, 1)$. Note that the sequence S_N is monotone in this case.

We also consider the cases where $a_N \sim C \log N$ or $a_N = O(\log N)$.

1. Introduction

Let $\mathbf{N}_0^d$ be the d-dimensional ($d \ge 1$) lattice whose elements have nonnegative integer coordinates. We introduce a partial order on $\mathbf{N}_0^d$:

$$i \le j \quad \text{for} \quad i = (i_1, \dots, i_d) \quad \text{and} \quad j = (j_1, \dots, j_d)$$

if and only if

$$i_k \le j_k \quad \text{for all } k = 1, \dots, d.$$

Let

$$(i, j] = \{k \in \mathbf{N}_0^d, i < k \le j\}, \qquad i, j \in \mathbf{N}_0^d,\ i \le j,$$

be a d-dimensional rectangle. The volume of the d-dimensional rectangle $(0, n]$ such that $n = (n_1, \dots, n_d)$ and $n_k \in \mathbf{N} \cup \{0\}$ is denoted by $|n| = \prod_{i=1}^d n_i$.

Consider a sequence X_n, $n \in \mathbf{N}_0^d$, of independent identically distributed random variables depending on d indices and with moment generating function $\phi(t) = \mathsf{E} \exp(tX_n)$ such that

$$\phi(t) < \infty \quad \text{for some } t > 0. \tag{1}$$

We further assume that

$$\mathsf{E} X_n = 0, \qquad \operatorname{Var} X_n = 1. \tag{2}$$

Consider a function $a(t)$, $t \ge 1$, $t \in \mathbf{R}$, such that

$$1 \le a(t) \le t, \qquad a(t) \text{ and } \frac{t}{a(t)} \text{ do not decrease.} \tag{3}$$

Put $a_N = [a(N)]$, where $[\cdot]$ stands for the integer part of a real number.

2000 *Mathematics Subject Classification.* Primary 60F15; Secondary 60K05.

Supported in part by the Ministry of Education of the Russian Federation under grants N E00-1.0-82 and "Leading scientific school" # 00-15-96019.

Let $S_N^\star$ be the maximum increment of a d-dimensional field of independent identically distributed random variables evaluated for d-dimensional rectangles

$$\{k\colon i < k \le j\}$$

such that $|j| \le N$ and $|j-i| = a_N$. If the rectangles satisfy the inequality $|j-i| \le a_N$ instead of the equality $|j-i| = a_N$, then the maximum increment is denoted by S_N:

$$S_{(i,j]} = \sum_{i<k\le j} X_k, \qquad (i,j] \subset \mathbf{N}_0^d,$$

$$S_N^\star = \max_{\substack{|j|\le N,\\ |j-i|=a_N}} S_{(i,j]}, \qquad S_N = \max_{\substack{|j|\le N,\\ |j-i|\le a_N}} S_{(i,j]}.$$

2. Some historical remarks

Csörgő and Révész [3] obtained limit theorems for increments of partial sums of independent identically distributed random variables in the case of $d = 1$ by using corresponding results for the Wiener process and the strong invariance principle. They also considered the asymptotic behavior of increments of the two parameter Wiener process.

Below we recall some results for random variables with multiindices.

Theorem 2.1 (Steinebach [1]). *Let X_n, $n \in \mathbf{N}_0^d$, be independent identically distributed random variables. Assume that conditions* (1) *and* (2) *hold. If*

$$\frac{a_N}{\log N} \to \infty, \qquad N \to \infty, \tag{4}$$

$$\frac{a_N}{N^\delta} \to 0, \quad N \to \infty, \quad \text{for all } \delta > 0, \tag{5}$$

$$a_N \quad \text{does not decrease}, \tag{6}$$

$$\frac{a_N}{N^{\delta_0}} \quad \text{does not increase for some } \delta_0, \tag{7}$$

then

$$\lim_{N\to\infty} \frac{D(N,a_N)}{(2a_N \log N)^{1/2}} = \lim_{N\to\infty} \frac{D^\star(N,a_N)}{(2a_N \log N)^{1/2}} = 1 \qquad a.s.,$$

where $D(N,a_N)$ is the maximum of increments evaluated on d-dimensional rectangles J, $|J| \le a_N$, belonging to the d-dimensional cube $(0, [N^{1/d}] \cdot e]$, and $e = (1,\dots,1)$:

$$D(N,a_N) = \max_{\substack{J \subset \left(0,[N^{1/d}]\cdot e\right],\\ |J|\le a_N}} S_J, \qquad D^\star(N,a_N) = \max_{\substack{J \subset \left(0,[N^{1/d}]\cdot e\right],\\ |J| = a_N}} S_J.$$

Note that the volume of the cube $\left(0, [N^{1/d}] \cdot e\right]$ is less than or equal to $N - dN^{1-1/d}$.

Theorem 2.2 (Steinebach [1]). *Assume that conditions* (5)–(7) *hold, and moreover*

$$a_N \sim C \log N, \qquad N \to \infty, \tag{8}$$

where C is a unique solution of the equation

$$\inf_{t\in[0,t_0)} \frac{\phi(t)}{\exp(\alpha t)} = \exp(-1/C) \tag{9}$$

for

$$\alpha \in \left\{ \frac{\phi'(t)}{\phi(t)} \colon t \in [0,t_0) \right\}$$

and $t_0 = \sup\{t\colon \phi(t) < \infty\}$. Then

$$\lim_{N\to\infty} \frac{D(N,a_N)}{(C \cdot a_N \log N)^{1/2}} = \lim_{N\to\infty} \frac{D^\star(N,a_N)}{(C \cdot a_N \log N)^{1/2}} = \alpha \qquad a.s.$$

Theorem 2.3 (Steinebach [1])**.** *Assume that* $\phi(t) < \infty$ *for all* $t > 0$*,*

$$\lim_{\tau\to+\infty} \log \phi''(\tau) = \gamma^2 > 0, \tag{10}$$

and the sequence a_N *satisfies conditions* (5)–(7) *and*

$$\frac{a_N}{\log N} \to 0, \qquad N \to \infty. \tag{11}$$

Then

$$\lim_{N\to\infty} \frac{D(N,a_N)}{(2\gamma^2 a_N \log N)^{1/2}} = \lim_{N\to\infty} \frac{D^\star(N,a_N)}{(2\gamma^2 a_N \log N)^{1/2}} = 1 \qquad a.s.$$

3. Main results

The main goal of this paper is to study the asymptotic behavior of the maximum of increments of a d-dimensional random field evaluated for rectangles whose volumes do not exceed a_N under the condition that all of them belong to a rectangle of volume N. In other words, we study the asymptotic behavior of random variables S_N and $S_N^\star$ introduced in Section 1.

Theorem 3.1. *Let conditions* (1)–(3) *hold. Assume that*

$$\frac{a_N}{\log N} \to \infty, \qquad N \to \infty. \tag{12}$$

Put $\delta_N = \{2a_N(\varepsilon_N + \beta_N)\}^{-1/2}$*, where*

$$\varepsilon_N = \log_+ \frac{N}{a_N}, \qquad \beta_N = d \log\log N,$$

and $\log_+ x = \log(x \vee e)$*. Then*

$$\limsup_{N\to\infty} S_N^\star \delta_N = \limsup_{N\to\infty} S_N \delta_N = 1 \qquad a.s.$$

If additionally

$$\frac{\varepsilon_N}{\beta_N} \to \infty, \qquad N \to \infty, \tag{13}$$

then

$$\lim_{N\to\infty} S_N^\star \delta_N = \lim_{N\to\infty} S_N \delta_N = \lim_{N\to\infty} D(N,a_N)\delta_N = \lim_{N\to\infty} D^\star(N,a_N)\delta_N = 1 \qquad a.s.$$

Remark 3.1. Condition (13) holds for all sequences $\{a_N\}$ such that

$$a_N = O\left(N^{\delta_0}\right), \quad N \to \infty, \quad \text{for some } \delta_0 \in (0,1). \tag{14}$$

Theorem 3.2. *Assume that*

$$a_N \sim C \log N, \qquad N \to \infty, \tag{15}$$

where the constant C *is defined in Theorem 2.2 (see* (9)*). Let*

$$\delta_N = \{2Ca_N \log N\}^{-1/2}.$$

Then

$$\lim_{N\to\infty} S_N \delta_N = \lim_{N\to\infty} S_N^\star \delta_N = \lim_{N\to\infty} D^\star(N,a_N)\delta_N = \lim_{N\to\infty} D(N,a_N)\delta_N = \alpha \qquad a.s.$$

Theorem 3.3. *Assume that* $\phi(t) < \infty$ *for all* $t > 0$,

$$\lim_{\tau \to +\infty} \log \phi''(\tau) = \gamma^2 > 0, \tag{16}$$

and

$$\frac{a_N}{\log N} \to 0, \qquad N \to \infty. \tag{17}$$

Put

$$\delta_N = \left\{2\gamma^2 a_N \log N\right\}^{-1/2}.$$

Then

$$\lim_{N\to\infty} S_N \delta_N = \lim_{N\to\infty} S_N^\star \delta_N = \lim_{N\to\infty} D^\star(N, a_N)\delta_N = \lim_{N\to\infty} D(N, a_N)\delta_N = 1 \qquad a.s.$$

4. Auxiliary results

We need some combinatorial estimates. In what follows let $d \geq 2$. We denote by P_{d-1} polynomials of degree $d-1$.

Lemma 4.1. *Put*

$$B_d(n) = \operatorname{Card}\left\{(0, n] \subset \mathbf{N}_0^d \colon |n| = N\right\}.$$

Then for all $\varepsilon > 0$

$$\sum_{n \leq N} B_d(n) = N \cdot P_{d-1}(\log N) + O\left(N^{(d-1)/(d+2)+\varepsilon}\right). \tag{18}$$

Remark 4.1. It is clear that $B_d(N)$ equals the total number of representations of the number N as a product of d integer factors. The proof of (18) can be found in [4] (also see [6]).

We also need an estimate for some coverings $A_{N,a_N}^{\square}$ of the set

$$A(N) = \bigcup_{|n| \leq N} (0, n]$$

by d-dimensional rectangles whose volumes are equal to a_N.

Lemma 4.2. *Let* $N \geq 1$ *and a vector* $i^0 \in \mathbf{N}_0^d$ *be such that* $|i^0| = a_N$. *Further let* $A_{N,a_N}^{\square}$ *be the minimal covering of* $A(N)$ *by disjoint rectangles obtained by parallel translations of the vector* i^0. *In other words,*

$$A_{N,a_N}^{\square} = \left\{\left(j(l) - i^0, j(l)\right] \subset A(N) \colon l \in \mathbf{N}_0^d,\ j_k(l_k) = l_k i_k^0,\ l_k \in \left\{1, \dots, \left[\frac{n_k}{i_k^0}\right] + 1\right\}\right\},$$

where n_k *is the maximum of the* k*th coordinates of vectors* n *such that* $i^0 \leq n$ *and* $|n| = N$, $k = 1, \dots, d$. *Then*

$$\operatorname{Card}\left(A_{N,1}^{\square}\right) = \sum_{n \leq N} B_d(n) \leq N \cdot P_{d-1}(\log N),$$

$$\operatorname{Card}\left(A_{N,a_N}^{\square}\right) \leq \frac{N}{a_N} \cdot P_{d-1}\left(\log_+ \frac{N}{a_N}\right).$$

Proof of Lemma 4.2. First we consider the case of $a_N = 1$. Applying Lemma 4.1 we obtain

$$\operatorname{Card}\left(A_{N,1}^{\square}\right) = \sum_{n \leq N} B_d(n) = N \cdot P_{d-1}(\log N) + O\left(N^{(d-1)/(d+2)+\varepsilon}\right).$$

In the case of $a_N > 1$ we consider the mapping $x_k = x_k / i_k^0$, $k = 1, \dots, d$. If a rectangle of $A_{N,a_N}^{\square}$ is of volume a_N, then this mapping transforms it into a cube of unit volume. Note also that the set $A(N)$ is transformed into the set $A(N/a_N)$ under this mapping, and thus $\operatorname{Card} A^{\square}_{N,a_N} = \operatorname{Card} A^{\square}_{N/a_N,1}$ and $\operatorname{Card}\bigl(A^{\square}_{N,a_N}\bigr) \le (N/a_N)P_{d-1}(\log_+(N/a_N))$. $\square$

The following result allows one to estimate the cardinality of the collection of rectangles of $A(N)$ whose volumes do not exceed a_N.

Lemma 4.3. *Let*

$$A_N = \bigl\{(i,j] \subset A(N)\colon |i-j| \le a_N\bigr\}.$$

Then

$$\operatorname{Card}(A_N) \le N \cdot a_N \cdot P_{d-1}(\log_+ a_N) \cdot P_{d-1}(\log N)$$

for all sufficiently large a_N.

Proof of Lemma 4.3. The total number of rectangles of $(0,l]$ whose volumes do not exceed N is equal to $\operatorname{Card}\bigl(A^{\square}_{N,1}\bigr)$. For any given l, there are at most $\operatorname{Card}\bigl(A^{\square}_{a_N,1}\bigr)$ rectangles $(l,k]$ such that $|k-l| = a_N$. Hence

$$\operatorname{Card}(A_N) = \operatorname{Card}\left(A^{\square}_{N,1}\right)\cdot \operatorname{Card}\left(A^{\square}_{a_{N,1}}\right) \le N\cdot a_N\cdot P_{d-1}(\log_+ a_N)\cdot P_{d-1}(\log N). \qquad \square$$

Denote by $\mathbf{R}_0^d$ the space of d-dimensional vectors with nonnegative real coordinates.

Lemma 4.4. *For all $\varrho > 1$ and all natural numbers $a_N > \varrho$ there is a finite set*

$$U_\varrho(a_N) \subset \mathbf{R}_0^d$$

such that

(1) *$|u| = \varrho^{d-1}\cdot a_N$ for all $u \in U_\varrho(a_N)$;*
(2) *for all $i \in \mathbf{N}_0^d$, $|i| \le a_N$, there exists $u \in U_\varrho(a_N)$ such that $i \le u$;*
(3) *$\operatorname{Card}(U_\varrho(a_N)) = \bigl[(\log^{d-1} a_N)/(\log^{d-1}\varrho)\bigr]$.*

Proof. Let $\varrho > 1$. For $i \in \mathbf{N}_0^d$ such that $|i| \le a_N$ we put

$$\begin{aligned} U_\varrho(a_N) = \Biggl\{ u_k = \left(u_{k_1},\dots,\frac{a_N}{\prod_{s=1}^{d-1} u_{k_s}}\right) = \left(\varrho^{k_1+1},\dots,\varrho^{k_{d-1}+1},\frac{a_N}{\varrho^{\sum_{s=1}^{d-1}k_s}}\right),\\ k_s \in \left\{1,\dots,\left[\frac{\log a_N}{\log\varrho}\right]\right\},\ s=1,\dots,d-1,\ |u_k| = \varrho^{d-1}\cdot a_N \Biggr\}. \end{aligned}$$

There exists $k = (k_1,\dots,k_{d-1}) \in \mathbf{N}_0^{d-1}$ such that $i \le u_k$. Indeed, let $k_s = [\log i_s/\log\varrho]$. Then $\varrho^{k_s} \le i_s < \varrho^{k_s+1}$, $s = 1,\dots,d-1$,

$$i_d \le \frac{a_N}{\sum_{s=1}^{d-1} i_s} \le \frac{a_N}{\varrho^{\sum_{s=1}^{d-1}k_s}}.$$

It is clear that all the assumptions of Lemma 4.3 are satisfied. $\square$

Lemma 4.5. *For all $\varrho > 1$, $0 < \nu < 1 - 1/\varrho$, and for all natural numbers $N > \rho$ one can construct a finite set $V_{\varrho,\nu}(N,a_N) \subset \mathbf{R}_0^d$ such that*

(1) *for all $v \in V_{\varrho,\nu}(N,a_N)$, the volume of v is equal to $\varrho^{2d-1}\cdot a_N$;*
(2) *for all $(i-j,i] \subset \mathbf{N}_0^d$, $|j| \le a_N$ and $|i| \le N$, there exists $v \in V_d(N,a_N)$ such that $(i-j,i] \subset v$;*
(3) *we have*

$$\operatorname{Card}\bigl(V_{\varrho,\nu}(N,a_N)\bigr) \le \frac{\log_N^{d-1}}{\log^{d-1}\varrho}\cdot\frac{N}{\nu^d\varrho^d a_N}\cdot P_{d-1}\left(\log_+\frac{N}{\nu^d\varrho^d a_N}\right).$$

Proof. The set $V_{\varrho,\nu}(N,a_N)$ is constructed in the following way:

$$V_{\varrho,\nu}(N,a_N)=\left\{v=\left(w(l)-\varrho^d\cdot u,w(l)\right],u\in U_\varrho(a_N),\,l\in\mathbf{N}_0^d,\right.$$
$$\left.w_k(l_k)=\nu\varrho l_k u_k,\,l_k\in\left\{1,\dots,\left[\frac{n_k}{\nu\varrho u_k}\right]+1\right\}\right\},$$

where n_k is the maximum of the kth coordinates of vectors $n\in U_\varrho(N)$ such that $u\le n$, $k=1,\dots,d$. The first assertion of the lemma is obvious.

Now we prove the second assertion. Let $(i-j,i]\subset\mathbf{N}_0^d$, $|j|\le a_N$ and $|i|\le N$. According to Lemma 4.4, there are $u\in U_\varrho(a_N)$ and $n\in U_\varrho(N)$ such that $j<u$ and $u\le n$, respectively.

Further, for any given i there are $w(l)\colon l=(l_1,\dots,l_d)\in\mathbf{N}_0^d$ such that

$$\nu\varrho(l_k-1)u_k\le i_k\le\nu\varrho l_k u_k=w_k(l_k).$$

Thus $\nu\varrho(l_k-1)u_k-u_k\le i_k-u_k\le\nu\varrho l_k u_k-u_k$. Now we check that

$$i_k-u_k\ge(\nu l_k-1)\varrho u_k.$$

This inequality holds if $\nu\varrho(l_k-1)u_k-u_k\ge(\nu l_k-1)\varrho u_k$. The latter condition is equivalent to $\nu\varrho\le\varrho-1$, which is true by construction.

Now we prove the third assertion of the lemma. Note that the cardinality of the set $V_{\varrho,\nu}(N,a_N)$ depends on the indices u and l. Since the sets of d-dimensional indices l used in the definitions of $A^{\square}_{N,a_N}$ and $V_{\varrho,\nu}(N,a_N)$ coincide,

$$\operatorname{Card}\bigl(V_{\varrho,\nu}(N,a_N)\bigr)\le\frac{\log^{d-1}a_N}{\log^{d-1}\varrho}\cdot\frac{N}{\nu^d\varrho^d a_N}\cdot P_{d-1}\left(\log_+\frac{N}{\nu^d\varrho^d a_N}\right).\qquad\square$$

5. Proofs of main results

Proof of Theorem 3.1. We split the proof into three steps.

Step 1. First we show that $\limsup S_N\delta_N\le1$ a.s. Consider an arbitrary $\varepsilon>0$. Using the above lemmas and applying Kolmogorov's inequality for random fields [5] we obtain

$$\mathsf{P}\left(\max_{(0,i]\colon|i|\le a_N}S_{(0,i]}\delta_N>1+2\varepsilon\right)\le2^d\sum_{i^0\colon i^0\in U_\varrho(a_N)}\mathsf{P}\left(S_{(0,i^0]}>(1+\varepsilon)\delta_N^{-1}\right).$$

Further

$$\begin{aligned}\mathsf{P}\bigl(S_N>(1+2\varepsilon)\delta_N^{-1}\bigr)&\le2^d\cdot\sum_{(j-i,j]\in V_{\varrho,\nu}(N,a_N)}\mathsf{P}\left(S_{(j-i,j]}>(1+\varepsilon)\delta_N^{-1}\right)\\&=2^d\cdot\sum_{(j-i,j]\in V_{\varrho,\nu}(N,a_N)}\mathsf{P}\left(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j-i,j]}>(1+\varepsilon)\delta_N^{-1}\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}\right).\end{aligned}\tag{19}$$

Note that

$$\delta_N^{-1}\cdot\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}=\sqrt{2}(\varepsilon_N+\beta_N).$$

Moreover

$$\frac{\varepsilon_N+\beta_N}{a_N}=\frac{\log N}{a_N}-\frac{\log a_N}{a_N}+\frac{d\log\log N}{a_N}\to0,\qquad N\to\infty.$$

Expanding the function

$$\log\phi\biggl(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}\cdot t\biggr)$$

in the Taylor series, we get

$$\lim_{N\to\infty} \frac{\varrho^{2d-1}\cdot a_N}{\varepsilon_N+\beta_N}\cdot \log\phi\left(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}\cdot t\right) = \frac{\varrho^{2d-1}\cdot t^2}{2}.$$

We recall a theorem due to Plachky and Steinebach [2].

Theorem 5.1. *Let $\{W_n\}_{n=1,2,\dots}$ be a sequence of random variables such that*

(1) $m_n(t) = \int \exp(tW_n)\,d\mathsf{P} < \infty$ *for all* $t \in (0, T_1)$, $T_1 > 0$,

(2) $\psi_n(t)/n \to c_0(t)$ *for all* $t \in (T_0, T_1)$, $0 \le T_0 < T_1$, *where* $\psi_n(t) = \log m_n(t)$.

Then

$$\lim(\mathsf{P}(W_n > na_n))^{1/n} = \inf_{t>0}\{\exp(c_0(t) - at)\}$$

for every sequence $\{a_n\}_{n=1,2,\dots}$, $a_n \in \mathbf{R}$, *such that*

$$a_n \to a \in A = \Big\{c_0'(h)\colon c_0'(h) \text{ exists, is right continuous,} \\ \text{and is strictly monotone for } h \in (T_0, T_1)\Big\}.$$

In what follows we need a more general result under a weaker condition than (2).

Remark 5.1. In the case under consideration

$$n = \varepsilon_N + \beta_N, \qquad W_n = \sqrt{\frac{\varepsilon_N+\beta_N}{a_N}} S_{(j-i,j]},$$

$$m_n(t) = \mathsf{E}\left(\exp\left(t\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j-i,j]}\right)\right), \qquad a_n = \sqrt{2}(1+\varepsilon),$$

$$c_2(t) = \frac{t^2}{2} = \liminf \psi_n(t)/n \le \limsup \psi_n(t)/n = \frac{\varrho^{2d-1}\cdot t^2}{2} = c_1(t).$$

Therefore condition (2) of Theorem 5.1 does not hold and the function $c_0(t)$ is not well defined. Nevertheless

$$\begin{aligned} A_2 &= \inf_{t>0}\left(\exp\left\{\frac{t^2}{2} - \sqrt{2}t(1+\varepsilon)\right\}\right) = \exp\left\{-(1+\varepsilon)^2\right\} \\ &= \liminf \mathsf{P}\left(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j-i,j]} > (1+\varepsilon)\sqrt{2}(\varepsilon_N+\beta_N)\right)^{\frac{1}{\varepsilon_N+\beta_N}} \\ &\le \limsup \mathsf{P}\left(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j-i,j]} > (1+\varepsilon)\sqrt{2}(\varepsilon_N+\beta_N)\right)^{\frac{1}{\varepsilon_N+\beta_N}} \\ &= \inf_{t>0}\left(\exp\left\{\frac{\varrho^{2d-1}\cdot t^2}{2} - \sqrt{2}t(1+\varepsilon)\right\}\right) = \exp\left\{-(1+\varepsilon)^2/\varrho^{2d-1}\right\} = A_1. \end{aligned}$$

Proof. Assume the opposite. Put $P_n = (\mathsf{P}(W_n > na_n))^{1/n}$. Note that the sequence $\psi_n'(t)/n$ is uniformly bounded for $t \in [0, T]$ and all sufficiently small $T > 0$. Indeed,

$$\psi_n'(t)/n \le \varrho^{2d-1} t \frac{\mathsf{E}\, Xe^{hX}}{h\,\mathsf{E}\, e^{hX}}, \qquad h = t\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}} \to 0.$$

Moreover

$$\begin{aligned} E(Xe^{hX}) &= \int_{-\infty}^0 x\left(e^{hx}-1\right)\,dF(x) + \int_0^\infty x\left(e^{hx}-1\right)\,dF(x) \\ &< h\left(\int_{-\infty}^0 x^2\,dF(x) + \int_0^\infty x^2 e^{hx}\,dF(x)\right), \\ &e^y - 1 < ye^y, \qquad e^{-y}-1 > -y, \qquad y > 0 \end{aligned}$$

by $\mathsf{E}\,X = 0$ and $\operatorname{Var} X = 1$. Therefore $\mathsf{E}(Xe^{hX}) \le c \cdot h$, $t \in [0, T]$, implying that $\psi_n'(t)/n$ is uniformly bounded.

Now we assume that there exists a subsequence P_{n_k} such that

$$\lim P_{n_k} \notin [A_2, A_1].$$

Consider an infinite collection of increasing uniformly bounded functions $\Psi = \{\psi_{n_k}'/n_k\}$, $\psi_{n_k}'/n_k \colon [0, T] \mapsto \mathbf{R}$. By Lemma 2, 4.VIII in [7] there exists a subcollection

$$\left\{\psi_{n_{k_m}}'/n_{k_m}\right\} \subset \Psi$$

such that the limit

$$\lim_{m\to\infty} \psi_{n_{k_m}}'/n_{k_m}(t) = c_\star(t)$$

exists for all $t \in [0, T]$ and the function $c_\star(t)$ is nondecreasing and right continuous. Moreover the function $c_\star(t)$ is strictly increasing, since the functions $\psi_n''(t)/n$ are positive and bounded away from zero and infinity for all sufficiently large n and $t \in [0, T]$. Put

$$c_0(t) = \lim_{n_{k_m}\to\infty} \int_0^t \psi_{n_{k_m}}'(x)/n_{k_m}\, dx.$$

By the Lebesgue theorem (see Theorem 16.3 in [8]) we get $c_0(t) = \int_0^t c_\star(x)\, dx$, whence $c_0'(t) = c_\star(t)$.

Thus by the Plachky and Steinebach theorem

$$\lim \Big(\mathsf{P}(W_{n_{k_p}} > n_{k_p} a)\Big)^{1/n_{k_p}} = \inf_{t>0}\big\{\exp(c_0(t) - at)\big\} \in [A_2, A_1]$$

and we arrive at a contradiction. $\square$

Using Lemma 4.5, Remark 5.1, and relation (19) we get

$$\begin{aligned}
\mathsf{P}\big(S_N\delta_N > 1 + 2\varepsilon\big) &\le 2^{d-1}\operatorname{Card}(V_\varrho(N, a_N)) \exp\left(-\frac{(1+\varepsilon/2)^2}{\varrho^{2d-1}}(\varepsilon_N + \beta_N)\right) \\
&\le \frac{\log^{d-1} a_N}{\nu^d \varrho^d \log^{d-1}\varrho} \cdot \left(\frac{N}{a_N}\right)^{1-(1+\varepsilon/2)^2/\varrho^{2d-1}} \\
&\quad \times P_{d-1}\left(\log_+ \frac{N}{\nu^d\varrho^d a_N}\right) \cdot \log^{-d(1+\varepsilon/2)^2/\varrho^{2d-1}} N.
\end{aligned}$$

Assume that $(1 + \varepsilon/2)^2 > \varrho^{2d-1}$. Note that the sequence

$$\left(\frac{N}{a_N}\right)^{1-(1+\varepsilon/2)^2/\varrho^{2d-1}} \cdot P_{d-1}\left(\log_+ \frac{N}{\nu^d\varrho^d a_N}\right)$$

is bounded.

Let $\vartheta > 1$ and $N = [\vartheta^r]$. The series

$$\sum_{r=1}^\infty \mathsf{P}\left(S_{[\vartheta^r]}\delta_{[\vartheta^r]} > 1 + 2\varepsilon\right)$$

converges, since its general term is of order

$$r^{-1-d\left((1+\varepsilon/2)^2/(\varrho^{2d-1})-1\right)}.$$

Letting $\varepsilon \to 0$ and $\varrho \to 1$ and applying the Borel–Cantelli lemma we obtain

$$\limsup S_{[\vartheta^r]}\delta_{[\vartheta^r]} \le 1 \quad \text{a.s.} \quad \text{for all } \vartheta > 1.$$

For any $N \in \mathbf{N}$, there exists r such that $[\vartheta^r] \le N < [\vartheta^{r+1}]$ and

$$S_N^\star \delta_N \le S_N \delta_N \le S_{[\vartheta^{r+1}]}\delta_{[\vartheta^{r+1}]}\frac{\delta_{[\vartheta^r]}}{\delta_{[\vartheta^{r+1}]}}, \qquad \limsup \frac{\delta_{[\vartheta^r]}}{\delta_{[\vartheta^{r+1}]}} = \sqrt{\vartheta}, \quad r \to \infty.$$

Thus

$$\limsup S_N^\star \delta_N \le \limsup S_N \delta_N \le 1 \quad \text{a.s.}$$

by letting $\vartheta \to 1$. This completes the proof of Step 1.

Step 2. We show that

$$\limsup_{N\to\infty} S_N \ge \limsup_{N\to\infty} S_N^\star \ge 1 \qquad \text{a.s.}$$

Choose ς such that $1 > \varsigma > 0$ and prove that

$$\limsup_{N\to\infty} S_N^\star \ge 1 - \varsigma \qquad \text{a.s.} \tag{20}$$

It is evident that the limit

$$\lim_{t\to\infty} a(t)/t = p$$

exists. First we consider the case of $p = 0$. We construct the sequence $\{N_k\}_{k\in\mathbf{N}}$ as follows: for $\varsigma/8 > \varepsilon > 0$ and

$$0 < \lambda < 2^{1-d}\frac{\varsigma/4}{1-\varsigma/4}$$

we choose N_1 in such a way that $a_N/N < \lambda$ for all $N > N_1$, and put

$$[N_k/\varepsilon] = N_{k+1} - a_{N_{k+1}}.$$

Now we construct a set of disjoint d-dimensional rectangles belonging to the domain $A(N_k) \setminus A(N_{k-1})$:

$$\begin{aligned}\Im_{N_k} = \Bigg\{ J = (j,i] \in \mathbf{R}_0^d\colon i = \left(\varrho^{s_1}, \dots, \varrho^{s_{d-1}} \frac{N_k}{\varrho^{\sum_{l=1}^{d-1} s_l}}\right), s_l \in \left\{1, \dots, \left[\frac{\log N_k}{\log \varrho}\right]\right\}, \\ s \in \mathbf{N}_0^{d-1},\, l = 1, \dots, d-1,\, j = \left(\varrho^{s_1-1}, \dots, \varrho^{s_{d-1}-1}, \frac{N_{k-1}}{\varrho^{\sum_{l=1}^{d-1} s_l - d + 1}}\right)\Bigg\},\end{aligned}$$

where

$$\varrho = 1 + (a_{N_k}/(N_k - a_{N_k}))^{1/(d-1)}.$$

Note that $|j| = N_{k-1}$ and $|i| = N_k$. Our current goal is to estimate $|i-j|$ for $(j,i] \in \Im_{N_k}$.

We have

$$
\begin{aligned}
|i-j| &= (\varrho-1)^{d-1}\left(\frac{N_k}{\varrho^{d-1}} - N_{k-1}\right)\\
&= \frac{a_{N_k}}{N_k - a_{N_k}}\left(\frac{N_k}{\left(1+\left(a_{N_k}/(N_k-a_{N_k})\right)^{1/(d-1)}\right)^{d-1}} - N_{k-1}\right)\\
&= \frac{a_{N_k}}{N_k - a_{N_k}}\left(\frac{N_k-a_{N_k}}{\left((1-a_{N_k}/N_k)^{1/(d-1)}+\left(a_{N_k}/N_k\right)^{1/(d-1)}\right)^{d-1}} - N_{k-1}\right)\\
&\geq a_{N_k}\left(\frac{1}{\left(1+\sum_{l=1}^{d-2} C_{d-1}^{l}(a_{N_k}/N_k)^{l/(d-1)}(1-a_{N_k}/N_k)^{(d-1-l)/(d-1)}\right)} - 2\varepsilon\right)\\
&> a_{N_k}\left(\frac{1}{1+\lambda 2^{d-1}} - 2\varepsilon\right) > a_{N_k}\left(\frac{1}{1+(\varsigma/4)/(1-\varsigma/4)} - 2\varepsilon\right) > a_{N_k}(1-\varsigma/2).
\end{aligned}
$$

Using the inequality $\log(1+x) < x$, we prove that

$$
\operatorname{Card}(\Im_{N_{k,1}}) \geq \frac{\log^{d-1} N_k}{2\log^{d-1}\left(1+\left(a_{N_k}/(N_k-a_{N_k})\right)^{1/(d-1)}\right)^{d-1}} \geq \frac{N_k - a_{N_k}}{2a_{N_k}}\log^{d-1} N_k.
$$

By Remark 5.1 we obtain $n = \varepsilon_N + \beta_N$, $(j,i] \in \Im_{N_k}$,

$$
W_n = \sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j,i]}, \qquad m_n(t) = \mathsf{E}\left(\exp\left(t\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}S_{(j,i]}\right)\right),
$$

$$
\lim_{N\to\infty} \frac{(1-\varsigma/2)\cdot a_N}{\varepsilon_N+\beta_N}\cdot \log\phi\left(\sqrt{\frac{\varepsilon_N+\beta_N}{a_N}}\cdot t\right) = \frac{(1-\varsigma/2)\cdot t^2}{2},
$$

$$
c_2(t) = \frac{(1-\varsigma/2)t^2}{2} = \liminf \psi_n(t)/n.
$$

Therefore

$$
\begin{aligned}
\mathsf{P}\left(\max_{J\in\Im_{N_k}} S_J\delta_{N_k} \geq 1-\varsigma\right) &\geq 1-\left(1-\mathsf{P}\left(\mathrm{S}_{\mathrm{J}\,J\in\Im_{N_k}}\delta_{N_k}\geq 1-\varsigma\right)\right)^{\operatorname{Card}(\Im_{N_k})}\\
&\geq \exp\{-(\varepsilon_{N_k}+\beta_{N_k})(1-\varsigma/2)\}\frac{N_k-a_{N_k}}{2a_{N_k}}\log^{d-1} N_k\\
&\geq \left(\frac{N_k}{2a_{N_k}}\log^{d} N_k\right)^{-(1-\varsigma/2)}\frac{N_k-a_{N_k}}{a_{N_k}}\log^{d-1} N_k\\
&\geq \frac{a_{N_k}}{N_k}\cdot\frac{N_k-a_{N_k}}{2a_{N_k}}\log^{d-1-d(1-\varsigma/2)} N_k\\
&= \frac{1}{2}\left(1-\frac{a_{N_k}}{N_k}\right)\log^{-1+\varsigma d/2} N_k\\
&> \frac{1}{2}(1-\lambda)\log^{-1+\varsigma d/2} N_k.
\end{aligned}
\tag{21}
$$

We also note that

$$
N_k = N_1\prod_{j=2}^{k}\frac{N_j}{N_{j-1}} \leq N_1\prod_{j=2}^{k}\frac{1}{\varepsilon(1-a_{N_j}/N_j)} \leq N_1\varepsilon^{-k+1}(1-\lambda)^{-k+1}.
$$

Thus $\log N_k \le (k-1)\,|\log(\varepsilon(1-\lambda))| + \log N_1$ and the series

$$\sum_{k=1}^{\infty} \mathsf{P}\left(\max_{J\in\Im_{N_k}} S_J\delta_{N_k} \ge 1-\varsigma\right)$$

diverges. The Borel–Cantelli lemma implies (20).

Now we consider the case of $p>0$. Let $0<\varsigma<p$. Put $N_k=[\theta^k]$, $p>\varepsilon>0$, $\theta=3(p^{1/(d-1)}+1)^{d-1}$, and let $m\in\mathbf{N}$. We construct the set of disjoint d-dimensional rectangles belonging to the domain $A(\theta^k)\setminus A(\theta^{k-1})$ as follows:

$$\Im_{\theta^k}=\left\{J=(j,i]\in\mathbf{R}_0^d\colon i=\left(\varrho^{s_1},\dots,\varrho^{s_{d-1}},\frac{\theta^k}{\varrho^{\sum_{l=1}^{d-1}s_l}}\right), s_l\in\left\{1,\dots,\left[\frac{k\log\theta}{\log\varrho}\right]\right\},\right.$$
$$\left. s\in\mathbf{N}_0^{d-1}, l=1,\dots,d-1, j=\left(\varrho^{s_1-1},\dots,\varrho^{s_{d-1}-1},\frac{\theta^{k-1}}{\varrho^{\sum_{l=1}^{d-1}s_l-d+1}}\right)\right\},$$

where $\varrho=1+p^{1/(d-1)}$ and $|i|=\theta^k$, $|j|=\theta^{k-1}$. The inequality $|i-j|>a_{N_{k-1}}$ holds if

$$(\varrho-1)^{d-1}\left(\frac{\theta}{\varrho^{d-1}}-1\right)>p,\quad \frac{\theta}{\left(p^{1/(d-1)}+1\right)^{d-1}}-1>1,\quad\text{and}\quad \theta>2\left(p^{1/(d-1)}+1\right)^{d-1}.$$

The latter condition is obvious.

Note that

$$\operatorname{Card}(\Im_{\theta^k})\ge\frac{k^{d-1}\log^{d-1}\theta}{2\log^{d-1}\varrho}.$$

Moreover $n=\varepsilon_{N_{k-1}}+\beta_{N_{k-1}}$ for $(j,i]\in\Im_{\theta^k}$ and

$$W_n=\sqrt{\frac{\varepsilon_{N_{k-1}}+\beta_{N_{k-1}}}{a_{N_{k-1}}}}S_{(j,i]},\qquad m_n(t)=\mathsf{E}\left(\exp\left(t\sqrt{\frac{\varepsilon_{N_{k-1}}+\beta_{N_{k-1}}}{a_{N_{k-1}}}}S_{(j,i]}\right)\right),$$
$$\lim_{N\to\infty}\frac{a_{N_{k-1}}}{\varepsilon_{N_{k-1}}+\beta_{N_{k-1}}}\cdot\log\phi\left(\sqrt{\frac{\varepsilon_{N_{k-1}}+\beta_{N_{k-1}}}{a_{N_{k-1}}}}\cdot t\right)=\frac{t^2}{2},$$
$$c_2(t)=\frac{t^2}{2}=\liminf\psi_n(t)/n.$$

Using the inequality $a_{N_{k-1}}/N_{k-1}>p-\varsigma$ we prove, similarly to (21), that

$$\mathsf{P}\left(\max_{J\in\Im_{\theta^k}}S_J\delta_{N_k}\ge1-\varsigma\right)\ge(p-\varsigma)\big((k-1)\log\theta\big)^{-d(1-\varsigma/2)^2}\frac{k^{d-1}\log^{d-1}\theta}{2\log^{d-1}\varrho}$$

for sufficiently large N_{k-1}. Thus the series $\sum_{k=2}^{\infty}\mathsf{P}\big(\max_{J\in\Im_{N_k}}S_J\delta_{N_k}\ge1-\varsigma\big)$ diverges, since its general term is greater than or equal to $(k-1)^{-1+d(1-(1-\varsigma/2)^2)}$.

An application of the Borel–Cantelli lemma completes the proof of Step 2.

Step 3. Suppose (13) holds. Since

$$\liminf S_N\delta_N\ge\liminf S_N^\star\delta_N\ge\liminf D(N,a_N)\delta_N\ge\liminf D^\star(N,a_N)\delta_N\quad\text{a.s.},$$

it is sufficient to prove that $\liminf D^\star(N,a_N)\delta_N\ge1$ almost surely.

We construct a partition of the d-dimensional cube $(0,[N^{1/d}]\cdot e]$ as follows:

$$\begin{aligned}\Im_N = \Big\{ & J=(i,j]\in \mathbf{N}_0^d\colon i=(i_1,\dots,i_d),\ j=(j_1,\dots,j_d),\\ & i_1\in\left\{0,a_N,\dots,\left(\left[\frac{N^{1/d}}{a_N}\right]-1\right)\cdot a_N\right\},\ j_1=i_1+a_N,\\ & i_k\in\{0,1,\dots,[N^{1/d}]-1\},\ j_k=i_k+1\ k=2,\dots,d\Big\}.\end{aligned} \tag{22}$$

The elements of this partition satisfy

$$J\in\Im_N \Longrightarrow |J|=a_N,\qquad \operatorname{Card}(\Im_N)=\exp\varepsilon_N(1-\zeta).$$

Choose an arbitrary $\zeta>0$. Since the random variables X_n are independent and the set $\Im_N$ consists of disjoint rectangles, we get

$$\begin{aligned}\mathsf{P}\left(\max_{J\in\Im_N} S_J\delta_N\le 1-\zeta\right) &= \big(\mathsf{P}(S_{J_1}\delta_N\le 1-\zeta)\big)^{\operatorname{Card}(\Im_N)}\\ &= O\left(\big(1-\mathsf{P}(S_{J_1}\delta_N>1-\zeta)\big)^{N/a_N}\right),\end{aligned} \tag{23}$$

where $J_1=(0,(a_N,1,\dots,1)]\in\Im$. It follows from Theorem 5.1 that

$$\lim_{N\to\infty}\mathsf{P}(S_{J_1}\delta_N>1-\zeta)^{1/(\varepsilon_N+\beta_N)}=\exp\left\{-(1-\zeta)^2\right\}.$$

To continue the estimation of (23) we use the inequality $1-x\le\exp(-x)$:

$$\begin{aligned}&O\left(\exp\left\{-\exp\left(-(\varepsilon_N+\beta_N)\cdot(1-\zeta)^2+(1-\zeta)\varepsilon_N\right)\right\}\right)\\ &\quad=O\left(\exp\left\{-\exp\left\{\beta_N\left(\frac{\varepsilon_N}{\beta_N}\cdot\left(1-\zeta-(1-\zeta)^2\right)-(1-\zeta)^2\right)\right\}\right\}\right)=O(N^{-\sigma})\end{aligned}$$

for some $\sigma>1$.

The series $\sum_{N=1}^{\infty}\mathsf{P}\big(\max_{J\in\Im_N}S_J\delta_N\le 1-\zeta\big)$ converges by (13). The Borel–Cantelli lemma completes the proof. □

Remark 5.2. We consider a wider class of increments in Theorems 3.2 and 3.3 as compared to Theorems 2.3 and 2.2. Nevertheless the limits coincide and the normalizing sequences are equivalent for both sets of the results. Therefore the proof of Theorems 3.2 and 3.3 reduces to the proof of the inequality for the upper limit.

Proof of Theorem 3.2. Since $\limsup_{N\to\infty}S_N^{\star}\delta_N\le\limsup_{N\to\infty}S_N\delta_N$, we only need to prove that

$$\limsup S_N\delta_N\le\alpha\quad\text{a.s.}$$

Let $\alpha'>\alpha$; then

$$\mathsf{P}(S_N\delta_N>\alpha')\le\operatorname{Card}(A_N)\cdot\mathsf{P}(S_J\delta_N>\alpha'),$$

where $J\subset(0,n]\subset\mathbf{N}_+^d\colon |n|\le N,\ |J|=a_N$. It follows from Theorem 5.1 that

$$\lim_{N\to\infty}\big(\mathsf{P}(S_J\delta_N>\alpha')\big)^{1/a_N}=\inf_{t\in(0,t_0)}\frac{\phi(t)}{\exp(t\alpha')}=\exp\left(-\frac{1}{C'}\right),$$

$$\lim_{N\to\infty}\delta_N^{-1}a_N^{-1}=1,$$

where C' is a unique positive solution of equation (9) with α' instead of α. Let

$$C''\in(C',C);$$

then

$$\mathsf{P}(S_N\delta_N > \alpha') \le \operatorname{Card}(A_N)\cdot\exp\left(-\frac{C}{C''}\log N\right) = \operatorname{Card}(A_N)\cdot\exp\left(-(1+\mu)\log N\right),$$

where $\mu = C/C'' - 1 > 0$. Further we use Lemma 4.3 and continue

$$= O\left(\log^{d-1} N\cdot a_N\log^{d-1} a_N\cdot N^{-\mu}\right)$$
$$= O\left(\log^{d} N\cdot \log^{d-1}\left(C\log N\right)\cdot N^{-\mu}\right).$$

Let $\vartheta > 1$ and $N = [\vartheta^r]$. The series $\sum_{r=1}^{\infty}\mathsf{P}(S_{[\vartheta^r]}\delta_{[\vartheta^r]} > \alpha)$ converges, whence the Borel–Cantelli lemma implies that $\limsup S_{[\vartheta^r]}\delta_{[\vartheta^r]} \le \alpha$ almost surely. The rest of the proof is the same as that of Theorem 3.1. □

Proof of Theorem 3.3. It is sufficient to show that $\limsup S_N\delta_N \le 1$ almost surely. Take an arbitrary $\varepsilon > 0$ and estimate the probability

$$\mathsf{P}\left(S_N > (1+\varepsilon)\delta_N^{-1}\right) = \mathsf{P}\left(\sqrt{\frac{\log N}{a_N}}S_N > (1+\varepsilon)\delta_N^{-1}\sqrt{\frac{\log N}{a_N}}\right).$$

Expanding the function $\log\phi$ into the Taylor series for sufficiently large τ_1 we prove that

$$\lim_{N\to\infty}\frac{a_N}{\log N}\log\phi\left(\sqrt{\frac{\log N}{a_N}}\cdot t\right) = \frac{\gamma^2t^2}{2}$$

in view of relations $\log N/a_N \to \infty$ and (16).

Note that

$$\delta_N^{-1}\cdot\sqrt{\frac{\log N}{a_N}} \sim \gamma\sqrt{2}\cdot\log N.$$

It follows from Theorem 5.1 that

$$\lim_{N\to\infty}(\mathsf{P}(S_J\delta_N > \alpha_N))^{1/\log N} = \inf_t \exp\left(\frac{\gamma^2t^2}{2} - (1+\varepsilon)\gamma\sqrt{2}\right) = \exp\left(-(1+\varepsilon)^2\right).$$

Thus

$$\mathsf{P}\left(S_N > (1+\varepsilon)\delta_N^{-1}\right) = O\left(\log_+^{d-1} N\cdot a_N\log_+^{d-1} a_N\cdot N^{1-(1+\varepsilon/2)^2}\right).$$

Let $\vartheta > 1$ and $N = [\vartheta^r]$. Then the series $\sum_{r=1}^{\infty}\mathsf{P}(S_{[\vartheta^r]}\delta_{[\vartheta^r]} > 1)$ converges. Now the Borel–Cantelli lemma implies that $\limsup S_{[\vartheta^r]}\delta_{[\vartheta^r]} \le 1$ a.s.

The rest of the proof is the same as that of Theorem 3.1. □

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Received 4/APR/2002

Translated by THE AUTHOR