

TWO-BOUNDARY PROBLEMS FOR A RANDOM WALK WITH NEGATIVE GEOMETRIC JUMPS

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ABSTRACT. Two-boundary problems for a random walk with negative geometric jumps are considered, and the corresponding results for a usual semicontinuous random walk are generalized for them. The following results are obtained: the probability distribution of ruin is found and expressed in terms of the lower and upper boundaries; formulas are given for the joint distribution of the infimum, supremum, and the walk itself at an arbitrary time instance; the transient probabilities and ergodic distribution are evaluated for the process describing the evolution of the random walk with two boundaries.

Two-boundary problems for random walks and stochastic processes have several applications in the queue theory, storage and inventory theories, reliability theory, and in many other fields.

Two-boundary problems have been studied for semicontinuous random walks and for semicontinuous stochastic processes. Several methods are known for solving those problems, namely combinatorial [1], resolvent [2]–[6], factorization [7], and renewal [8] methods.

In this paper we solve two-boundary problems for random walks with negative geometric jumps. This model is a generalization of a usual model of semicontinuous random walks.

1. MAIN NOTATION

Let $\alpha \in \{0, 1, \dots\}$ be a nonnegative integer-valued random variable,

$$\mathbb{E}[\theta^\alpha] = \sum_{i=0}^{\infty} a_i \theta^i, \quad a_i = \mathbb{P}[\alpha = i], \quad \mathbb{P}[\alpha > 1] > 0, \quad |\theta| \leq 1.$$

Consider the random variable

$$\xi = \alpha - \beta, \quad \xi \in \{0, \pm 1, \dots\} = \mathbf{Z},$$

where $\beta \in \{1, 2, \dots\}$ is a positive integer-valued random variable distributed geometrically with parameter $b \in [0, 1)$, that is,

$$\mathbb{P}[\beta = n] = (1 - b)b^{n-1}, \quad n > 0.$$

It is clear that

$$\mathbb{E}[\theta^\xi] = \mathbb{E}[\theta^\alpha] \frac{(1 - b)/\theta}{1 - b/\theta} \stackrel{\text{def}}{=} \sum_{i=-\infty}^{\infty} p_i \theta^i, \quad |\theta| = 1.$$

An easy evaluation yields the distribution of the random variable ξ :

$$(1) \quad p_i = \mathbb{P}[\xi = i] = (1 - b)b^{-i-1} \mathbb{E}[b^\alpha; \alpha > i], \quad i \in \mathbf{Z}.$$

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Some of the properties of the distribution p_i , $i \in \mathbf{Z}$, are listed in the following result.

Lemma 1. For all $i \in \mathbf{Z}$,

$$(2) \quad \begin{aligned} bp_i - p_{i-1} &= \begin{cases} 0, & i < 0, \\ -(1-b)a_i, & i \geq 0, \end{cases} \\ b\mathbf{P}[\xi \geq i] - \mathbf{P}[\xi \geq i-1] &= -(1-b)\mathbf{P}[\alpha \geq i], \\ b\mathbf{P}[\xi \leq i] - \mathbf{P}[\xi \leq i-1] &= -(1-b)\mathbf{P}[\alpha \leq i], \end{aligned}$$

where $\mathbf{P}[\alpha = m] = 0$ for $m < 0$.

These equalities follow from the explicit expression (1) for the distribution of the random variable ξ .

Let $\{\xi, \xi'_i\}$, $i > 0$, be a sequence of independent identically distributed random variables and let ξ_n , $n \geq 0$, be a random walk such that

$$\xi_0 = 0, \quad \xi_n = \xi'_1 + \cdots + \xi'_n, \quad n > 0.$$

The moment generating function of increments of the random walk ξ_n , $n \geq 0$, satisfies

$$\sum_{n=0}^{\infty} t^n \mathbf{E}[\theta^{\xi_n}] = \frac{1}{1-t\mathbf{E}[\theta^\xi]} = \left\{ 1 - t\mathbf{E}[\theta^\alpha] \frac{(1-b)/\theta}{1-b/\theta} \right\}^{-1}, \quad |\theta| = 1, \quad t \in [0, 1].$$

The equation

$$1 - \frac{b}{\theta} - (1-b)\frac{t}{\theta}\mathbf{E}[\theta^\alpha] = 0$$

has a unique root $\theta = c(t)$ in the domain $|\theta| < 1$. Moreover the root is positive and $b < c(t) < 1$. This result is well known for semicontinuous random walks [9]; the proof for the case under consideration is similar.

Definition 1. A sequence $R_k(t)$, $k > 0$, is called the resolvent of the random walk ξ_n , $n \geq 0$, if

$$(3) \quad R(\theta, t) = \sum_{k=1}^{\infty} \theta^k R_k(t) = \left[\frac{b}{\theta} + (1-b)\frac{t}{\theta}\mathbf{E}[\theta^\alpha] - 1 \right]^{-1}, \quad |\theta| < c(t).$$

Since the right-hand side of (3) is analytic for $|\theta| < c(t)$, it can be expanded in a power series. In what follows we need the first term of the sequence $R_k(t)$, $k > 0$:

$$R_1(t) = (b + (1-b)t\mathbf{P}[\alpha = 0])^{-1}.$$

Definition 2. A sequence R_k , $k > 0$, is called the potential of the random walk ξ_n , $n \geq 0$, if

$$(4) \quad R(\theta) = \sum_{k=1}^{\infty} \theta^k R_k = \left[\frac{b}{\theta} + (1-b)\frac{1}{\theta}\mathbf{E}[\theta^\alpha] - 1 \right]^{-1}, \quad |\theta| < c(1) \leq 1.$$

The following two equalities are used later when solving two-boundary problems.

Lemma 2. For all $k > 0$,

$$(5) \quad R_k(t) + t(1-b)b^{-k-1} \sum_{i=1}^k R_i(t)b^i \mathbf{E}[b^\alpha; \alpha \leq k-i] = b^{-k},$$

$$(6) \quad S_k(t) + t(1-b)b^{-k-1} \sum_{i=1}^k S_i(t)b^i \mathbf{E}[b^\alpha; \alpha \leq k-i] = \frac{1}{1-b}[b^{-k} - 1],$$

where

$$S_k(t) = \sum_{i=1}^k R_i(t).$$

Proof. Denote the left-hand side of (5) by A_k , $k > 0$. Then

$$\sum_{k=1}^{\infty} \theta^k A_k = R(\theta, t) + t(1-b) \frac{1}{b} R(\theta, t) \frac{\mathbf{E}[\theta^\alpha]}{1-\theta/b}, \quad |\theta| < b.$$

It follows from the definition of the resolvent (3) that

$$R(\theta, t) \left[\frac{b}{\theta} + (1-b) \frac{t}{b} \mathbf{E}[\theta^\alpha] - 1 \right] = 1, \quad |\theta| < c(t),$$

whence

$$\sum_{k=1}^{\infty} \theta^k A_k = \frac{\theta}{b} \left\{ 1 - \frac{\theta}{b} \right\}^{-1}, \quad |\theta| < b.$$

Comparing the coefficients for θ^k , $k > 0$, on both sides of the latter equality, we get $A_k = b^{-k}$, $k > 0$. Equality (6) is proved similarly. \square

Now we turn to solving the two-boundary problems.

2. RUIN PROBLEM

Fix an integer $N > 1$ and put $\mathbf{P}[\xi_0 = k] = 1$, $k \in \{1, \dots, N-1\}$. Let

$$\tau_k = \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}\}$$

be the first moment when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N-1\}$. We also introduce the first moments when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N-1\}$ through the upper and lower boundary, respectively:

$$\begin{aligned} \tau_k^+ &= \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \geq N\}, \\ \tau_k^- &= \inf\{n > 0: \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \leq 0\}. \end{aligned}$$

Theorem 1. *Let*

$$\mathbf{E} \left[t^{\tau_k^-} \right] = \mathbf{E} [t^{\tau_k}; \xi_{\tau_k} \leq 0], \quad \mathbf{E} \left[t^{\tau_k^+} \right] = \mathbf{E} [t^{\tau_k}; \xi_{\tau_k} \geq N]$$

be the moment generating functions of the “ruin” times τ_k^- and τ_k^+ , $k \in \{1, \dots, N-1\}$. Then

$$(7) \quad \mathbf{E} \left[t^{\tau_k^-} \right] = \frac{b^N}{1-b} \frac{R_{N-k}(t)}{\hat{R}_N(b, t)},$$

$$\mathbf{E} \left[t^{\tau_k^+} \right] = 1 - \frac{R_{N-k}(t)}{\hat{R}_N(b, t)} \left[\frac{b^N}{1-b} + (1-b)(1-t) \hat{S}_N(b, t) \right] + (1-b)(1-t) S_{N-k}(t),$$

where $R_k(t)$, $k > 0$, is the resolvent (3) of the random walk ξ_n , $n \geq 0$,

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t), \quad S_k(t) = \sum_{i=1}^k R_i(t), \quad \hat{S}_N(b, t) = \sum_{i=N}^{\infty} b^i S_i(t).$$

Corollary 1. *Let*

$$\mathbf{P}_k^- = \mathbf{P}[\xi_{\tau_k} \leq 0] \quad \text{and} \quad \mathbf{P}_k^+ = \mathbf{P}[\xi_{\tau_k} \geq N]$$

be the “ruin” probabilities corresponding to the lower and upper boundaries, respectively. Denote by $E[\tau_k]$ the mean “ruin” time. Then

$$P_k^- = \frac{b^N R_{N-k}}{1 - b \hat{R}_N(b)}, \quad P_k^+ = 1 - \frac{b^N R_{N-k}}{1 - b \hat{R}_N(b)} = 1 - P_k^-,$$

$$E[\tau_k] = b^N \frac{R_{N-k}}{\hat{R}_N(b)} S_{N-1} + R_{N-k} - (1-b)S_{N-k},$$

where R_k , $k > 0$, is the potential of the random walk ξ_n , $n \geq 0$,

$$\hat{R}_N(b) = \sum_{i=N}^{\infty} b^i R_i, \quad S_k = \sum_{i=1}^k R_i.$$

Proof of Theorem 1. Let $P[\xi_0 = N - k] = 1$, $k \in \{1, \dots, N - 1\}$. By the full probability formula, the functions

$$\varphi_k(t) = E\left[t^{\tau_{N-k}^-}\right], \quad k \in \{1, \dots, N - 1\}, \quad t \in [0, 1),$$

satisfy the following system of equations:

$$(8) \quad \varphi_k(t) = t P[\xi \leq k - N] + t \sum_{i=1}^{N-1} \varphi_i(t) p_{k-i}, \quad k \in \{1, \dots, N - 1\}.$$

System (8) is regular, that is,

$$t \sum_{i=1}^{N-1} p_{k-i} < 1 - t b^{N-2} E[b^\alpha].$$

Therefore it has a unique solution (see [10]). We are going to find this solution in an explicit form. When solving this problem we follow a method that was useful for solving other two-boundary problems.

The functions $b\varphi_{k+1}(t)$ are such that

$$b\varphi_{k+1}(t) = t b P[\xi \leq k + 1 - N] + t b \sum_{i=1}^{N-1} \varphi_i(t) p_{k+1-i}, \quad k \in \{0, \dots, N - 2\}.$$

Subtracting this equation from (8) and applying (2) we get

$$(9) \quad b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b) \sum_{i=1}^{k+1} \varphi_i(t) a_{k+1-i}, \quad k \in \{1, \dots, N - 2\}.$$

Consider equation (9) for all $k \geq 1$:

$$(10) \quad b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b) \sum_{i=1}^{k+1} \varphi_i(t) a_{k+1-i}, \quad k \geq 1,$$

and solve it by using the moment generating function method. Put

$$\Phi(\theta, t) = \sum_{k=1}^{\infty} \theta^k \varphi_k(t), \quad |\theta| < c(t).$$

Rewriting equation (10) for the corresponding moment generating functions we obtain

$$\Phi(\theta, t) \left(\frac{b}{\theta} + (1-b) \frac{t}{\theta} E[\theta^\alpha] - 1 \right) = \varphi_1(t) \frac{1}{R_1(t)}, \quad |\theta| < c(t).$$

By the definition of the resolvent (3) we get from the latter equation that

$$(11) \quad \varphi_k(t) = \varphi_1(t) \frac{R_k(t)}{R_1(t)}, \quad k \geq 1.$$

The function $\varphi_k(t)$ in (11) satisfies equation (10) for all $k \geq 1$, thus it is a solution of equation (9) for $k \in \{1, \dots, N - 2\}$. A solution of initial equation (8) belongs to the set of solutions of equation (9) for which the function $\varphi_1(t)$ is a free parameter. It remains to find a function $\varphi_1^*(t)$ such that $\varphi_k(t)$ defined by (11) satisfies equation (8). Substituting the function $\varphi_k(t)$ from (11) into the initial equation (8) and using Lemma 2 we get

$$\varphi_1^*(t) = \frac{b^N R_1(t)}{1 - b \hat{R}_N(b, t)}, \quad \hat{R}_N(b, t) = \sum_{i=N}^{\infty} R_i(t) b^i.$$

Thus

$$\varphi_k(t) = \mathbb{E} \left[t^{\tau_{N-k}} \right] = \frac{b^N R_k(t)}{1 - b \hat{R}_N(b, t)}, \quad k \in \{1, \dots, N - 1\},$$

and the first equality of Theorem 1 is proved.

Now we prove the second equality. Let $\mathbb{P}[\xi_0 = N - k] = 1, k \in \{1, \dots, N - 1\}$. By the full probability formula, the functions

$$\psi_k(t) = \mathbb{E} \left[t^{\tau_{N-k}}; \xi_{\tau_{N-k}} \geq N \right], \quad t \in [0, 1),$$

satisfy the system of equations

$$(12) \quad \psi_k(t) = t \mathbb{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \psi_i(t) p_{k-i}, \quad k \in \{1, \dots, N - 1\}.$$

The equation for $b\psi_{k+1}(t)$ is given by

$$b\psi_{k+1}(t) = tb \mathbb{P}[\xi \geq k + 1] + tb \sum_{i=1}^{N-1} \psi_i(t) p_{k+1-i}, \quad k \in \{0, \dots, N - 2\}.$$

Subtracting this equation from (12) we obtain that for all $k \geq 1$,

$$(13) \quad b\psi_{k+1}(t) - \psi_k(t) = -t(1 - b) \sum_{i=1}^{k+1} \psi_i(t) a_{k+1-i} - t(1 - b) \mathbb{P}[\alpha \geq k + 1], \quad k \geq 1.$$

Put

$$\Psi(\theta, t) = \sum_{k=1}^{\infty} \theta^k \psi_k(t), \quad |\theta| < c(t).$$

Rewriting equation (13) for the corresponding moment generating functions and using the definition of the resolvent (3) we get

$$\Psi(\theta, t) = \frac{\theta}{1 - \theta} + \frac{\psi_1(t) - 1}{R_1(t)} R(\theta, t) + (1 - b)(1 - t) \frac{\theta}{1 - \theta} R(\theta, t), \quad |\theta| < c(t).$$

Comparing the coefficients for $\theta^k, k > 0$, on both sides of the latter equality we obtain

$$(14) \quad \psi_k(t) = 1 + \frac{\psi_1(t) - 1}{R_1(t)} R_k(t) + (1 - b)(1 - t) S_{k-1}(t), \quad k \geq 1.$$

To find a function $\psi_1^*(t)$ such that $\psi_k(t)$ defined by (14) satisfies the initial system of equations (12), we substitute the right-hand side of (14) into the system of equations (12) and apply Lemma 2. Then we obtain the following relation:

$$\frac{\psi_1^*(t) - 1}{R_1(t)} = -\frac{1}{\hat{R}_N(b, t)} \left[\frac{b^N}{1 - b} + (1 - b)(1 - t) \hat{S}_N(b, t) \right] + (1 - b)(1 - t),$$

where

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t), \quad \hat{S}_N(b, t) = \sum_{i=N}^{\infty} b^i S_i(t).$$

Therefore

$$\psi_k(t) = 1 - \frac{R_k(t)}{\hat{R}_N(b, t)} \left[\frac{b^N}{1-b} + (1-b)(1-t)\hat{S}_N(b, t) \right] + (1-b)(1-t)S_k(t),$$

$$k \in \{1, \dots, N-1\},$$

and the second equality of Theorem 1 is proved.

Further

$$\sum_{n=0}^{\infty} t^n \mathbb{P}[\tau_{N-k} > n] = \frac{1 - \varphi_k(t) - \psi_k(t)}{1-t} = \frac{(1-b)\hat{S}_N(b, t)}{\hat{R}_N(b, t)} R_k(t) - (1-b)S_k(t).$$

Since

$$(1-b)\hat{S}_N(b, t) = \hat{R}_N(b, t) + b^N S_{N-1}(t),$$

we have

$$\sum_{n=0}^{\infty} t^n \mathbb{P}[\tau_{N-k} > n] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} S_{N-1}(t) + R_k(t) - (1-b)S_k(t),$$

whence equality (7) follows. Corollary 1 follows from Theorem 1 for $t = 1$. \square

3. THE JOINT DISTRIBUTION OF $\inf\{\xi_0, \dots, \xi_n\}$, $\sup\{\xi_0, \dots, \xi_n\}$, AND ξ_n , $n \geq 0$

Assume that $\mathbb{P}[\xi_0 = 0] = 1$ and consider the random variables

$$\mu_n^+ = \sup\{\xi_0, \dots, \xi_n\}, \quad \mu_n^- = \inf\{\xi_0, \dots, \xi_n\}.$$

Let k and r be positive integers, $R_m(t) = 0$, $m \leq 0$.

Theorem 2. *The joint distribution*

$$\mathbb{P}_{(-r, k)}^n[i] = \mathbb{P}[-r < \mu_n^-, \xi_n = i, \mu_n^+ < k], \quad -r < i < k,$$

of the random variables μ_n^- , ξ_n , and μ_n^+ satisfies

$$\sum_{n=0}^{\infty} t^n \mathbb{P}_{(-r, k)}^n[i] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} R_{i+r}(t) + b R_{i+1}(t) - R_i(t), \quad N = k + r.$$

Proof of Theorem 2. Fix $N > 1$ and assume that $\mathbb{P}[\xi_0 = N - k] = 1$, $k \in \{1, \dots, N - 1\}$.

Let

$$\tau_{N-k} = \inf\{n > 0: \xi_n \notin \{1, \dots, N - 1\}\}$$

be the first moment when the random walk ξ_n , $n \geq 0$, exits the set $\{1, \dots, N - 1\}$.

Denote by

$$Q_{k,d}(n) = \mathbb{P}[\xi_n = N - d, \tau_{N-k} > n], \quad d \in \{1, \dots, N - 1\},$$

the transient probability of the random walk ξ_n , $n > 0$, on the interval $0 \leq n < \tau_{N-k}$.

By the full probability formula, the moment generating functions

$$Q_{k,d}^t = \sum_{n=0}^{\infty} t^n Q_{k,d}(n), \quad k, d \in \{1, \dots, N - 1\}, \quad t \in [0, 1),$$

satisfy the system of equations

$$(15) \quad Q_{k,d}^t = \delta_{k,d} + t \sum_{i=1}^{N-1} p_{k-i} Q_{i,d}^t, \quad k \in \{1, \dots, N - 1\},$$

where $\delta_{k,r}$ is the Kronecker symbol. The corresponding equation for $bQ_{k+1,d}^t$ is

$$bQ_{k+1,d}^t = b\delta_{k+1,d} + tb \sum_{i=1}^{N-1} p_{k+1-i} Q_{i,d}^t, \quad k \in \{0, \dots, N-2\}.$$

Subtracting this equation from (15) we obtain that for all $k > 0$,

$$(16) \quad bQ_{k+1,d}^t - Q_{k,d}^t = b\delta_{k+1,d} - \delta_{k,d} - t(1-b) \sum_{i=1}^{k+1} a_{k+1-i} Q_{i,d}^t.$$

Consider the moment generating function

$$Q_d^t(\theta) = \sum_{k=1}^{\infty} \theta^k Q_{k,d}^t, \quad |\theta| < c(t).$$

Rewriting (16) for the corresponding moment generating functions we obtain

$$Q_d^t(\theta) = \left(\frac{Q_{1,d}^t}{R_1(t)} - b\delta_{1,d} \right) R(\theta, t) + b\theta^{d-1} R(\theta, t) - \theta^d R(\theta, t), \quad |\theta| < c(t),$$

where $R(\theta, t)$ is the moment generating function of resolvent (3) for the random walk ξ_n , $n > 0$. Comparing the coefficients for θ^k , $k > 0$, and putting $R_m(t) = 0$, $m \leq 0$, we get

$$(17) \quad Q_{k,d}^t = \left(\frac{Q_{1,d}^t}{R_1(t)} - b\delta_{1,d} \right) R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t).$$

We look for a function $Q_{1,d}^t = \tilde{Q}_{1,d}^t$ for which solution (17) of the infinite system (16) is also a solution of the initial system. Substituting expression (17) into the initial system (15) we get by Lemma 2 that

$$\frac{\tilde{Q}_{1,d}^t}{R_1(t)} - b\delta_{1,d} = b^N \frac{R_{N-d}(t)}{\hat{R}_N(b, t)}.$$

Combining this expression and (17) we obtain

$$(18) \quad Q_{k,d}^t = b^N \frac{R_{N-d}(t)}{\hat{R}_N(b, t)} R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t), \quad k, d \in \{1, \dots, N-1\}.$$

Now put $P[\xi_0 = 0] = 1$, and let k and r be positive integers, $N = k + r$. Let

$$\tau = \inf\{n > 0: \xi_n \notin \{-r + 1, \dots, k - 1\}\}$$

be the first moment when the random walk ξ_n , $n > 0$, exits the set

$$\{-r + 1, \dots, k - 1\}.$$

Denote by

$$P_{(-r,k)}^n[i] = P[\xi_n = i, \tau > n], \quad -r < i < k,$$

the transient probability for the random walk ξ_n on the interval $0 \leq n < \tau$. Since the random walk ξ_n , $n \geq 0$, is homogeneous, we find from (18) that

$$\sum_{n=0}^{\infty} t^n P_{(-r,k)}^n[i] = b^N \frac{R_k(t)}{\hat{R}_N(b, t)} R_{i+r}(t) + bR_{i+1}(t) - R_i(t), \quad -r < i < k.$$

Since

$$P[\tau > n] = P[-r < \mu_n^-, \mu_n^+ < k],$$

we have

$$P_{(-r,k)}^n[i] = P[-r < \mu_n^-, \xi_n = i, \mu_n^+ < k], \quad -r < i < k,$$

whence Theorem 2 follows. □

4. A RANDOM WALK ξ_n , $n \geq 0$, AND TWO ABSORBING BOUNDARIES

Fix a positive integer N and consider a random sequence $X_n \in \{0, 1, \dots, N\}$, $n \geq 0$, whose one-step transient probabilities are given by

$$(19) \quad k \rightarrow \begin{cases} i, & \text{P}[\xi = i - k], \quad i \in \{1, \dots, N - 1\}, \\ 0, & \text{P}[\xi \leq -k], \\ N, & \text{P}[\xi \geq N - k] \end{cases}$$

for all $k \in \{0, \dots, N\}$.

The homogeneous (discrete time) Markov process X_n , $n \geq 0$, describes the evolution of the random walk ξ_n , $n \geq 0$, in the strip $0, \dots, N$ with two reflecting boundaries with absorption (the upper boundary is at the level N , while the lower one is at the level 0).

If the process X_n , $n \geq 0$, reaches the boundary 0 at a moment n_0 , it stays there for a random time $\nu(0)$ until the first positive jump of the random walk ξ_{n-n_0} , $n \geq n_0$, occurs. Then the process is “reflected” and appears at a state $i \in \{1, \dots, N - 1\}$ with probability $\text{P}[\xi = i]$ (the probability of its appearing on the upper boundary is $\text{P}[\xi \geq N]$). It is clear that $\nu(0)$ is a geometric random variable and

$$\text{P}[\nu(0) = n] = \text{P}[\xi > 0](\text{P}[\xi \leq 0])^{n-1}, \quad n > 0.$$

If the process X_n , $n \geq 0$, reaches the upper boundary N at a moment n_N , it stays there for a random time $\nu(N)$ until the first negative jump of the random walk ξ_{n-n_N} , $n \geq n_N$, occurs. Then the process is “reflected” and appears at a state $i \in \{1, \dots, N - 1\}$ with probability $\text{P}[\xi = i - N]$ (the probability of its appearing on the lower boundary is $\text{P}[\xi \leq -N]$). It is clear that $\nu(N)$ is a geometric random variable and

$$\text{P}[\nu(N) = n] = \text{P}[\xi < 0](\text{P}[\xi \geq 0])^{n-1}, \quad n > 0.$$

Random walks with two reflecting boundaries are useful for queueing systems having a bounded number of waiting places as well as in storage and inventory theories, and in other applied probability fields.

4.1. Hitting the boundaries. Let $\text{P}[X_0 = k] = 1$, and let

$$\tau_k^0 = \inf\{n > 0: X_n = 0\}, \quad k \in \{1, \dots, N\},$$

be the first time when the process X_n , $n \geq 0$, reaches the lower boundary 0. Also let

$$\tau_k^N = \inf\{n > 0: X_n = N\}, \quad k \in \{0, \dots, N - 1\},$$

be the first time when the process X_n , $n \geq 0$, reaches the upper boundary N .

Theorem 3. *The moment generating functions of the random variables τ_k^0 and τ_k^N and their means $\text{E}[\tau_k^0]$ and $\text{E}[\tau_k^N]$ satisfy*

$$\begin{aligned} \text{E} \left[t^{\tau_k^0} \right] &= \frac{b^N}{1 - b} \frac{1 + (1 - b)(1 - t)S_{N-k}(t)}{b^N/(1 - b) + (1 - b)(1 - t)\hat{S}_N(b, t)}, \\ \text{E} \left[\tau_k^0 \right] &= (1 - b) \left[b^{-N} \hat{R}_N(b) + S_{N-1} - S_{N-k} \right], \\ \sum_{n=0}^{\infty} t^n \text{P} \left[\tau_k^N > n \right] &= (1 - b) \left\{ \frac{\hat{R}_{N+1}(b, t)}{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)} R_{N-k}(t) - S_{N-k}(t) \right\}, \\ \text{E} \left[\tau_k^N \right] &= (1 - b) \left\{ \frac{\hat{R}_{N+1}(b)}{\hat{R}_{N+1}(b) - b\hat{R}_N(b)} R_{N-k} - S_{N-k} \right\}, \end{aligned}$$

where

$$\begin{aligned}\hat{R}_N(b, t) &= \sum_{i=N}^{\infty} b^i R_i(t), & \hat{R}_N(b) &= \sum_{i=N}^{\infty} b^i R_i = \hat{R}_N(b, 1), \\ \hat{S}_N(b, t) &= \sum_{i=N}^{\infty} b^i S_i(t), & S_k(t) &= \sum_{i=1}^k R_i(t).\end{aligned}$$

Proof of Theorem 3. Let $\mathbb{P}[X_0 = N - k] = 1$. Using the transient probabilities (19) of the process X_n , $n \geq 0$, we see that the moment generating function

$$\tilde{\varphi}_k(t) = \mathbb{E} \left[t^{\tau_{N-k}^0} \right], \quad t \in (0, 1), \quad k \in \{0, \dots, N - 1\},$$

satisfies the system of equations

$$(20) \quad \begin{aligned}\tilde{\varphi}_k(t) &= t \mathbb{P}[\xi \leq k - N] + t \tilde{\varphi}_0(t) \mathbb{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \tilde{\varphi}_i(t) p_{k-i}, \\ & \quad k \in \{0, \dots, N - 1\},\end{aligned}$$

by the full probability formula. System (20) is strongly regular, that is,

$$t \mathbb{P}[\xi \geq k] + t \sum_{i=1}^{N-1} \mathbb{P}[\xi = k - i] = t \mathbb{P}[\xi > k - N] < 1 - t b^{N-1} \mathbb{E}[b^\alpha].$$

Therefore it has a unique solution (see [10]).

Using the transient probabilities (19) of the process X_n , $n \geq 0$, we prove that the moment generating function

$$\tilde{\psi}_k(t) = \mathbb{E} \left[t^{\tau_{N-k}^N} \right], \quad k \in \{1, \dots, N\}, \quad t \in [0, 1),$$

satisfies the system of equations

$$(21) \quad \tilde{\psi}_k(t) = t \mathbb{P}[\xi \geq k] + t \mathbb{P}[\xi \leq k - N] \tilde{\psi}_N(t) + t \sum_{i=1}^N \tilde{\psi}_i(t) p_{k-i}, \quad k \in \{1, \dots, N\},$$

by the full probability formula. The latter system is strongly regular, since

$$t \mathbb{P}[\xi < k - N] + t \mathbb{P}[k - N \leq \xi \leq k - 1] < t \mathbb{P}[\xi \leq N - 1] < 1 - t \mathbb{E} [1 - b^{\alpha-N}; \alpha > N].$$

Thus it has a unique solution (see [10]).

Applying to systems (20) and (21) the method used in the preceding sections for solving corresponding systems of equations, we complete the proof of Theorem 3. \square

4.2. Transient probabilities and ergodic distribution of the process X_n , $n \geq 0$.

Let $N > 0$ and let ν_t be a geometric random variable with parameter t : $\mathbb{P}[\nu_t = n] = (1 - t)t^n$, $n \geq 0$, $t \in [0, 1)$.

Theorem 4. *Let $\mathbb{P}[X_0 = k] = 1$, $k \in \{0, \dots, N - 1\}$. Then*

$$\begin{aligned}\mathbb{P}[X_{\nu_t} = i] &= \frac{b^N}{1 - b} \frac{R_{i+1}(t) - R_i(t)}{\hat{R}_{N+1}(b, t)} [1 + (1 - b)(1 - t)S_{N-k}(t)] \\ & \quad + (1 - t) [bR_{i-k+1}(t) - R_{i-k}(t)], \quad i \in \{0, \dots, N - 1\},\end{aligned}$$

$$\mathbb{P}[X_{\nu_t} = N] = \frac{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)}{(1 - b)\hat{R}_{N+1}(b, t)} [1 + (1 - b)(1 - t)S_{N-k}(t)] - (1 - t)R_{N-k}(t).$$

Moreover there exists the ergodic distribution

$$\Pi_i = \lim_{n \rightarrow \infty} \mathbb{P}[X_n = i], \quad i \in \{0, \dots, N\},$$

of the process X_n , $n \geq 0$, such that

$$\Pi_i = \frac{b^{N+1} R_{i+1} - R_i}{1 - b \hat{R}_{N+1}(b)}, \quad i \in \{0, \dots, N-1\}, \quad \Pi_N = \frac{\hat{R}_{N+1}(b) - b\hat{R}_N(b)}{(1-b)\hat{R}_{N+1}(b)}.$$

Proof of Theorem 4. Let $\mathbb{P}[X_0 = N - k] = 1$, $k \in \{0, \dots, N\}$. Denote by

$$\mathbb{P}_{kr}(n) = \mathbb{P}[X_n = N - r], \quad r \in \{0, \dots, N\},$$

the transient probability of the process X_n , $n \geq 0$. By the full probability formula and (19), the functions

$$\mathbb{P}_{k,r}^t = \sum_{n=0}^{\infty} t^n \mathbb{P}_{kr}(n), \quad t \in [0, 1],$$

satisfy the system

$$(22) \quad \mathbb{P}_{k,r}^t = \delta_{k,r} + t \mathbb{P}_{N,r}^t \mathbb{P}[\xi < k - N] + t \mathbb{P}_{0,r}^t \mathbb{P}[\xi \geq k] + t \sum_{i=1}^N \mathbb{P}_{i,r}^t p_{k-i},$$

$$k, r \in \{0, \dots, N\}.$$

This problem is the most complicated among similar problems considered in this paper. Nevertheless the method used for solving the preceding two-boundary problems works in this case, too.

The functions $b \mathbb{P}_{k+1,r}^t$, $k \in \{-1, \dots, N-1\}$, satisfy the following equations:

$$b \mathbb{P}_{k+1,r}^t = b\delta_{k+1,r} + tb \mathbb{P}_{N,r}^t \mathbb{P}[\xi < k + 1 - N] + tb \mathbb{P}_{0,r}^t \mathbb{P}[\xi \geq k + 1] + tb \sum_{i=1}^N \mathbb{P}_{i,r}^t p_{k+1-i}.$$

Subtracting (22) from the latter equality we get that for $k \in \{0, \dots, N-1\}$,

$$(23) \quad b \mathbb{P}_{k+1,r}^t - \mathbb{P}_{k,r}^t = b\delta_{k+1,r} - \delta_{k,r} - t(1-b) \mathbb{P}_{0,r}^t \mathbb{P}[\alpha \geq k + 1] - t(1-b) \sum_{i=1}^{k+1} \mathbb{P}_{i,r}^t a_{k+1-i}.$$

Put

$$\mathbb{P}_r(\theta, t) = \sum_{k=0}^{\infty} \theta^k \mathbb{P}_{k,r}^t, \quad |\theta| < c(t).$$

Solving equations (23) by the moment generating function method we obtain that for all $k \geq 0$,

$$\mathbb{P}_r(\theta, t) = \frac{\mathbb{P}_{0,r}^t}{1-\theta} [1 + (1-b)(1-t)R(\theta, t)] + b(1-\delta_{r,0})\theta^{r-1}R(\theta, t) - \theta^r R(\theta, t),$$

$$|\theta| < c(t).$$

Comparing the coefficients for θ^k , $k \geq 0$, on both sides of this equality, we see that

$$(24) \quad \mathbb{P}_{k,r}^t = \mathbb{P}_{0,r}^t [1 + (1-b)(1-t)S_k(t)] + b(1-\delta_{r,0})R_{k+1-r}(t) - R_{k-r}(t),$$

$$R_m(t) = 0, \quad m \leq 0.$$

Now we find a function $\mathbb{P}_{0,r}^*(t)$ such that $\mathbb{P}_{k,r}^t$ defined by (24) for $k \in \{0, \dots, N\}$ is a solution of the initial system (22). Substituting the right-hand side of (24) into (22) and applying Lemma 2 we prove that

$$\mathbb{P}_{00}^*(t) = \frac{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)}{(1-b)(1-t)\hat{R}_{N+1}(b, t)},$$

$$\mathbb{P}_{0r}^*(t) = \frac{b^{N+1} R_{N+1-r}(t) - R_{N-r}(t)}{1-b(1-t)\hat{R}_{N+1}(b, t)}, \quad r \in \{1, \dots, N\},$$

where

$$\hat{R}_N(b, t) = \sum_{i=N}^{\infty} b^i R_i(t).$$

Substituting $P_{0r}^*(t)$ into (24) we find that for all $k, r \in \{0, \dots, N\}$,

$$\begin{aligned} P_{k,r}^t &= \frac{b^{N+1} R_{N+1-r}(t) - R_{N-r}(t)}{1-b} \frac{\hat{R}_{N+1}(b, t)}{(1-t)\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_k(t)] \\ &\quad + bR_{k+1-r}(t) - R_{k-r}(t), \quad r \in \{1, \dots, N\}, \\ (25) \quad P_{k,0}^t &= \frac{\hat{R}_{N+1}(b, t) - b\hat{R}_N(b, t)}{(1-t)(1-b)\hat{R}_{N+1}(b, t)} [1 + (1-b)(1-t)S_k(t)] - R_k(t). \end{aligned}$$

Since

$$P[X_{\nu_t} = i] = (1-t)P_{N-k, N-i}^t, \quad k, i \in \{0, \dots, N\},$$

the first two identities of Theorem 4 follow from (25).

The process X_n , $n \geq 0$, is a Markov chain with a finite set of states $\{0, \dots, N\}$. If $P[\alpha > 1] > 0$, then there exists n_0 such that $\min_{k,r} P_{k,r}(n_0) > 0$. Then (see [11]) there exists the ergodic distribution

$$\Pi_i = \lim_{n \rightarrow \infty} P[X_n = i], \quad i \in \{0, \dots, N\}.$$

To get the ergodic distribution in an explicit form, one needs to evaluate the limits

$$\Pi_i = \lim_{t \rightarrow 1} (1-t)P_{k, N-i}^t, \quad i \in \{0, \dots, N\}.$$

Multiplying equalities (25) by $(1-t)$ and passing to the limits as $t \rightarrow 1$, we get by changing $r \rightarrow N-i$ that

$$\Pi_i = \frac{b^{N+1} R_{i+1} - R_i}{1-b} \frac{R_{i+1} - R_i}{\hat{R}_{N+1}(b)}, \quad i \in \{0, \dots, N-1\}, \quad \Pi_N = \frac{\hat{R}_{N+1}(b) - b\hat{R}_N(b)}{(1-b)\hat{R}_{N+1}(b)}$$

where R_k , $k > 0$, is potential (4) of the random walk ξ_n , $n \geq 0$. Theorem 4 is proved. \square

Remark 1. Putting $b = 0$ in our theorems and corollaries we obtain the corresponding results for a lower semicontinuous random walk ξ_n , $n \geq 0$, whose increments have the moment generating function

$$\sum_{n=0}^{\infty} t^n E[\theta^{\xi_n}] = \left\{ 1 - \frac{t}{\theta} E[\theta^{\alpha}] \right\}^{-1}, \quad |\theta| = 1, \quad t \in (0, 1).$$

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