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QUASI-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH A FRACTIONAL BROWNIAN COMPONENT

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Abstract. The paper is devoted to stochastic differential equations with a fractional Brownian component. The fractional Brownian motion is constructed on the white noise space with the help of "forward" and "backward" fractional integrals. The fractional white noise and Wick products are considered. A similar construction for the "complete" fractional integral is considered by Elliott and van der Hoek. We consider two possible approaches to the existence and uniqueness of solutions of stochastic differential equation with a fractional Brownian motion.

1. Introduction

We consider quasi-linear stochastic differential equations with a fractional Brownian component.

The paper is organized as follows. Section 2 contains the construction of the "forward" and "backward" fractional Brownian motions on the white noise space (a similar construction for the "two-sided" fractional Brownian motion is described in [1, 2]).

In Section 3, the fractional Brownian noise related to the "forward" fractional Brownian motion is considered. It is proved that the fractional Brownian noise belongs to the Hida space S^*.

Section 4 contains the conditions on the process Y under which the Wick product $Y \diamond W_H$ with a fractional Brownian noise W_H is S^*-integrable. It is also proved that the S^*-integral with a nonrandom integrand and a "forward" fractional Brownian motion as the integrator can be reduced to the ordinary Itô integral with a nonrandom integrand that is a kernel of the Volterra type, and with a Wiener process as the integrator.

Two possible methods of solving stochastic differential equations with a fractional Brownian noise are described in Section 5. The first method is based on the Lipschitz and growth conditions for negative norms of coefficients (note however that those conditions are rather restrictive). The second method is applied only to quasi-linear equations and is based on the Giessing lemma [3]. Both methods show that fractional stochastic differential equations on the white noise space are very close in their properties to ordinary stochastic differential equations with respect to the Wiener process.

2. "Forward" and "backward" fractional Brownian motion

Consider a probability white noise space. More precisely, let $S(\mathbf{R})$ be the Schwartz space of real smooth rapidly decreasing functions on $\mathbf{R}$, and let $S'(\mathbf{R})$ be the dual space of tempered distributions equipped with the weak-star topology. We treat $S'(\mathbf{R})$ as the

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probability space Ω with the σ-field F of Borel sets. According to the Bochner–Minlos theorem, there exists a probability measure P on (Ω, F) such that

$$\mathsf{E}\exp\{i\langle f,\omega\rangle\} = \exp\left\{-\frac{1}{2}\|f\|_2^2\right\} \tag{1}$$

for all functions $f \in S(\mathbf{R})$, where $\|f\|_2^2 = \int_{\mathbf{R}} |f(x)|^2\,dx$.

Note that $\mathsf{E}\langle f,\omega\rangle = 0$ and $\mathsf{E}\langle f,\omega\rangle^2 = \|f\|_2^2$ for $f \in S(\mathbf{R})$ by (1) and thus the duality $\langle f,\omega\rangle$ can be extended by isometry to the whole space $L_2(\mathbf{R})$.

For $H \in (\frac{1}{2}, 1)$, consider fractional integrals of the form

$$M_H f(x) = C_H \int_x^\infty (t-x)^{H-3/2} f(t)\,dt, \qquad M_H^* f(x) = C_H \int_{-\infty}^x (x-t)^{H-3/2} f(t)\,dt,$$

where

$$C_H = \left(\sin(\pi H)\Gamma(2H+1)\right)^{1/2}\left(\left(C_H^1\right)^2 + \left(C_H^2\right)^2\right)^{-1/2},$$

$$C_H^1 = \pi\left(2\cos\left(\left(\frac{3}{4} - \frac{H}{2}\right)\pi\right)\Gamma\left(\frac{3}{2} - H\right)\right)^{-1},$$

$$C_H^2 = \pi\left(2\sin\left(\left(\frac{3}{4} - \frac{H}{2}\right)\pi\right)\Gamma\left(\frac{3}{2} - H\right)\right)^{-1}.$$

We set $M_{1/2} f(x) = M_{1/2}^* f(x) = f(x)$ for $H = \frac{1}{2}$.

According to [4], these operators are defined on $L_p(\mathbf{R})$ for $1 \le p < \alpha^{-1}$, where

$$\alpha = H - \frac{1}{2}.$$

If $1 < p < \frac{1}{2}$, then the operators are bounded and act from $L_p(\mathbf{R})$ to $L_q(\mathbf{R})$, where $q = p(1-p\alpha)^{-1}$. Let $f_1 \in L_2(\mathbf{R})$ and $f_2 \in L_{1/H}(\mathbf{R})$. Then $M_H f_1 \in L_{1/(1-H)}(\mathbf{R})$ and $M_H f_2 \in L_2(\mathbf{R})$, so that one can introduce an inner product in $L_2(\mathbf{R})$ by putting

$$(f_1, M_H f_2) = \int_{\mathbf{R}} f_1(x) M_H f_2(x)\,dx.$$

It follows from the Fubini theorem that

$$\begin{aligned}(f_1, M_H f_2) &= C_H \int_{\mathbf{R}} f_1(x)\left(\int_x^\infty (t-x)^{H-3/2} f_2(t)\,dt\right)dx \\ &= C_H \int_{\mathbf{R}} f_2(t)\left(\int_{-\infty}^t f_1(x)(t-x)^{H-3/2}\,dx\right)dt = (M_H^* f_1, f_2).\end{aligned} \tag{2}$$

Relation (2) means that the operators M_H and M_H^* are, in some sense, conjugate. Note that M_H and M_H^* transform $S(\mathbf{R})$ into $S(\mathbf{R})$.

Let $I_{[0,t]}(s)$, $0 \le t$, be the indicator function. For $t < 0$ we put $I_{[0,t]}(s) = -I_{[t,0]}(s)$. Set $M_{H,t}(x) = M_H I_{[0,t]}(x)$ and $M_{H,t}^*(x) = M_H^* I_{[0,t]}(x)$. Consider two stochastic processes

$$B_H(t)(\omega) = \langle M_{H,t}, \omega\rangle \quad \text{and} \quad B_H^*(t)(\omega) = \left\langle M_{H,t}^*, \omega\right\rangle, \qquad t \in \mathbf{R}. \tag{3}$$

Note that B_H and B_H^* are Gaussian stochastic processes such that

$$\mathsf{E}\,B_H(t) = \mathsf{E}\,B_H^*(t) = 0.$$

It follows from the Parseval equality that

$$\mathsf{E}\,B_H(t)B_H(s) = \int_{\mathbf{R}} M_{H,t}(x) M_{H,s}(x)\,dx = \frac{1}{2\pi}\int_{\mathbf{R}} \widehat{M_{H,t}}(\lambda)\overline{\widehat{M_{H,s}}(\lambda)}\,d\lambda,$$

where $\hat{f}(\lambda) = \int_{\mathbf{R}} e^{-i\lambda y} f(y)\,dy$ is the Fourier transform. The "fractional" Fourier transform is of the form

$$\widehat{M_H f}(\lambda) = C_H \left(C_H^1 + i \cdot \operatorname{sgn}\lambda \cdot C_H^2\right) |\lambda|^{1/2-H} \hat{f}(\lambda),$$
$$\widehat{M_H^* f}(\lambda) = C_H \left(C_H^1 - i \cdot \operatorname{sgn}\lambda \cdot C_H^2\right) |\lambda|^{1/2-H} \hat{f}(\lambda)$$

(see [4]). In the particular case of $f(x) = I_{[0,t]}(x)$ we have

$$\widehat{M_{H,t}}(\lambda) = C_H \left(C_H^1 + i \cdot \operatorname{sgn}\lambda \cdot C_H^2\right) |\lambda|^{1/2-H} \frac{1 - e^{-i\lambda t}}{i\lambda},$$
$$\widehat{M_{H,t}^*}(\lambda) = C_H \left(C_H^1 - i \cdot \operatorname{sgn}\lambda \cdot C_H^2\right) |\lambda|^{1/2-H} \frac{1 - e^{-i\lambda t}}{i\lambda}.$$

Therefore

$$\begin{aligned} \mathsf{E}\, B_H(t) B_H(s) &= \frac{1}{2\pi} C_H^2 \left(\left(C_H^1\right)^2 + \left(C_H^2\right)^2\right) \int_{\mathbf{R}} |\lambda|^{-1-2H} \left(1 - e^{-i\lambda t}\right)\left(1 - e^{i\lambda s}\right) d\lambda \\ &= \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \end{aligned} \tag{4}$$

for all $t, s \in \mathbf{R}$.

It follows from (3) and (4) that the processes $B_H(t)$ and $B_H^*(t)$ are fractional Brownian motions and have stationary increments. Moreover, representation (3) allows one to express $B_H(t)$ and $B_H^*(t)$ in terms of a standard Brownian motion $B(t) = B_{1/2}(t) = \langle I_{[0,t]}, \omega\rangle$:

$$B_H(t) = \int_{\mathbf{R}} M_{H,t}(u)\,dB(u), \qquad B_H^*(t) = \int_{\mathbf{R}} M_{H,t}^*(u)\,dB(u)$$

or

$$B_H(t) = C_H \left(H - \frac{1}{2}\right)^{-1} \int_{-\infty}^{t} \left((t-u)^{H-1/2} - (-u)_+^{H-1/2}\right) dB(u), \tag{5}$$

$$B_H^*(t) = C_H \left(H - \frac{1}{2}\right)^{-1} \int_{t}^{\infty} \left(u_+^{H-1/2} - (u-t)^{H-1/2}\right) dB(u). \tag{6}$$

It is seen from representations (5)–(6) that B_H is a "forward" fractional Brownian motion, since it depends only on the "past", namely on $\{B(u), -\infty < u \le t\}$. Similarly, B_H^* is a "backward" fractional Brownian motion, since it depends on the "future", namely on $\{B(u), t \le u < \infty\}$. In what follows we consider only stochastic equations with $B_H(t)$, since the solutions are adapted and depend on $\sigma\{B(u), u \le t\}$ in this case.

Note that the so-called "two-sided" fractional Brownian motion is considered in [1]. This process depends on the whole trajectory $\{B(t), t \in \mathbf{R}\}$. This property creates difficulties when calculating the mutual correlation of the processes B_H and B, since $B(t) - B(s)$, $s > u$, depends on $B_H(u)$.

Now we construct a linear combination of our operators,

$$M f(x) = \sum_{k=1}^{m} \sigma_k M_{H_k} f(x), \qquad H_k \in \left[\frac{1}{2}, 1\right), \ \sigma_k > 0,$$

and the corresponding linear combination of fractional Brownian motions with different Hurst indices,

$$B_M(t) = \sum_{k=1}^{m} \sigma_k B_{H_k}(t) = \left\langle M I_{[0,t]}, \omega \right\rangle.$$

3. Fractional white noise and its representation in terms of Hermite functions

In what follows we adopt the following notation of [1, 5]. Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, I be the set of all finite multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbf{N}_0$. Put $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1!\,\alpha_2! \cdots \alpha_n!$. Consider the Hermite polynomials

$$h_n(x) = (-1)^n \exp\left\{\frac{x^2}{2}\right\} \frac{d^n}{dx^n} \exp\left\{-\frac{x^2}{2}\right\}, \qquad n \ge 0,$$

and the Hermite functions

$$\tilde{h}_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} \cdot h_{n-1}\left(\sqrt{x}\right) \exp\left\{-\frac{x^2}{2}\right\}, \qquad n \ge 1.$$

It is well known that the functions $\tilde{h}_n(x)$, $n \ge 1$, form an orthonormal basis in $L_2(\mathbf{R})$. Let $H_\alpha(\omega) = \prod_{i=1}^n h_{\alpha_i}\left(\langle \tilde{h}_i, \omega\rangle\right)$, and assume that

$$F = F(\omega) \in L_2(S'(\mathbf{R}), \mathcal{F}, \mathsf{P}) = L_2(\Omega)$$

for a random variable F. Then

$$F(\omega) = \sum_{\alpha \in I} c_\alpha H_\alpha(\omega) \tag{7}$$

by [4], and

$$\|F\|^2_{L_2(\Omega)} = \sum_{\alpha \in I} \alpha!\, c_\alpha^2 < \infty.$$

Consider the following dual spaces.

(a) S: $F \in S$ if the coefficients of expansion (7) are such that

$$\|F\|_k^2 = \sum_{\alpha \in I} (\alpha!)^2 c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty$$

for all $k \ge 1$, where $(2\mathbb{N})^\gamma = \prod_{j=1}^m (2j)^{\gamma_j}$ and $\gamma = (\gamma_1, \dots, \gamma_m) \in I$.

(b) S^*: $F \in S^*$ if F admits a formal expansion (7) with a finite negative norm

$$\|F\|_{-q}^2 = \sum_{\alpha \in I} c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty$$

for at least one $q \in \mathbf{N}$ (we write $F \in S_{-q}$ in this case).

For

$$F = \sum_\alpha c_\alpha H_\alpha \in S, \qquad G = \sum_\alpha d_\alpha H_\alpha \in S^*$$

we set

$$\langle\!\langle F, G \rangle\!\rangle = \sum_\alpha \alpha!\, c_\alpha d_\alpha.$$

Further we define the spaces

$$L_M^2(\mathbf{R}) = \{f \colon Mf \in L_2(\mathbf{R})\} = \{f \colon \widehat{Mf} \in L_2(\mathbf{R})\},$$
$$L_{M^*}^2(\mathbf{R}) = \{f \colon M^* f \in L_2(\mathbf{R})\}.$$

The inner product in $L_M^2(\mathbf{R})$ and $L_{M^*}^2(\mathbf{R})$ is introduced by

$$\langle f, g\rangle_M = \int_{\mathbf{R}} Mf \cdot Mg\, dt, \qquad \langle f, g\rangle_{M^*} = \int_{\mathbf{R}} M^* f \cdot M^* g\, dt.$$

We also define the inverse operator M^{-1} in terms of the Fourier transform. Let

$$g(x) = M^{-1} f(x);$$

then

$$\hat{f}(\lambda) = \hat{g}(\lambda) \cdot \sum_{k=1}^{m} \sigma_k D_{H_k} |\lambda|^{1/2-H_k},$$

where $D_{H_k} = C_{H_k}(C^1_{H_k} + i \cdot \operatorname{sgn} \lambda \cdot C^2_{H_k})$. Thus

$$(\widehat{M^{-1}f})(\lambda) = \left(\sum_{k=1}^{m} \sigma_k D_{H_k} |\lambda|^{1/2-H_k} \right)^{-1} \hat{f}(\lambda).$$

It is clear that the functions $e_k = M^{-1}\tilde{h}_k$, $k \geq 1$, form an orthonormal basis in the space $L^2_M(\mathbf{R})$. Now we represent the linear combination of the fractional Brownian motions $B_M(t)$ in terms of $\tilde{h}_k$, $k \geq 1$, by using the conjugacy property of M and M^*.

Lemma 1. *The following representation holds:*

$$B_M(t) = \sum_{k=1}^{\infty} \int_0^t M^* \tilde{h}_k(x)\, dx \left\langle \tilde{h}_k, \omega \right\rangle, \qquad t \in \mathbf{R},\ \omega \in S'(\mathbf{R}), \tag{8}$$

and the series converges in $L_2(\Omega)$.

Proof. First assume that $\omega \in S(\mathbf{R})$. Using relation (2) we obtain

$$B_M(t) = \left\langle M I_{[0,t]}, \omega \right\rangle = \left\langle I_{[0,t]}, M^* \omega \right\rangle,$$

where $M^*\omega \in S(\mathbf{R})$. Since $I_{[0,t]} \in L^2_M(\mathbf{R})$, we have $I_{[0,t]} = \sum_{k=1}^{\infty} \left\langle I_{[0,t]}, e_k \right\rangle_M \cdot e_k$, where the series converges in $L^2_M(\mathbf{R})$. Then $\left\langle I_{[0,t]}, M^*\omega \right\rangle = \sum_{k=1}^{\infty} \left\langle I_{[0,t]}, e_k \right\rangle_M \cdot \langle e_k, M^*\omega \rangle$ and the series converges in $L_2(\Omega)$. Further,

$$\begin{aligned}
\sum_{k=1}^{\infty} \left\langle I_{[0,t]}, e_k \right\rangle_M \cdot \langle e_k, M^*\omega \rangle &= \sum_{k=1}^{\infty} \int_{\mathbf{R}} M I_{[0,t]}(x) M e_k(x)\, dx \cdot \langle M e_k, \omega \rangle \\
&= \sum_{k=1}^{\infty} \int_{\mathbf{R}} I_{[0,t]}(x) M^* \widetilde{h}_k(x)\, dx \cdot \left\langle \widetilde{h}_k, \omega \right\rangle \\
&= \sum_{k=1}^{\infty} \int_0^t M^* \widetilde{h}_k(x)\, dx \cdot \left\langle \widetilde{h}_k, \omega \right\rangle
\end{aligned}$$

and (8) is proved for $\omega \in S(\mathbf{R})$. Now we can extend (8) to the whole space $\Omega = S'(\mathbf{R})$, since $S(\mathbf{R})$ is tight in $S'(\mathbf{R})$ in the weak-star topology generating the weak convergence. Finally

$$\left\langle \widetilde{h}_k, \omega \right\rangle = H_{\varepsilon_k}(\omega), \qquad \varepsilon_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots),$$

and

$$\sum_{k=1}^{\infty} \left| \int_0^t M^* \widetilde{h}_k(x)\, dx \right|^2 (\varepsilon_k!)^2 = \sum_{k=1}^{\infty} \left| \int_0^t M^* \widetilde{h}_k(x)\, dx \right|^2 < \infty,$$

whence it follows that series (8) converges in $L_2(\Omega)$. □

Now we define the fractional white noise $W_H(t)$ as the formal series

$$W_H(t)(\omega) = \sum_{k=1}^{\infty} M^*_H \widetilde{h}_k(x)\, dx \cdot \left\langle \widetilde{h}_k, \omega \right\rangle.$$

The linear combination of fractal noises is given by

$$W_H(t)(\omega) := \sum_{k=1}^{\infty} M^* \widetilde{h}_k(x)\, dx \cdot \left\langle \widetilde{h}_k, \omega \right\rangle.$$

Lemma 2. *The fractional noises $W_H(t)$ and $W_M(t)$ belong to the space S^* for any $t \in \mathbf{R}$.*

Proof. It is sufficient to consider the case of $W_H(t)$. In what follows we denote by C the constants whose exact values have no importance for our consideration. Using the Fourier transform we obtain

$$\left|M^* \tilde{h}_k(x)\right| = C \left| \int_{\mathbf{R}} e^{ixt} \widehat{\tilde{h}_k(t)} |t|^{1/2-H}\, dt \right| \le C \left| \int_{|t|\le 1} \right| + C \left| \int_{|t|>1} \right|.$$

It is known that

$$\widehat{\tilde{h}_k(t)} = C\tilde{h}_k(t),$$
$$|\tilde{h}_k(t)| \le \begin{cases} Ck^{-1/12} & \text{if } |t| \le 2\sqrt{k}, \\ Ce^{-\gamma t^2} & \text{if } |t| > 2\sqrt{k}, \end{cases}$$

where the constants $C > 0$ and $\gamma > 0$ do not depend on k and $t \in \mathbf{R}$ (see [1, 5]). Therefore

$$\begin{aligned} \left|M_H^* \tilde{h}_k(x)\right| &\le C \Bigg(\int_{|t|\le 1} k^{-1/12} |t|^{1/2-H}\, dt + \int_{1<t\le 2\sqrt{k}} |t|^{1/2-H} k^{-1/12}\, dt \\ &\qquad + \int_{|t|>2\sqrt{k}} |t|^{1/2-H} e^{-\gamma t^2}\, dt \Bigg) \\ &\le C \left(k^{-1/12} + k^{-1/12} \cdot k^{3/4-H/2} + e^{-2\gamma\sqrt{k}} \right) \le C \left(k^{2/3-H/2} + k^{-1/2} \right) \\ &\le Ck^{2/3-H/2}, \end{aligned} \tag{9}$$

whence

$$\|W_H(t)\|_{-q}^2 = \sum_{k=1}^{\infty} \left|M^* \tilde{h}_k(t)\right|^2 (2k)^{-q} \le C \sum_{k=1}^{\infty} k^{4/3-H-q} < \infty$$

for $q > \frac{7}{3} - H$. Since $H \ge \frac{1}{2}$, we get that for $q > \frac{11}{6}$,

$$\|W_H(t)\|_{-q}^2 < \infty$$

for all $t \in \mathbf{R}$ and $H \ge \frac{1}{2}$. The proof is complete. $\square$

4. Wick products, integrals, and some representations

Let $F(\omega) = \sum_{\alpha \in I} c_\alpha H_\alpha(\omega)$ and $G(\omega) = \sum_{\alpha \in I} d_\alpha H_\alpha(\omega)$. Then the Wick product is defined by

$$(F \diamond G)(\omega) = \sum_{\alpha, \beta \in I} c_\alpha d_\beta H_{\alpha+\beta}(\omega).$$

According to [5], $F \diamond G \in S$ for $F, G \in S$, and $F \diamond G \in S^*$ for $F, G \in S^*$.

Let $Z \colon \mathbf{R} \to S^*$ and

$$\langle\!\langle Z(t), F \rangle\!\rangle \in L_1(\mathbf{R}), \qquad t \in \mathbf{R},$$

for all $F \in S$. We define the S^*-integral $\int_{\mathbf{R}} Z(t)\, dt$ as the unique element of S^* such that

$$\left\langle\!\!\left\langle \int_{\mathbf{R}} Z(t)\, dt, F \right\rangle\!\!\right\rangle = \int_{\mathbf{R}} \langle\!\langle Z(t), F \rangle\!\rangle\, dt.$$

Theorem 1. *Suppose $Y(t) \in S^*$ is represented as $Y(t) = \sum_{\alpha \in I} c_\alpha(t) H_\alpha(\omega)$, $t \in \mathbf{R}$, where the coefficients c_α satisfy $K := \sup_\alpha \left\{ \alpha! \cdot \|c_\alpha\|_{L_1(\mathbf{R})}^2 (2\mathbb{N})^{-q\alpha} \right\} < \infty$ for some $q > 0$.*

Then the Wick product $Y(t) \diamond W_M(t)$ is S^*-integrable and

$$\int_{\mathbf{R}} Y(t) \diamond W_M(t)\,dt = \sum_{\alpha,k} \int_{\mathbf{R}} c_\alpha(t) M^* \tilde{h}_k(t)\,dt \cdot H_{\alpha+\varepsilon_k}(\omega). \tag{10}$$

Proof. We consider the case of $W_H(t)$; the proof for $W_M(t)$ is the same. It is clear that $Y(t) \diamond W_H(t) \in S^*$ and the Wick product equals $\sum_{\alpha,k} c_\alpha(t) M^* \tilde{h}_k(t) H_{\alpha+\varepsilon_k}(\omega)$. We apply Lemmas 2.5.6 and 2.5.7 of [5] to prove that $Y(t) \diamond W_H(t)$ is S^*-integrable. According to these results, the S^*-integrability follows from

$$\sum_{\beta \in I} \beta! \cdot \left\| \sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} c_\alpha(t) M_H^* \tilde{h}_k(t) \right\|^2_{L_1(\mathbf{R})} (2\mathbb{N})^{(-p\beta)} < \infty$$

for some $p > 0$. We proved in Lemma 2 that $\left| M_H^* \tilde{h}_k(t) \right| < Ck^{2/3-H/2} < Ck^{5/12}$ for all $k \geq 1$ and some $C > 0$. Therefore

$$\int_{\mathbf{R}} \left| c_\alpha(t) M_H^* \tilde{h}_k(t) \right| dt \leq Ck^{5/12} \|c_\alpha\|_{L_1(\mathbf{R})}$$

and

$$\begin{aligned} \left\| \sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} c_\alpha(t) M_H^* \tilde{h}_k(t) \right\|^2_{L_1(\mathbf{R})} &\leq \left(\sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} \left\| c_\alpha(t) M_3 \tilde{h}_k(t) \right\|_{L_1(\mathbf{R})} \right)^2 \\ &< C \left(\sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} k^{5/12} \|c_\alpha\|_{L_1(\mathbf{R})} \right)^2. \end{aligned}$$

Further

$$\begin{aligned} S &= \sum_{\beta \in I} \beta! \cdot \left(\sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} k^{(5/12)} \|c_\alpha(t)\|_{L_1(\mathbf{R})} \right)^2 (2\mathbb{N})^{(-p\beta)} \\ &\leq \sum_{\beta \in I} \beta! \cdot (l(\beta))^{5/6} \cdot \left(\sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} \|c_\alpha(t)\|_{L_1(\mathbf{R})} \right)^2 (2\mathbb{N})^{(-p\beta)} \end{aligned}$$

where $l(\beta)$ equals the subscript of the last nonzero element in the index β (the length of the index β). It is shown in the proof of Lemma 2.5.7 in [5] that for all α and β there exists at most one number k such that $\alpha + \varepsilon_k = \beta$, whence

$$\left(\sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} \|c_\alpha(t)\|_{L_1(\mathbf{R})} \right)^2 \leq l(\beta)^2 \sum_{\alpha,k:\ \alpha+\varepsilon_k=\beta} \|c_\alpha\|^2_{L_1(\mathbf{R})}.$$

Thus

$$\begin{aligned} S &\leq \sum_{\alpha,k} (\alpha+\varepsilon_k)!\, (l(\alpha+\varepsilon_k))^{17/6} \|c_\alpha\|^2_{L_1(\mathbf{R})} (2\mathbb{N})^{-p\alpha} (2\mathbb{N})^{-p\varepsilon_k} \\ &\leq K \sum_{\alpha,k} \frac{(\alpha+\varepsilon_k)!}{\alpha!} (l(\alpha+\varepsilon_k))^3 (2\mathbb{N})^{-(p-q)\alpha} (2\mathbb{N})^{-p\varepsilon_k} \\ &\leq K \sum_{\alpha,k} (|\alpha|+1)^4 2^{-|\alpha|(p-q)} k^{-p} < \infty \end{aligned}$$

for $p > q + 1$. This result means that $Y(t) \diamond W_H(t)$ is S^*-integrable.

By the definitions of the S^*-integral and Wick product,

$$\begin{aligned}\left\langle\!\!\left\langle \int_{\mathbf{R}} Y(t)\diamond W_H(t)\,dt, F\right\rangle\!\!\right\rangle &= \int_{\mathbf{R}}\left\langle\!\!\left\langle \sum_{\alpha,k} c_\alpha(t) M_H^*\tilde{h}_k(t) H_{\alpha+\varepsilon_k}(\omega), F\right\rangle\!\!\right\rangle dt \\ &= \int_{\mathbf{R}}\sum_{\alpha,k}(\alpha+\varepsilon_k)!\, c_\alpha(t)\, d_{\alpha,k} M_H^*\tilde{h}_k(t)(\omega)\,dt\end{aligned} \tag{11}$$

for all $F\in S$, $F=\sum_{\beta,k} H_{\beta+\varepsilon_k}(\omega)$. Note that

$$\sum_{\alpha,k}(\alpha+\varepsilon_k)!\,|d_{\alpha,k}|^2(2\mathbb{N})^{2q(\alpha+\varepsilon_k)} = C_q<\infty$$

for all $q\in\mathbb{N}$, whence

$$\begin{aligned}\sum_{\alpha,k}\int_{\mathbf{R}}(\alpha+\varepsilon_k)!\,|c_\alpha(t)|\cdot|d_{\alpha,k}| M_H^*\tilde{h}_k(t)\,dt &\le \sum_{\alpha,k}(\alpha+\varepsilon_k)!\,|d_{\alpha,k}| k^{5/12}\|c_\alpha(t)\|_{L_1(\mathbf{R})} \\ &\le \sum_{\alpha,k}(\alpha+\varepsilon_k)!\,|d_{\alpha,k}|^2(2\mathbb{N})^{2q(\alpha+\varepsilon_k)}\sum_{\alpha,k}k^{5/6}\|c_\alpha\|^2_{L_1(\mathbf{R})}(\alpha+\varepsilon_k)!\,(2\mathbb{N})^{-2q(\alpha+\varepsilon_k)} \\ &\le C_q\cdot K\sum_{\alpha,k}k^{5/6}(|\alpha|+1)(2\mathbb{N})^{-q|\alpha|}k^{-2q}<\infty\end{aligned}$$

for $q>\frac{11}{12}$. Interchanging the sum and the integral in (11) we obtain

$$\begin{aligned}\left\langle\!\!\left\langle \int_{\mathbf{R}} Y(t)\diamond W_H(t)\,dt, F\right\rangle\!\!\right\rangle &= \sum_{\alpha,k}(\alpha+\varepsilon_k)!\,d_{\alpha,k}\int_{\mathbf{R}} c_\alpha(t)M_H^*\tilde{h}_k(t)(\omega)\,dt \\ &= \left\langle\!\!\left\langle \sum_{\alpha,k}\int_{\mathbf{R}} c_\alpha(t)M_H^*\tilde{h}_k(t)(\omega)\,dt, F\right\rangle\!\!\right\rangle,\end{aligned}$$

and (10) follows. □

Corollary 1. *Let $Y(t)=\sum_\alpha c_\alpha(t)H_\alpha(\omega)\in S^*$ be a stochastic process such that*

$$\int_0^T \mathsf{E}\,Y^2(t)\,dt<\infty$$

for some $T>0$. Then

$$\sum_\alpha \alpha!\int_0^T c_\alpha^2(t)\,dt = \int_0^T \mathsf{E}\,Y^2(t)\,dt<\infty,$$

whence $K := \sup_\alpha\{\alpha!\,\|\overline{c}_\alpha\|^2_{L_1(\mathbf{R})}(2\mathbb{N})^{-q\alpha}\}<\infty$ for all $q>0$ (we use the notation $\overline{c}_\alpha = c_\alpha(t)I_{[0,T]}(t)$). By Theorem 1 this means that $Y(t)\diamond W_M(t)$ is S^*-integrable and (10) holds.

Corollary 2. *Let $Y(t)\equiv 1$. Then it follows from Corollary 1 that*

$$\int_0^T W_M(t)\,dt = \sum_{\alpha,k}\int_0^T M^*\tilde{h}_k(t)\,dt\cdot H_{\alpha+\varepsilon_k}(\omega) = \sum_k\int_0^T M^*\tilde{h}_k(t)\,dt\cdot\left\langle \tilde{h}_k,\omega\right\rangle = B_M(T).$$

In this sense, we say that a fractional noise is the S^*-derivative of a fractional Brownian motion.

Consider the case where $Y(t) \in L_2(\mathbf{R})$ is a nonrandom function. Then $c_\alpha(t) = Y(t)$ for $\alpha = 0$ and $c_\alpha(t) = 0$ otherwise. It follows from Theorem 1 that

$$\begin{aligned} \int_0^T Y(t) \diamond W_M(t)\,dt &= \sum_k \int_0^T Y(t) M^* \tilde{h}_k(t)\,dt \cdot \left\langle \tilde{h}_k, \omega \right\rangle \\ &= \sum_k \int_{\mathbf{R}} M\overline{Y}_T(t)\tilde{h}_k(t)\,dt \cdot \left\langle \tilde{h}_k, \omega \right\rangle \\ &= \sum_k \int_{\mathbf{R}} M\overline{Y}_T(t)\tilde{h}_k(t)\,dt \cdot H_{\varepsilon_k}(\omega), \\ \overline{Y}_T(t) &= Y(t) I_{[0,T]}(t). \end{aligned} \tag{12}$$

The right-hand side of (12) is the same as that of equality (2.5.22) in [5]. Thus the left-hand sides also are equal, and

$$\begin{aligned} \int_0^T Y(t) \diamond W_M(t)\,dt &= \int_{\mathbf{R}} M\overline{Y}_T(t) \diamond W_M(t)\,dt = \int_{\mathbf{R}} M\overline{Y}_T(t) \cdot W_M(t)\,dt \\ &= \int_{-\infty}^T \sum_{k=1}^m \left(\sigma_k c_{H_k} \int_{0\vee t}^T Y(x)(x-t)^{H_k - 3/2}\,dx \right) W_M(t)\,dt. \end{aligned} \tag{13}$$

Now let $Y(t)$, $t \in \mathbb{R}$, be a stochastic process such that $Y(t) \diamond W_M(t)$ is S^*-integrable on any interval $[0, T]$. Let

$$\int_0^T Y(t)\,dB_M(t) := \int_0^T Y(t) \diamond W_M(t)\,dt, \qquad T > 0.$$

Then it follows from (13) and Theorems 2.5.4 and 2.5.9 in [5] that

$$\begin{aligned} \int_0^T Y(t)\,dB_M(t) &= \int_{\mathbf{R}} M\overline{Y}_T(t) \cdot W_M(t)\,dt = \int_{\mathbf{R}} M\overline{Y}_T(t)\,\delta B(t) \\ &= \int_{\mathbf{R}} M\overline{Y}_T(t)\,dB(t) \end{aligned} \tag{14}$$

for nonrandom $Y \in L_2(\mathbf{R})$, where the symbols δ and d stand for the Skorokhod and Itô integrals, respectively. Therefore, the S^*-integral $\int_0^T Y(t) \diamond W_M(t)\,dt$ is an ordinary Itô integral with nonrandom integrand

$$C_H \int_{0\vee t}^T Y(x)(x-t)^{H-3/2}\,dx$$

of Volterra type if $Y \in L_2(\mathbf{R})$ is nonrandom. Note that

$$\begin{aligned} \left\| M_H \overline{Y}_T \right\|_{L_2(\mathbf{R})}^2 &= c_H^2 \int_{-\infty}^0 \left(\int_0^T Y(x)(x-t)^{H-3/2} dx \right)^2 dt \\ &\quad + c_H^2 \int_0^T \left(\int_t^T Y(x)(x-t)^{H-3/2}\,dx \right)^2 dt \\ &\le \|Y\|_{L_2[0,T]}^2 \cdot c_H^2 \cdot (1-H)^{-1} \cdot \int_{-\infty}^0 \left((T-t)^{2H-2} - (-t)^{2H-2} \right) dt < \infty, \end{aligned}$$

so that the Itô integral is well defined.

Note also that it is proved in [1] that the integrals $\int_{\mathbf{R}} Y(t)\,dB_M(t)$ and $\int_{\mathbf{R}} MY(t)\,dB(t)$ coincide in the case of a nonrandom Y and a "two-sided" fractional Brownian motion.

5. Stochastic differential equations with a fractional white noise

Consider two possible methods for solving stochastic differential equations with a fractional noise.

5.1. **Lipschitz and growth conditions posed on negative norms of coefficients.** Consider a stochastic differential equation of the form

$$X(t) = X_0 + \int_0^t a(s, X(s))\, ds + \sum_{k=1}^m \int_0^t b_k(s, X(s)) \diamond W_{H_k}(s)\, ds, \qquad 0 \le t \le T, \tag{15}$$

where $H_k \in [\frac{1}{2}, 1)$ for all $k \le m$, and $H_i \ne H_j$, $i \ne j$. The case of equation (15) with a usual (nonfractional) white noise is considered in [6]. Note that the proof presented in [6] does not use the structure of a white noise; it is based on the fact that the white noise belongs to the space S^*, and this is true for fractional noises as well. If $F \in S_{-r}$, $G \in S_{-q}$, and $r < q - 1$, then

$$\|F \diamond G\|_{-r} \le C_{r,q} \|F\|_{-r} \|G\|_{-q}$$

by Theorem 1 in [6]. According to Lemma 2, $W_{H_k}(t) \in S_{-q}$ for all $q > \frac{11}{6}$. In particular, $W_{H_k}(t) \in S_{-2}$. Moreover, $\sup_{t\ge 0} \|W_{H_k}(t)\|_{-2} \le C$ for some $C > 0$. Thus

$$\|F \diamond W_{H_k}(t)\|_{-r} \le C\|F\|_{-r}$$

for $r < -3$, $F \in S_{-r}$, and $t > 0$.

Assume that the coefficients a and b and initial value X_0 of equation (15) satisfy

(A) for all $T > 0$, $1 \le k \le m$, and some $r > 3$

$$a, b_k \colon [0, T] \times S_{-r} \to S_{-r},$$

$X_0 \in S_{-r}$, and the functions $a(t, X(t))$ and $b_k(t, X(t))$, $1 \le k \le m$, are strongly measurable on $[0, T]$ for all $X \in C([0, T], S_{-r})$;

(B) The Lipschitz and growth conditions on negative norms of coefficients a and b_k, namely

$$\|a(t, x) - a(t, y)\|_{-r} + \sum_{k=1}^m \|b_k(t, x) - b_k(t, y)\|_{-r} \le C\|x - y\|_{-r}, t \le T;$$

$$\|a(t, x)\|_{-r} + \sum_{k=1}^m \|b_k(t, x)\|_{-r} \le C(1 + \|x\|_{-r}), t \le T.$$

Since b_k are strongly measurable, it follows from Theorem 6 of [6] that

$$b_k(t, X(t)) \diamond W_{H_k}(t)$$

also is strongly measurable. The existence of the integrals

$$\int_0^t a(s, X(s))\, ds$$

and

$$\int_0^t b_k(s, X(s)) \diamond W_{H_k}(s)\, ds$$

follows from condition (B). The latter integrals can be viewed as Bochner integrals in S_{-r} if $X \in C([0, T], S_{-r})$.

The following result can be proved by the standard successive approximation method (a similar proof for a white noise can be found in [6]).

Theorem 2. *Assume conditions (A) and (B). Then equation* (15) *has a unique solution on* $[0, T]$*, and this solution belongs to* $C([0, T], S_{-r})$.

5.2. **Quasi-linear stochastic differential equations with a fractional noise.** The Lipschitz and growth conditions on negative norms of coefficients together are very restrictive (see [6]). Consider a stochastic differential equation for which one can drop these conditions, namely the following quasi-linear equation:

$$X_t = X_0 + \int_0^t a(s, X(s))\, ds + \sum_{k=1}^{m} \int_0^t \sigma_k(s) X(s) \diamond W_{H_k}(s)\, ds, \tag{16}$$

where $H_k \in [\frac{1}{2}, 1)$ for all $k \le m$, and the coefficients and initial value X_0 are such that

(C) $\sigma_k(s)$, $1 \le k \le m$, are nonrandom functions, $\sigma_k \in L_{1/H_k}[0,T]$;

(D) the function $a(s,x,w)\colon [0,T] \times \mathbf{R} \times S'(\mathbf{R}) \to \mathbf{R}$ is measurable in all its arguments,

$$|a(s,x,w)| \le c(1+|x|), \qquad w \in S'(\mathbf{R}),\ 0 \le s \le T,\ x \in \mathbf{R},$$
$$|a(s,x,w) - a(s,y,w)| \le c|x-y|, \qquad x,y \in \mathbf{R},\ w \in S'(\mathbf{R}),\ 0 \le s \le T;$$

(E) $X_0 \in L_p(\Omega)$ for some $p > 0$.

Theorem 3. *Assume conditions (C)–(E). Then equation* (16) *has a unique solution X on $[0,T]$, and moreover $X \in L_{p'}(\mathsf{P})$ for all $p' < p$.*

Proof. For simplicity we give the proof for the case of $m = 1$ and $H = H_1 > \frac{1}{2}$. Consider the differential form of equation (16),

$$\frac{dX(t)}{dt} = a(t, X(t)) + \sigma(t) X(t) \diamond W_H(t), \qquad X(0) = X_0. \tag{17}$$

Set $\sigma_t(s) = \sigma(s) I_{[o,t]}(s)$ and let

$$J_\sigma(t) = \exp^\diamond \left(-\int_0^t \sigma(s)\, dB_H(s) \right),$$

where

$$\exp^\diamond X = \sum_{n=0}^\infty \frac{X^{\diamond n}}{n!}$$

is the Wick exponential. It follows from (14) that

$$J_\sigma(t) = \exp^\diamond \left\{ -\int_{\mathbf{R}} M_H \sigma_t(s)\, dB(s) \right\}.$$

Put $Z(t) := J_\sigma(t) \diamond X(t)$. By the rules of stochastic differentiation given in [5],

$$\frac{dZ(t)}{dt} = J_\sigma(t) \diamond \frac{dX(t)}{dt} - \frac{dJ_\sigma(t)}{dt} \diamond X(t) \diamond W_H(t) \cdot \sigma(t).$$

Taking the Wick product of both sides with $\frac{dJ_\sigma(t)}{dt}$, we obtain from (17) that

$$\frac{dZ(t)}{dt} = \frac{dJ_\sigma(t)}{dt} \diamond a(t, X(t)). \tag{18}$$

Now we apply the Giessing lemma:

$$\frac{dJ_\sigma(t)}{dt} \diamond a(t, X(t), w) = \frac{dJ_\sigma(t)}{dt} \cdot a\left(t, T_{-M_H\sigma_t} X(t), w - M_H \sigma_t\right), \tag{19}$$

where T is the shift operator and $T_{w_0} f(w) = f(w + w_0)$. Similarly,

$$Z(t) = J_\sigma(t) \cdot T_{-M_H\sigma_t} X(t).$$

By (18)–(19), $Z(t)$ is a solution of the ordinary differential equation

$$\frac{dZ(t)}{dt} = \frac{dJ_\sigma(t)}{dt} \cdot a\left(t, J_\sigma^{-1}(t) \cdot Z(t), w - M_H \sigma_t\right) \tag{20}$$

for every $w \in S'(\mathbf{R})$, $Z(0) = X_0$.

The only difference between equations (20) and (3.6.15) in [5] is that the function $M_H\sigma_t$ is involved in (20) instead of σ_t in (3.6.15). Since the structures of $M_H\sigma_t$ and σ_t are the same, equation (20) has a unique solution on $[0,T]$ for all $w \in \Omega = S'(\mathbf{R})$ (here we take conditions (C)–(D) into account).

Now we estimate the moments of the solution $X(t)$. It follows from condition (C) that

$$\begin{aligned}|Z(t)| &\le |X_0| + \int_0^t J_\sigma(s)\left|a\left(s, J_\sigma^{-1}(s)Z(s), w - M_H\sigma_s\right)\right| ds\\ &\le |X_0| + C\int_0^t J_\sigma(s)\left(1 + J_\sigma^{-1}(s)|Z(s)|\right) ds\\ &\le |X_0| + C\int_0^t J_\sigma(s)\,ds + C\int_0^t |Z(s)|\,ds,\end{aligned}$$

whence

$$|Z(t)| \le \left(|X_0| + C\int_0^T J_\sigma(s)\,ds\right)\exp\{CT\},$$

$$\mathsf{E}\,|Z(t)|^p \le \exp\{pCT\}2^p\left(\mathsf{E}\,|X_0|^p + CE\int_0^T |J_\sigma(s)|^p\,ds\right) \tag{21}$$

by the Gronwall inequality. Since

$$\mathsf{E}\,|J_\sigma(s)|^p = \mathsf{E}\exp^\diamond\left\{-p\int_{\mathbf{R}}(M_H\sigma_t)(s)\,dB(s)\right\} = \exp\left\{p^2\|M_H\sigma_t\|_2^2\right\},$$

we obtain from (21) that

$$\mathsf{E}\,|Z(t)|^p < \infty$$

as condition (C) implies that $M_H\sigma_t \in L_2(\mathbf{R})$. Further,

$$T_{-M_H\sigma_t}X(t) = Z(t)J_\sigma^{-1}(t),$$

and $\mathsf{E}\,|J_\sigma^{-1}(t)|^q < \infty$ for all $q > 0$. Therefore

$$T_{-M_H\sigma_t}X(t) \in L_{p'}(\Omega)$$

for all $p' < p$. Since $M_H\sigma_t \in L_2(\mathbf{R})$, we have $X \in L_{p'}(\Omega)$ for all $p' < p$ by Corollary 2.10.5 in [5]. $\square$

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