

## AN ESSAY ON GNEDENKO'S THEOREM

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ABSTRACT. A local central limit theorem is established for a Markov chain on a lattice under recurrence type assumptions and a simple additional assumption on conditional distributions of the process. The main result extends, in particular, the classical theorem by B. V. Gnedenko for the case of independent identically distributed random variables (1948). Sufficient conditions for recurrence assumptions are provided.

### 1. INTRODUCTION

In the famous textbook on probability by Boris Vladimirovich Gnedenko there is a local central limit theorem for random variables on a lattice (see [3, §43, Theorem] or [2]), which generalizes the local central limit theorem for Bernoulli random variables and some other local theorems. This is a classical result obtained at the time of intensive studies of independent and identically distributed random variables. (See [13, Chapter 7] for some generalizations.) Nowadays, recurrent Markov and stationary mixing processes play an extremely important role in mathematical models. Hence it is natural to ask what could be a version of the Gnedenko theorem today, in the setting of dependent random variables, as a direct development of his ideas. This is the explanation of the title of the paper.

In Section 2 we recall the Gnedenko result, Theorem 1, for the case of independent identically distributed random variables. Section 3 contains the model of a Markov chain and the main result, Theorem 2 (the local central limit theorem), in this setting; naturally, this is an analogue of Theorem 1 for ergodic Markov chains. Section 4 is devoted to the proof of the Theorem 2. Section 5 contains results on mixing and the (integral) central limit theorem for ergodic Markov chains, mostly from [15]; this section is independent of the other parts of the paper. The main result, Theorem 2, states that once a Markov process satisfies the central limit theorem, under certain additional assumptions it must also satisfy the local central limit theorem with a corresponding maximal span of the lattice. We present its simplest version, namely for  $\mathbf{R}^1$  and on a fixed lattice (this version corresponds to Gnedenko's Theorem 1). More challenging are versions on a lattice with a small span depending on a parameter; we do not discuss this problem here. The key point of the method is a regeneration of the Markov chain. It is not clear whether this approach works for non-Markov stationary sequences with mixing. Again we present the simplest version with a regeneration at one (initial) state. This assumption can be relaxed by using the coupling (splitting) technique under wider assumptions.

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The results of Section 5 on the central limit theorem and mixing from [15] are based on the coupling method; in a certain nonstrict sense this method itself arose from the renewal or regeneration theory. Hence this version of the central limit theorem, Proposition 4 below, relates to the results of Gnedenko's (with coauthors) classical monographs on queueing [5] and reliability theory [4].

Among the literature on local limit theorems for Markov chains we mention [8] concerning the case of a finite state space, and [11] concerning uniformly ergodic, and hence with an exponential convergence, infinite lattice case; the second paper uses essentially Gnedenko's ideas. Our case, roughly speaking, is nonuniform polynomial ergodic.

## 2. GNEDENKO'S THEOREM

Let  $h > 0$  be a fixed real value and  $(\xi_n, n \geq 0)$  a sequence of independent identically distributed random variables with values in the lattice  $Z_{h,a}^1 = \{a, a \pm h, a \pm 2h, \dots\}$ , and put  $\zeta_n = \sum_{k=1}^n \xi_k$ . It is a standing assumption that  $h$  is the maximal span of the distribution of  $\xi_1$  (cf. [3]). Put  $P_n(k) = \mathbb{P}(\zeta_n = na + kh)$  and

$$z_{nk} = \frac{an + kh - A_n}{B_n}, \quad A_n = \mathbb{E} \zeta_n, \quad B_n^2 = \text{Var}(\zeta_n).$$

It is assumed that  $B_1^2 > 0$ .

**Theorem 1** (Gnedenko, 1948). *Let  $\mathbb{E} \xi_n^2 < \infty$ . Then the relation*

$$\frac{B_n}{h} P_n(k) - \frac{1}{\sqrt{2\pi}} \exp(-z_{nk}^2/2) \rightarrow 0$$

*holds uniformly in  $-\infty < k < \infty$  as  $n \rightarrow \infty$  if and only if the distribution span  $h$  is maximal.*

## 3. ERGODIC MARKOV PROCESS SETTING

For simplicity we assume that  $a = 0$  and  $X_0 = 0$ . The main example is as follows. Consider  $(X_n)$  defined by a stochastic difference equation

$$(1) \quad X_{n+1}^h = X_n^h + f_h(X_n^h) + W_{n+1}^h, \quad n \geq 0,$$

with  $X_0 = 0$ ,  $f_h: Z_{h,0} \rightarrow Z_{h,0}$ , and where  $W_n^h$  are independent identically distributed random variables with values in  $Z_{h,0}$ . Put  $S_n = \sum_{k=1}^n X_k$ ,  $P_n(k) = \mathbb{P}(S_n = kh)$ , and

$$z_{nk} = \frac{kh - A_n}{B_n}, \quad A_n = \mathbb{E} S_n, \quad B_n^2 = \text{Var}(S_n)$$

(that is, both  $\mathbb{E}$  and  $\text{Var}$  are taken with respect to the initial measure  $\delta(\{0\})$ ). It can be shown that both normalizing constants can be calculated with respect to the invariant measure instead, since the difference disappears in the asymptotic sense. However, from the point of view of the proof, our choice is more convenient. As usual in central limit theorems for dependent random variables, it is assumed that

$$(H_{\text{sigma}}) \quad \lim_{n \rightarrow \infty} B_n^2/n = \sigma^2 > 0.$$

Let  $\tau = \inf(t \geq 1: X_t = 0)$ ,  $\tau_1 = \tau$ , and  $\tau_{n+1} = \inf(t \geq \tau_n + 1: X_t = 0)$ ,  $n \geq 1$ . The following is the main recurrence assumption: there exist  $\kappa > 0$ ,  $b > \frac{1}{2}$ , and  $C > 0$  such that

$$(H_{\text{re}}) \quad \mathbb{P}(\tau_{[\kappa n]} > n) \leq C/n^b \quad \text{for all } n \geq 1.$$

This can be relaxed to  $\sqrt{n} \mathbb{P}(\tau_{[\kappa n]} > n) \rightarrow 0$  as  $n \rightarrow \infty$ ; we use  $(H_{\text{re}})$  because this condition is satisfied for all our examples (see Section 5).

The following is the main assumption concerning the span: for any  $k > 1$  and any  $\varepsilon > 0$

$$(H_{\text{span}}) \quad \sup_{\varepsilon < |\lambda| < 2\pi/h - \varepsilon} |\mathbb{E}(e^{i\lambda S_\tau} \mid \tau = k)| < 1.$$

This condition essentially means that for each  $k > 1$ ,  $S_\tau \mathbb{1}(\tau = k)$  has the maximal span  $h$ . (Notice that  $\tau = 1$  means  $X_1 = 0$ , hence, there is no maximal span in this case.) Condition  $(H_{\text{span}})$  can be relaxed to the same inequality for *some*  $k > 1$  satisfying

$$P(\tau = k) > 0$$

(see the proof). In the model (1), for  $(H_{\text{span}})$  it suffices that  $P(W_1 = mh) > 0$  for all  $m = 0, \pm 1, \pm 2, \dots$ .

**Theorem 2.** *Let  $(H_{\text{span}})$  and  $(H_{\text{sigma}})$  hold true, and  $S_n$  satisfy the central limit theorem, that is,  $(S_n - A_n)/B_n$  converges weakly to  $\mathcal{N}(0, \sigma^2)$  with  $\sigma^2 > 0$ . If the bounds  $(H_{\text{re}})$  hold true, then the relation*

$$\frac{B_n}{h} P_n(k) - \frac{1}{\sqrt{2\pi}} \exp(-z_{nk}^2/2) \rightarrow 0$$

holds uniformly in  $-\infty < k < \infty$  as  $n \rightarrow \infty$ .

#### 4. PROOF OF THEOREM 2

Let  $\bar{S}_n = S_n - A_n$  and  $\phi_n(\lambda) = \mathbb{E} \exp(i\lambda \bar{S}_n)$ . Then  $\mathbb{E} \exp(i\lambda \bar{S}_n/B_n) = \phi_n(\lambda/B_n)$ . Since  $S_n$  satisfies the central limit theorem, for any  $\lambda \in \mathbf{R}^1$

$$\phi_n(\lambda/B_n) \rightarrow \exp(-\lambda^2/2), \quad n \rightarrow \infty.$$

Similarly to Gnedenko's proof, we use the backward Fourier transform. Since

$$\phi_n(\lambda) = \sum_k P_n(k) e^{i\lambda(kh - A_n)}$$

and  $kh = B_n z_{nk} + A_n$ , we get

$$\begin{aligned} P_n(k) &= \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \phi_n(\lambda) e^{-i\lambda kh + i\lambda A_n} d\lambda = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \phi_n(\lambda) e^{-iB_n z_{nk} \lambda} d\lambda \\ &= \frac{h}{2B_n \pi} \int_{-B_n \pi/h}^{B_n \pi/h} \phi_n(\lambda/B_n) e^{-i\lambda z_{nk}} d\lambda \end{aligned}$$

(change of variables  $\lambda \mapsto \lambda/B_n$ ). We split the latter integral (without  $h/(2B_n \pi)$ ) into three parts, namely

$$\begin{aligned} J_1 &= \int_{-A}^A e^{-iz_{nk}\lambda} \phi_n(\lambda/B_n) d\lambda, \\ J_2 &= \int_{A < |\lambda| < \varepsilon B_n} e^{-iz_{nk}\lambda} \phi_n(\lambda/B_n) d\lambda, \\ J_3 &= \int_{\varepsilon B_n < |\lambda| < \pi B_n/h} e^{-iz_{nk}\lambda} \phi_n(\lambda/B_n) d\lambda. \end{aligned}$$

Here  $A$  and  $\varepsilon > 0$  are constants to be chosen later.

$J_1$ . With any  $A$ , due to the central limit theorem for  $S_n$  and uniformly with respect to  $k$ ,

$$J_1 - \int_{-A}^A e^{-iz_{nk}\lambda} e^{-\lambda^2/2} d\lambda \rightarrow 0, \quad n \rightarrow \infty.$$

If  $A$  is large enough, then this integral is close to

$$\int_{-\infty}^{\infty} e^{-iz_{nk}\lambda} e^{-\lambda^2/2} d\lambda = \int_{-\infty}^{\infty} e^{-(\lambda+iz_{nk})^2/2} e^{-z_{nk}^2/2} d\lambda = \sqrt{2\pi} e^{-z_{nk}^2/2}$$

uniformly with respect to  $z_{nk}$ . Indeed,

$$\int_{|\lambda|>A} \exp(-\lambda^2/2) d\lambda \rightarrow 0, \quad A \rightarrow \infty.$$

It remains to show that  $J_2$  and  $J_3$  tend to zero as  $n \rightarrow \infty$ .

$J_3$ . We represent  $\phi_n(\lambda/B_n)$  in the following form, with  $\theta := \lambda/B_n$ :

$$\mathbf{E} e^{i\theta \bar{S}_n} = \mathbf{E} e^{i\theta \bar{S}_n} \mathbb{1}(\tau_{[\kappa n]} \leq n) + \mathbf{E} e^{i\theta \bar{S}_n} \mathbb{1}(\tau_{[\kappa n]} > n).$$

According to  $(H_{\text{re}})$ ,

$$\left| \mathbf{E} e^{i\theta \bar{S}_n} \mathbb{1}(\tau_{[\kappa n]} > n) \right| \leq Cn^{-b}$$

where  $b > \frac{1}{2}$ . We use the notation

$$\bar{S}_{m,n} = \sum_{k=m}^n (X_k - \mathbf{E} X_k) \quad \text{and} \quad F_{\tau_j, \infty} = \sigma\{\tau_j; X_k, \tau_j \leq k < \infty\}.$$

Next,

$$\begin{aligned} \mathbf{E} e^{i\theta \bar{S}_n} \mathbb{1}(\tau_{[\kappa n]} \leq n) &= \mathbf{E} e^{i\theta \bar{S}_{\tau_{[\kappa n]}}} e^{i\theta(\bar{S}_n - \bar{S}_{\tau_{[\kappa n]})}} \mathbb{1}(\tau_{[\kappa n]} \leq n) \\ &= \mathbf{E} e^{i\theta \bar{S}_{\tau_1+1, \tau_{[\kappa n]}}} e^{i\theta(\bar{S}_n - \bar{S}_{\tau_{[\kappa n]})}} \mathbb{1}(\tau_{[\kappa n]} \leq n) \left( \mathbf{E} \left( e^{i\theta \bar{S}_{\tau_1}} \mid F_{\tau_1, \infty} \right) \right). \end{aligned}$$

Indeed, all inner terms except one are measurable with respect to the  $\sigma$ -field  $F_{\tau_1, \infty}$ . Now use the fact that the  $\sigma$ -fields  $F_{\tau_k}$  and  $F_{\tau_k, \infty}$  are independent for any  $k$ , given  $\tau_k$ . Thus,

$$\mathbf{E} \left( e^{i\theta \bar{S}_{\tau_1}} \mid F_{\tau_1, \infty} \right) = \mathbf{E} \left( e^{i\theta \bar{S}_{\tau_1}} \mid \tau_1 \right) =: \psi_\theta(\tau_1).$$

For any  $\tau_1 > 1$ ,

$$\sup_{\varepsilon \leq \theta \leq (2\pi/h) - \varepsilon} |\psi_\theta(\tau_1)| < 1;$$

see  $(H_{\text{span}})$ . This implies that

$$(2) \quad \sup_{\varepsilon \leq \theta \leq (2\pi/h) - \varepsilon} \mathbf{E} |\psi_\theta(\tau_1)|^2 =: \exp(-c_\varepsilon) < 1.$$

By induction, we get

$$\begin{aligned} &\mathbf{E} \exp \{ i\theta \bar{S}_{1, \tau_{[\kappa n]}} \} \exp \{ i\theta (\bar{S}_n - \bar{S}_{\tau_{[\kappa n]}}) \} \mathbb{1}(\tau_{[\kappa n]} \leq n) \\ &= \mathbf{E} \exp \{ i\theta (\bar{S}_n - \bar{S}_{\tau_{[\kappa n]}}) \} \mathbb{1}(\tau_{[\kappa n]} \leq n) \left( \prod_{k=1}^{[\kappa n]} \psi_\theta(\tau_k - \tau_{k-1}) \right). \end{aligned}$$

(We set  $\tau_0 = 0$  here.) Therefore

$$\begin{aligned}
 (3) \quad \left| \mathbb{E} e^{i\theta \bar{S}_n} \right| &\leq Cn^{-b} + \left| \mathbb{E} \exp \{ i\theta (\bar{S}_n - \bar{S}_{\tau_{[\kappa n]}}) \} \mathbb{1} (\tau_{[\kappa n]} \leq n) \prod_{k=1}^{[\kappa n]} \psi_\theta(\tau_k - \tau_{k-1}) \right| \\
 &\leq Cn^{-b} + \left( \mathbb{E} \prod_{k=1}^{[\kappa n]} |\psi_\theta(\tau_k - \tau_{k-1})|^2 \right)^{1/2} \\
 &= Cn^{-b} + \left( \prod_{k=1}^{[\kappa n]} \mathbb{E} |\psi_\theta(\tau_k - \tau_{k-1})|^2 \right)^{1/2} \\
 &= Cn^{-b} + (\mathbb{E} |\psi_\theta(\tau)|^2)^{[\kappa n]/2},
 \end{aligned}$$

since  $(\tau_k - \tau_{k-1})$  are independent and identically distributed.

Finally, due to (2),

$$\left| \mathbb{E} e^{i\theta \bar{S}_n} \right| \leq \exp(-c_\varepsilon[\kappa n]/2) + Cn^{-b}.$$

Since  $b > \frac{1}{2}$ , this implies that

$$|J_3| \leq C\sqrt{n} (\exp(-c_\varepsilon[\kappa n]/2) + Cn^{-b}) \rightarrow 0, \quad n \rightarrow \infty.$$

$J_2$ . We use the same estimation method, now with  $|\lambda| \leq \varepsilon B_n$ . Let  $\theta = \lambda/B_n$  again. Let  $\bar{A}_k = \mathbb{E}(\bar{S}_k | \tau = k)$  and  $d_k = \mathbb{E}((\bar{S}_k - \bar{A}_k)^2 | \tau = k)$ . With  $|\theta| \leq \varepsilon$  and sufficiently small  $\varepsilon$ , for any  $k > 1$ ,

$$\begin{aligned}
 \psi_\theta(k) &= \mathbb{E} \left( e^{i\theta \bar{S}_\tau} | \tau = k \right) = e^{i\theta \bar{A}_k} \mathbb{E} \left( e^{i\theta(\bar{S}_k - \bar{A}_k)} | \tau = k \right) \\
 &= e^{i\theta \bar{A}_k} (1 - \theta^2 d_k/2 + o(\theta^2)), \quad |\theta| \rightarrow 0.
 \end{aligned}$$

So (always for  $k > 1$ ), for  $|\theta| \leq \varepsilon_k$ ,

$$|\psi_\theta(k)| = \left| \mathbb{E} \left( e^{i\theta(\bar{S}_k - \bar{A}_k)} | \tau = k \right) \right| \leq 1 - \theta^2 d_k^2/3.$$

Thus  $\mathbb{P}(\tau = k_0) > 0$  for some  $k_0 > 1$  (otherwise  $\tau \equiv 1$  and there is no randomness, in particular,  $(H_{\text{sigma}})$  fails), whence

$$\begin{aligned}
 \mathbb{E} |\psi_\theta(\tau)|^2 &= \mathbb{E} \left| \mathbb{E} \left( e^{i\theta \bar{S}_\tau} | \tau \right) \right|^2 \leq \mathbb{P}(\tau = k_0) (1 - \theta^2 d_2^2/3) + \sum_{k \neq k_0} \mathbb{P}(\tau = k) \\
 &\leq 1 - \theta^2 d \leq \exp(-\theta^2 d)
 \end{aligned}$$

for  $|\theta| \leq \varepsilon$  with  $\varepsilon = \varepsilon_2$  and  $d = d_2^2 \mathbb{P}(\tau = k_0) > 0$ . By virtue of (3) we get

$$\begin{aligned}
 \left| \mathbb{E} e^{i\theta \bar{S}_n} \right| &\leq Cn^{-b} + \exp(-c_\varepsilon[\kappa n]\theta^2 d/2) \\
 &= Cn^{-b} + \exp(-c_\varepsilon([\kappa n]/B_n^2) \lambda^2 d/2) \\
 &\leq Cn^{-b} + \exp(-c'_\varepsilon \lambda^2)
 \end{aligned}$$

for  $|\theta| \leq \varepsilon$ . Finally,

$$|J_2| \leq 2 \int_A^{\varepsilon B_n} (\exp(-c'_\varepsilon \lambda^2) + C/n^b) d\lambda \leq Cn^{-b+1/2} + \int_A^\infty e^{-c'_\varepsilon \lambda^2} d\lambda.$$

The right-hand side is small if  $A$  and  $n$  are sufficiently large. This completes the proof of the theorem.

## 5. MIXING AND THE CENTRAL LIMIT THEOREM

In this section we consider the main example and obtain the central limit theorem for this case under certain ergodic assumptions. To use the general results from [7], we establish a certain  $\beta$ -mixing rate. Ibragimov and Linnik [7] use the  $\alpha$ - and  $\phi$ -mixing; the latter is uniform and so does not fit our setting, while the former does. Indeed, it is well known that (for the stationary regime, as in [7]) the  $\alpha$ -coefficient

$$\alpha(t) = \sup_{A \in F_0^X, B \in F_{\geq t}^X} (\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B))$$

is less than or equal to the  $\beta$ -coefficient, that is, to our  $\beta^\mu(t)$  (see below the explanation of what  $F_{\geq t}^X$  is). Hence, once we get any bound for  $\beta^\mu(t)$ , the same or better bound holds for the  $\alpha$ -mixing, and the general theorems from [7] are applicable. This method is used in [15].

Let

$$(H_{\bar{D},loc}) \quad \mathbb{P}(W_1 = kh) > 0$$

for each integer  $k$ . This is a rather strong form of a local Doeblin type condition (see below), which may be relaxed. The main result of this section is mixing bounds and a central limit theorem for dependent random variables  $X_n$ . We assume also that there exists  $m_0 > 0$  such that

$$(H_W) \quad \mathbb{E}|W_1|^{m_0} < \infty.$$

The ergodic assumption is as follows:

$$(H_{erg}) \quad \mathbb{E}W_1 = 0, \quad \lim_{|x| \rightarrow \infty} \left( \frac{|x + f(x)|}{|x|} - 1 \right) |x|^2 = -\infty.$$

Assumptions  $(H_{\bar{D},loc})$  and  $(H_{erg})$  can be relaxed: in fact, we do not need  $-\infty$  as the limit, but we do need that  $\limsup$  is a large negative number;  $(H_{\bar{D},loc})$  could be changed to  $(D_l)$ ; see below. Under these assumptions it follows that  $(X_n)$  is ergodic,  $\beta$ -mixing (see below), and possesses a unique invariant measure  $\mu$ . If the initial value is nonrandom or it is a random variable with the distribution  $\mu$ , that is,

$$(H_{stat}) \quad \text{Law}(X_0) = \mu,$$

then the Markov chain  $(X_n)$  is stationary. Both versions are actually used in the proof of Proposition 4 which can be found in [15].

Denote by  $X_t^x$  the solution of (1) with the initial data  $x \in \mathbf{Z}^1$ , and let  $\mu_t^x = \mathcal{L}(X_t^x)$  be its marginal distribution and  $\mu$  the invariant probability measure. Recall the definition of the  $\beta$ -mixing coefficient:

$$\beta^x(t) = \sup_{s \geq 0} \mathbb{E}_x \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_x(B | F_{\leq s}^X) - \mathbb{P}_x(B)),$$

where  $F_I^X$  is the  $\sigma$ -field generated by the values  $\{X_s, s \in I\}$ , and  $\mathbb{E}_x$  means the expectation for the process with the fixed initial data  $x$ . We also use the stationary version of this coefficient, namely

$$\begin{aligned} \beta^\mu(t) &= \sup_{s \geq 0} \mathbb{E}_\mu \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_\mu(B | F_{\leq s}^X) - \mathbb{P}_\mu(B)) \\ &= \mathbb{E}_\mu \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_\mu(B | F_{\leq s}^X) - \mathbb{P}_\mu(B)), \end{aligned}$$

where  $\mathbb{E}_\mu$  means the expectation for the process with the initial data  $X_0$  whose distribution coincides with  $\mu$ , and the coefficient

$$\bar{\beta}(t) = \int \sup_{s \geq 0} \mathbb{E}_x \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_x(B | F_{\leq s}^X) - \mathbb{P}_x(B)) \mu(dx) = \int \beta^x(t) \mu(dx)$$

(the version of the  $\beta$ -mixing coefficient introduced implicitly in [14] and [15]). Note that

$$\beta^\mu(t) \leq \bar{\beta}(t) + \int \text{Var}(\mu^x(t) - \mu) \mu(dx),$$

since

$$\begin{aligned} & \sup_{s \geq 0} \int \mathbb{E}_x \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_x(B | F_{\leq s}^X) - \mathbb{P}_\mu(B)) \mu(dx) \\ & \leq \int \sup_{s \geq 0} \mathbb{E}_x \text{Var}_{F_{\geq t+s}^X} (\mathbb{P}_x(B | F_{\leq s}^X) - \mathbb{P}_\mu(B)) \mu(dx). \end{aligned}$$

To formulate the central limit theorem we need the  $\beta$ -mixing and convergence rate bounds:

$$(4) \quad \text{Var}(\mu^x(t) - \mu) \leq C(1 + |x|^m)(1 + t)^{-(k+1)}$$

for some  $C, m$ , and  $k > 0$  (see the exact statements in what follows);

$$(5) \quad \beta^x(t) \leq C(1 + |x|^m)(1 + t)^{-(k+1)},$$

and

$$(6) \quad \bar{\beta}(t) \leq C(1 + t)^{-(k+1)}.$$

Let  $B_R = \{x \in \mathbf{R}^1 : |x| \leq R\}$  and  $\tau = \tau_R = \inf(t \geq 1 : |X_t| \leq R)$  (the  $d$ -dimensional setting is similar, but we deal with the case of  $d = 1$  in this paper). Auxiliary bounds worth mentioning are

$$(7) \quad \mathbb{E}_x \tau^{(k+1)} \leq C(1 + |x|^m)$$

with some  $R > 0$  and  $k, m > 0$ ,

$$(8) \quad \sup_{t \geq 0} \mathbb{E}_x |X_t|^m \mathbb{1}(t \leq \tau) \leq C(1 + |x|^m),$$

and

$$(9) \quad \int |x|^{m'} \mu(dx) < \infty$$

for each  $m' > 0$ .

Consider the process  $(X_t, t = 0, 1, \dots)$  which satisfies (1) and condition  $(H_{\text{erg}})$ . Let  $B \subset \mathbf{R}^d$  and  $\tau_1^B := \inf(t \geq 0 : X_t \in B)$  and  $\tau_{n+1}^B := \inf(t \geq \tau_n^B + 1 : X_t \in B)$ . We omit the superscript  $\tau$  if  $B$  is fixed. Put also  $\hat{\tau} = \hat{\tau}^B = \inf(t \geq 1 : X_t \in B)$ . Define the “process on  $B$ ”,  $X_t^B := X_{\tau_t^B}$ . Denote by  $P^B(n, x, dx')$  the  $n$ -step transition probability of  $(X_t^B)$ .

We say that the *local Doeblin condition* holds for the process  $(X_t)$  if for any sufficiently large  $R \geq 0$  the process on  $B = B_R := \{y \in \mathbf{R}^d : |y| \leq R\}$  satisfies the following condition: there exists an integer  $n_0 = n_0(R) > 0$  such that

$$(D_l) \quad \inf_{x, x' \in B} \int \min \left\{ \frac{P^B(n_0, x, dy)}{P^B(n_0, x', dy)}, 1 \right\} P^B(n_0, x', dy) =: q(R, n_0) > 0,$$

where  $P(dy)/P'(dy)$  means the derivative of the absolutely continuous part of  $P$  with respect to  $P'$ . The singular part may also exist. The assumption  $(D_l)$  requires, in particular, that the singular part is not close to 1. We recall that a simple sufficient condition for  $(D_l)$  is  $(H_{\bar{D}, \text{loc}})$ .

We assume that  $(X_t)$  satisfies the local Doeblin condition  $(D_l)$ . This implies the irreducibility (see [10]). Let  $r_m := (m - 1) \mathbb{E}|W_1|^2/2$ .

**Proposition 1.** *Assume that the process  $(X_t)$  satisfies conditions (1),  $(H_{\text{erg}})$ , and the local Doeblin condition  $(D_l)$ . If  $m_0 \geq 2$  and  $r > r_m$  with  $2 \leq m \leq m_0$ , then the (unique) invariant measure  $\mu$  exists and (8) is satisfied for this  $m$ . If  $m_0 > 2$ ,  $2 < m \leq m_0$ ,  $r > r_2$ , and  $k < (m-2)/2$ , then (7) holds and (9) is satisfied for  $m' = m-2$ . If  $m_0 > 4$ ,  $2 < m \leq m_0 - 2$ , and  $k < (m-2)/2$ , then (4) holds.*

Let  $\mu^m(dy) := (1 + |y|^m) \mu(dy)$  and  $\mu_t^{x,m}(dy) := (1 + |y|^m) \mu_t^x(dy)$ .

**Proposition 2.** *Let  $m_0 > 4$  and  $(H_{\text{erg}})$  be satisfied. Then for any*

$$2 < m < m_1 - 2 \leq m_0 - 2$$

and  $k < (m-2)/2$  there exist  $C$  and  $r_0 > 0$  such that

$$(10) \quad \text{Var}(\mu_t^{x,m} - \mu^m) \leq C(1 + |x|^{m_1})(1+t)^{-(k+1)}$$

for all  $r \geq r_0$ .

This estimate is a generalization of (4) under a more restrictive assumption on  $m_0$ .

**Proposition 3.** *Let  $m_0 > 4$  and  $(H_{\text{erg}})$  be satisfied. Then for any*

$$2 < m < m_1 - 2 \leq m_0 - 2$$

there exist  $C$  and  $r_0 > 0$  such that

$$(11) \quad \sup_t \mathbf{E}_x |X_t|^m \leq C(1 + |x|^{m_1})$$

for all  $r \geq r_0$ . Moreover, for any  $2 < m \leq m_0 - 2$  and  $k < (m-2)/2$  there exist  $r$  and  $C > 0$  such that inequalities (5) and (6) hold.

Let  $\xi_k = g(X_k)$  where the Borel function  $g$  is such that

$$|g(x)| \leq C_g(1 + |x|^2).$$

Put

$$S_n^g := \sum_{i=0}^{n-1} (\xi_k - \bar{g})$$

where  $\bar{g} = \int g(x) \mu(dx)$ . Note that  $\bar{g}$  is finite.

**Proposition 4.** *Let  $m_0 > 4$  and  $(H_{\text{erg}})$  be satisfied. Then  $S_n^g$  satisfies the central limit theorem, that is,*

$$(12) \quad n^{-1/2} S_n^g \Rightarrow \mathcal{N}(0, s^2)$$

where the variance  $s^2 = \sum_{i=-\infty}^{\infty} \text{cov}(\xi_0, \xi_i)$  is nonnegative and finite, and ‘‘cov’’ means the covariance calculated in the stationary regime.

Hence we may apply Proposition 4 for  $g(x) \equiv x$  and then use this function in the local central limit theorem.

Finally, we say a few words about the bounds  $(H_{\text{re}})$ . These are the technical estimates in the limit theorems of recurrence and large deviation theory. Often it is possible to show more, namely, that for any  $\varepsilon > 0$  and any  $b > 0$  there exists a constant  $C > 0$  such that

$$\tau_n/n \rightarrow \ell \equiv \mathbf{E}_\mu \tau, \quad n \rightarrow \infty,$$

$$\mathbf{P}(|\tau_n/n - \ell| > \varepsilon) \leq C/n^b \quad \text{for all } n \geq 1.$$

However we do not need this stronger form of the assumption and will not discuss it here.

**Lemma 1.** *Suppose the bound (7) holds for some  $k > 0$  (see Proposition 1). Then  $(H_{\text{re}})$  is satisfied for  $b = (1+k)/2$  and some  $C > 0$ .*

Though the lemma is standard, we give its short proof for completeness.



*Proof.* We use the estimate for independent identically distributed random variables  $(\xi_j)$  with  $E\xi_j = 0$ ,  $b > 1$ ,

$$E \left| \sum_{j=1}^m \xi_j \right|^b \leq Cm^{b/2} E|\xi_1|^b$$

(cf. [13, (3.5.15)–(3.5.16)]). Applying the latter estimate to  $\xi_j = \tau_{j+1} - \tau_j - c$  for  $c = E\tau_1$  and  $b = 1 + k$ , we get

$$E|\tau_n - cn|^{1+k} \leq Cn^{(1+k)/2} E|\tau_1 - c|^{1+k}.$$

Now, due to the Bienaimé–Chebyshev inequality we get

$$P(\tau_n - cn > \varepsilon n) \leq (\varepsilon n)^{-(1+k)} Cn^{(1+k)/2} E|\tau_1 - c|^{1+k} = Cn^{-(1+k)/2}$$

for all  $\varepsilon > 0$ . If  $\kappa < 1/c$ , then the desired inequality is satisfied. Lemma 1 is proved.  $\square$

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