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VERSIONS OF A COMPOUND POISSON PROCESS

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Abstract. We consider two versions of an oscillating compound Poisson process with reflections from two boundaries. The versions are constructed from an upper continuous compound Poisson process $\xi(t)$ and two functionals of it, namely the exit time from an interval and first upcrossing or downcrossing times from the upper or lower boundaries, respectively. The basic characteristics of the processes considered in the paper are given in terms of the potential and resolvent of the process $\xi(t)$ introduced earlier by V. S. Korolyuk.

Risk problems for compound Poisson processes with an upper reflecting boundary are considered in [1, 2]. We study other versions of the classical risk process

$$\xi_u(t) = u + \xi(t), \qquad u > 0,\ t \geq 0, \qquad \xi(0) = 0,$$

where $\xi(t)$ is a compound Poisson process with the cumulant

$$\psi(\alpha) = i\alpha a + \int_{-\infty}^{0} (e^{i\alpha x} - 1)\,\Pi(dx),$$
$$\int_{-1}^{0} |x|\,\Pi(dx) < \infty, \qquad a > 0,$$
$$\mathsf{E}\, e^{i\alpha\xi(t)} = e^{t\psi(\alpha)}, \qquad \operatorname{Im}\alpha = 0.$$

We introduce two versions of the risk process by using the reflections of $\xi_u(t)$ from the boundaries $x = 0$ and $x = B$ inside the interval $[0; B]$. To construct these versions we use the exit functionals from the interval $[0; B]$ for the process $\xi_u(t)$:

$$\begin{cases} \tau_B(u) = \inf\{t\colon \xi_u(t) \overline{\in} [0; B]\}, & 0 \leq u \leq B, \\ \tau^+(v) = \inf\{t\colon \xi(t) \geq v\}, & v = B - u, \\ \tau^-(-u) = \inf\{t\colon \xi(t) \leq -u\}. \end{cases} \tag{1}$$

The first version $\zeta_{B,u}(t)$ of the process with reflections from two boundaries is defined as an oscillating process

$$\zeta_{B,u}(t) \doteq \begin{cases} \xi_u(t), & \tau_B(u) > t, \\ \zeta_{B,u}(t - \tau^+(v)), & \tau_B^+(u) = \tau^+(v) < \tau^-(-u), \\ \zeta_{B,0}(t - \tau^-(-u)), & \tau_B^-(u) = \tau^-(-u) < \tau^+(v) \end{cases} \tag{2}$$

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for $u \in [0; B]$. It follows from the definition (2) that $\zeta_{B,u}(t)$ is permanently reflecting from B to 0 after the moment when the process assumes the value 0 for the first time. This case is partially treated in [3].

The second version $\zeta^u_{B,u}(t)$ of the process with reflections from two boundaries is defined by

$$
\zeta^u_{B,u}(t) \doteq \begin{cases} \xi_u(t), & \tau_B(u) > t, \\ \zeta^u_{B,u}(t-\tau^+(v)), & \tau^+(u) < \tau^-(-u), \\ \zeta^u_{B,0}(t-\tau^-(-u)), & \tau^-(u) < \tau^+(v), \end{cases} \tag{3}
$$

where the superscript u for $\zeta^u_{B,u}$ and $\zeta^u_{B,0}$ means that the reflection from B to the initial state $u > 0$ is permanent and that $\zeta^u_{B,0}(t)$ is given by

$$
\zeta^u_{B,0}(t) \doteq \begin{cases} \xi(t), & \tau_B(0) > t, \\ \zeta^u_{B,u}(t-\tau^+(B)), & \tau^+(B) < \tau^-(-0), \\ \zeta^u_{B,0}(t-\tau^-(-0)), & \tau^-(-0) < \tau^+(B). \end{cases} \tag{4}
$$

Using relations (3)–(4) we derive a pair of integral equations for characteristic functions

$$
\varphi_{t,B,u}(\alpha) = \mathsf{E}\, e^{i\alpha\zeta^u_{B,u}(t)}, \qquad \varphi_{t,B,0}(\alpha) = \mathsf{E}\, e^{i\alpha{}^u_{B,0}(t)}.
$$

These are convolution type equations, namely

$$
\begin{aligned}
\varphi_{t,B,u}(\alpha) = \mathsf{E}&\left[e^{i\alpha\xi_u(t)},\, \tau_B(u) > t\right] \\
&+ \int_0^t \mathsf{E}\left[e^{i\alpha\zeta^u_{B,u}(t-y)},\, \tau^+(v) \in dy, \tau^-(-u) > y\right] \\
&+ \int_0^t \mathsf{E}\left[e^{i\alpha\zeta^u_{B,0}(t-y)},\, \tau^-(-u) \in dy, \tau^+(v) > y\right],
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\varphi^u_{t,B,0}(\alpha) = \mathsf{E}&\left[e^{i\alpha\xi(t)},\, \tau_B(0) > t\right] \\
&+ \int_0^t \mathsf{E}\left[e^{i\alpha\zeta^u_{B,u}(t-\tau^+(B))}, \tau^+(B) \in dy,\, \tau^-(-0) > y\right] \\
&+ \int_0^t \mathsf{E}\left[e^{i\alpha\zeta^u_{B,0}(t-\tau^-(-0))}, \tau^-(-0) \in dy, \tau^+(B) > y\right].
\end{aligned} \tag{6}
$$

We denote by

$$
\xi^\pm(t) = \sup_{0\le u\le t}(\inf)\xi(u), \qquad \xi^\pm = \sup_{0\le t<\infty}(\inf)\xi(u)
$$

the extrema of $\xi(t)$, and let θ_s be an exponential random variable, independent of $\xi(t)$:

$$
\mathsf{P}\{\theta_s > t\} = \exp\{-st\}, \qquad t \ge 0,\ s > 0.
$$

Further, we introduce the following notation for the distributions:

$$
\begin{gathered}
P(s,x) = \mathsf{P}\{\xi(\theta_s) < x\}, \qquad -\infty < x < \infty; \\
\overline{P}(s,x) = 1 - \mathsf{P}(s,x); \qquad P_\pm(s,x) = \mathsf{P}\{\xi^+(\theta_s) < x\}, \quad \pm x > 0,
\end{gathered}
$$

and for the corresponding characteristic functions:

$$
\varphi(s,\alpha) = \mathsf{E}\, e^{i\alpha\xi(\theta_s)} = \frac{s}{s-\psi(\alpha)}, \qquad \varphi_\pm(s,\alpha) = \mathsf{E}\, e^{i\alpha\xi^\pm(\theta_s)}.
$$

Let $\rho(s)$ be a root of the equation $\psi(-i\rho) = s$,

$$\rho(s) = P'(s,+0)\overline{P}^{-1}(s,0);$$
$$\Pi(s) = \int_{-\infty}^{x} \Pi(dy), \qquad x < 0,$$
$$K(s,x) = \rho(s)\int_{-\infty}^{x} e^{-\rho(s)(x-y)}\Pi(y)dy, \qquad x \le 0,$$
$$\mathcal{K}(s,\alpha) = \int_{-\infty}^{0} e^{i\alpha x} K(s,x)\,dx, \qquad k(s,\alpha) = \int_{-\infty}^{0} \left(e^{i\alpha x} - 1\right)\, dK(s,x).$$

Here are three auxiliary results.

Lemma 1. *Let $\xi(t)$ be an upper continuous process with the cumulant* (1). *Then the following main factorization equality holds for the characteristic function $\varphi(s,\alpha)$:*

$$\varphi(s,\alpha) = \varphi_+(s,\alpha)\varphi_-(s,\alpha), \qquad \operatorname{Im}\alpha = 0.$$

The components of this equality are defined by

$$\varphi_+(s,\alpha) = \frac{\rho(s)}{\rho(s)+i\alpha}; \qquad \rho(s)p_-(s) = \frac{s}{a}, \tag{7}$$

$$\varphi_-(s,\alpha) = \left[1 - s^{-1}k(s,\alpha)\right]^{-1}, \qquad p_-(s) = \mathsf{P}\{\xi^-(\theta_s) = 0\} > 0. \tag{8}$$

If $m_1 = \mathsf{E}\,\xi(1) > 0$, then $\rho(s) \underset{s\to 0}{\longrightarrow} 0$, $\rho'(0) = m_1^{-1} > 0$, and the absolute minimum ξ^- has a nondegenerate distribution such that

$$p_- = \mathsf{P}\{\xi^- = 0\} > 0$$

and the characteristic function is given by

$$\varphi_-(\alpha) = \mathsf{E}\,e^{i\alpha\xi^-} = \left[1 - \rho'(0)\int_{-\infty}^{0} (e^{i\alpha x} - 1)\Pi(x)\,dx\right]^{-1}. \tag{9}$$

Note that the characteristic function $\varphi_-(s,\alpha)$ of an upper continuous process $\xi(t)$ can be expressed in terms of the distribution of negative values of $\xi(t)$ (see relation (3), Lemma 1 in [1]) or in terms of $k(s,\alpha)$ defined in (8). The limit value of $\varphi_-(s,\alpha)$ as $s \to 0$ determines the characteristic function of the absolute minimum according to relation (9), whence one can easily obtain the Pollaczek–Khinchin formula.

Following [4, 5, 6] we denote by $\tau(x,T)$ the first exit time of the process $\xi(t)$ from the interval $[x-T, x]$, where $0 < x < T$,

$$\tau(x,T) = \begin{cases} \tau^+(x,T) = \tau^+(x), & A_+(x), \\ \tau^-(x,T) = \tau^-(x-T), & A_-(x), \end{cases}$$
$$A_\pm(x) = \left\{\omega\colon \tau^+(x) \lesseqgtr \tau^-(x-T)\right\}.$$

The corresponding characteristic functions of the process before the exit and at the exit time from the interval are denoted by

$$V(s,\alpha,x,T) = \mathsf{E}\left[e^{i\alpha\xi(\theta_s)}, \tau(x,T) > \theta_s\right],$$
$$V_+(s,\alpha,x,T) = \mathsf{E}\left[e^{i\alpha\xi(\tau^+(x,T)) - s\tau^+(x)}, A_+(x)\right],$$
$$V_-(s,\alpha,x,T) = \mathsf{E}\left[e^{i\alpha\xi(\tau^-(x,T)) - s\tau^-(x)}, A_-(x)\right].$$

For an upper continuous process $\xi(t)$ we make use of the representation for its resolvent

$$(10)\qquad \begin{gathered} R_s(x) = s^{-1}\rho(s)\int_{-0}^{x} e^{\rho(s)(x-y)}\, d\,\mathsf{P}\{-\xi^-(\theta_s) < y\}, \\ R(x) = \lim_{s\to 0} R_s(x), \qquad x > 0 \end{gathered}$$

(see [4, 5, 6]; the definition of the potential and resolvent can be found in [4, 5]).

The following result is a corollary of the projection relations obtained in [7].

Lemma 2. *If $\xi(t)$ is an upper continuous process with the cumulant* (1)*, then the following relations (called projection relations) hold:*

$$(11)\qquad \begin{cases} V_-(s,\alpha,x,\tau) = \varphi_-^{-1}(s,\alpha) \\ \qquad\qquad \times \left[\varphi_-(s,\alpha)(1-Q^T(s,T-x))e^{i\alpha x}\right]_{(-\infty,x-T]}, & \operatorname{Im}\alpha \le 0, \\ V(s,\alpha,x,\tau) = \varphi_+(s,\alpha) \\ \qquad\qquad \times \left[\varphi_-(s,\alpha)(1-Q^T(s,T-x))e^{i\alpha x}\right]_{[x-T,\infty)}, & \operatorname{Im}\alpha = 0, \end{cases}$$

where $\varphi_\pm(s,\alpha)$ are defined by (7)–(8) *and*

$$Q^T(s,T-x) = V_+(s,0,x,T) = \mathsf{E}\left[e^{-s\tau^+(x,T)}, A_+(x)\right].$$

We use the notation related to the walk of the process $\xi_u(t)$ in the interval $[0;B]$ for $0 < u \le B$ and $v = B-u$:

$$\begin{gathered} \tau_B(u) = \begin{cases} \tau_B^+(u) = \tau^+(v) < \tau^-(-u), & A_B^+, \\ \tau_B^-(u) = \tau^-(-u) < \tau^+(v), & A_B^-, \end{cases} \\ A_B^\pm = \left\{\omega\colon \tau^+(v) \lessgtr \tau^-(-u)\right\}. \end{gathered}$$

Let

$$\begin{gathered} Q^B(s,u) = \mathsf{E}\left[e^{-s\tau_B^+(u)}, A_B^+\right], \\ Q_B(s,u) = \mathsf{E}\left[e^{-s\tau_B^-(u)}, A_B^-\right], \\ Q(B,s,u) = \mathsf{E}\, e^{-s\tau_B(u)} = Q^B(s,u) + Q_B(s,u), \\ P(B,s,u) = \mathsf{P}\{\tau_B(u) > \theta_s\} = 1 - Q(B,s,u) \end{gathered}$$

be the moment generating functions of the first exit time from the interval. The following lemma follows from Korolyuk's results [4] and Lemma 2.

Lemma 3. *If $\xi(t)$ is an upper continuous process, then*

$$(12)\qquad \begin{cases} Q^B(s,u) = R_s(u)R_s^{-1}(B), & 0 \le u \le B, \\ Q_B(s,u) = \mathsf{P}\{\xi^-(\theta_s) < -u\} - Q^B(s,u)\,\mathsf{P}\{\xi^-(\theta_s) < -B\}, \end{cases}$$

$$(13)\qquad Q(B,s,u) = \begin{cases} 1 - s\left(Q^B(s,u)\int_0^B R_s(y)\,dy - \int_0^u R_s(y)\,dy\right), \\ \mathsf{P}\{\xi^-(\theta_s) < -u\} + Q^B(s,u)\,\mathsf{P}\{\xi^-(\theta_s) \in [-B,0]\}, \end{cases}$$

$$(14)\qquad P(B,s,u) = \mathsf{P}\{-\xi^-(\theta_s) \in [0,u]\} - Q^B(s,u)\,\mathsf{P}\{-\xi^-(\theta_s) \in [0,B]\},$$

$$(15)\qquad \mathsf{E}\left[e^{i\alpha\xi_u(\theta_s)}, \tau_B(u) > \theta_s\right] = \frac{\rho(s)e^{i\alpha u}}{\rho(s) - iu}\left[\varphi_-(s,\alpha)(1 - Q^B(s,u)e^{i\alpha v})\right]_{[-u,v]}.$$

The density of the distributions of $\xi(\theta_s)$ and $\xi_u(\theta_s)$ until the first exit from the interval is given by

$$\frac{\partial}{\partial x}\mathsf{P}\{\xi(\theta_s)<x,\tau_B(u)>\theta_s\}=sQ^B(s,u)R_s(v-x)-sR_s(-x)\delta(x<0),$$
$$\frac{\partial}{\partial x}P\{\xi_u(\theta_s)<x,\tau_B(u)>\theta_s\}=sQ^B(s,u)R_s(B-x)-sR_s(u-x)\delta(u>x). \tag{16}$$

Let

$$\varphi_{B,u}(s,\alpha)=\mathsf{E}\,e^{i\alpha\zeta_{B,u}(\alpha)},$$
$$p_{B,u}(s,x)=\frac{\partial}{\partial x}\mathsf{P}\{\zeta_{B,u}(\theta_s)<x\},\qquad 0\le x\le B,$$

be the characteristic function and the density of the distribution for the first version of the process $\zeta_{B,u}(t)$ (see (2)).

Theorem 1. *Let $\xi(t)$ be an upper continuous process with the cumulant* (1). *Then the characteristic functions of $\varphi_{B,0}(s,\alpha)$ and $\varphi_{B,u}(s,\alpha)$ are given by*

$$\begin{aligned}\varphi_{B,0}(s,\alpha)&=\mathsf{E}\left[e^{i\alpha\xi(\theta_s)},\tau_B(0)>\theta_s\right]P^{-1}(B,s,0)\\&=\varphi_+(s,\alpha)\\&\quad\times\left\{p_-(s)-(aR_s(B))^{-1}\,\mathsf{E}\left[e^{i\alpha(\xi^-(\theta_s)+B)},\xi^-(\theta_s)\in[-B,0]\right]\right\}\\&\quad\times P^{-1}(B,s,0),\end{aligned}\tag{17}$$

$$\begin{aligned}\varphi_{B,u}(s,\alpha)&=\left(1-Q^B(s,u)\right)^{-1}\\&\quad\times\left\{Q_B(s,u)\varphi_{B,0}(s,\alpha)\right.\\&\qquad\left.+\frac{\rho(s)e^{i\alpha u}}{\rho(s)-i\alpha}\left[\varphi_-(s,\alpha)\left(1-Q^B(s,u)e^{i\alpha v}\right)\right]\right\}_{[-u,v]}\end{aligned}\tag{18}$$

where $Q^B(s,u)$, $Q_B(s,x)$, and $P(B,s,u)$ are defined in Lemma 3, while $\varphi_\pm(s,\alpha)$ is defined in Lemma 2. The densities of the distributions of $\zeta_{B,0}(\theta_s)$ and $\zeta_{B,u}(\theta_s)$ are defined as follows:

$$p_{B,0}(s,x)=sQ^B(s,0)R_s(B-x)P^{-1}(B,s,0),\qquad 0<x<B,\tag{19}$$

$$\begin{aligned}p_{B,u}(s,x)&=\left(1-Q^B(s,u)\right)^{-1}\\&\quad\times\left\{Q_B(s,u)P_{B,0}(s,x)\right.\\&\qquad\left.+sQ^B(s,u)R_s(B-x)-sR_s(u-x)\delta(x<u)\right\}.\end{aligned}\tag{20}$$

Proof. Relation (2) for $\varphi_{B,u}(s,\alpha)$ implies that

$$\varphi_{B,u}(s,\alpha)=\mathsf{E}\left[e^{i\alpha\xi_u(\theta_s)},\tau_B(u)>\theta_s\right]+\varphi_{B,0}(s,\alpha)Q_B(s,u)+\varphi_{B,u}(s,\alpha)Q^B(s,u),\tag{21}$$

whence it follows for $u=0$ that

$$\varphi_{B,0}(s,\alpha)\left(1-Q_B(s,0)-Q^B(s,0)\right)=\mathsf{E}\left[e^{i\alpha\xi(\theta_s)},\tau_B(0)>\theta_s\right].$$

The latter equation together with relation (15) for $u=0$ imply (17). Inverting with respect to α and using the first relation in (16) and (17) we find the density $p_{B,0}(s,x)$ (see (19)). Similarly, inverting (18) with respect to α we obtain relation (20) for $p_{B,u}(s,x)$.

Now we introduce the notation for the characteristic functions in the case of the second version of the process $\zeta^u_{B,u}(t)$ (see (3)–(4)):

$$\varphi^u_{B,u}(s,\alpha) = \mathsf{E}\left[e^{i\alpha\zeta^u_{B,u}(\theta_s)}\right], \qquad \varphi^u_{B,0}(s,\alpha) = \mathsf{E}\left[e^{i\alpha\zeta^u_{B,0}(\theta_s)}\right],$$
$$\tilde{h}_s(\alpha,u,B) = \mathsf{E}\left[e^{i\alpha\xi_u(\theta_s)}, \tau_B(u) > \theta_s\right],$$
$$\tilde{h}_s(\alpha,0,B) = \mathsf{E}\left[e^{i\alpha\xi_u(\theta_s)}, \tau_B(0) > \theta_s\right].$$

The following notation is useful when evaluating the characteristic functions $\varphi^u_{B,u}(s,\alpha)$ and $\varphi^u_{B,0}(s,\alpha)$ from integral transforms of equations (5)–(6):

$$\begin{aligned}
\Delta(s,u,B) &= \begin{vmatrix} 1-Q_B(s,0) & -Q^B(s,0) \\ -Q_B(s,u) & 1-Q_B(s,u) \end{vmatrix} \\
&= (1-Q_B(s,0))P(B,s,u) + Q_B(s,u)P(B,s,0), \\
\tilde{\Delta}_1(s,\alpha,u,B) &= \begin{vmatrix} \tilde{h}_s(\alpha,0,B) & -Q^B(s,0) \\ \tilde{h}_s(\alpha,u,B) & 1-Q^B(s,u) \end{vmatrix} \\
&= \tilde{h}_s(\alpha,0,B)(1-Q^B(s,u)) + \tilde{h}_s(\alpha,u,B)Q^B(s,0), \\
\tilde{\Delta}_2(s,\alpha,u,B) &= \begin{vmatrix} 1-Q_B(s,0) & \tilde{h}_s(\alpha,0,B) \\ -Q_B(s,u) & \tilde{h}_s(\alpha,u,B) \end{vmatrix} \\
&= \tilde{h}_s(\alpha,u,B)(1-Q^B(s,0)) + \tilde{h}_s(\alpha,0,B)Q^B(s,u).
\end{aligned}$$

An application of equations (5)–(6) completes the proof. □

Theorem 2. *If $\xi(t)$ is an upper continuous process with the cumulant* (1), *then the characteristic functions $\varphi^u_{B,0}(s,\alpha)$ and $\varphi^u_{B,u}(s,\alpha)$ are given by*

$$\text{(22)} \qquad \begin{cases} \varphi^u_{B,0}(s,\alpha) = \tilde{\Delta}_1(s,\alpha,u,B)\Delta^{-1}(s,u,B), \\ \varphi^u_{B,u}(s,\alpha) = \tilde{\Delta}_2(s,\alpha,u,B)\Delta^{-1}(s,u,B). \end{cases}$$

The densities of the distributions $\zeta^u_{B,0}(\theta_s)$ and $\zeta^u_{B,u}(\theta_s)$ are obtained by inverting relations (22) *with respect to α as follows:*

$$\text{(23)} \qquad \begin{cases} \frac{\partial}{\partial x}\mathsf{P}\{\zeta^u_{B,u}(\theta_s) < x\} \\ \quad = \Delta^{-1}(s,u,B)\left\{h_s(x,0,B)\left(1-Q^B(s,u)\right) + h_s(x,u,B)Q^B(s,0)\right\}, \\ \frac{\partial}{\partial x}\mathsf{P}\{\zeta^u_{B,0}(\theta_s) < x\} \\ \quad = \Delta^{-1}(s,u,B)\left\{h_s(x,u,B)\left(1-Q^B(s,0)\right) + h_s(x,0,B)Q^B(s,u)\right\}, \\ \quad 0 < x < B. \end{cases}$$

The densities of the distributions $h_s(x,0,B)$ and $h_s(x,u,B)$ are expressed in terms of the resolvent $R_s(\cdot)$ according to (16) *and Corollary 1 of* [3]*:*

$$\text{(24)} \qquad \begin{cases} h_s(x,0,B) = \frac{\partial}{\partial x}\mathsf{P}\{\xi(\theta_s) < x, \tau_B(0) > \theta_s\} \\ \qquad\qquad = sQ^B(s,0)R_s(B-x) - sR_s(-x)\delta(x<0), \\ h_s(x,u,B) = \frac{\partial}{\partial x}\mathsf{P}\{\xi_u(\theta_s) < x, \tau_B(u) > \theta_s\} \\ \qquad\qquad = sQ^B(s,u)R_s(B-x) - sR_s(u-x)\delta(u>x). \end{cases}$$

Proof. The integral convolution type equations obtained from (3)–(4) (see (5)–(6)) become, after the integral Laplace–Carson transform,

$$\begin{cases} \varphi^u_{B,u}(s,\alpha) = \tilde{h}_s(\alpha,u,B) + \varphi^u_{B,u}(s,\alpha)Q^B(s,u) + \varphi^u_{B,0}(s,\alpha)Q_B(s,u); \\ \varphi^u_{B,0}(s,\alpha) = \tilde{h}_s(\alpha,0,B) + \varphi^u_{B,u}(s,\alpha)Q^B(s,0) + \varphi^u_{B,0}(s,\alpha)Q_B(s,0), \end{cases}$$

whence we obtain the system of equations

$$
(25)\qquad \begin{cases}\varphi^u_{B,0}(s,\alpha)(1-Q_B(s,0))-\varphi^u_{B,u}(s,\alpha)Q^B(s,0)=\tilde h_s(\alpha,0,B),\\ -\varphi^u_{B,0}(s,\alpha)Q_B(s,u)+\varphi^u_{B,u}(s,\alpha)(1-Q^B(s,u))=\tilde h_s(\alpha,u,B).\end{cases}
$$

A solution of system (25) can be obtained with the help of relations (22), which are easily inverted with respect to α. Inverting $\tilde\Delta_{1,2}(s,\alpha,u,B)$ we get the densities of the distributions (23). Theorem 2 is proved. □

Applying the truncated Fourier transform to (24) we obtain relations determining $\tilde h_s(\dots)$:

$$
(26)\qquad \begin{cases}\tilde h_s(\alpha,0,B)=\int_0^B h_s(x,0,B)\,dx=sQ^B(s,0)e^{i\alpha B}r_s^B(-\alpha),\\ e^{i\alpha B}r_s^B(-\alpha)=\int_0^B e^{i\alpha x}R_s(B-x)\,dx,\\ \tilde h_s(\alpha,u,B)=\int_0^B h_s(x,u,B)\,dx=sQ^B(s,u)e^{i\alpha B}r_s^B(-\alpha)-se^{i\alpha u}r_s^u(-\alpha).\end{cases}
$$

The functions (26) allow one to find the limit density of the distribution and its characteristic function.

Below are corollaries of Theorem 1 (depending to the sign of $\mathsf E\,\xi(1)$).

Corollary 1. *Let the assumptions of Theorem 1 hold. If*

$$
m_1=\mathsf E\,\xi(1)>0,
$$

then the distribution of the absolute minimum is nondegenerate (see relation (9) *in Lemma 1). Moreover the potential is determined by the limit relation*

$$
(27)\qquad R(x)=R_+(x)=\lim_{s\to0}R_s(x)=m_1^{-1}\,\mathsf P\{-\xi^-<x\},\qquad x>0.
$$

In addition, the characteristics

$$
(28)\qquad \begin{cases}\lim_{s\to0}s^{-1}P(B,s,u)=\mathsf E\,\tau_B(u)=Q^B(u)\int_0^B R(y)\,dy-\int_0^u R(y)\,dy,\\ Q^B(u)=\frac{R(u)}{R(B)}=\frac{\mathsf P\{-\xi^-<u\}}{\mathsf P\{\xi^-<B\}},\qquad \mathsf E\,\tau_B(0)=Q^B(0)\int_0^B R(y)\,dy,\end{cases}
$$

and limit characteristic function

$$
(29)\qquad \begin{aligned}\varphi_{B,0}(\alpha)&=\lim_{s\to0}\mathsf E\,e^{i\alpha\zeta_{B,0}(\theta_s)}=\varphi^u_{B,u}(\alpha)\\ &=(-i\alpha m_1\,\mathsf E\,\tau_B(0))^{-1}\left\{p_- -Q_B(0)\,\mathsf E\left[e^{i\alpha(\xi^-+B)},\xi^-\ge -B\right]\right\}\end{aligned}
$$

are expressed in terms of the distribution of the absolute minimum. The limit densities are given by

$$
(30)\qquad \begin{cases}p_{B,0}(x)=\lim_{s\to0}p_{B,0}(s,x)=Q^B(0)R(B-x)[\mathsf E\,\tau_B(0)]^{-1},\\ p_{B,u}(x)=p_{B,0}(x)=R(B-x)\left[\int_0^B R(y)\,dy\right]^{-1},\qquad 0\le x<B.\end{cases}
$$

Proof. Relation (27) follows from equalities (10). It determines the potential, thus the limit characteristics in (28) can be expressed in terms of the potential. Relation (29) follows from (17) since

$$
\lim_{s\to0}s^{-1}\varphi_+(s,\alpha)=\lim_{s\to0}\frac{\rho(s)s^{-1}}{\rho(s)-i\alpha}=-(i\alpha m_1)^{-1},\qquad p_-(s)\underset{s\to0}{\longrightarrow}p_->0
$$

for $m_1>0$. Relation (30) follows from equalities (23) as $s\to0$. Relation (29) is equivalent to the relation obtained from (21) for $u=0$, and as $s\to0$ we have

$$
\varphi_{B,0}(\alpha)=\lim_{s\to0}\tilde h_s(\alpha,0,B)P^{-1}(B,s,0)=\tilde h_0'(\alpha,0,B)[\mathsf E\,\tau_B(0)]^{-1},
$$

where

$$\tilde{h}_0(\alpha,0,B)=\lim_{s\to 0}s^{-1}\tilde{h}_s(\alpha,0,B)=m_1^{-1}Q^B(0)\int_0^B e^{i\alpha x}\,\mathsf{P}\{\xi^->B-x\}\,dx.$$

Finally, all the characteristics in (28)–(30) are expressed in terms of the distribution of the absolute minimum; in particular

$$Q^B(0)=p_-[\mathsf{P}\{-\xi^-<B\}]^{-1},\qquad p_-=\mathsf{P}\{\xi^-=0\}>0,$$

$$\mathsf{E}\,\tau_B(0)=\frac{p_-}{m_1\,\mathsf{P}\{-\xi^-<B\}}\int_0^B \mathsf{P}\{-\xi^-<y\}\,dy,$$

$$P_{B,u}(x)=\mathsf{P}\{-\xi^-<B-x\}\left[\int_0^B \mathsf{P}\{-\xi^-<y\}\,dy\right]^{-1}. \qquad \square$$

Corollary 2. *Let all the assumptions of Theorem 1 hold. If* $m_1=0$, *then*

$$\lim_{s\to 0}\rho(s)s^{-1/2}=\sqrt{\frac{2}{\mathcal{D}\xi(1)}}=k_+>0,$$

$$\lim_{s\to 0}\varphi_-(s,\alpha)s^{-1/2}=\frac{1}{k_+\tilde{\Pi}(0)}\frac{1}{1-\varphi_-^*(\alpha)}, \tag{31}$$

where

$$\varphi_-^*(\alpha)=\tilde{\Pi}(0)^{-1}\int_{-\infty}^0 e^{i\alpha x}\Pi(x)\,dx,\qquad \Pi(x)=\int_{-\infty}^x \Pi(dy),\quad x<0,$$

$$\tilde{\Pi}(u)=\int_{-\infty}^0 e^{ux}\Pi(x)\,dx,\qquad \tilde{\Pi}(0)=\int_{-\infty}^0 \Pi(x)\,dx.$$

The potential $R(x)$ *is given by*

$$R(x)=R_0(x)=\lim_{s\to 0}R_s(x)=\tilde{\Pi}(0)^{-1}H_*(x),\qquad x>0, \tag{32}$$

where

$$H_*(x)=\sum_{k=0}^{\infty}F_*(x)^{*k}$$

is the renewal function for the distribution with the density

$$p_*(x)=\tilde{\Pi}(0)^{-1}\Pi(-x),\qquad x>0.$$

The limit characteristics $Q_B(u)$, $Q^B(u)$, *and* $\mathsf{E}\,\tau_B(u)$ *are expressed in terms of the renewal function* $H_*(x)$. *In particular,*

$$\begin{cases} Q^B(u)=H_*(u)H_*^{-1}(B),\\ \mathsf{E}\,\tau_B(u)=\left[Q^B(u)\int_0^B H_*(y)\,dy-\int_0^u H_*(y)\,dy\right]\tilde{\Pi}(0)^{-1},\\ p_{B,0}(x)=p_{B,u}(x)=H_*(B-x)\left[\int_0^B H_*(y)\,dy\right]^{-1}, \qquad 0\le x<B. \end{cases} \tag{33}$$

Proof. It follows from relation (8) that

$$\lim_{s\to 0}s^{-1/2}\varphi_-(s,\alpha)=\lim_{s\to 0}\left(\sqrt{s}-s^{-1/2}k(s,\alpha)\right)^{-1}.$$

At the same time,

$$\lim_{s\to 0}s^{-1/2}k(s,\alpha)=\lim_{s\to 0}\frac{\rho(s)}{\sqrt{s}}\int_{-\infty}^0\left(e^{i\alpha x}-1\right)\Pi(x)\,dx$$
$$=k_+\tilde{\Pi}(0)\int_{-\infty}^0(e^{i\alpha x}-1)p_*(-x)\,dx=-k_+\tilde{\Pi}(0)(1-\varphi_-^*(\alpha)).$$

The latter two relations imply (31). Approaching the limit in (10) and (31) as $s \to 0$ we obtain relation (32) for the potential in terms of the renewal function. In its turn, the renewal function determines the limit density in (33) and other limit characteristics by approaching the limit in (19)–(20) as $s \to 0$. ☐

Corollary 3. *Let all the assumptions of Theorem 1 hold. If $m_1 < 0$, then the distribution of the absolute maximum is a nondegenerate exponential distribution, namely*

$$
\begin{cases}
\lim_{s\to 0} \varphi_+(s,\alpha) = \mathsf{E}\, e^{i\alpha\xi^+} = \frac{\rho}{\rho+i\alpha}, \\
\qquad \rho = \lim_{s\to 0}\rho(s) > 0, \quad \mathsf{P}\{\xi^+ > x\} = e^{-\rho x}, \qquad x>0, \\
\lim_{s\to 0} s^{-1}\,\mathsf{P}\{\xi^-(\theta_s) > -y\} = \mathsf{E}\,\tau^-(-y) = G(y), \quad y>0.
\end{cases} \tag{34}
$$

Passing to the limit, the potential is obtained as a convolution

$$
\lim_{s\to 0} R_s(x) = R_-(x) = \rho \int_{-0}^{x} e^{\rho(x-y)}\, dG(y), \qquad x>0. \tag{35}
$$

The limit characteristics $Q^B(u)$, $Q_B(u)$, $E\tau_B(u)$, and limit density are expressed in terms of the convolution (35)*; in particular*

$$
\begin{cases}
Q^B(u) = R_-(u) R_-^{-1}(B), & 0 \le u < B, \\
\mathsf{E}\,\tau_B(u) = Q^B(u)\int_0^B R_-(y)\,dy - \int_0^u R_-(y)\,dy, & \\
p_{B,0}(x) = p_{B,u}(x) = R_-(B-x)\left[\int_0^B R_-(x)\,dx\right]^{-1}, & 0\le u < B.
\end{cases} \tag{36}
$$

Proof. It is well known that the distribution of the absolute maximum of an upper continuous process is nondegenerate and exponential in the case of $m_1 < 0$. The last relation in (34) is obtained by passing to the limit:

$$
\begin{aligned}
\lim_{s\to 0}\frac{1}{s}\,\mathsf{P}\{\xi^-(\theta_s) > -y\} &= \lim_{s\to 0}\int_0^\infty e^{-st}\,\mathsf{P}\{\xi^-(t) > -y\}\,dt \\
&= \lim_{s\to 0}\int_0^\infty e^{-st}\,\mathsf{P}\{\tau^-(-y) > t\}\,dt = \mathsf{E}\,\tau^-(-y).
\end{aligned}
$$

The potential $R_-(x)$ defined in (35) is obtained from (10) and (34) in terms of a convolution. All other limit characteristics in (36) are also expressed in terms of this convolution. ☐

For an upper continuous process with the cumulant (1), the limit results of Corollaries 1–3 are expressed in terms of the representations for the potential $R(x)$, $x>0$, depending on the sign of m_1 (see relation (27) for $m_1 > 0$, (32) for $m_1 = 0$, and (35) for $m_1 < 0$), namely

$$
R(x) = \begin{cases}
R_+(x) = \frac{1}{m_1}\,\mathsf{P}\{-\xi^- < x\}, & m_1 > 0, \\
R_0(x) = [\tilde\Pi(0)]^{-1} H_*(x), & m_1 = 0, \\
R_-(x) = \rho\int_{-0}^{x} e^{\rho(x-y)}\,dG(y), & m_1 < 0,\ \rho > 0.
\end{cases} \tag{37}
$$

Theorem 2 for the second version of the process with reflections from two boundaries (see (2)–(3)) imply results similar to those in Corollaries 1–3.

Corollary 4. *Let all the assumptions of Theorem 2 hold. Then the limit distribution of the processes $\zeta^u_{B,0}(t)$ and $\zeta^u_{B,u}(t)$ exists as $t\to\infty$ and determines the characteristic function*

$$
\begin{aligned}
\varphi^u_{B,0}(\alpha) &= \lim_{s\to 0}\varphi^u_{B,0}(s,\alpha) = \lim_{s\to 0}\varphi^u_{B,u}(s,\alpha) = \varphi^u_{B,u}(\alpha) \\
&= \frac{\tilde h_0'(\alpha,0,B)Q_B(u) + \tilde h_0'(\alpha,u,B)Q^B(0)}{\mathsf{E}\,\tau_B(u)Q^B(0) + \mathsf{E}\,\tau_B(0)Q_B(u)},
\end{aligned} \tag{38}
$$

where

$$(39)\qquad \begin{cases} \tilde{h}'_0(\alpha,0,B) = \lim_{s\to 0} s^{-1}\tilde{h}_s(\alpha,0,B) = Q^B(0)e^{i\alpha B}\int_0^B e^{-i\alpha x}R(x)\,dx,\\ \tilde{h}'_0(\alpha,0,B) = \lim_{s\to 0} s^{-1}\tilde{h}_s(\alpha,u,B)\\ \qquad\qquad = Q^B(u)e^{i\alpha B}\int_0^B e^{-i\alpha x}R(x)\,dx - e^{i\alpha u}\int_0^u e^{-i\alpha x}R(x)\,dx. \end{cases}$$

The limit densities are defined in terms of the potential $R(x)$ with the help of representation (37) *by*

$$(40)\qquad \begin{aligned} \lim_{s\to 0}\frac{\partial}{\partial x}\mathsf{P}\{\zeta^u_{B,u}(\theta_s)<x\} &= \lim_{s\to 0}\frac{\partial}{\partial x}\mathsf{P}\{\zeta^u_{B,0}(\theta_s)<x\}\\ &= \left[\mathsf{E}\,\tau_B(u)Q^B(0)+\mathsf{E}\,\tau_B(0)Q_B(u)\right]^{-1}\\ &\quad\times\Big\{Q^B(0)R(B-x)Q_B(u)\\ &\qquad + \left[Q^B(u)R(B-x)-R(u-x)\delta(u>x)\right]Q_B(0)\Big\}. \end{aligned}$$

If $m_1>0$, then the limit characteristics

$$(41)\qquad \begin{cases} Q^B(u) = R(u)R^{-1}(B), \quad Q_B(u) = 1-Q^B(u),\\ \mathsf{E}\,\tau_B(u) = Q^B(u)\int_0^B R(y)\,dy - \int_0^u R(y)\,dy, \qquad 0\le u<B, \end{cases}$$

and limit density (40) *are expressed in terms of the distribution of the absolute minimum according to the representation of the potential in* (37)*, namely:*

If $m_1>0$, then (40) *is expressed according to the first relation in* (37)*.*

If $m_1=0$, then (40) *and* (41) *are expressed in terms of the renewal function in* (32)*.*

If $m_1<0$, then (40) *and* (41) *are expressed in terms of convolution* (35)*.*

Proof. Passing to the limit in relations (23) as $s\to 0$ we obtain (38) because

$$(42)\qquad \begin{gathered} \lim_{s\to 0} s\Delta^{-1}(s,u,B) = Q^B(0)\,\mathsf{E}\,\tau_B(u) + Q_B(u)\,\mathsf{E}\,\tau_B(0),\\ \lim_{s\to 0} s^{-1}\tilde{h}_s(\alpha,0,B) = Q^B(0)e^{i\alpha B}\int_0^B e^{-i\alpha x}R(x)\,dx,\\ \lim_{s\to 0} s^{-1}\tilde{h}(\alpha,u,B) = Q^B(u)e^{i\alpha B}\int_0^B e^{-i\alpha x}R(x)\,dx - e^{i\alpha u}\int_0^u e^{-i\alpha x}R(x)\,dx \end{gathered}$$

according to equalities (26). The potential $R(x)$ in relations (40) and (41) is chosen from (37) according to the sign of m_1: if $m_1>0$, then

$$R(x) = R_+(x)$$

(see (27)); if $m_1=0$, then

$$R(x) = R_0(x)$$

(see (32)); and if $m_1<0$, then

$$R(x) = R_-(x)$$

(see (35)). □

To deal with the case where the process $\xi(t)$ is a mixture of a Poisson process with cumulant (1) and a diffusion component, we consider a different kind of reflection from both upper an lower boundaries to an initial state $u>0$. The version of the process with reflection from two boundaries is defined as follows. Let

$$\xi_u(t) = u+\xi(t), \qquad t>0,\ u>0,\ \xi(0)=0,$$

where $\xi(t)$ is an upper continuous process whose cumulant is

$$\psi(\alpha) = i\alpha a - \frac{\sigma^2}{2}\alpha^2 + \int_{-\infty}^{0} (e^{i\alpha x} - 1)\,\Pi(dx), \qquad \int_{-\infty}^{0} |x|\,\Pi(dx) < \infty. \tag{43}$$

Then the process $\eta_{B,\sigma}^u(t)$ is given by

$$\eta_{B,\sigma}^u(t) \doteq \begin{cases} \xi_u(t), & \tau_B(u) > t,\ 0 < u < B, \\ \eta_{B,\sigma}^u(t - \tau^+(v)), & \tau_B(u) = \tau^+(v) < \tau^-(-u),\ \tau^+(v) < t, \\ \eta_{B,\sigma}^u(t - \tau^-(-u)), & \tau_B(u) = \tau^-(-u) < \tau^+(v),\ \tau^-(-u) < t. \end{cases} \tag{44}$$

Theorem 3. *If $\xi(t)$ is an upper continuous process with cumulant* (43)*, then the characteristic function of the process* (44)

$$\varphi_{B,\sigma}^u(s,\alpha) = \mathsf{E}\, e^{i\alpha\eta_{B,\sigma}^u(\theta_s)}, \qquad 0 < u < B,$$

is defined by

$$\varphi_{B,\sigma}^u(s,\alpha) = \tilde{h}_s(\alpha,u,B)\left[1 - \mathsf{E}\, e^{-s\tau_B(u)}\right]^{-1} \tag{45}$$

where

$$\tilde{h}_s(\alpha,u,B) = sQ^B(s,u)\int_0^B e^{i\alpha x} R_s(B-x)\,dx - s\int_0^u e^{i\alpha x} R_s(u-x)\,dx.$$

The limit characteristic function of the process $\eta_{B,\sigma}^u(t)$ as $t \to \infty$ is given by

$$\varphi_{B,\sigma}^u(\alpha) = \lim_{s\to 0} \varphi_{B,\sigma}^u(s,x) = \frac{h_0'(\alpha,u,B)}{E\tau_B(u)} \tag{46}$$

where

$$\begin{cases} \tilde{h}_0'(\alpha,u,B) = Q^B(u)\int_0^B e^{i\alpha x} R(B-x)\,dx - \int_0^u e^{i\alpha x} R(u-x)\,dx, \\ \mathsf{E}\,\tau_B(u) = Q^B(u)\int_0^B R(x)\,dx - \int_0^u R(x)\,dx \end{cases} \tag{47}$$

according to (42)*. The limit density of the distribution of this process with reflections from two boundaries is*

$$\frac{\partial}{\partial x}\mathsf{P}\left\{\eta_{B,\sigma}^u(\infty) < x\right\} = \left[Q^B(u)R(B-x) - R(u-x)\delta(u > x)\right]\left[\mathsf{E}\,\tau_B(u)\right]^{-1} \tag{48}$$

if $0 < u < B$.

Proof. The following equality for the characteristic function of the modified process $\eta_{B,\sigma}^u$ is a consequence of (44):

$$\mathsf{E}\, e^{i\alpha\eta_{B,\sigma}^u(t)} = \mathsf{E}\left[e^{i\alpha\xi_u(t)}, \tau_B(u) > t\right] + \int_0^t \mathsf{E}\left[e^{i\alpha\eta_{B,\sigma}^u(t-y)}, \tau_B(u) = \tau^+(v) \in dy\right] + \int_0^t \mathsf{E}\left[e^{i\alpha\eta_{B,\sigma}^u(t-y)}, \tau_B(u) = \tau^-(-u) \in dy\right].$$

Taking the Laplace–Carson transform for the characteristic function

$$\varphi_{B,\sigma}^u(s,\alpha) = s\int_0^\infty e^{-st}\,\mathsf{E}\left[e^{i\alpha\eta_{B,\sigma}^u(t)}\right]dt = \mathsf{E}\, e^{i\alpha\eta_{B,\sigma}^u(\theta_s)}$$

we obtain from the latter equation that

$$\varphi_{B,\sigma}^u(s,\alpha) = \tilde{h}_s(\alpha,u,B) + \varphi_{B,\sigma}^u(s,\alpha)\left(Q_B(s,u) + Q^B(s,u)\right),$$

whence (45) follows. Passing to the limit as $s \to 0$, the limit characteristic function (46) is determined from (45). By inverting we get the density (48) according to relations (24). In so doing we apply the corresponding representation in (37) for $R(x)$ according to the sign of m_1. Namely, we substitute to the right-hand sides of (46)–(48)

$$R(x) = R_+(x)$$

if $m_1 > 0$ (the first case in (37)),

$$R(x) = R_0(x)$$

if $m_1 = 0$ (the second case in (37)), and

$$R(x) = R_-(x)$$

if $m_1 < 0$ (the third case in (37)). Note that the sign "$-$" in the lower bound of integration of the representation (10) disappears in the case of $\sigma > 0$, since the distribution of the minimum of $\xi^-(\theta_s)$ has no atom in this case. □

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