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A CONJECTURE ON THE BEHAVIOR OF TAILS OF FIXED POINTS OF THE SHOT NOISE TRANSFORM

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Abstract. We show, under some restrictions on the response function depending on a parameter α, that the tails of fixed points of transforms of a Poisson shot noise process are proportional at infinity to the exponential function of order $-\alpha$ if $\alpha \in (1,2)$. We advance an argument in support of the conjecture that this result remains true for $\alpha \geq 2$.

1. Introduction, the result, and conjecture

Let $\mathcal{P}^+$ be the set of probability measures on Borel subsets of $[0,+\infty)$, τ_i the points of a Poisson process with intensity $\lambda > 0$, and ξ_i, $i = 1, 2, \dots$, independent copies of a random variable ξ independent of the Poisson process. Assume that

$$h\colon (0,\infty) \to [0,\infty)$$

is a Borel function; in what follows it is called the *response function*. By $\mathcal{L}(\cdot)$ we denote the probability distribution of the random variable written in the parentheses. The shot noise transform $\mathbb{T}_{h,\lambda}$ is introduced in [4] for fixed h and λ. It is defined on the set

$$\mathcal{P}_h^+ := \left\{ \mu \in \mathcal{P}^+ \colon \int_0^\infty \int_0^\infty [1 \wedge h(s)y] \, ds \, \mu(dy) < \infty \right\},$$

assumes values in $\mathcal{P}^+$, and is given by

$$\mathbb{T}_{h,\lambda}(\mathcal{L}(\xi)) = \mathcal{L}\left(\sum_{i=1}^\infty \xi_i h(\tau_i) \right).$$

We study the behavior of tails of some fixed points nondegenerate at 0 of the shot noise transform $\mathbb{T}_{h,\lambda}$. Namely we study distributions such that

$$\mu = \mathbb{T}_{h,\lambda}(\mu).$$

Without loss of generality, we assume that the response function h does not increase and is right continuous. Thus the generalized inverse function $h^{\leftarrow}$ is well defined by

$$h^{\leftarrow}(z) := \begin{cases} \inf\{u \colon h(u) < z\} & \text{if } z < h(0^+); \\ 0 & \text{otherwise.} \end{cases}$$

Note that $h^{\leftarrow}$ does not increase.

If

$$\lambda \int_0^\infty h(u) \, du = 1,$$

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then $\rho(dz) := -\lambda z\, h^{\leftarrow}(dz)$ is a probability distribution without atom at zero (see Remark 3.1 in [4]).

Let μ be a fixed point with the mean m, and let A, η, and η_{sb} be independent random variables with distributions ρ, μ, and μ_{sb}, respectively. The distribution μ_{sb} is such that $\mu_{sb}(dx) := m^{-1}x\,\mu(dx)$. Then (see Proposition 2.1 in [3])

$$\eta_{sb} \overset{d}{=} A\eta_{sb} + \eta. \tag{1}$$

The existence of a fixed point with a finite mean in the proposition and lemma below follows from Propositions 1.1(a) and 2.1(a) in [3] since

$$\mathsf{E} \ln A = \lambda \int_0^\infty h(u) \ln h(u)\, du < 0.$$

This inequality follows from $\lambda \int_0^\infty h^{\alpha-\varepsilon}(u)\, du < 1$, $\varepsilon \in (0, \alpha - 1)$, which is a consequence of relation (2) and the Jensen inequality.

The following is the main result in this paper.

Proposition. *If*

$$\lambda \int_0^\infty h(u)\, du = \lambda \int_0^\infty h^\alpha(u)\, du = 1, \tag{2}$$
$$\int_0^\infty h^\alpha(u) \ln^+ h(u)\, du < \infty$$

for some $1 < \alpha < 2$*, then*

$$\lim_{x\to\infty} x^\alpha \mu(x, \infty) = C(m, \alpha) := C > 0$$

for all fixed points μ *of the shot noise transform* $\mathbb{T}_{h,\lambda}$ *with a finite mean* $m > 0$.

Taking (1) into account, we can apply Theorem 2 of [2] to study fixed points of shot noise transforms. In particular, a theorem of Grinčevicjus allows one to use the condition $\alpha > 1$ instead of $1 < \alpha < 2$ in the case of nonarithmetic distributions $\mathcal{L}(\ln A)$. Thus it is reasonable to conjecture that this can be done in the general case, too.

Conjecture. *The proposition holds for all* $\alpha > 1$.

Note that the Grinčevicjus theorem applied to arithmetic distributions $\mathcal{L}(\ln A)$ implies that $x^\alpha \mu(x, \infty)$ is bounded away from zero and infinity. Thus the proposition is new only in the case of arithmetic distributions. On the other hand, it has the advantage that it does not depend on the type of the distribution $\mathcal{L}(\ln A)$.

The tails of the distributions of random variables

$$\zeta \overset{d}{=} \sum_{i=1}^{M} v_i \zeta_i, \tag{3}$$

where M is a random variable, and ζ_i, $1 \le i \le M$, are random variables conditionally independent of $(M, v_1, v_2, \dots)$, are studied in [6]. Conditions in [6] are similar to those used in our proposition. The case of arithmetic distributions $\mathcal{L}(\ln v_1)$ in [6] is, in fact, the corresponding part of the Grinčevicjus theorem. If the support of h is bounded (the random variable M in (3) has the Poisson distribution), then our proposition improves Theorem 2.2 of [6], which correspond to the special case $v_i = h(\tau_i)$.

2. Proof

Lemma. *Let all the assumptions of the proposition hold. Let φ be the Laplace–Stieltjes transform of a fixed point μ with finite mean $m > 0$. Then the function $\omega(s) := \varphi(s) - 1 + ms$ is regularly varying at zero with index α.*

Proof. The function φ is such that

$$\varphi(s) = \exp\left(\lambda \int_0^\infty (\varphi(\mathrm{h}(u)\,\mathrm{s}) - 1)\,du\right). \tag{4}$$

Without loss of generality we assume that $m = 1$. Consider a positive function whose derivative $\phi(s) := s^{-1}\omega(s)$ is completely monotone. It follows from (4) that

$$\ln\varphi(s) + s = \lambda \int_0^\infty (\varphi(\mathrm{h}(u)\,\mathrm{s}) - 1 + \mathrm{h}(u)\,\mathrm{s})\,du.$$

Using the Maclaurin series with the Peano remainder term, we get

$$\lim_{s\to+0} \frac{\ln\varphi(s) + s}{\varphi(s) - 1 + s} = 1,$$

whence

$$1 = \lim_{s\to+0} \int_0^\infty \frac{\phi(sz)}{\phi(s)}\,\rho(dz), \tag{5}$$

where ρ is a probability measure whose moment of order $\alpha - 1$ equals 1.

Using the same arguments as those in the proof of Lemma 3.3 in [4] we prove that there exist a positive function Λ whose derivative is completely monotone and, by the selection principle, a sequence $t_n \to 0$, $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{\phi(t_n z)}{\phi(t_n)} = \Lambda(z) \quad \text{everywhere in } (0, \infty).$$

Note that $\Lambda(0) = 0$ and $\Lambda(1) = 1$. For a fixed $v > 0$ we have

$$\lim_{n\to\infty} \frac{\phi(t_n vz)}{\phi(t_n v)} = \frac{\Lambda(vz)}{\Lambda(v)}, \qquad z \in (0, \infty). \tag{6}$$

Now we show that (5) and (6) imply

$$1 = \lim_{n\to\infty} \int_0^\infty \frac{\phi(t_n vz)}{\phi(t_n v)}\,\rho(dz) = \int_0^\infty \frac{\Lambda(vz)}{\Lambda(v)}\,\rho(dz), \qquad v > 0.$$

Since the tail is monotone, it follows from Theorem 2 in [2] that there are C_1 and C_2 such that $C_1, C_2 \in (0, +\infty)$ and

$$C_1 1_{[1,\infty)}(y) \le y^{\alpha-1}\mu_{sb}(y,\infty) \le y^{\alpha-1}\mu_{sb}(y,\infty) 1_{[0,\infty)}(y) \le C_2$$

where $\mu_{sb} = \mathcal{L}(\eta_{sb})$ is defined by equality (1).

Put $\phi_\alpha(z) := z^{1-\alpha}\phi(z)$. The latter inequality implies that

$$\begin{aligned}
\phi_\alpha(z) = z^{1-\alpha}\left(1 - z^{-1}(1 - \varphi(z))\right) &= z^{1-\alpha} \int_0^\infty \frac{1 - e^{-zy}}{y^\alpha} y^\alpha \mu(y,\infty)\,dy \\
&\le z^{1-\alpha} \int_0^\infty \frac{1 - e^{-zy}}{y^\alpha} y^{\alpha-1}\mu_{sb}(y,\infty)\,dy \le \frac{C_2\Gamma(2-\alpha)}{\alpha - 1}
\end{aligned}$$

for all $z > 0$. Since

$$y^{\alpha-1}\mu_{sb}(y,\infty) = y^\alpha\mu(y,\infty) + y^{\alpha-1}\int_y^\infty \mu(z,\infty)\,dz \le 2y^\alpha\mu(y,\infty)$$

for $y \geq 1$, we have $C_3 := C_1/2 \leq y^\alpha \mu(y,\infty)$ for $y \geq 1$. Now we get

$$\phi_\alpha(z) \geq z^{1-\alpha} \int_1^\infty \frac{1-e^{-zy}}{y^\alpha} y^\alpha \mu(y,\infty)\, dy \geq C_3 \int_z^\infty \frac{1-e^{-y}}{y^\alpha}\, dy,$$

whence

$$\varliminf_{z\to+0} \phi_\alpha(z) \geq \frac{C_3\Gamma(2-\alpha)}{\alpha-1}.$$

The Lebesgue dominated convergence theorem yields that

$$\lim_{n\to\infty} \int_0^\infty \frac{\phi(t_n vz)}{\phi(t_n v)}\, \rho(dz) = \lim_{n\to\infty} \int_0^\infty \frac{\phi_\alpha(t_n vz)}{\phi_\alpha(t_n v)} z^{\alpha-1}\, \rho(dz) = \int_0^\infty \frac{\Lambda(vz)}{\Lambda(v)}\, \rho(dz),$$

since $z^{\alpha-1}\,\rho(dz)$ is a probability measure, and

$$\frac{\phi_\alpha(t_n vz)}{\phi_\alpha(t_n v)} \leq \frac{C_3}{C_1}$$

for a fixed $v > 0$ and sufficiently large $n \in \mathbb{N}$. Making the change $z := e^{-u}$ in the integral

$$\int_0^\infty \Lambda(vz)\,\rho(dz) = \Lambda(v), \qquad v > 0,$$

we obtain the integro-differential Cauchy equation (with respect to $\Lambda(e^{-v})$). In view of relation (1.5) in [5] and the continuity of Λ, this Cauchy equation has a solution

$$\Lambda(v) = p_1(v) + p_2(v)v^{\alpha-1} \quad \text{for all } v > 0; \tag{7}$$

$$p_i(v) = p_i(vw) \geq 0 \quad \text{for all } w \in \operatorname{supp}(\rho), \qquad i = 1, 2. \tag{8}$$

The assumptions of the lemma imply that there is $w \in \operatorname{supp}(\rho)$ such that $w \in (0,1)$. Fix $v_0 > 0$. It follows from (7) and (8) that

$$\Lambda(v_0w^n) = p_1(v_0w^n) + p_2(v_0w^n)(v_0w^n)^{\alpha-1} = p_1(v_0) + p_2(v_0)(v_0w^n)^{\alpha-1}$$

for all $n \in \mathbb{N}$. Since $\Lambda(0) = 0$, the latter relation yields $p_1(v_0) = 0$. Thus

$$p_2(v) = \Lambda(v)v^{1-\alpha} \quad \text{for all } v > 0,$$

and $p_2'(v) = v^{1-\alpha}((1-\alpha)v^{-1}\Lambda(v) + \Lambda'(v))$. Now we show that $p_2(v) = \text{const}$.

Since $p_2(v)$ is differentiable and periodic, there exists $v_1 > 0$ such that $p_2'(v_1) = 0$ and

$$p_2'(u^n v_0) = 0 \quad \text{for } u \in \operatorname{supp}(\rho) \text{ and } n \in \mathbb{N}. \tag{9}$$

On the other hand, the functions $v^{-1}\Lambda(v)$ and $\Lambda'(v)$ are nonnegative, convex, and nonincreasing. This implies that for $1 < \alpha < 2$ the equality $(\alpha-1)v^{-1}\Lambda(v) = \Lambda'(v)$ either is an identity or holds for at most two points (the graphs of the left- and right-hand sides either coincide or intersect at most at two points). Then $p_2'(v) = 0$ for at most two points contradicting (9). Thus $(\alpha-1)v^{-1}\Lambda(v) \equiv \Lambda'(v)$, whence $p_2(v) = \text{const}$.

On the other hand, $\Lambda(1) = 1$ and thus $\Lambda(v) = v^{\alpha-1}$. This implies

$$\lim_{n\to\infty} \frac{\omega(t_n v)}{\omega(t_n)} = v\Lambda(v) = v^\alpha \quad \text{for all } v \geq 0.$$

We repeat the same arguments as those after relation (6), but now for a general sequence. This gives us the desired result:

$$\lim_{s\to+0} \frac{\omega(sz)}{\omega(s)} = z^\alpha \quad \text{for all } z \geq 0.$$

The lemma is proved. $\square$

Proof of the proposition. Equality (1) holds under the assumptions of the proposition; moreover the distribution of $\mathcal{L}(\eta_{sb})$ is nondegenerate and $\mathsf{E}A^{\alpha-1} = \lambda \int_0^\infty h^\alpha(u)\,du = 1$,

$$\mathsf{E}A^{\alpha-1}\ln^+ A = \lambda \int_0^\infty h^\alpha(u)\ln^+ h(u)\,du < \infty.$$

Since $\mathsf{E}A^{\alpha-1-\varepsilon} < 1$ for an arbitrary $\varepsilon \in (0, \alpha-1)$, Proposition 2(a2) of [3] implies $\mathsf{E}\eta^{\alpha-\varepsilon} < \infty$. Thus Theorem 2 of [2] can be applied. In particular,

a) if $\mathcal{L}(\ln A)$ is a nonarithmetic distribution, then

$$\lim_{x\to\infty} x^{\alpha-1}\mathsf{P}\{\eta_{sb} > x\} = c_\alpha > 0$$

and

$$\lim_{x\to\infty} x^{\alpha}\mathsf{P}\{\eta > x\} = c_\alpha m(\alpha-1)/\alpha := C > 0$$

by Theorem 1.6.5 in [1];

b) if $\mathcal{L}(\ln A)$ is an arithmetic distribution with step γ, then

$$\lim_{n\to\infty} R_\alpha(e^{h+\gamma n}) = c(h) > 0 \quad \text{for all } h \in \mathbb{R} \tag{10}$$

where $R_\alpha(x) := x^{\alpha-1}\mu_{sb}(x,\infty)$ and $\mu_{sb} = \mathcal{L}(\eta_{sb})$. It is known that the function

$$\int_0^\infty e^{-sx}\,\mu(dx) - 1 + ms$$

is regularly varying at zero with index α. Thus $\mu(x,\infty)$ and $\mu_{sb}(x,\infty)$ are regularly varying at ∞ with indices $(-\alpha)$ and $1-\alpha$, respectively (the result for the first tail is due to Theorem 8.1.6 in [1], while the result for the second tail follows from Theorem 1.6.5 in [1], since the first tail is regularly varying). Therefore $R_\alpha(x)$ is regularly varying at ∞.

The latter result and (10) complete the proof. Indeed,

$$1 = \lim_{n\to\infty} \frac{R_\alpha(e^{h+\gamma n})}{R_\alpha(e^{\gamma n})} = \frac{c(h)}{c(0)}$$

for all $h \in \mathbb{R}$, whence

$$\lim_{n\to\infty} R_\alpha(e^{h+\gamma n}) = c(0). \tag{11}$$

Now we check that

$$\lim_{x\to\infty} R_\alpha(x) = c(0). \tag{12}$$

For $u > 1$ and $s \ge e^\gamma$ let $n(s) := [\gamma^{-1}\log s]$ where $[\cdot]$ is the integer part of a number. Then $e^{\gamma n(s)} \le s < e^{\gamma(n(s)+1)}$ and

$$\frac{us}{e^{\gamma n(s)}} = e^\gamma \frac{us}{e^{\gamma(n(s)+1)}} \in [1, ue^\gamma].$$

Since the convergence in relation (11) is locally uniform, we have

$$\lim_{s\to\infty} R_\alpha(us) = \lim_{s\to\infty} R_\alpha\left(\frac{us}{e^{\gamma n(s)}}e^{\gamma n(s)}\right) = c(0),$$

which is equivalent to relation (12).

Therefore, similarly to the nonarithmetical case,

$$\lim_{x\to\infty} x^{\alpha-1}\mathsf{P}\{\eta_{sb} > x\} = c(0).$$

By Theorem 1.6.5 in [1],

$$\lim_{x\to\infty} x^{\alpha}\mathsf{P}\{\eta > x\} = c(0)m(\alpha-1)/\alpha$$

and the proposition is proved. □

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Bibliography

1. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1989. MR1015093 (90i:26003)
2. A. K. Grinčevicjus, *On a limit distribution for a random walk on a line*, Lit. Mat. Sbornik **15** (1975), no. 4, 79–91. (Russian) MR0448571 (56:6877)
3. A. M. Iksanov, *On perpetuities related to the size-biased distributions*, Theory Stoch. Proc. **8**(**24**) (2002), no. 1–2, 128–135. MR2028745 (2004m:60146)
4. A. M. Iksanov and Z. J. Jurek, *On fixed points of Poisson shot noise transforms*, Adv. Appl. Prob. **34** (2002), 798–825. MR1938943 (2003i:60021)
5. K. S. Lau and C. R. Rao, *Solution to the integrated Cauchy functional equation on the whole line*, Sankhyā **A46** (1984), 311–318. MR0798038 (86m:39017)
6. Q. Liu, *On generalized multiplicative cascades*, Stoch. Proc. Appl. **86** (2000), 263–286. MR1741808 (2001b:60102)

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