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AN EXACT SOLUTION OF THE RISK EQUATION WITH A STEP CURRENT RESERVE FUNCTION

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D. P. KEREKESHA

Abstract. We obtain explicitly the solution of the risk equation for a step current reserve function.

Consider the risk process

$$R_t = u - \sum_{i=1}^{N_t} U_i + \int_0^t p(R_s)\,ds. \tag{1}$$

Here U_i is a sequence of nonnegative identically distributed random variables with the distribution function $B(t) = \mathsf{P}(U_1 < t)$ and finite mathematical expectation $\mu = \mathsf{E}\,U_1$, N_t is a Poisson process with intensity β, and $p(u)$ is the current reserve function.

The ruin probability is defined by

$$\psi(u) = \mathsf{P}\left(\inf_{t\geq 0} R_t < 0 \mid R_0 = u\right).$$

It is shown in [1] that if $g(x)$ is such that

$$\psi(x) = \int_x^\infty g(t)\,dt,$$

then

$$\frac{p(u)}{\beta} g(u) = \gamma_0 \overline{B}(u) + \int_0^u \overline{B}(x-t) g(t)\,dt. \tag{2}$$

Here

$$\bar{B} = 1 - B, \qquad \gamma_0 = 1 - \psi(+0). \tag{3}$$

Since the expectation is finite, $\bar{B}(u) \in L_1(0,\infty)$.

In the general case, the solution of equation (2) is not known for the general case. There are particular cases (see [1, 2]) where equation (2) can be solved explicitly. For example, the case of

$$\frac{p(u)}{\beta} = \begin{cases} p_1, & u \in [0, u_1], \\ p_2, & u \in (u_1, \infty), \end{cases}$$

where p_1 and p_2 are positive constants, is studied in [1].

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In this paper we consider a more general case, namely

$$\frac{p(u)}{\beta} = \begin{cases} p_1, & u \in [0, u_1], \\ p_2, & u \in (u_1, u_2), \\ p_i, & u \in (u_{i-1}, u_i), \\ p_n, & u \in (u_n, \infty), \end{cases} \tag{4}$$

where p_i, $i = 1, \ldots, n$, are positive constants such that

$$p_i \geq \beta, \quad i = 1, \ldots, n-1, \qquad p_n > \beta. \tag{5}$$

Condition (5) means that the intensity of claims changes with the current reserve function, and R_t is a step function. This property can be explained in some economic applications by an increase of competition on the market. Another explanation is the payout of dividends where the premium paid to shareholders depends on the reserve of a company.

To generalize the result of [1] we develop a new approach to solving equation (2). In what follows we assume that $\overline{B}(u) \in L_2(0, \infty)$ and look for a solution of equation (2) in the space $L_2(0, \infty)$.

Definition 1. Denote by L_{2+} (L_{2-}) the space of functions φ_+ (φ_-) in L_2 that vanish for $x < 0$ ($x > 0$).

Definition 2. We say that a function $\Phi(z)$ ($z = x + iy$) belongs to the Hardy space H_2^+ if it is analytic in the half-plane $y > 0$ and

$$\sup_{y>0} \int_{-\infty}^{\infty} |\Phi(x+iy)|^2 \, dx < \infty.$$

Definition 3. We say that a function $\Phi(z)$ ($z = x + iy$) belongs to the Hardy space H_2^- if it is analytic in the half-plane $y < 0$ and

$$\sup_{y<0} \int_{-\infty}^{\infty} |\Phi(x+iy)|^2 \, dx < \infty.$$

It is known (see [3]) that the limit

$$\lim_{z \to x} \Phi(z)$$

exists for almost all $x \in \mathbb{R}$ if $\Phi \in H_2^+$ and $\operatorname{Im} z > 0$. Moreover, if we denote the limit by $\Phi(x)$, then $\Phi(x) \in L_2(-\infty, \infty)$ in this case. A similar result holds for functions in the space H_2^-.

Theorem 1. *A function φ_+ (φ_-) belongs to the class L_{2+} (L_{2-}) if and only if its Fourier transform Φ^+ (Φ^-) is the limit of a function of the space H_2^+ (H_2^-).*

The proof of Theorem 1 is based on some results of the book [4].

Let

$$M = \bigcup_{k=1}^{n} (a_i, b_i)$$

where $0 \leq a_0 < b_0 \leq a_1 < b_1 \leq \cdots \leq a_n < b_n \leq \infty$.

Definition 4. By $L_2\{M\}$ we denote the space of functions φ_+ that belong to L_{2+} and vanish on $\mathbb{R}^+ \setminus M$.

Definition 5. By $L_2^+\{M\}$ we denote the space of functions Φ that are the Fourier transforms of functions of the space $L_2\{M\}$.

Theorem 2. *In order that $\Phi(x) \in L_2^+\{(a,b)\}$ for $a,b>0$ it is necessary and sufficient that the function $\Phi(x)$ be such that:*

(1) $\Phi(x) = e^{ixa}\Phi_1^+(x)$ *for some* $\Phi_1^+(z) \in H_2^+$ *and* $\Phi_1^+(x) = \lim_{z\to x}\Phi_1^+(z)$ *almost everywhere;*
(2) $\Phi(x) = e^{ixb}\Phi_2^-(x)$ *for some* $\Phi_2^-(z) \in H_2^-$ *and* $\Phi_2^-(x) = \lim_{z\to x}\Phi_2^-(z)$ *almost everywhere.*

Proof. Necessity. Let $\Phi \in L_2^+\{(a,b)\}$. Then the function can be represented as follows:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{ixt}\varphi_+(t)\,dt \tag{6}$$

for some $\varphi(t) \in L_{2+}\{M\}$ (see Definition 5). Putting $\tau = t-a$ in (6), we obtain

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}e^{ixa}\int_0^{b-a} e^{ixt}\varphi_{1+}(t)\,dt, \qquad \varphi_{1+}(t) = \varphi(t+a), \tag{7}$$

where

$$\Phi_1^+(x) = \frac{1}{\sqrt{2\pi}}\int_0^{b-a} e^{ixt}\varphi_{1+}(t)\,dt$$

is the limit value of the function $\Phi_1^+(z)$ that belongs to the space H_2^+.

Theorem 1 implies that

$$\Phi(x) = \exp\{ixa\}\Phi_1^+(x).$$

Thus the function $\Phi(x)$ satisfies the assumptions of Theorem 1. A similar reasoning proves that the function $\Phi(x)$ satisfies condition (2).

Sufficiency. Let $\Phi(x) = \exp\{ixa\}\Phi_1^+(x)$. Then

$$V^{-1}(\Phi)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt}\Phi(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x-a)}\Phi_1(t)\,dt,$$

whence $V^{-1}(\Phi)(x) = 0$ for $x < a$. Similarly we prove that

$$V^{-1}(\Phi)(x) = 0$$

for $x > b$. Thus

$$\Phi \in L_2^+\{(a,b)\}$$

(see Definition 3). □

Corollary 1. *In order that $\varphi \in L_2\{(a,b)\}$ for $a,b>0$ it is necessary and sufficient that the function $\Phi(x)$ be such that:*

(1) $\Phi(x) = e^{ixa}\Phi_1^+(x)$, *and*
(2) $\Phi(x) = e^{ixb}\Phi_2^-(x)$,

where $\Phi_1^+(x)$ and $\Phi_2^-(x)$ are limit values of some functions in the spaces H_2^+ and H_2^-, respectively.

Theorem 3. *In order that $\Phi(x) \in L_2^+\{(a,\infty)\}$ for $a>0$ it is necessary and sufficient that the function $\Phi(x)$ be such that*

$$\Phi(x) = e^{ixa}\Phi_1^+(x)$$

where $\Phi_1^+(x)$ is the limit value of a function $\Phi_1^+(z) \in H_2^+$.

Proof. Necessity. Let $\Phi \in L_2^+\{(a,\infty)\}$. Then

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_a^{\infty} e^{ixt}\varphi_+(t)\,dt \tag{8}$$

for some $\varphi(t) \in L_{2+}\{M\}$ (see Definition 5). Putting $\tau = t - a$ in (8), we obtain

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{ixa} \int_0^\infty e^{ixt} \varphi_{1+}(t)\, dt, \qquad \varphi_{1+}(t) = \varphi(t+a). \tag{9}$$

Theorem 1 implies that $\Phi(x) = \exp\{ixa\}\Phi_1^+(x)$ where $\Phi_1^+(x)$ is the limit value of a function $\Phi_1^+(z) \in H_2^+$.

Sufficiency. Let $\Phi(x) = \exp\{ixa\}\Phi_1^+(x)$. Then

$$V^{-1}(\Phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ixt}\Phi(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{it(x-a)}\Phi_1(t)\, dt,$$

whence $V^{-1}(\Phi)(x) = 0$ for $x < a$. Then

$$\Phi \in L_2^+\{(a,\infty)\}$$

(by Definition 5). □

Corollary 2. *In order that $\varphi \in L_2\{(a,\infty)\}$ for $a > 0$ it is necessary and sufficient that the function $\Phi(x)$ be such that*

$$\Phi(x) = e^{ixa}\Phi_1^+(x), \qquad \Phi_1^+(z) \in H_2^+, \quad z = x + iy.$$

Theorem 4. *In order that $\varphi \in L_2\{(a,b)\}$ for $a, b > 0$ it is necessary and sufficient that $\Phi \in L_2^+\{(a,b)\}$.*

Necessity follows from Definition 5, while sufficiency follows from Theorem 3.

Theorem 5. *In order that $\varphi \in L_2\{(a,\infty)\}$ it is necessary and sufficient that*

$$\Phi \in L_2^+\{(a,\infty)\}.$$

Necessity follows from Definition 5, while sufficiency follows from Theorem 4.

Definition 6. Denote by P^+ the projector of the function $F(x) \in L_2(-\infty,\infty)$ defined by

$$P^+F(x) = F^+(x), \qquad F^+(z) \in H_2^+ \ (z = x + iy).$$

Definition 7. By $P^+_{\{M\}}$ we denote the operator projecting the space $L_2(-\infty,\infty)$ onto the subspace $L_2^+\{M\}$.

Theorem 6. *In order that $\varphi \in L_2\{M\}$ it is necessary and sufficient that $\Phi \in L_2^+\{M\}$.*

Proof. Necessity follows from Definition 5.

Sufficiency. Let $\Phi \in L_2^+\{M\}$. Then it follows from results of [5, 6] that

$$P^+_{\{M\}}(\Phi) = P^+_{\{(a_1,b_1)\}}(\Phi) \oplus \cdots \oplus P^+_{\{(a_n,b_n)\}}(\Phi). \tag{10}$$

Applying the inverse Fourier transform to (10), we obtain

$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_n. \tag{11}$$

It follows from Theorems 4 and 5 that

$$\varphi_i \in L_{2+}\{(a_i,b_i)\}, \qquad i = 1, 2, \ldots, n. \tag{12}$$

Then we obtain from (11) and (12) that $\varphi \in L_{2+}\{(M)\}$. □

Theorem 7. *Let $\Phi \in L_2^+\{M_1\}$ and $\Phi \in L_2^+\{M_2\}$. Then $\Phi \in L_2^+\{M_1 \cap M_2\}$.*

Since P^+ is a selfadjoint operator, it follows from [7] that

$$P^+_{\{M_1\}} \cdot P^+_{\{M_2\}} = P^+_{\{M_1 \cap M_2\}}. \tag{13}$$

The assumptions of the theorem yield that $P_{\{M_1\}}\Phi = P_{\{M_2\}}\Phi$. Thus we obtain from (13) that

$$P_{\{M_1 \cap M_2\}}\Phi = \left(P_{\{M_2\}}P_{\{M_1\}}\right)\Phi = \left(P_{\{M_2\}}P_{\{M_2\}}\right)\Phi = P_{\{M_2\}}\Phi = \Phi.$$

Lemma 1. *Let $\Phi \in L_2^+\{M\}$. Assume that G^+ is an analytic function in the upper half-plane, and let G^+ be continuous on the whole closed axis. Then*

$$\Phi G \in L_2^+\{M\}.$$

The assertion follows from Definition 4, since the function G is bounded for $\operatorname{Im} z \geq 0$.

In order to apply the Fourier transform to equation (2) with p_i satisfying conditions (4) and (5) we consider new unknown functions f_i such that

$$f_i \in L_{2+}\{(u_{i-1}, u_i)\}, \qquad i = 1, \ldots, n-1, \ u_0 = 0, \ f_n \in L_{2+}\{(u_n, \infty)\}. \tag{14}$$

Then equation (2) can be rewritten as a system of equations

$$p_i g(u) = \gamma_0 \overline{B}(u) + \int_0^u \bar{B}(u-t)g(t)\,dt + f_i(u), \qquad i = 1, \ldots, n. \tag{15}$$

Applying the Fourier transform to system (15) we obtain

$$G(x)\left(p_i - \sqrt{2\pi}\hat{\bar{B}}(x)\right) = \gamma_0 \hat{\bar{B}}(x) + F_i(x), \qquad i = 1, \ldots, n, \tag{16}$$

where $F_i \in L_2^+\{(u_{i-1}, u_i)\}$, $i = 1, \ldots, n-1$, and $F_n \in L_2^+\{(u_n, \infty)\}$. Here

$$\hat{\bar{B}}(x) = (V\bar{B})(x), \qquad G(x) = (Vg)(x), \qquad F_i(x) = (Vf_i)(x), \quad i = 1, \ldots, n,$$

and G and F_i are unknown functions.

System (16) can easily be rewritten in the following form:

$$\begin{cases} F_{i-1}(x)\frac{p_i - \sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i-1} - \sqrt{2\pi}\hat{\bar{B}}(x)} = F_i(x) + \frac{\gamma_0 \hat{\bar{B}}(x)(p_{i-1} - p_i)}{p_{i-1} - \sqrt{2\pi}\hat{\bar{B}}(x)}, \quad i = 2, \ldots, n, \\ G(x)(p_n - \sqrt{2\pi}\hat{\bar{B}}(x)) = \gamma_0 \hat{\bar{B}}(x) + F_n(x). \end{cases} \tag{17}$$

Using conditions (5), results of [5], and properties of the function $\bar{B}(x)$, we show that

$$\frac{p_i - \sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i-1} - \sqrt{2\pi}\hat{\bar{B}}(x)}, \qquad i = 2, \ldots, n,$$

are continuous functions on the whole closed axis and moreover they can be continuously extended to the upper half-plane.

Next we apply Theorems 6 and 7 and Lemma 1 to prove that

$$F_k(x) = -\sum_{i=1}^{k-1} P_{(0,u_i)}\left(\frac{\gamma_0 \hat{\bar{B}}(x)(p_i - p_{i+1})}{p_i - \sqrt{2\pi}\hat{\bar{B}}(x)} \frac{p_k - \sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i+1} - \sqrt{2\pi}\hat{\bar{B}}(x)}\right) + R_k(x) \tag{18}$$
$$k = 2, \ldots, n,$$

where

$$R_i \in L_2^+\{(u_i, \infty)\}, \qquad i = 2, \ldots, n-1, \ R_n(x) \equiv 0. \tag{19}$$

Now we use equalities (18) and system (17) to show that

$$G(x) = \frac{\gamma_0 \hat{\bar{B}}(x) - \sum_{i=1}^{n-1} P_{(0,u_i)}\left(\frac{\gamma_0 \hat{\bar{B}}(x)(p_i - p_{i+1})}{p_i - \sqrt{2\pi}\bar{B}(x)} \frac{p_n - \sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i+1} - \sqrt{2\pi}\hat{\bar{B}}(x)}\right)}{p_n - \sqrt{2\pi}\hat{\bar{B}}(x)}. \tag{20}$$

Applying the inverse Fourier transform to (20), we obtain a solution of equation (2)

$$g(u) = V^{-1}\left(\frac{\gamma_0\hat{\bar{B}}(x) - \sum_{i=1}^{n-1} P_{(0,u_i)}\left(\frac{\gamma_0\hat{\bar{B}}(x)(p_i-p_{i+1})}{p_i-\sqrt{2\pi}\hat{\bar{B}}(x)}\frac{p_n-\sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i+1}-\sqrt{2\pi}\hat{\bar{B}}(x)}\right)}{p_n-\sqrt{2\pi}\hat{\bar{B}}(x)}\right)(u).$$

Now the ruin probability is given by

$$\psi(u) = \int_u^\infty g(t)\,dt \tag{21}$$

where $g(u)$ is defined by (21).

Finally, γ_0 is determined from condition (4), namely:

$$\gamma_0 = \frac{1}{1+\int_0^\infty V^{-1}\left(\frac{\gamma_0\hat{\bar{B}}(x)-\sum_{i=1}^{n-1} P_{(0,u_i)}\left(\frac{\gamma_0\hat{\bar{B}}(x)(p_i-p_{i+1})}{p_i-\sqrt{2\pi}\hat{\bar{B}}(x)}\frac{p_n-\sqrt{2\pi}\hat{\bar{B}}(x)}{p_{i+1}-\sqrt{2\pi}\hat{\bar{B}}(x)}\right)}{p_n-\sqrt{2\pi}\hat{\bar{B}}(x)}\right)(x)\,dx}. \tag{22}$$

Theorem 8. *Let*

$$\bar{B}(u) \in L_2(0,\infty)\cap L_1(0,\infty),$$

and let the current reserve function $p(u)$ *satisfy conditions* (4)–(5). *Then equation* (2) *has a unique solution* $g(u) \in L_2(0,\infty)$ *given by relation* (21). *The ruin probability* ψ *is then determined by relations* (21)–(22).

Example. Let $\bar{B} = e^{-\delta u}$ and $n = 2$. Applying Theorem 8, we obtain

$$\psi = \begin{cases} 0, & 0 < u, \\ \frac{1}{p_1\delta}e^{-\gamma_1 u}, & 0 < u < u_1, \\ \frac{\gamma_2}{p_2\gamma_1\delta}e^{-\gamma_2 u}e^{-u_1(\gamma_1-\gamma_2)}, & u_1 < u < \infty, \end{cases} \tag{23}$$

where

$$\gamma_i = \delta - \frac{1}{p_i}, \qquad i = 1, 2.$$

This solution of the risk equation can be checked explicitly.

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Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, Kyiv National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

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