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JUSTIFICATION OF THE FOURIER METHOD FOR HYPERBOLIC EQUATIONS WITH RANDOM INITIAL CONDITIONS

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Abstract. Conditions for the existence of a twice differentiable solution of a hyperbolic type partial differential equation with random strongly $\operatorname{Sub}_\varphi(\Omega)$ initial conditions are found in the multidimensional case.

1. Introduction

We consider a boundary problem for a homogeneous hyperbolic partial differential equation with random strongly $\operatorname{Sub}_\varphi(\Omega)$ initial conditions in the multidimensional case. The main aim of the paper is to propose a new approach for studying partial differential equations with random initial conditions and to apply this approach for the justification of the Fourier method for solving hyperbolic type problems. We also find estimates for the distribution of the supremum of a solution of such a problem.

Similar problems for the vibration of a homogeneous string, square membrane, and round membrane with random strongly sub-Gaussian initial conditions are considered in [1, 9, 8]. Further references can be found in [1].

2. Stochastic processes of the space $\operatorname{Sub}_\varphi(\Omega)$

Definition 2.1 ([1]). Let T be a nonempty set. A function $\rho\colon T\times T\to[0,\infty)$ is called a pseudometric if

(1) $\rho(t,s)=\rho(s,t)$, $t,s\in T$;
(2) $\rho(t,s)\le\rho(t,v)+\rho(v,s)$, $t,s,v\in T$;
(3) $\rho(t,s)=0$ if $t=s$.

The pair (T,ρ) is called a pseudometric space.

Definition 2.2 ([1]). Let (T,ρ) be a nonempty metric space and let $\varepsilon>0$. Denote by $N_\rho(T,\varepsilon)$ the minimum number of points of an ε-net of the set T with respect to the pseudometric ρ. The function $(N_\rho(T,\varepsilon),\varepsilon>0)$ is called the massiveness of the set T with respect to the pseudometric ρ.

Definition 2.3 ([2]). A continuous even function $u(x)$, $x\in\mathbf{R}^1$, such that $u(0)=0$, $u(x)>0$ for $x\ne0$, and $\lim_{x\to0}u(x)/x=0$, $\lim_{x\to\infty}u(x)/x=\infty$ is called an N-function.

Definition 2.4 ([1]). We say that an N-function u satisfies the g-condition if there exist constants $z_0>0$, $k>0$, and $A>0$ such that

$$u(x)u(y)\le Au(kxy)$$

for all $x>z_0$ and $y>z_0$.

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Example 2.1. The function $u(x) = B|x|^\alpha$ for $B > 0$ and $\alpha > 1$ as well as

$$u(x) = \exp\{A|x|^\alpha\} - 1$$

for $A > 0$ and $\alpha > 1$ are examples of N-functions.

The function $u(x) = \exp\{\varphi(x)\} - 1$ also is an N-function if $\varphi(x)$ is an N-function.

Lemma 2.1. *Let $u(x)$ be an N-function. Then*

(1) $u(\alpha x) \le \alpha u(x)$ *for* $0 \le \alpha \le 1$ *and* $x \in \mathbf{R}$*;*
(2) $u(\alpha x) \ge \alpha u(x)$ *for* $\alpha > 1$ *and* $x \in \mathbf{R}$*;*
(3) $u(|x| + |y|) \le u(x) + u(y)$ *for* $x, y \in \mathbf{R}$*;*
(4) *the function* $u(x)/x$ *is nondecreasing for* $x > 0$*.*

Lemma 2.2. *Let $u^{(-1)}(x)$ be the inverse to an N-function $u(x)$ for $x > 0$. Then $u^{(-1)}(x)$ is a convex increasing function such that*

(1) $u^{(-1)}(\alpha x) \le \alpha u^{(-1)}(x)$ *for* $0 \le \alpha \le 1$ *and* $x \in \mathbf{R}$*;*
(2) $u^{(-1)}(\alpha x) \ge \alpha u^{(-1)}(x)$ *for* $\alpha > 1$ *and* $x \in \mathbf{R}$*;*
(3) $u^{(-1)}(|x| + |y|) \le u^{(-1)}(x) + u^{(-1)}(y)$ *for* $x, y \in \mathbf{R}$*;*
(4) *the function* $u^{(-1)}(x)/x$ *is nonincreasing for* $x > 0$*.*

Definition 2.5 ([2])**.** Let $u(x)$ be an N-function. The function

$$u^*(x) = \sup_{y \in \mathbf{R}} (xy - u(y))$$

is called the Young–Fenchel transform of the function $u(x)$.

The function $u^*(x)$ also is an N-function.

The Orlicz space of random variables generated by an N-function $u(x)$. Let $\{\Omega, \Im, \mathsf{P}\}$ be a probability space.

Definition 2.6 ([2, 1])**.** The set of random variables $\xi = \xi(\omega)$, $\omega \in \Omega$, is called the Orlicz space $L_u(\Omega)$ of random variables generated by the N-function $u(x)$ if for any $\xi \in L_u(\Omega)$ there exists a constant r_ξ such that $\mathsf{E}\, u(\xi/r_\xi) \le \infty$.

The Orlicz space $L_u(\Omega)$ is a Banach space with respect to the norm

$$\|\xi\|_{L_u} = \inf\left\{ r > 0\colon\ \mathsf{E}\, u\left(\frac{\xi}{r}\right) \le 1 \right\}.$$

Definition 2.7. A stochastic process $X = \{X(t), t \in T\}$ belongs to the Orlicz space $L_u(\Omega)$ if the random variable $X(t)$ belongs to $L_u(\Omega)$ for all $t \in T$.

The space $\mathrm{Sub}_\varphi(\Omega)$.

Definition 2.8 ([1])**.** Let $\varphi(x)$ be an N-function for which there exist constants $x_0 > 0$ and $c > 0$ such that $\varphi(x) = cx^2$ for $|x| < x_0$. The set of random variables $\xi(\omega)$, $\omega \in \Omega$, is called the space $\mathrm{Sub}_\varphi(\Omega)$ generated by the N-function $\varphi(x)$ if $\mathsf{E}\,\xi = 0$ and there exists a constant a_ξ such that

$$\mathsf{E} \exp\{\lambda \xi\} \le \exp\{\varphi(\lambda a_\xi)\}$$

for all $\lambda \in \mathbf{R}^1$.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi) = \sup_{\lambda \ne 0} \frac{\varphi^{(-1)}(\ln \mathsf{E} \exp\{\lambda \xi\})}{|\lambda|}$$

(see [1] for the proof).

Definition 2.9. A stochastic process $X = \{X(t), t \in T\}$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ ($X \in \mathrm{Sub}_\varphi(\Omega)$) if $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ for all $t \in T$.

Example 2.2. A Gaussian stochastic process $X(t)$ with zero mean belongs to $\mathrm{Sub}_\varphi(\Omega)$ where $\varphi(x) = x^2/2$ and $\tau(X(t)) = (\mathsf{E}(X(t))^2)^{1/2}$.

A family of strongly $\mathrm{Sub}_\varphi(\Omega)$ random variables and a family of strongly $\mathrm{Sub}_\varphi(\Omega)$ stochastic processes.

Lemma 2.3 ([1]). *If $\xi \in \mathrm{Sub}_\varphi(\Omega)$, then there exists a constant $C > 0$ such that*

$$(\mathsf{E}(\xi)^2)^{1/2} \leq C\tau_\varphi(\xi).$$

Definition 2.10. A random variable $\xi \in \mathrm{Sub}_\varphi(\Omega)$ is called strongly $\mathrm{Sub}_\varphi(\Omega)$ random variable if $\tau_\varphi(\xi) = \left(\mathsf{E}\,\xi^2\right)^{1/2}$. The space of strongly $\mathrm{Sub}_\varphi(\Omega)$ random variables is denoted by $\mathrm{SSub}_\varphi(\Omega)$.

Properties and applications of $\mathrm{SSub}_\varphi(\Omega)$ random variables and stochastic processes can be found in [1].

Definition 2.11 ([3]). A family Δ of random variables ξ of the space $\mathrm{Sub}_\varphi(\Omega)$ is called $\mathrm{SSub}_\varphi(\Omega)$ family if

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) = \left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}$$

for all $\lambda_i \in \mathbf{R}^1$ where I is at most countable and $\xi_i \in \Delta$, $i \in I$.

Theorem 2.1 ([3]). *Let Δ be a strongly $\mathrm{Sub}_\varphi(\Omega)$ family of random variables. Then the linear closure $\bar{\Delta}$ of the family Δ in the space $L_2(\Omega)$ and in the mean square sense is a strongly $\mathrm{Sub}_\varphi(\Omega)$ family.*

Definition 2.12. A stochastic process $X_i = \{X_i(t), t \in T, i \in I\}$ is called an $\mathrm{SSub}_\varphi(\Omega)$ process if the family of random variables $X_i = \{X_i(t), t \in T, i \in I\}$ is a $\mathrm{SSub}_\varphi(\Omega)$ family.

Theorem 2.2 ([3]). *Let $X_i = \{X_i(t), t \in T, i \in I\}$ be a family of jointly $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes. Then (T, O, μ) is a measurable space. If*

$$\{\varphi_{k_i}(t), i \in I, k = 1, \ldots, \infty\}$$

is a family of measurable functions in (T, O, μ) and the integral

$$\xi_{k_i} = \int_T \varphi_k(t) X_j(t)\, d\mu(t)$$

is well defined in the mean square sense, then the family of random variables

$$\Delta_\xi = \{\xi_{k_i}, i \in I, k = 1, \ldots, \infty\}$$

is an $\mathrm{SSub}_\varphi(\Omega)$ family.

Remark. A Gaussian stochastic process with zero mean is an $\mathrm{SSub}_\varphi(\Omega)$ process for

$$u(x) = x^2/2.$$

3. Conditions for the weak convergence in $C(T)$ of $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes

Theorem 3.1. *Let T be a set of parameters, let $X_n = \{X_n(t), t \in T\}$ be a sequence of stochastic processes belonging to the Orlicz space $L_u(\Omega)$, let the N-function u satisfy the g-condition, and let $\|\cdot\|_{L_u}$ be the norm in $L_u(\Omega)$. Assume that*

(1)

$$\sup_{n\ge 1}\sup_{s,t\in T} \|X_n(t) - X_n(s)\|_{L_u} < \infty. \tag{3.1}$$

(2) *A pseudometric ρ is defined on T as follows:*

$$\rho(t,s) = \sup_{n\ge 1} \rho_n(t,s)$$

where $(t,s) \in T$ and

$$\rho_n(t,s) = \|X_n(t) - X_n(s)\|_{L_u}.$$

(3) *The space (T,ρ) is compact and every process $X_n(t)$ is separable on (T,ρ).*
(4) *$N(\varepsilon) = N_\rho(T,\varepsilon)$ is the metric massiveness of the space (T,ρ).*
(5)

$$\int_{0+} U^{(-1)}(N(\varepsilon))\, d\varepsilon < \infty \tag{3.2}$$

where $U^{(-1)}(x)$ is the inverse to $U(x)$ for $x > 0$.

Then for all $\delta > 0$

$$\lim_{\varepsilon\to 0}\sup_{n\ge 1} \mathsf{P}\left\{\sup_{\substack{t,s\in T\\ \rho(t,s)<\varepsilon}} |X_n(t) - X_n(s)| > \delta\right\} = 0. \tag{3.3}$$

Proof. Theorem 3.1 is a variant of Theorem 3.5.1 and Corollary 3.5.2 in [1] for processes in the Orlicz spaces. □

Theorem 3.2. *Let (T,d) be a compact metric space, let*

$$X_n = \{X_n(t), t \in T\}$$

be a sequence of stochastic process belonging to the Orlicz space $L_u(\Omega)$, and let the function u satisfy the g-condition. Assume that all the processes $X_n(t)$ are separable in (T,d) and

$$\rho_n(t,s) = \|X_n(t) - X_n(s)\|_{L_u}.$$

Assume also that

$$\rho(t,s) = \sup_{n\ge 1} \rho_n(t,s) \le Z(d(t,s)) \tag{3.4}$$

where $Z = \{Z(x) > 0, x > 0\}$ is a function such that $Z(x) \to 0$ as $x \to 0$. Denote by $N_\rho(\varepsilon)$ the metric massiveness of the space (T,ρ). If

$$\int_{0+} U^{(-1)}(N_\rho(\varepsilon))\, d\varepsilon < \infty, \tag{3.5}$$

then

$$\lim_{\varepsilon\to 0}\sup_{n\ge 1} \mathsf{P}\left\{\sup_{\substack{t,s\in T\\ d(t,s)<\varepsilon}} |X_n(t) - X_n(s)| > \delta\right\} = 0$$

for all $\delta > 0$.

Proof. Condition (3.4) implies condition (3.1). Since (3.4) holds, the processes $X_n(t)$ are separable in the space (T, ρ) (see [1] for the proof). Thus (3.5) implies (3.3). Condition (3.4) yields that for any $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ such that $\rho(t,s) < \gamma(\varepsilon)$ if $d(t,s) < \varepsilon$. Moreover, $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ in this case. Thus

$$\mathsf{P}\left\{ \sup_{\substack{t,s\in T \\ d(t,s)<\varepsilon}} |X_n(t) - X_n(s)| > \delta \right\} \le \mathsf{P}\left\{ \sup_{\substack{t,s\in T \\ \rho(t,s)<\gamma(\varepsilon)}} |X_n(t) - X_n(s)| > \varepsilon \right\}. \tag{3.6}$$

Therefore (3.6) completes the proof of Theorem 3.2. □

Lemma 3.1 ([1])**.** *Let (T, ρ_1) and (T, ρ_2) be two pseudometric spaces such that ρ_1 and ρ_2 are equivalent, that is, there are constants $c_1 > 0$ and $c_2 > 0$ for which*

$$c_1 \rho_1(t,s) \le \rho_2(t,s) \le c_2 \rho_1(t,s).$$

Then

$$N_1\left(\frac{u}{c_1}\right) \le N_2(u) \le N_1\left(\frac{u}{c_2}\right) \tag{3.7}$$

where $N_i(u)$ is the metric massiveness of the space (T, ρ_i), $i = 1, 2$.

Corollary 3.1. *Conditions* (3.7) *and*

$$\int_{0+} U^{(-1)}\left(N_1(\varepsilon)\right) d\varepsilon < \infty$$

hold for N-functions $N_1(u)$ and $N_2(u)$ if and only if

$$\int_{0+} U^{(-1)}\left(N_2(\varepsilon)\right) d\varepsilon < \infty.$$

Theorem 3.3. *Let (T, d) be a compact metric space, and*

$$X_n = \{X_n(t), t \in T\}$$

a sequence of stochastic processes belonging to the space $\mathrm{Sub}_\varphi(\Omega)$. Assume that all these processes are separable on (T, d). Further let

$$m_n(t,s) = \tau_\varphi\left(X_n(t) - X_n(s)\right)$$

and

$$m(t,s) = \sup_{n\ge 1} m_n(t,s) \le z\left(d(t,s)\right) \tag{3.8}$$

where $z = \{z(x) > 0, x > 0\}$ is a function such that $z(x) \to 0$ as $x \to 0$. Denote by $N_m(u)$ the metric massiveness of the space (T, m). If

$$\int_{0+} \Psi\left(N_m(u)\right) du < \infty \tag{3.9}$$

where $\Psi(z) = z/\varphi^{(-1)}(z)$ and $\varphi^{(-1)}(u)$ is the inverse to $\varphi(u)$ for $u > 0$, then

$$\lim_{\varepsilon\to 0} \sup_{n\ge 1} \mathsf{P}\left\{ \sup_{\substack{t,s\in T \\ d(t,s)<\varepsilon}} |X_n(t) - X_n(s)| > \delta \right\} = 0 \tag{3.10}$$

for all $\delta > 0$.

Proof. Consider the Orlicz space $L_u(\Omega)$ for the function $u(x) = \exp\{\varphi^*(x)\} - 1$ where $\varphi^*(x)$ is the Young–Fenchel transform of the function $\varphi(x)$. It is shown in [1] that $\xi \in \mathrm{Sub}_\varphi(\Omega)$ if and only if $\xi \in L_u(\Omega)$. Moreover the norms $\tau_\varphi(\xi)$ and $\|\xi\|_{L_u}$ are equivalent, that is, there are constants $b_1 > 0$ and $b_2 > 0$ such that

$$b_1 \|\xi\|_{L_u} \le \tau_\varphi(\xi) \le b_2 \|\xi\|_{L_u}. \tag{3.11}$$

Thus all stochastic processes $X_n(t)$ belong to the space $L_u(\Omega)$. Put

$$\rho_n(t,s) = \|X_n(t) - X_n(s)\|_{L_u}, \qquad \rho(t,s) = \sup_n \rho_n(t,s).$$

It follows from (3.11) that the pseudometrics $m(t,s)$ and $\rho(t,s)$ are equivalent, that is

$$b_1\rho(t,s) \le m(t,s) \le b_2\rho(t,s) \tag{3.12}$$

where b_1 and b_2 are defined in (3.11). Relations (3.12) and (3.8) imply

$$\rho(t,s) \le b_1^{-1} z(d(t,s)).$$

Note that the g-condition holds for the function $u(x) = \exp\{\varphi^*(x)\} - 1$ (see [1]).

Let $N_\rho(u)$ be the massiveness of the space (T,ρ). According to Theorem 3.2, if

$$\int_{0+} U^{(-1)}(N_\rho(u))\, du < \infty, \tag{3.13}$$

then Theorem 3.3 is proved.

According to Corollary 3.1, condition (3.13) holds if and only if

$$\int_{0+} U^{(-1)}(N_m(u))\, du < \infty \tag{3.14}$$

where $N_m(u)$ is the massiveness of the space (T,m). If condition (3.14) holds, then

$$U^{(-1)}(x) = \varphi^{(-1)*}(\ln(x+1))$$

where $\varphi^{(-1)*}(x)$ is the inverse function to $\varphi^*(x)$ for $x > 0$. Thus, condition (3.14) holds if

$$\int_{0+} \varphi^{(-1)*}(\ln(N_m(u)+1))\, du < \infty.$$

In its turn, the latter integral converges if

$$\int_{0+} \varphi^{(-1)*}(H_m(u))\, du < \infty \tag{3.15}$$

where $H_m(u) = \ln N_m(u)$.

It follows from Lemma 3.4.1 of [1] that (3.15) holds if and only if (3.9) holds. □

Theorem 3.4. *Let $\mathbf{R}^k$ be the k-dimensional space, $d(t,s) = \max_{1\le i\le k} |t_i - s_i|$,*

$$T = \{0 \le t_i \le T_i, i = 1, 2, \dots, k\}, \qquad T_i > 0,$$

and let $X_n = \{X_n(t), t \in T\} \in \mathrm{Sub}_\varphi(\Omega)$. Let the processes $X_n(t)$ be separable. Assume that

$$\sup_{d(t,s)\le h} \tau_\varphi(X_n(t) - X_n(s)) \le \sigma(h) \tag{3.16}$$

where $\sigma(h)$ is a monotone increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$. If

$$\int_{0+} \Psi\left(\ln \frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty \tag{3.17}$$

where $\Psi(u) = u/\varphi^{(-1)}(u)$ *and* $\sigma^{(-1)}(\varepsilon)$ *is the inverse function to* $\sigma(\varepsilon)$*, then*

$$\lim_{\varepsilon\to 0}\sup_{n\geq 1}\mathsf{P}\left\{\sup_{\substack{t,s\in T\\ d(t,s)<\varepsilon}}|X_n(t)-X_n(s)|>\delta\right\}=0$$

for all $\delta > 0$.

Proof. Since $m(t,s) = \sup_n \tau_\varphi\left(X_n(t) - X_n(s)\right)$, it follows from [1] that

$$N_m(\varepsilon) \leq \prod_{i=1}^{k}\left(\frac{T_i}{2\sigma^{(-1)}(\varepsilon)}+1\right).$$

For sufficiently small ε, we have

$$N_m(\varepsilon) \leq \left(\frac{\tilde{T}}{\sigma^{(-1)}(\varepsilon)}\right)^k$$

where $\tilde{T} = \max_{1\leq i\leq k} T_i$. Thus condition (3.9) follows from

$$\int_{0+}\Psi\left(k\ln\frac{\tilde{T}}{\sigma^{(-1)}(\varepsilon)}\right)d\varepsilon<\infty. \tag{3.18}$$

It is easy to show that (3.18) follows from (3.17), and therefore Theorem 3.3 implies Theorem 3.4. □

Theorem 3.5 ([1])**.** *Let* (T, d) *be a compact space and* $C(T)$ *the Banach space of continuous functions equipped with the uniform norm. Let* $\{X_n(t), t \in T\}$*,* $n \geq 1$*, be a sequence of random variables of the space* $C(T)$*. The sequence* $X_n(t)$ *converges in probability in the space* $C(T)$ *if*

(1) *the sequence* $(X_n(t), n \geq 1)$ *converges in probability for all* $t \in T_s$ *where* T_s *is an arbitrary set dense in* T*;*
(2) *for all* $\delta > 0$

$$\lim_{\varepsilon\to 0}\sup_{n\geq 1}\mathsf{P}\left\{\sup_{\substack{t,s\in T\\ d(t,s)<\varepsilon}}|X_n(t)-X_n(s)|>\delta\right\}=0.$$

The next result follows from Theorems 3.4 and 3.5.

Theorem 3.6. *Let* $\mathbf{R}^k$ *be the k-dimensional space,* $d(t,s) = \max_{1\leq i\leq k}|t_i - s_i|$,

$$T=\{0\leq t_i\leq T_i, 1=1,2,\dots,k\},\quad T_i>0,\qquad X_n=\{X_n(t), t\in T\}\in \mathrm{Sub}_\varphi(\Omega).$$

Assume that the process $X_n(t)$ *is separable and*

$$\sup_{d(t,s)\leq h}\tau_\varphi(X_n(t)-X_n(s))\leq\sigma(h)$$

where $\sigma(h)$ *is a monotone increasing continuous function such that* $\sigma(h) \to 0$ *as* $h \to 0$*. We also assume that*

$$\int_{0+}\Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right)d\varepsilon<\infty$$

where $\Psi(u) = u/\varphi^{(-1)}(u)$ *and* $\sigma^{(-1)}(\varepsilon)$ *is the inverse function to* $\sigma(\varepsilon)$*. If the processes* $X_n(t)$ *converge in probability to the process* $X(t)$ *for all* $t \in T$*, then* $X_n(t)$ *converge in probability in the space* $C(T)$*.*

Consider some sufficient conditions for the existence of continuous partial derivatives of a random field of the space $\mathrm{SSub}_\varphi(\Omega)$.

Theorem 3.7 ([4])**.** *Let $\xi(X)$ be an almost sure continuous random field such that $\mathsf{E}\,\xi(X)=0$ for $X\in T$ where $T=\{a_i\le x_i\le b_i, i=1,\dots,m\}$. Let*

$$B(X,Y)=\mathsf{E}\,\xi(X)\xi(Y)$$

be the correlation function of the field $\xi(X)$, and let there exist partial derivatives

$$B_{ii}(X,Y)=\frac{\partial^2 B(X,Y)}{\partial X_i\partial Y_i},\qquad i=1,\dots,m,$$

where $B_{ii}(X,Y)$ are the correlation functions of square mean derivatives $\partial\xi(X)/\partial x_i$. If there is a version of the field $\partial\xi(X)/\partial x_i$, $i=1,\dots,m$, that is a continuous random field, then this version is an ordinary partial derivative of the random field $\xi(X)$.

The following result contains conditions for the continuity of random fields of the space $\mathrm{Sub}_\varphi(\Omega)$.

Theorem 3.8 ([5])**.** *Let (T,ρ) be a compact metric space, $X(t)$, $t\in T$, a separable stochastic process, and $X(t)\in\mathrm{Sub}_\varphi(\Omega)$. Assume that there exists a monotone increasing continuous function $\sigma=(\sigma(h),h>0)$ such that $\sigma(h)\to 0$ as $h\to 0$ and*

$$\sup_{\rho(t,s)<h}\tau_\varphi\left(X(t)-X(s)\right)\le\sigma(h).$$

Further assume that

$$\int_0^\varepsilon \Psi\left(H\left(\sigma^{(-1)}(u)\right)\right)du<\infty$$

for all $\varepsilon>0$ where $\Psi(v)=v/\varphi^{(-1)}(v)$ for $v>0$. Denote by $H(\varepsilon)$ the metric entropy of the space (T,ρ). Then the stochastic process $X(t)$, $t\in T$, is continuous with probability one.

The following result contains conditions for the existence of partial derivatives for stochastic processes of the space $\mathrm{SSub}_\varphi(\Omega)$.

Theorem 3.9. *Let $T=\{a_i\le x_i\le b_i, i=1,\dots,m\}$, and let $\xi(X)$, $X\in T$, be a separable random field such that $\xi(X)\in\mathrm{SSub}_\varphi(\Omega)$. Put $B_{0000}(X,Y)=\mathsf{E}\,\xi(X)\xi(Y)$ and assume that the partial derivatives $B_{i0i0}(X,Y)=\partial^2B(X,Y)/\partial x_i\partial y_i$, $i=1,\dots,m$, and*

$$B_{ikik}(X,Y)=\frac{\partial^4 B(X,Y)}{\partial x_i\partial y_i\partial x_k\partial y_k},\qquad i=1,\dots,m,\ k=1,\dots,m,$$

exist. Let there exist a monotone increasing continuous function $\sigma_z(h)>0$, $h>0$, such that $\sigma_z(h)\to 0$ as $h\to 0$ for $z=(0,0,0,0)$, $z=(i,0,i,0)$, $i=1,\dots,m$; and $z=(i,k,i,k)$, $i=1,\dots,m$, $k=1,\dots,m$. Assume that

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}}\left(B_z(X,X)+B_z(Y,Y)-2B_z(X,Y)\right)^{1/2}\le\sigma_z(h). \tag{3.19}$$

If

$$\int_0^\varepsilon \Psi\left(\ln\left(\sigma_z^{(-1)}(u)\right)\right)du<\infty \tag{3.20}$$

for all z and for sufficiently small $\varepsilon>0$ where $\Psi(u)=u/\varphi^{(-1)}(u)$, then with probability one the partial derivatives

$$\frac{\partial\xi(X)}{\partial x_i},\qquad\frac{\partial^2\xi(X)}{\partial x_i\partial x_j},\qquad i,j=1,\dots,m,$$

exist and are continuous.

Proof. It is sufficient to provide the proof of the theorem only for the first derivative. If the derivative $B_{i0i0}(X,Y)$, $i=1,\dots,k$, exists and is continuous, then the partial derivative $\partial\xi(X)/\partial x_i$, $i=1,\dots,m$, exists in the mean square sense. The process $\xi(x)$ belongs to the space $\mathrm{SSub}_\varphi(\Omega)$, thus Theorem 2.1 implies that the process $\partial\xi(X)/\partial x_i$ belongs to the space $\mathrm{SSub}_\varphi(\Omega)$, too. Hence

$$\begin{aligned}\tau_\varphi\left(\frac{\partial\xi(X)}{\partial x_i}-\frac{\partial\xi(Y)}{\partial x_i}\right)&=\left(\mathsf{E}\left(\frac{\partial\xi(X)}{\partial x_i}-\frac{\partial\xi(Y)}{\partial x_i}\right)^2\right)^{1/2}\\&=\left[\mathsf{E}\left(\frac{\partial\xi(X)}{\partial x_i}\right)^2+\mathsf{E}\left(\frac{\partial\xi(Y)}{\partial x_i}\right)^2-2\,\mathsf{E}\,\frac{\partial\xi(X)}{\partial x_i}\frac{\partial\xi(Y)}{\partial x_i}\right]^{1/2}\\&=\left(B_{i0i0}(X,X)+B_{i0i0}(Y,Y)-2B_{i0i0}(X,Y)\right)^{1/2}.\end{aligned}$$

Thus it follows from (3.19) and (3.20) that the assumption of Theorem 3.8 holds for $\xi(X)$ (recall that $\xi(X)$ is separable) and for a separable version of $\partial\xi(X)/\partial x_i$. Hence $\xi(X)$ is continuous with probability one and there exists a continuous version of $\partial\xi(X)/\partial x_i$. Theorem 3.7 implies that this version is a usual partial derivative of the field $\xi(X)$. □

4. The justification of the Fourier method for a partial differential equation with random initial conditions

Consider the equation

$$\frac{\partial^2 u}{\partial t^2}=L(u) \tag{4.1}$$

for

$$L(u)=\sum_{i=1}^n\sum_{j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial u}{dx_j}\right)-a(X)u.$$

The coefficients of the operator L are defined in a finite connected domain G of dimension n; let

$$X=(x_1,x_2,\dots,x_n)$$

be an arbitrary point of G. Assume that

$$a(X)=0,\qquad a_{ij}=a_{ji},\qquad \sum_{i,j=1}^n a_{ij}\gamma_i\gamma_j\ge\alpha\sum_{i=1}^n\gamma_i^2,\qquad \alpha>0,$$

in the domain G.

Consider the following problem for equation (4.1): solve equation (4.1) in the cylinder $Q_T=G[0<t<T]$ for the initial conditions

$$u|_{t=0}=\xi(X),\qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=\eta(X), \tag{4.2}$$

and the boundary condition

$$u\big|_S=0,\qquad t\in[0,T], \tag{4.3}$$

where S is the boundary of the domain G. Assume that the initial conditions

$$(\xi(X),X\in G)\quad\text{and}\quad(\eta(X),X\in G)$$

are jointly $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes.

When solving similar problems by using the Fourier method, regardless of whether initial conditions are random or nonrandom, we look for a solution of the form

$$u(X,t)=\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)v_k(X) \tag{4.4}$$

where

$$A_k=\int_G \xi(X)v_k(X)\,dX, \qquad B_k=\frac{1}{\sqrt{\lambda_k}}\int_G \eta(X)v_k(X)\,dX,$$

and the λ_k and $v_k(X)$ are eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$L(v)+\lambda v=0$$

(see, for example, [6]).

Theorem 4.1. *Let $\xi(X)$, $X\in G$, and $\eta(X)$, $X\in G$, be jointly* $\mathrm{SSub}_\varphi(\Omega)$ *stochastic processes. In order that a twice continuously differentiable solution of problem* (4.1)–(4.3) *exist with probability one in the domain $0\le x_i\le S_i$, $0\le t\le T$, (T is a positive constant), and be represented in the form of a uniformly convergent in probability series* (4.4), *it is sufficient that*

(1) *the continuous derivatives*

$$\frac{\partial^2\xi(X)}{\partial x_i\partial x_j}, \qquad \frac{\partial\eta(X)}{\partial x_i}$$

exist with probability one;

(2) *for all $X\in G$ and $t\in[0,T]$ series* (4.4) *and the series*

$$\sum_{k=1}^{\infty}\sqrt{\lambda_k}\left(-A_k\sin\sqrt{\lambda_k}t+B_k\cos\sqrt{\lambda_k}t\right)v_k(X), \tag{4.5}$$

$$\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)\frac{\partial v_k(X)}{\partial x_i}, \qquad i=1,\dots,m, \tag{4.6}$$

$$\sum_{k=1}^{\infty}\lambda_k\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)v_k(X), \tag{4.7}$$

$$\sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)\frac{\partial^2 v_k(X)}{\partial x_i\partial x_j}, \qquad i,j=1,\dots,m, \tag{4.8}$$

converge uniformly in probability.

Proof. Note that there exist subsequences of partial sums of series (4.4)–(4.8) that converge uniformly in probability. The rest of the proof is the same as in the nonrandom case. □

Lemma 4.1. *Let initial conditions*

$$(\xi(X),X\in G) \quad \text{and} \quad (\eta(X),X\in G)$$

be jointly $\mathrm{SSub}_\varphi(\Omega)$ *stochastic processes and assume that the hypotheses of Theorem* 4.1 *hold. Then the random series* (4.4)–(4.8) *also are jointly* $\mathrm{SSub}_\varphi(\Omega)$ *stochastic processes.*

Proof. It follows from Theorem 2.2 that the family of random variables A_k, B_k, $k\ge 1$, is a jointly $\mathrm{Sub}_\varphi(\Omega)$ family. According to Theorem 2.1 random series (4.4)–(4.8) are jointly $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes. □

For $n \geq 0$ put

$$S_n^{(0)} = \sum_{k=1}^{n} \Big(A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t\Big) v_k(X),$$

$$S_n^{(1)} = \sum_{k=1}^{n} \sqrt{\lambda_k}\,\Big(-A_k \sin\sqrt{\lambda_k}t + B_k \cos\sqrt{\lambda_k}t\Big) v_k(X),$$

$$S_n^{(2)} = \sum_{k=1}^{n} \Big(A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t\Big) \frac{\partial v_k(X)}{\partial x_i}, \qquad i = 1, \dots, m,$$

$$S_n^{(3)} = \sum_{k=1}^{n} \lambda_k\,\Big(A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t\Big) v_k(X),$$

$$S_n^{(4)} = \sum_{k=1}^{n} \Big(A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t\Big) \frac{\partial^2 v_k(X)}{\partial x_i \partial x_j}, \qquad i, j = 1, \dots, m.$$

Theorem 4.2. *Let $\xi(X)$, $X \in G$, and $\eta(X)$, $X \in G$, be jointly $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes. In order that a twice continuously differentiable solution of problem (4.1)–(4.3) exist in the domain of variables $(t, x_1, \dots, x_n)$ such that $0 \leq t \leq T$,*

$$G = \{0 \leq x_i \leq S_i, i = 1, \dots, m\}$$

(T is a positive constant), and be represented in the form of series (4.4), uniformly convergent in probability, it is sufficient that

(1) *the derivatives*

$$\frac{\partial^2 \xi(X)}{\partial x_i \partial x_j}, \qquad \frac{\partial \eta(X)}{\partial x_i}$$

exist and are continuous with probability one;

(2) *for all $X \in G$ and $t \in [0, t]$, the series*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} v_k(X)\, v_l(X) \Big[\mathsf{E}\, A_k A_l \cos\sqrt{\lambda_k}t \cos\sqrt{\lambda_l}t + \mathsf{E}\, B_k B_l \sin\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t + 2\,\mathsf{E}\, A_k B_l \cos\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\partial v_k(X)}{\partial x_i} \frac{\partial v_l(X)}{\partial x_j} \Big[\mathsf{E}\, A_k A_l \cos\sqrt{\lambda_k}t \cos\sqrt{\lambda_l}t + \mathsf{E}\, B_k B_l \sin\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t + 2\,\mathsf{E}\, A_k B_l \cos\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sqrt{\lambda_k}\sqrt{\lambda_l} v_k(X) v_l(X) \Big[\mathsf{E}\, A_k A_l \sin\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t + \mathsf{E}\, B_k B_l \cos\sqrt{\lambda_k}t \cos\sqrt{\lambda_l}t - 2\,\mathsf{E}\, A_k B_l \cos\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\partial^2 v_k(X)}{\partial x_i \partial x_j} \frac{\partial^2 v_l(X)}{\partial x_i \partial x_j} \Big[\mathsf{E}\, A_k A_l \cos\sqrt{\lambda_k}t \cos\sqrt{\lambda_l}t + \mathsf{E}\, B_k B_l \sin\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t + 2\,\mathsf{E}\, A_k B_l \cos\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \lambda_k \lambda_l v_k(X) v_l(X) \Big[\mathsf{E}\, A_k A_l \cos\sqrt{\lambda_k}t \cos\sqrt{\lambda_l}t + \mathsf{E}\, B_k B_l \sin\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t + 2\,\mathsf{E}\, A_k B_l \cos\sqrt{\lambda_k}t \sin\sqrt{\lambda_l}t\Big]$$

converge;

(3) *for $n \ge 1$ and $k = 0, 1, 2, 3, 4$,*

$$\sup_{\substack{|x_i - y_i| \le h \\ |t-s| \le h}} \left(\mathsf{E} \left| S_n^{(k)}(X,t) - S_n^{(k)}(Y,s) \right|^2 \right)^{1/2} \le \sigma_k(h)$$

where $\sigma_k(h)$ is a monotone increasing continuous function such that $\sigma_k(h) \to 0$ as $h \to 0$; moreover

$$\int_{0+} \Psi \left(\ln \frac{1}{\sigma_k^{(-1)}(\varepsilon)} \right) d\varepsilon < \infty \tag{4.9}$$

where $\Psi(u) = u/\varphi^{(-1)}(u)$ and $\sigma_k^{(-1)}(\varepsilon)$ is the inverse function to $\sigma_k(\varepsilon)$.

Proof. Condition (2) implies that series (4.4) and (4.5)–(4.8) converge in the mean square sense. According to Theorem 3.6 and Lemma 4.1, series (4.4)–(4.8) converge in probability in the space $C(G \times [0,T])$.

Now Theorem 4.2 follows from Theorem 4.1. □

Remark 4.1. Condition (1) of Theorem 4.2 holds if Theorem 3.9 holds for stochastic processes $\xi(X)$ and $\eta(X)$.

Example 4.1. Assume that $\xi(X)$ and $\eta(X)$ are jointly $\mathrm{SSub}_\varphi(\Omega)$ stochastic processes. Then Theorems 2.2 and 2.1 (also see Lemma 4.1) imply that $S_n^{(k)}(t,X)$, $k = 1, \dots, 5$, are jointly $\mathrm{Sub}_\varphi(\Omega)$ stochastic processes. Let $\varphi(x)$ be a function such that $\varphi(x) = |x|^p$ for some $p > 1$ and all $|x| > 1$. Then $\Psi(x) = x^{1-1/p}$ for $x > 1$ and condition (4.9) holds for all $\varepsilon > 0$:

$$\int_{0+} \left(\ln \frac{1}{\sigma_k^{(-1)}(u)} \right)^{1-1/p} du < \infty. \tag{4.10}$$

Condition (4.10) holds if $\sigma_k(h) = C_k / |\ln |h||^\delta$ for $\delta > 1 - 1/p$ and $C_k > 0$, $k = 1, \dots, 5$. In this case, assumption (3) of Theorem 4.2 is satisfied if for $k = 1, \dots, 5$ there exist constants $C_k > 0$ such that

$$\left(\mathsf{E} \left| S_n^{(k)}(t) - S_n^{(k)}(s) \right|^2 \right)^{1/2} \le \frac{C_k}{|\ln |h||^\delta} \tag{4.11}$$

for $\delta > 1 - 1/p$, all $n = 1, 2, \dots$, and sufficiently small $|h|$.

Lemma 4.2. *Let*

$$G_n(X,t) = \sum_{l=1}^{n} \left(\xi_l \cos \sqrt{\lambda_l} t + \eta_l \sin \sqrt{\lambda_l} t \right) Z_l(X), \qquad X \in G, \ t \in [0,T],$$

let $Z_l(X)$ be a continuous function, and let ξ_l and η_l be random variables such that $\mathsf{E}\, \eta_l^2 < \infty$ and $\mathsf{E}\, \xi_l^2 < \infty$. If

$$\sup_{X \in G} |Z_l(X)| \le \delta_l, \tag{4.12}$$

$$\sup_{\substack{|x_i - y_i| \le h \\ i=1,\dots,m}} |Z_l(X) - Z_l(Y)| \le z_l \frac{1}{|\ln |h||^\delta}, \qquad \delta > 0, \ |h| < 1, \tag{4.13}$$

$$\sum_{l=1}^{\infty} \left(\left(\mathsf{E}\, \xi_l^2 \right)^{1/2} + \left(\mathsf{E}\, \eta_l^2 \right)^{1/2} \right) \left(z_l + \delta_l (\ln \lambda_l)^\delta \right) < \infty, \tag{4.14}$$

then

$$\sup_{\substack{|x_i-y_i|\le h\\ |t-s|\le h\\ i=1,\dots,m}} \left(\mathsf{E}\,|G_n(X,t)-G_n(Y,t)|^2\right)^{1/2} \le \frac{C}{|\ln|h||^\delta} \tag{4.15}$$

for $|h|<1$ *where*

$$C=\sum_{l=1}^{\infty}\left(\left(\mathsf{E}\,\xi_l^2\right)^{1/2}+\left(\mathsf{E}\,\eta_l^2\right)^{1/2}\left(z_l+\delta_l\left(\ln\left(\frac{\sqrt{\lambda_l}}{2}+e^\delta\right)\right)^\delta\right)\right).$$

Proof. It is clear that

$$\begin{aligned}&\left(\mathsf{E}(G_n(X,t)-G_n(Y,t))^2\right)^{1/2}\\ &\quad\le\sum_{l=1}^{n}\Big[\left(\mathsf{E}\,\xi_l^2\right)^{1/2}\left|\cos\sqrt{\lambda_l}tZ_l(X)-\cos\sqrt{\lambda_l}sZ_l(Y)\right|\\ &\qquad+(\mathsf{E}\,\eta_l^2)^{1/2}\left|\sin\sqrt{\lambda_l}tZ_l(X)-\sin\sqrt{\lambda_l}sZ_l(Y)\right|\Big].\end{aligned} \tag{4.16}$$

Further,

$$\begin{aligned}&\left|\cos\sqrt{\lambda_l}tZ_l(X)-\cos\sqrt{\lambda_l}sZ_l(Y)\right|\\ &\quad\le|Z_l(X)|\left|\cos\sqrt{\lambda_l}t-\cos\sqrt{\lambda_l}s\right|+\left|\cos\sqrt{\lambda_l}s\right||Z_l(X)-Z_l(Y)|\\ &\quad\le\delta_l2\left|\sin\frac{\sqrt{\lambda_l}(t-s)}{2}\right|+z_l\frac{1}{|\ln|h||^\delta}.\end{aligned}$$

The inequality

$$|\sin uv|\le\frac{\left(\ln\left(|v|+e^\delta\right)\right)^\delta}{(|\ln|h||)^\delta},\qquad\delta>0 \tag{4.17}$$

(see [7]) together with (4.16) implies that

$$\left|\cos\sqrt{\lambda_l}tZ_l(X)-\cos\sqrt{\lambda_l}tZ_l(Y)\right|\le\frac{1}{(|\ln|h||)^\delta}\left(z_l+\delta_l2\left(\ln\frac{\sqrt{\lambda_l}}{2}+e^\delta\right)^\delta\right). \tag{4.18}$$

Thus

$$\left|\sin\sqrt{\lambda_l}tZ_l(X)-\cos\sqrt{\lambda_l}tZ_l(Y)\right|\le\frac{1}{(|\ln|h||)^\delta}\left(z_l+\delta_l2\left(\ln\frac{\sqrt{\lambda_l}}{2}+e^\delta\right)^\delta\right). \tag{4.19}$$

Now (4.16), (4.18), and (4.19) complete the proof of the lemma. □

As a corollary we obtain a set of conditions for the existence of a solution of problem (4.1)–(4.3).

Theorem 4.3. *Let* $\xi(X)$, $X\in G$, *and* $\eta(X)$, $X\in G$, *be* $\mathrm{SSub}_\varphi(\Omega)$ *stochastic processes where* $\varphi(x)$ *is a function such that* $\varphi(x)=|x|^p$ *for some* $p>1$ *and all* $|x|>1$. *Set*

$$\begin{aligned}B(X,Y)&=B_{0000}(X,Y)=\mathsf{E}\,\xi(X)\xi(Y),\\ R(X,Y)&=R_{0000}(X,Y)=\mathsf{E}\,\eta(X)\eta(Y).\end{aligned}$$

In order that a twice continuously differentiable solution of problem (4.1)–(4.3) *exist with probability one in the domain* $0\le t\le T$, $G=\{0\le x_i\le S_i, i=1,\dots,m\}$, *and be represented in the form of series* (4.4), *uniformly convergent in probability, it is sufficient that*

1. The partial derivatives

$$B_{i0i0}(X,Y)=\frac{\partial^2 B(X,Y)}{\partial x_i\partial y_i},\qquad i=1,\dots,m,$$
$$B_{ikik}(X,Y)=\frac{\partial^4 B(X,Y)}{\partial x_i\partial y_i\partial x_k\partial y_k},\qquad i,k=1,\dots,m,$$
$$R_{i0i0}(X,Y)=\frac{\partial^2 R(X,Y)}{\partial x_i\partial y_i},\qquad i=1,\dots,m,$$

exist for $X,Y\in G$ and are continuous, and

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}}\left(B_z(X,X)+B_z(Y,Y)-2B_z(X,Y)\right)^{1/2}\le\frac{C_z}{|\ln h|^{\delta}},\tag{4.20}$$

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}}\left(R_{z_1}(X,X)+R_{z_1}(Y,Y)-2R_{z_1}(X,Y)\right)^{1/2}\le\frac{C_{z_1}}{|\ln h|^{\delta}}\tag{4.21}$$

for sufficiently small h where $\delta>1-1/p$; $z=(0,0,0,0)$; $(i,0,i,0)$, $i=1,\dots,m$, (i,k,i,k), $i,k=1,\dots,m$, $z_1=(0,0,0,0),(i,0,i,0)$, $i=1,\dots,m$.

2. The series

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}r_kr_l\left[|\mathsf{E}\,A_kA_l|+|\mathsf{E}\,B_kB_l|+2\,|\mathsf{E}\,A_kB_l|\right]\tag{4.22}$$

converges where $r_k=\max_{i,j=1,\dots,m}(\lambda_kv_k,v_{ki},v_{kij})$ and

$$v_k=\sup_{X\in G}|v_k(X)|,\qquad v_{ki}=\sup_{X\in G}\left|\frac{\partial v_k(X)}{\partial x_i}\right|,\qquad v_{kij}=\sup_{X\in G}\left|\frac{\partial v_k(X)}{\partial x_i}\right|.$$

3. $\sup_{X\in G}|v_l(X)|\le\delta_{l00}$,

$$\sup_{X\in G}\frac{\partial v_l(X)}{\partial x_i}\le\delta_{li0},\qquad\sup_{X\in G}\frac{\partial^2 v_l(X)}{\partial x_i\partial x_j}\le\delta_{lij},$$

and

$$\sup_{\substack{|x_k-y_k|<h\\ k=1,\dots,m}}|v_l(X)-v_l(Y)|\le\gamma_{l00}\frac{1}{|\ln h|^{\delta}},$$
$$\sup_{\substack{|x_k-y_k|<h\\ k=1,\dots,m}}\left|\frac{\partial v_l(X)}{\partial x_i}-\frac{\partial v_l(Y)}{\partial x_i}\right|\le\gamma_{li0}\frac{1}{|\ln h|^{\delta}},\qquad i=1,\dots,m,$$
$$\sup_{\substack{|x_k-y_k|<h\\ k=1,\dots,m}}\left|\frac{\partial^2 v_l(X)}{\partial x_i\partial x_j}-\frac{\partial^2 v_l(Y)}{\partial x_i\partial x_j}\right|\le\gamma_{lij}\frac{1}{|\ln h|^{\delta}},\qquad i,j=1,\dots,m,$$
$$\sum_{l=1}^{\infty}\left(\left(\mathsf{E}\,A_l^2\right)^{1/2}+\left(\mathsf{E}\,B_l^2\right)^{1/2}\right)\lambda_l\left(\delta_{l00}+(\ln\,\lambda_l)^{\delta}\gamma_{l00}\right)<\infty,$$
$$\sum_{l=1}^{\infty}\left(\left(\mathsf{E}\,A_l^2\right)^{1/2}+\left(\mathsf{E}\,B_l^2\right)^{1/2}\right)\left(\delta_{li0}+(\ln\,\lambda_l)^{\delta}\gamma_{li0}\right)<\infty,\qquad i=1,\dots,m,$$
$$\sum_{l=1}^{\infty}\left(\left(\mathsf{E}\,A_l^2\right)^{1/2}+\left(\mathsf{E}\,B_l^2\right)^{1/2}\right)\left(\delta_{lij}+(\ln\,\lambda_l)^{\delta}\gamma_{lij}\right)<\infty,\qquad i,j=1,\dots,m,$$

for arbitrary $\delta>1-1/p$ and $|h|<1$.

Proof. Condition 1 of Theorem 4.3 implies condition (1) of Theorem 4.2. According to Example 4.1, conditions of Theorem 3.9 hold for the processes $\xi(X)$ and $\eta(X)$ if

$$\sigma_z(h) = \frac{C_z}{|\ln|h||^{\delta}}, \qquad \delta > 1 - \frac{1}{p}.$$

It is clear that the series in condition (2) of Theorem 4.2 converge if so do the series in (4.22). Example 4.1 and Lemma 4.1 imply that condition (3) of Theorem 4.2 follows from condition 3 of Theorem 4.3. □

Remark 4.2. Let $\mathrm{CS\,Sub}_\varphi(\Omega)$ be the space of random variables of $\mathrm{Sub}_\varphi(\Omega)$ such that

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) \le C\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}$$

for an arbitrary countable set I, $\xi_i \in \mathrm{CS\,Sub}_\varphi(\Omega)$, $i \in I$, and for $\lambda_i \in \mathbf{R}^1$ where C is a positive constant. It is easy to see that $\mathrm{CS\,Sub}_\varphi(\Omega) = \mathrm{SSub}_{\varphi_1}(\Omega)$ for

$$\varphi_1(x) = \varphi(Cx).$$

In the case of problem (4.1)–(4.3), if the processes $\xi(X)$ and $\eta(X)$ belong to the space $\mathrm{CS\,Sub}_\varphi(\Omega)$, then we may treat them as elements of $\mathrm{SSub}_{\varphi_1}(\Omega)$ for $\varphi_1(x) = \varphi(Cx)$.

5. Estimates of the distribution of the supremum of a solution of the boundary value problem

Theorem 5.1. *Let T be a set of parameters, let $X = \{X(t), t \in T\}$ be a stochastic process such that $X \in \mathrm{Sub}_\varphi(\Omega)$, and let $\rho(t,s) = \tau_\varphi(X(t) - X(s))$. Assume that (T,ρ) is a pseudometric compact space and the process X is separable on (T,ρ). Denote by $N(\varepsilon)$ the metric massiveness of the space (T,ρ) and $H(\varepsilon) = \ln N(\varepsilon)$. If*

$$\varepsilon_0 = \sup_{t\in T}\tau_\varphi(X(t)) < \infty, \qquad \int_0^{\varepsilon_0}\Psi(H(\varepsilon))\,d\varepsilon < \infty \tag{5.1}$$

where $\Psi(u) = u/\varphi^{(-1)}(u)$ and $\varphi^{(-1)}(u)$ is the inverse function to $\varphi(u)$ for $u > 0$, then

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)| \ge \varepsilon\right\} \le 2A(u,\theta)$$

for all $u > 2I_\varphi(\theta\varepsilon_0)/\theta(1-\theta)$ where

$$A(u,\theta) = \exp\left\{-\varphi^*\left(\frac{1}{\varepsilon_0}\left[u(1-\theta) - \frac{2}{\theta}I_\varphi(\theta\varepsilon)\right]\right)\right\}, \qquad I_\varphi(y) = \int_0^y \Psi(H(\varepsilon))\,d\varepsilon.$$

Corollary 5.1. *Let $\mathbf{R}^k$ be the k-dimensional space,*

$$d(t,s) = \max_{1\le i\le k}|t_i - s_i|, \qquad T = \{0 \le t_i \le T_i, n = 1,2,\dots,k\}, \quad T_i > 0.$$

Assume that $X = \{X(t), t \in T\}$ is separable and $X \in \mathrm{Sub}_\varphi(\Omega)$. If

$$\sup_{d(t,s)\le h}\tau_\varphi(X(t) - X(s)) \le \sigma(h)$$

where $\sigma(h)$ is a monotone increasing continuous function such that $\sigma(h) \to 0$ as $h \to \infty$, and

$$\int_{0+}\Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right)d\varepsilon < \infty$$

where $\Psi(u) = u/\varphi^{(-1)}(u)$, then

$$\mathsf{P}\left\{\sup_{t\in T}|X(t)| > u\right\} \le 2\tilde{A}(u,\theta)$$

for all $0<\theta<1$ *and* $u>2I_\varphi(\theta\varepsilon_0)/(\theta(1-\theta))$ *where*

$$\tilde{A}\,(u,\theta)=\exp\left\{-\varphi^*\left(\frac{1}{\tilde{\varepsilon}_0}\left[u\,(1-\theta)-\frac{2}{\theta}\tilde{I}_\varphi\,(\theta\tilde{\varepsilon})\right]\right)\right\},$$

$$\tilde{\varepsilon}_0=\sup_{t\in T}\left(\mathsf{E}\,|X(t)|^2\right)^{1/2},\qquad \tilde{I}_\varphi\,(\delta)=\int_0^\delta\Psi\left(\sum_{i=1}^k\ln\left(\frac{T_i}{2\varsigma^{(-1)}(\varepsilon)}+1\right)\right)d\varepsilon.$$

Proof. Corollary 5.1 follows from Theorem 5.1 for $\varepsilon_0=\tilde{\varepsilon}_0$. It remains to prove that $I_\varphi(y)\le\tilde{I}_\varphi(y)$. This inequality holds, since

$$N(\varepsilon)\le\prod_{i=1}^k\left(\frac{T_i}{2\sigma^{(-1)}(\varepsilon)}+1\right)$$

by the assumptions of Theorem 5.1.

Put

$$u_n(X,t)=\sum_{k=n}^\infty\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)v_k(X).$$

As in Theorem 4.2, assume that

$$\left(\mathsf{E}\left|S_n^{(0)}(X,t)-S_n^{(0)}(Y,t)\right|^2\right)^{1/2}\le\sigma_0(h)$$

where $\sigma_0(h)$ is a monotone increasing continuous function such that $\sigma_0(h)\to0$ as $h\to0$, and

$$\int_{0+}\Psi\left(\ln\frac{1}{\sigma_0^{(-1)}(\varepsilon)}\right)d\varepsilon<\infty$$

where $\Psi(u)=u/\varphi^{(-1)}(u)$ and $\sigma_0^{(-1)}(\varepsilon)$ is the inverse function to $\sigma_0(\varepsilon)$. Then

$$\mathsf{P}\left\{\sup_{t\in T}|u_n\,(X,t)|>u\right\}\le2\tilde{A}(u,\theta)$$

where

$$\tilde{A}\,(u,\theta)=\exp\left\{-\varphi^*\left(\frac{1}{\tilde{\varepsilon}_0}\left[u\,(1-\theta)-\frac{2}{\theta}\tilde{I}_\varphi\,(\theta\tilde{\varepsilon})\right]\right)\right\},$$

$\tilde{\varepsilon}_0=\sup_{\substack{t\in[0,T]\\x_i\in[0,S_i]}}\left(\mathsf{E}\,|u_n(X,t)|^2\right)^{1/2}$, and

$$I_\varphi(\theta,\tilde{\varepsilon_0})=\int_0^{\theta\tilde{\varepsilon_0}}\Psi\left(\ln\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)}+1\right)+\sum_{i=1}^k\ln\left(\frac{S_i}{2\sigma^{(-1)}(\varepsilon)}+1\right)\right)d\varepsilon.\qquad\square$$

Remark. Since

$$\begin{aligned}\tilde{\varepsilon}_0&=\sup_{\substack{t\in[0,T]\\x_i\in[0,S_i]}}\left(\mathsf{E}\,|u_n(X,t)|^2\right)^{1/2}\\&=\sup_{\substack{t\in[0,T]\\x_i\in[0,S_i]}}\left(\mathsf{E}\left|\sum_{k=n}^\infty\left(A_k\cos\sqrt{\lambda_k}t+B_k\sin\sqrt{\lambda_k}t\right)v_k(X)\right|^2\right)^{1/2}\\&\le\left(\sum_{k=n}^\infty\sum_{l=n}^\infty|v_k(X)|\cdot|v_l(X)|\,(|\mathsf{E}\,A_kA_l|+|\mathsf{E}\,B_kB_l|+2\,|\mathsf{E}\,A_kB_l|)\right)^{1/2}=\hat{\varepsilon},\end{aligned}$$

one has

$$\tilde{A}(u,\theta) = \exp\left\{-\varphi^*\left(\frac{1}{\hat{\varepsilon}}\left[u(1-\theta) - \frac{2}{\theta}\hat{I}_\varphi(\theta\hat{\varepsilon})\right]\right)\right\}.$$

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