Теорія Ймовір. та Матем. Статист.
Вип. 69, 2003

Theor. Probability and Math. Statist.
No. 69, 2004, Pages 85–94
S 0094-9000(05)00616-2
Article electronically published on February 8, 2005

A CRITERION FOR TESTING HYPOTHESES ABOUT THE COVARIANCE FUNCTION OF A GAUSSIAN STATIONARY PROCESS

UDC 519.21

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Abstract. New upper and lower bounds for distributions of quadratic forms of Gaussian random variables as well as those for the limits of quadratic forms are found in this paper. Based on these estimates, a criterion is proposed to test a hypothesis about the covariance function $\rho(\tau)$ of a Gaussian stochastic process.

1. Introduction

In this paper, we consider the space $SG_\Xi(\Omega)$ of square Gaussian random variables and obtain new upper and lower bounds for distributions of quadratic forms of square Gaussian random variables and bounds for distributions of limits of quadratic forms. The upper estimates improve some results of [1, 2].

The inequalities obtained in this paper allow one to construct confidence sets for estimators of the covariance function of a Gaussian stochastic process.

Using these inequalities we propose a criterion to test a hypothesis about the covariance function $\rho(\tau)$ of a Gaussian stochastic process.

2. The space of square Gaussian random variables

Definition. Let $\Xi = \{\xi(t), t \in T\}$ be a family of jointly Gaussian random variables (for example, $\xi(t)$ is a Gaussian stochastic process) such that $\mathsf{E}\,\xi(t) = 0$. The set of random variables ζ that either can be represented in the form

$$\zeta = \overline{\xi}^T A \overline{\eta} - \mathsf{E}\,\overline{\xi}^T A \overline{\eta} \tag{1}$$

or are the mean square limits of random variables represented in the form of (1) is called the space $SG_\Xi(\Omega)$ of square Gaussian random variables; here $\overline{\xi} = (\xi_1, \dots, \xi_d)^T$ and $\overline{\eta} = (\eta_1, \dots, \eta_d)^T$ are Gaussian random vectors with $\mathsf{E}\,\overline{\xi} = 0$ and $\mathsf{E}\,\overline{\eta} = 0$; the random variables ξ_i, η_i, $i = 1, \dots, d$, belong to Ξ; A is a symmetric matrix.

It is shown in [1] that

(i) $SG_\Xi(\Omega)$ is a Banach space with respect to the norm $\|\zeta\| = \sqrt{\mathsf{E}\,\zeta^2}$;
(ii) $SG_\Xi(\Omega)$ is a subspace of the Orlicz space generated by the function

$$U(x) = \exp\{|x|\} - 1;$$

(iii) the norm $\|\zeta\|_{L_u(\Omega)}$ on $SG_\Xi(\Omega)$ is equivalent to the norm $\sqrt{\mathsf{E}\,\zeta^2}$.

The following result holds for random variables of the space $SG_\Xi(\Omega)$.

2000 *Mathematics Subject Classification.* Primary 60G17; Secondary 60G07.

Lemma 2.1 ([1]). *Let ζ_i, $i = 1, 2, \ldots, n$, be random variables of the space $SG_\Xi(\Omega)$. Then*

$$\mathsf{E}\exp\left\{\frac{s}{\sqrt{2}}\frac{\zeta}{(\operatorname{Var}\zeta)^{1/2}}\right\} \le R(|s|) \tag{2}$$

for all $|s| < 1$ and all $\lambda_i \in \mathbf{R}^1$, $i = 1, 2, \ldots, n$, where $\zeta = \sum_{i=1}^n \lambda_i\zeta_i$ and

$$R(s) = \exp\{-s/2\}(1-s)^{-1/2}.$$

3. Distributions of quadratic forms of random variables of the space $SG_\Xi(\Omega)$

The following result improves Lemma 3 in [2].

Lemma 3.1. *Let $\overline{\zeta}^T = (\zeta_1, \ldots, \zeta_d)$ be a random vector such that $\zeta_i \in SG_\Xi(\Omega)$. Let A be a symmetric positive definite matrix. Then*

$$\mathsf{E}\cosh\left(\sqrt{\frac{t^2\overline{\zeta}^T A\overline{\zeta}}{\mathsf{E}(\overline{\zeta}^T A\overline{\zeta})}}\right) \le R(\sqrt{2}t) \tag{3}$$

for all $0 \le t < 2^{-1/2}$, where $R(t)$ is defined by (2).

Proof. First we consider the case $A = I$, where I is the identity matrix and $\overline{\zeta}$ is such that the random variables ζ_i are orthogonal, that is, $\operatorname{Var}(\sum_{i=1}^d \lambda_i\zeta_i) = \sum_{i=1}^d \lambda_i^2\,\mathsf{E}\,\zeta_i^2$. Put $\sigma_i^2 = \mathsf{E}\,\zeta_i^2$, $i = 1, 2, \ldots, d$. It follows from (2) that

$$\mathsf{E}\exp\left\{\frac{s\sum_{i=1}^d \lambda_i\zeta_i}{\sqrt{2}\left(\sum_{i=1}^d \lambda_i^2\sigma_i^2\right)^{1/2}}\right\} \le R(|s|) \tag{4}$$

for all $\lambda_i \in \mathbf{R}$, $i = 1, 2, \ldots, d$.

Put

$$u = \frac{s}{\sqrt{2}\sqrt{\sum_{i=1}^d \lambda_i^2\sigma_i^2}}.$$

Inequality (4) implies that

$$\mathsf{E}\exp\left\{u\sum_{i=1}^d \lambda_i\zeta_i\right\} \le R\left(\sqrt{2}|u|\sqrt{\sum_{i=1}^d \lambda_i^2\sigma_i^2}\right) \tag{5}$$

for

$$|u| < \left(2\sum_{i=1}^d \lambda_i^2\sigma_i^2\right)^{-1/2}.$$

Put $s_i = u\lambda_i\sigma_i$. Then

$$\sum_{i=1}^d s_i^2 = u^2\sum_{i=1}^d \lambda_i^2\sigma_i^2 = \frac{s^2}{2}$$

and $\sum_{i=1}^d s_i^2 < \frac{1}{2}$. It follows from (5) that

$$\mathsf{E}\exp\left\{\sum_{i=1}^d s_i\frac{\zeta_i}{\sigma_i}\right\} \le R\left(\sqrt{2\sum_{i=1}^d s_i^2}\right) \tag{6}$$

for all s_i such that $\sum_{i=1}^d s_i^2 < \frac{1}{2}$.

Applying inequality (6) we get

$$\begin{aligned}
\mathsf{E}\prod_{i=1}^{d}\cosh\left(\frac{s_i\zeta_i}{\sigma_i}\right) &= \mathsf{E}\prod_{i=1}^{d}\frac{\exp\{s_i\zeta_i/\sigma_i\}+\exp\{-s_i\zeta_i/\sigma_i\}}{2} \\
&= \frac{1}{2^d}\,\mathsf{E}\prod_{i=1}^{d}\left(\exp\left\{\frac{s_i\zeta_i}{\sigma_i}\right\}+\exp\left\{-\frac{s_i\zeta_i}{\sigma_i}\right\}\right) \\
&= \frac{1}{2^d}\sum\mathsf{E}\prod_{i=1}^{d}\exp\left\{\frac{s_i\zeta_i\delta_i}{\sigma_i}\right\} = \frac{1}{2^d}\sum\mathsf{E}\exp\left\{\sum_{i=1}^{d}\frac{s_i\zeta_i\delta_i}{\sigma_i}\right\} \\
&\le \frac{1}{2^d}\sum R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right) = R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right)
\end{aligned}$$

where $\delta_i = \pm 1$. Therefore

$$\mathsf{E}\prod_{i=1}^{d}\cosh\left(\frac{s_i\zeta_i}{\sigma_i}\right) \le R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right).$$

Put $f(z) = \ln\cosh\sqrt{z}$, $z > 0$. The function $f(z)$ is concave since $f(0) = \ln\cosh 0 = 0$ and $f''(z) < 0$. Thus

$$\sum_{i=1}^{d} f(z_i) \ge f\left(\sum_{i=1}^{d} z_i\right)$$

for all $z_i > 0$, $i = 1, 2, \ldots, d$. This means that

$$\prod_{i=1}^{d}\cosh\sqrt{z_i} \ge \cosh\sqrt{\sum_{i=1}^{d} z_i}, \qquad z_i > 0.$$

Therefore

$$\begin{aligned}
\mathsf{E}\cosh\sqrt{\sum_{i=1}^{d}\frac{s_i^2\zeta_i^2}{\sigma_i^2}} &\le \mathsf{E}\prod_{i=1}^{d}\cosh\sqrt{\frac{s_i^2\zeta_i^2}{\sigma_i^2}} = \mathsf{E}\prod_{i=1}^{d}\cosh\left\{\frac{|s_i\zeta_i|}{\sigma_i}\right\} = \mathsf{E}\prod_{i=1}^{d}\cosh\left\{\frac{s_i\zeta_i}{\sigma_i}\right\} \\
&\le R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right)
\end{aligned}$$

if $\sum_{i=1}^{d} s_i^2 < \frac{1}{2}$. Set

$$s_i^2 = \frac{\sigma_i^2 t^2}{\sum_{i=1}^{d}\sigma_i^2}.$$

Then the latter inequality implies that

$$\mathsf{E}\cosh\left(\sqrt{\frac{t^2\sum_{i=1}^{d}\zeta_i^2}{\sum_{i=1}^{d}\sigma_i^2}}\right) \le R(\sqrt{2}t) \tag{7}$$

for $0 \le t < 2^{-1/2}$.

Now we turn to the general case. Let B be a symmetric matrix such that $BB^T = B^2 = A$. Let $R = \operatorname{cov}\overline{\zeta}$, and let O be the orthogonal matrix that reduces BRB to the diagonal form, namely

$$OBRBO^T = D = \operatorname{diag}\left(d_k^2\right)_{k=1}^{d}.$$

Let $\overline{\theta} = OB\overline{\zeta}$. Then

$$\overline{\theta}^{\,T}\overline{\theta} = \overline{\zeta}^{\,T}BO^TOB\overline{\zeta} = \overline{\zeta}^{\,T}A\overline{\zeta}$$

and $\operatorname{cov}\overline{\theta} = OB\operatorname{cov}\zeta BO^T = D$. Since $\theta_i \in SG_\Xi(\Omega)$, inequality (7) holds for $\overline{\theta}$ instead of ζ where $\overline{\theta}^{\,T} = (\theta_1, \dots, \theta_d)$. Since

$$\overline{\theta}^{\,T}\overline{\theta} = \sum_{i=1}^{d}\theta_i^2 = \overline{\zeta}^{\,T}A\overline{\zeta},$$

we get

$$\cosh\sqrt{\frac{t^2\overline{\theta}^{\,T}\overline{\theta}}{\mathsf{E}\,\overline{\theta}^{\,T}\overline{\theta}}} = \cosh\sqrt{\frac{t^2\overline{\zeta}^{\,T}A\overline{\zeta}}{\mathsf{E}\,\overline{\zeta}^{\,T}A\overline{\zeta}}}.$$

The lemma is proved. □

Corollary 3.1. *Let the assumptions of Lemma 3.1 hold. If*

$$\eta = \operatorname*{l.i.m.}_{n\to\infty}\overline{\zeta}_n^{\,T}A_n\overline{\zeta}_n,$$

then

$$\mathsf{E}\cosh\left(\sqrt{\frac{t^2\eta}{\mathsf{E}\,\eta}}\right) \le R(\sqrt{2}t).$$

Corollary 3.1 follows from the Fatou lemma.

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Then*

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \le \frac{2^{1/4}x^{1/4}}{\cosh\left(\sqrt{x/2} - \frac{1}{2}\right)} \tag{8}$$

for $x > \frac{1}{2}$ where either $\eta = \overline{\zeta}_n^{\,T}A_n\overline{\zeta}_n$ or $\eta = \operatorname{l.i.m.}_{n\to\infty}\overline{\zeta}_n^{\,T}A_n\overline{\zeta}_n$.

Proof. It follows from the Chebyshev inequality and (3) in the case of $\eta = \overline{\zeta}_n^{\,T}A_n\overline{\zeta}_n$, or from Corollary 3.1 in the case of $\eta = \operatorname{l.i.m.}_{n\to\infty}\overline{\zeta}_n^{\,T}A_n\overline{\zeta}_n$, that

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \le \frac{\mathsf{E}\cosh\sqrt{t^2\eta/\,\mathsf{E}\,\eta}}{\cosh\sqrt{t^2x}} \le \frac{R(\sqrt{2}t)}{\cosh\sqrt{t^2x}}$$

for $x > 0$ and $0 \le t < 2^{-1/2}$. Put $t = 2^{-1/2} - (2x^{1/2})^{-1}$ for $x > \frac{1}{2}$. Then

$$\frac{R(\sqrt{2}t)}{\cosh\sqrt{t^2x}} = \frac{(2x)^{1/4}\exp\left\{(2\sqrt{2x})^{-1} - \frac{1}{2}\right\}}{\cosh\left(\sqrt{x/2} - \frac{1}{2}\right)}.$$

Since $\exp\left\{(2(2x)^{1/2})^{-1} - \frac{1}{2}\right\} < 1$ for $x > \frac{1}{2}$, we get

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \le \frac{2^{1/4}x^{1/4}}{\cosh\left(\sqrt{x/2} - \frac{1}{2}\right)}.$$

Lemma 3.2 is proved. □

Lemma 3.3. *Let $\xi_1, \xi_2, \dots, \xi_m$ be independent normal random variables such that*

$$\mathsf{E}\,\xi_k = 0 \quad \textit{and} \quad \mathsf{E}\,\xi_k^2 = \sigma_k^2,$$

and let $c_k = \pm 1$ and $s > 0$. Then

$$\left|\mathsf{E}\exp\left\{i\frac{s\sum_{k=1}^m\xi_k^2c_k}{2(\sum_{k=1}^m\sigma_k^4)^{1/2}}\right\}\right| \le \frac{1}{(1+s^2)^{1/4}}.$$

Proof. We have

$$\mathsf{E}\exp\left\{i\frac{\sum_{k=1}^m \xi_k^2 c_k}{r}\right\} = \mathsf{E}\prod_{k=1}^m \exp\left\{\frac{i\xi_k^2 c_k}{r}\right\}. \tag{9}$$

Taking the equality

$$\mathsf{E}\exp\left\{is\xi_k^2\right\} = \left(1-2is\sigma_k^2\right)^{-1/2}$$

into account, we rewrite (9) in the following form:

$$\mathsf{E}\exp\left\{i\frac{\sum_{k=1}^m \xi_k^2 c_k}{r}\right\} = \prod_{k=1}^m \left(1-2i\frac{\sigma_k^2 c_k}{r}\right)^{-1/2}.$$

Thus

$$\left|\mathsf{E}\exp\left\{i\frac{\sum_{k=1}^m \xi_k^2 c_k}{r}\right\}\right| = \left|\prod_{k=1}^m \left(1-2i\frac{\sigma_k^2 c_k}{r}\right)^{-1/2}\right| = \prod_{k=1}^m \left|1-2i\frac{\sigma_k^2 c_k}{r}\right|^{-1/2}$$
$$= \prod_{k=1}^m \left(1+\left(2\frac{\sigma_k^2 c_k}{r}\right)^2\right)^{-1/4} = \prod_{k=1}^m \left(1+\frac{4\sigma_k^4}{r^2}\right)^{-1/4}.$$

Put

$$I = \prod_{k=1}^m \left(1+\frac{4\sigma_k^4}{r^2}\right)^{-1/4}.$$

Then

$$\ln I = -\frac{1}{4}\sum_{k=1}^m \ln\left(1+\frac{4\sigma_k^4}{r^2}\right). \tag{10}$$

Consider the function $f(x)=\ln(1+x)$ for $x>0$. It is clear that $f(0)=0$, $f(x)$ is concave, and thus

$$f\left(\sum_{k=1}^m x_k\right) \le \sum_{k=1}^m f(x_k), \qquad x_k \ge 0.$$

Furthermore

$$-\sum_{k=1}^m f(x_k) \le -f\left(\sum_{k=1}^m x_k\right),$$

whence

$$\ln I \le -\frac{1}{4}\ln\left(1+\frac{4}{r^2}\sum_{k=1}^m \sigma_k^4\right)$$

and

$$I \le \left(1+\frac{4}{r^2}\sum_{k=1}^m \sigma_k^4\right)^{-1/4}$$

in view of (10).

Let

$$r = \frac{2\left(\sum_{k=1}^m \sigma_k^4\right)^{1/2}}{s}, \qquad s>0.$$

Then $I \le \left(1+s^2\right)^{-1/4}$ for $s>0$.

Lemma 3.3 is proved. □

Theorem 3.1. *Let A be a symmetric matrix and $\overline{\xi}^T = (\xi_1, \xi_2, \dots, \xi_m)$ a random vector whose coordinates ξ_k are normal random variables such that $\mathsf{E}\,\xi_k = 0$ and $\mathsf{E}\,\xi_k^2 = \sigma_k^2$. Then*

$$\left| \mathsf{E}\exp\left\{ i\frac{s\left(\overline{\xi}^T A\overline{\xi} - \mathsf{E}\,\overline{\xi}^T A\overline{\xi}\right)}{\sqrt{2}\sqrt{D\overline{\xi}^T A\overline{\xi}}} \right\} \right| \le \frac{1}{(1+s^2)^{1/4}}, \qquad s > 0.$$

Theorem 3.1 easily follows from Lemma 3.3.

Lemma 3.4. *Let $\overline{\zeta}^T = (\zeta_1, \dots, \zeta_n)$ be a random vector whose coordinates ζ_j, $1 \le j \le n$, belong to the space $SG_\Xi(\Omega)$. Let A be a symmetric positive definite matrix. Then*

$$\mathsf{E}\exp\left\{ -\frac{u^2}{2}\frac{\overline{\zeta}^T A\overline{\zeta}}{\mathsf{E}\,\overline{\zeta}^T A\overline{\zeta}} \right\} \le g(u)$$

where

$$g(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{ds}{(1+u^2s^2)^{1/4}}, \qquad 0 < u < 1.$$

Proof. First let ζ_j be orthogonal square Gaussian random variables such that $\mathsf{E}\,\zeta_j^2 = \sigma_j^2$ and $\sigma_j^2 > 0$. Let $\lambda_j \in \mathbf{R}^1$. Then Lemma 3.1 and the Fatou lemma imply

$$\left| \mathsf{E}\exp\left\{ i\frac{s}{\sqrt{2}} \frac{\sum_{j=1}^n \lambda_j\zeta_j}{\left(\sum_{j=1}^n \lambda_j^2\sigma_j^2\right)^{1/2}} \right\} \right| \le \frac{1}{\left(1+s^2\right)^{1/4}}. \tag{11}$$

We rewrite the left-hand side of (11) in the following form:

$$\left| \mathsf{E}\exp\left\{ i\frac{s}{\sqrt{2}} \frac{\sum_{j=1}^n \frac{\zeta_j}{\sigma_j}(\lambda_j\sigma_j)}{\left(\sum_{j=1}^n \lambda_j^2\sigma_j^2\right)^{1/2}} \right\} \right|.$$

Put

$$s_j = s\frac{\lambda_j\sigma_j}{\left(\sum_{j=1}^n \lambda_j^2\sigma_j^2\right)^{1/2}}, \qquad s^2 = \sum_{j=1}^n s_j^2.$$

Then (11) can be rewritten in the following form:

$$\left| \mathsf{E}\exp\left\{ i\frac{1}{\sqrt{2}}\sum_{j=1}^n \frac{s_j\zeta_j}{\sigma_j} \right\} \right| \le \frac{1}{\left(1+\sum_{j=1}^n s_j^2\right)^{1/4}}. \tag{12}$$

For $t_j > 0$ we have

$$\begin{aligned} \int_{\mathbf{R}^n}\cdots\int \mathsf{E}\exp\left\{ i\frac{1}{\sqrt{2}}\sum_{j=1}^n s_j\frac{\zeta_j}{\sigma_j} \right\} \prod_{j=1}^n \frac{1}{\sqrt{2\pi}t_j} \exp\left\{-\frac{s_j^2}{2t_j^2}\right\} ds_1\cdots ds_n \\ = \mathsf{E}\exp\left\{ -\sum_{j=1}^n \frac{\zeta_j^2 t_j^2}{2\sigma_j^2} \right\}, \end{aligned}$$

whence

$$\mathsf{E}\exp\left\{ -\sum_{j=1}^n \frac{\zeta_j^2 t_j^2}{2\sigma_j^2} \right\} \le \int_{\mathbf{R}^n}\cdots\int \prod_{j=1}^n \left(\frac{1}{\sqrt{2\pi}t_j}\right) \exp\left\{-\frac{s_j^2}{2t_j^2}\right\} \frac{ds_1\cdots ds_n}{\left(1+\sum_{j=1}^n s_j^2\right)^{1/4}}$$

in view of inequality (12). Setting $s_j/t_j = u_j$ we obtain

$$\begin{aligned} &\mathsf{E}\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2 t_j^2}{2\sigma_j^2}\right\} \\ &\le \int_{\mathbf{R}^n}\cdots\int\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}u_j^2\right\}\frac{du_1\cdots du_n}{\left(1+\sum_{j=1}^{n}t_j^2u_j^2\right)^{1/4}}. \end{aligned} \tag{13}$$

Put

$$t_j^2 = \sigma_j^2\frac{u^2}{\sum_{j=1}^{n}\sigma_j^2}, \qquad \sum_{j=1}^{n}t_j^2 = u^2.$$

Since the function $f(u) = \frac{1}{4}\ln(1+u)$ is concave and $f(0) = 0$, we have

$$\frac{1}{4}\ln\left(1+\sum_{i=1}^{n}\alpha_i x_i\right) \ge \sum_{i=1}^{n}\alpha_i\left(\frac{1}{4}\ln(1+x_i)\right)$$

for $\alpha_i > 0$ such that $\sum_{i=1}^{n}\alpha_i = 1$. Further

$$-\frac{1}{4}\ln\left(1+\sum_{i=1}^{n}\alpha_i x_i\right) \le \sum_{i=1}^{n}\alpha_i\left(-\frac{1}{4}\ln(1+x_i)\right).$$

Thus

$$\frac{1}{(1+\sum_{i=1}^{n}\alpha_i x_i)^{1/4}} \le \prod_{i=1}^{n}\frac{1}{(1+x_i)^{(1/4)^{\alpha_i}}}.$$

Therefore

$$\frac{1}{\left(1+\sum_{i=1}^{n}t_j^2u_j^2\right)^{1/4}} = \frac{1}{\left(1+\sum_{i=1}^{n}\frac{t_j^2}{u^2}u_j^2u^2\right)^{1/4}} \le \prod_{j=1}^{n}\frac{1}{\left(1+u_j^2u^2\right)^{t_j^2/(4u^2)}},$$

since $\sum_{j=1}^{n}t_j^2/u^2 = 1$. This inequality together with (13) yields

$$\mathsf{E}\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2 t_j^2}{2\sigma_j^2}\right\} \le \prod_{j=1}^{n}\mathsf{E}\left(\left(\frac{1}{\left(1+\xi_j^2u^2\right)^{1/4}}\right)^{t_j^2/u^2}\right)$$

where ξ_j are Gaussian $N(0,1)$ independent random variables. Applying the inequality $\mathsf{E}\,|\xi|^\alpha \le (\mathsf{E}\,|\xi|)^\alpha$, $\alpha < 1$, we obtain

$$\begin{aligned} \mathsf{E}\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2 t_j^2}{2\sigma_j^2}\right\} &\le \left(\mathsf{E}\left(\frac{1}{\left(1+\xi^2u^2\right)^{1/4}}\right)\right)^{\sum_{j=1}^{n}t_j^2/u^2} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{\left(1+s^2u^2\right)^{1/4}} = g(u). \end{aligned}$$

Therefore

$$\mathsf{E}\exp\left\{-\frac{u^2}{2}\frac{\sum_{j=1}^{n}\zeta_j^2}{\sum_{j=1}^{n}\sigma_j^2}\right\} \le g(u)$$

where

$$g(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{(1+s^2u^2)^{1/4}}.$$

Now we consider the general case. Let B be a matrix such that $BB^T = B^2 = A$, and let $R = \operatorname{cov}\overline{\zeta}$. Let O be the orthogonal matrix that reduces the matrix BRB to the diagonal form, namely

$$OBRBO^T = D = \operatorname{diag}(d_k^2)_{k=1}^n.$$

Putting $\overline{\theta} = OB\overline{\zeta}$ we get

$$\overline{\theta}^T\overline{\theta} = \overline{\zeta}^T BO^T OB\overline{\zeta} = \overline{\zeta}^T A\overline{\zeta}$$

and $\operatorname{cov}\overline{\theta} = OB\operatorname{cov}\overline{\zeta}BO^T = D$. Since $\theta_i \in SG_\Xi(\Omega)$ and $\overline{\theta}^T = (\theta_1, \dots, \theta_n)$, the lemma holds for $\overline{\theta}$ instead of ζ. Thus

$$\overline{\theta}^T\overline{\theta} = \sum_{i=1}^n \theta_i^2 = \overline{\zeta}^T A\overline{\zeta}$$

and

$$\mathsf{E}\exp\left\{-\frac{u^2}{2}\frac{\overline{\zeta}^T A\overline{\zeta}}{\mathsf{E}\,\overline{\zeta}^T A\overline{\zeta}}\right\} \le g(u).$$

The lemma is proved. □

Theorem 3.2. *Let the assumptions of Lemmas 3.2 and 3.4 hold. Then*

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \ge 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\} \tag{14}$$

for all $1 > u > 0$ *and* $x < -2\ln g(u)/u^2$. *Moreover*

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \le \frac{2^{1/4}x^{1/4}}{\cosh\left(\sqrt{x/2} - \frac{1}{2}\right)} \tag{15}$$

for $x > \frac{1}{2}$ *where*

$$g(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{(1+s^2u^2)^{1/4}}$$

and either $\eta = \overline{\zeta}_n^T A_n\overline{\zeta}_n$ *or* $\eta = \operatorname{l.i.m.}\overline{\zeta}_n^T A_n\overline{\zeta}_n$, $\overline{\zeta}^T = (\zeta_1, \dots, \zeta_d)$, $\zeta_i \in SG_\Xi(\Omega)$.

Proof. According to Lemma 3.4

$$\mathsf{E}\exp\left\{-\frac{u^2\eta}{2\,\mathsf{E}\,\eta}\right\} \le g(u).$$

Put $\theta = \eta/\mathsf{E}\,\eta$. Then

$$\begin{aligned}\mathsf{P}\{\theta < x\} &= \int_0^x dF(v) = \int_0^x \frac{\exp\{-u^2v/2\}}{\exp\{-u^2v/2\}}\,dF(v) \le \frac{1}{\exp\{-u^2x/2\}}\,\mathsf{E}\exp\left\{-\frac{u^2\theta}{2}\right\}\\ &\le \frac{g(u)}{\exp\{-u^2x/2\}} = g(u)\exp\left\{\frac{u^2x}{2}\right\},\end{aligned}$$

whence

$$\mathsf{P}\{\theta > x\} \ge 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\}.$$

Using the initial notation we have

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} > x\right\} \ge 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\}.$$

Inequality (15) is proved in Lemma 3.2. □

4. The construction of a criterion to test hypotheses

Let $\xi(t)$ be a Gaussian stochastic process with $\mathsf{E}\,\xi(t)=0$ and the correlation function

$$\mathsf{E}\,\xi(t+\tau)\xi(t)=\rho(\tau).$$

Let

$$\hat\rho(\tau)=\frac{1}{T}\int_0^T \xi(t+\tau)\xi(t)\,dt,\qquad 0\le\tau\le T,$$

be an estimator of the covariance function $\rho(\tau)$. Put $\chi(\tau)=\hat\rho(\tau)-\rho(\tau)$. It is obvious that $\chi(\tau)$ is a square Gaussian random variable.

Further let

$$\eta=\int_0^B \left(\hat\rho(\tau)-\rho(\tau)\right)^2 d\tau.$$

It is obvious that inequalities (14)–(15) hold for η where

$$\begin{aligned}\mathsf{E}\,\eta&=\mathsf{E}\int_0^B \left(\hat\rho(\tau)-\rho(\tau)\right)^2 d\tau\\&=\frac{1}{T^2}\int_0^B\int_0^T\int_0^T \left[\rho^2(t-s)+\rho(t+\tau-s)\rho(t-\tau-s)\right]\,dt\,ds\,d\tau.\end{aligned}$$

Put

$$f(x)=\frac{2^{1/4}x^{1/4}}{\cosh\left(\sqrt{x/2}-\frac12\right)},\qquad s(x,u)=1-g(u)\exp\left\{\frac{u^2x}{2}\right\}$$

for $g(u)$ defined as in Theorem 3.2.

It follows from (14) and (15) that

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta}\notin[x;y]\right\}\le 1-s(x,u)+f(y),$$

whence

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta}\in[x;y]\right\}\ge s(x,u)-f(y)$$

for $1>u>0$, $x<-2\ln g(u)/u^2$, and $y>\frac12$.

Let H be the hypothesis that the covariance function of a Gaussian stochastic process equals $\rho(\tau)$ for $0\le\tau\le T$. We regard $\hat\rho(\tau)$ as the estimator of the function $\rho(\tau)$.

To test the hypothesis H one can use the following criterion.

Criterion. Given α, $0<\alpha<1$, one should determine x_α and y_α such that

$$1-s(x_\alpha,u)+f(y_\alpha)=\alpha.$$

The hypothesis H is accepted if

$$x_\alpha<\frac{\int_0^B \left(\hat\rho(\tau)-\rho(\tau)\right)^2 d\tau}{\mathsf{E}\int_0^B \left(\hat\rho(\tau)-\rho(\tau)\right)^2 d\tau}<y_\alpha.$$

Otherwise the hypothesis is rejected.

We now show how to find a and b such that

$$\mathsf{P}\left\{a\le\frac{\eta}{\mathsf{E}\,\eta}\le b\right\}\ge 1-\alpha$$

if α is given. In other words we want to find a and b such that

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta}\notin[a;b]\right\}\le\alpha$$

if α is given. The latter inequality follows from

$$\mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} \le a\right\} \le \alpha\gamma \quad \text{and} \quad \mathsf{P}\left\{\frac{\eta}{\mathsf{E}\,\eta} \ge b\right\} \le \alpha(1-\gamma)$$

for some $0 < \gamma < 1$.

We choose the constant γ to minimize the difference $b_\gamma - a_\gamma$ where a_γ and b_γ are solutions of the equations

$$g(u)\exp\left\{\frac{u^2 a}{2}\right\} = \alpha\gamma,$$

$$\frac{2^{1/4} b^{1/4}}{\cosh\left(\sqrt{b/2} - \frac{1}{2}\right)} = \alpha(1-\gamma).$$

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Received 19/DEC/2002

Translated by OLEG KLESOV