Теорія Ймовір. та Матем. Статист.
Вип. 69, 2003

Theor. Probability and Math. Statist.
No. 69, 2004, Pages 115–127
S 0094-9000(05)00619-8
Article electronically published on February 8, 2005

ABELIAN AND TAUBERIAN THEOREMS FOR RANDOM FIELDS ON TWO-POINT HOMOGENEOUS SPACES

UDC 519.21

A. A. MALYARENKO

Abstract. We consider centered mean-square continuous random fields for which the variance of increments between two points depends only on the distance between these points. Relations between the asymptotic behavior of the variance of increments near zero and the asymptotic behavior of the spectral measure of the field near infinity are investigated. We prove several Abelian and Tauberian theorems in terms of slowly varying functions.

1. Introduction

Let $\xi(x)$ be a centered mean-square continuous random field defined on the space $\mathbb{R}^n$ and whose covariance function is $B(x,y) = \mathsf{E}\,\xi(x)\xi(y)$. The classical definition of a homogeneous isotropic random field can be formulated in two different but equivalent ways (see [11]). Here are these two definitions.

Definition 1. A random field $\xi(x)$ is called *homogeneous and isotropic* if its correlation function $B(x,y)$ is invariant with respect to the isometry group G of the space $\mathbb{R}^n$.

Definition 2. A random field $\xi(x)$ is called *homogeneous and isotropic* if its correlation function $B(x,y)$ depends only on the distance $\rho(x,y)$ between points x and y.

An easy proof of the equivalence of Definitions 1 and 2 is based on the following important property of the space $\mathbb{R}^n$. The isometry group of $\mathbb{R}^n$ is *transitive on equidistant pairs of points*. In other words, whenever $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ are such that $\rho(x_1,y_1) = \rho(x_2,y_2)$, there is an isometry g such that $g(x_1) = x_2$ and $g(y_1) = y_2$.

Any metric space (X,ρ) possessing this property is called a *two-point homogeneous space*. We are concerned with the case where X is both a two-point space and a connected Riemann manifold. The complete classification of such spaces is known (see [4]). We present this classification in Tables 1 and 2.

In Table 1, S^n, $n \geq 1$, denotes the n-dimensional sphere, $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$, $n \geq 2$, stand for the n-dimensional projective spaces over fields of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$, respectively, while $\mathbb{C}\mathrm{a}P^2$ means the projective plane over Cayley numbers. Similarly in Table 2, $\mathbb{R}^n$, $n \geq 1$, denotes the n-dimensional Euclidean space, $\mathbb{R}\Lambda^n$, $\mathbb{C}\Lambda^n$, and $\mathbb{H}\Lambda^n$, $n \geq 2$, stand for the n-dimensional hyperbolic spaces over the fields of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and quaternions $\mathbb{H}$, respectively, while $\mathbb{C}\mathrm{a}\Lambda^2$ means the hyperbolic plane over Cayley numbers.

2000 *Mathematics Subject Classification.* Primary 60G60, 60G10; Secondary 40E05.
Key words and phrases. Random field, Abelian theorem, Tauberian theorem, two-point homogeneous space.
This work is supported in part by the Foundation for Knowledge and Competence Development.

TABLE 1. Compact two-point homogeneous spaces.

X	G	K	α	β
S^n	$\mathrm{SO}(n+1)$	$\mathrm{SO}(n)$	$(n-2)/2$	$(n-2)/2$
$\mathbb{R}P^n$	$\mathrm{SO}(n+1)$	$\mathrm{O}(n)$	$(n-2)/2$	$-1/2$
$\mathbb{C}P^n$	$\mathrm{SU}(n+1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	$n-1$	0
$\mathbb{H}P^n$	$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(1)$	$2n-1$	1
$\mathbb{C}\mathrm{a}P^2$	$\mathrm{F}_{4(-52)}$	$\mathrm{Spin}(9)$	7	3

TABLE 2. Noncompact two-point homogeneous spaces.

X	G	K	M	α	β
$\mathbb{R}^n$	$\mathrm{ISO}(n)$	$\mathrm{SO}(n)$	$\mathrm{SO}(n-1)$	$(n-2)/2$	$(n-2)/2$
$\mathbb{R}\Lambda^n$	$\mathrm{SO}_0(n,1)$	$\mathrm{SO}(n)$	$\mathrm{SO}(n-1)$	$(n-2)/2$	$-1/2$
$\mathbb{C}\Lambda^n$	$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1)^*)$	$n-1$	0
$\mathbb{H}\Lambda^n$	$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(1)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(1)^*$	$2n-1$	1
$\mathbb{C}\mathrm{a}\Lambda^2$	$\mathrm{F}_{4(-20)}$	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$	7	3

One must distinguish between the dimension n of the space X considered over some number system, and the *topological* dimension N of the manifold X. We have $N = n$ for the case of real numbers, $N = 2n$ for complex numbers, $N = 4n$ for quaternions, while $N = 8n = 16$ for Cayley numbers.

In what follows the symbol X stands for an arbitrary space in the first columns of Tables 1 and 2. The second column in these tables presents the connected component G of the identity element of the corresponding isometry group. We use the standard notation for Lie groups; see [18].

Let K denote the stationary group of a fixed point $o \in X$, namely

$$K = \{g \in G\colon go = o\}.$$

This group is described in the third column in Tables 1 and 2. Then the space X can be represented as the homogeneous space G/K.

Definition 3 ([21])**.** A random field $\xi(x)$ on a space $X = G/K$ is called *homogeneous* if its correlation function is invariant with respect to the group G.

Unfortunately, the commonly accepted terminology differs from that introduced in Definition 3 for two cases. First, if $X = \mathbb{R}^n = \mathrm{ISO}(n)/\mathrm{SO}(n)$, then according to Definition 1 the fields possessing the property of Definition 3 are called *homogeneous and isotropic*. Second, if

$$X = S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n),$$

then those fields are called *isotropic* [11]. In both cases we use the commonly accepted terminology.

In what follows $\xi(x)$ denotes a homogeneous random field on X. The correlation function $B(x,y)$ of a random field $\xi(x)$ can be represented in the form $B(x,y) = B(r)$ where r denotes the distance between the points x and y. Homogeneous random fields on compact two-point homogeneous spaces are studied in [13]. The case of noncompact spaces is presented in [7].

Consider the variance of an increment of a random field $\xi(x)$:

$$\sigma^2(x,y) = \mathsf{E}(\xi(x) - \xi(y))^2.$$

It is easy to see that $\sigma^2(x,y) = \sigma^2(r)$ where $r = \rho(x,y)$. Indeed,

$$\sigma^2(x,y) = 2(B(0) - B(r)). \tag{1}$$

Note that the converse is not true. A random field $\eta(x)$ on a space X is not necessarily homogeneous even though the variance of the increments depends only on the distance between points.

Consider an example. Let $W_1(t)$ and $W_2(t)$, $t \in [0,\infty)$, be two independent Wiener processes. Let the stochastic process $\eta(t)$ be defined by

$$\eta(t) = \begin{cases} W_1(t), & t > 0, \\ W_2(-t), & t \le 0. \end{cases}$$

The variance of increments of the process $\eta(t)$ is $\sigma^2(s,t) = |s-t|$; however the process is not homogeneous.

Random fields $\eta(x)$ whose variance of increments between two points depends only on the distance between these points were studied in [17] for various spaces X under an additional assumption that $\eta(o) = 0$. We are interested in the cases where X is one of the spaces listed in Tables 1 and 2.

It is easy to check that the variance of increments of the random field $\eta(x) = \xi(x) - \xi(o)$ depends only on the distance between points. According to [17], this is the only possibility in the case of compact spaces. Nevertheless there are other possibilities in the case of noncompact spaces. For simplicity we consider only the case of $X = \mathbb{R}^n$, which corresponds to the first row of Table 2.

Throughout the paper $\eta(x)$ denotes a centered mean square continuous random field on $X = \mathbb{R}^n$ whose variance of increments between two points depends only on the distance between these points.

Many problems on local properties of Gaussian and sub-Gaussian random fields require the asymptotic behavior of the function $\sigma^2(r)$ near zero (see [2, 12]). In this paper, we prove several theorems concerning relations between the asymptotic behavior of the spectral measure of random fields $\xi(x)$ and $\eta(x)$ at infinity and the asymptotic behavior of the variance of increments near zero. Other kinds of Abelian and Tauberian theorems for homogeneous and isotropic random fields are studied in [5, 6, 8, 11, 19].

In Section 2, we give statements of results. Section 3 is devoted to the proofs. Concluding remarks are given in Section 4.

The author is indebted to Professor N. H. Bingham for helpful discussions of Abelian and Tauberian theorems.

2. Results

To formulate our results, we need some preparation.

Consider a homogeneous random field $\xi(x)$. By π we denote the mapping from G onto X that sends $g \in G$ to the corresponding coset $gK \in X$. Recall that a subgroup K of a group G is called *massive* (see [3]) if any irreducible unitary representation U of the group G contains at most one copy of the trivial representation of the group K. A representation U is called a *representation of class* 1 with respect to a group K if it contains exactly one copy of the trivial representation of the group K.

The group K is massive for all spaces listed in Tables 1 and 2. Thus according to [21], the correlation function $B(x,y)$ of the random field $\xi(x)$ is of the form

$$B(x,y) = \int_{\hat{G}_K} T_{00}\left(\lambda; g_2^{-1} g_1\right)\, d\mu(\lambda) \tag{2}$$

where $\hat{G}_K$ denotes the set of equivalence classes of the irreducible unitary representations of class 1 of the group G with respect to the group K, $T_{00}(\lambda; g_2^{-1}g_1)$ denotes the zonal spherical function (see [21]) that corresponds to an element $\lambda \in \hat{G}_K$, g_1 is an arbitrary element of the set $\pi^{-1}x$, g_2 is an arbitrary element of the set $\pi^{-1}y$, and μ is a finite measure on $\hat{G}_K$. The measure μ is called the *spectral measure* of the random field $\xi(x)$.

The set $\hat{G}_K$ can be identified with the set $\mathbb{Z}_+$ of nonnegative integers in the case of compact spaces listed in Table 1 (see [13]). We have $\hat{G}_K = [0, \infty)$ for the space $\mathbb{R}^n$. For all other noncompact spaces in Table 2, the set $\hat{G}_K$ can be identified with the set $\{i\rho\} \cup [0, is_0] \cup (0, \infty)$ where the parameters ρ and s_0 are given by

$$\rho = \alpha + \beta + 1, \qquad s_0 = \min\{\rho, \alpha - \beta + 1\}$$

(see [16]).

Consider a random field $\eta(x)$, $x \in \mathbb{R}^n$, for which the variance of increments depends only on the distance between points. Assume that $\eta(0) = 0$. According to [17], the correlation function of the field $\eta(x)$ is of the form

$$\sigma^2(r) = cr^2 + \int_0^\infty \left(1 - 2^{(n-2)/2}\Gamma(n/2)\frac{J_{(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}}\right) d\mu(\lambda) \tag{3}$$

where $c \geq 0$ is a constant, $J_\nu(z)$ is the Bessel function of order ν, and μ denotes the so-called *Lévy–Khinchin measure* on the interval $(0, \infty)$, that is, a measure such that

$$\int_0^\infty \frac{\lambda^2\, d\mu(\lambda)}{1 + \lambda^2} < \infty.$$

Therefore all the cases above deal with the asymptotic behavior of the spectral measure μ at infinity.

The class of random fields considered above has an alternative description. Recall that a random field $\eta(x)$ has *homogeneous increments* if the variance $\sigma^2(x, y)$ of its increments is such that

$$\sigma^2(x + z, y + z) = \sigma^2(x, y), \qquad z \in \mathbb{R}^n.$$

Definition 4 ([11]). A random field $\eta(x)$ is called *isotropic* if the expectation $\mathsf{E}\,\eta(x)$ depends only on the norm $\|x\|$ and

$$B(gx, gy) = B(x, y)$$

for all $g \in \mathrm{SO}(n)$.

Lemma 1. *Let $\eta(x)$, $x \in \mathbb{R}^n$, be a centered mean square continuous random field. Then the following two conditions are equivalent:*

(1) *the variance of increments of the random field $\eta(x)$ depends only on the distance between points;*
(2) *$\eta(x)$ is an isotropic field with homogeneous increments.*

Let $\alpha \geq \beta > -1$ be arbitrary real numbers and let $P_m^{(\alpha,\beta)}(r)$ be the Jacobi polynomials, that is, polynomials orthogonal on the interval $[-1, 1]$ with respect to the measure

$$(1 - r)^\alpha (1 + r)^\beta\, dr.$$

Further let

$$R_m^{(\alpha,\beta)}(r) = \frac{P_m^{(\alpha,\beta)}(r)}{P_m^{(\alpha,\beta)}(1)}.$$

We write $f(r) \sim g(r)$ as $r \downarrow 0$ if

$$\lim_{r \downarrow 0} \frac{f(r)}{g(r)} = 1.$$

Similarly we write $a_m \sim b_m$ as $m \to \infty$ if

$$\lim_{m\to\infty} \frac{a_m}{b_m} = 1.$$

Lemma 2. *Let $\sigma^2(r)$ denote the variance of increments of either a centered mean square continuous homogeneous random field on a space X or a centered mean square continuous isotropic random field with homogeneous increments on the space $\mathbb{R}^n$. In both cases, there are a sequence of real numbers a_m, $m \geq 1$, depending only on the spectral measure μ, and real numbers α' and β' depending only on the space X such that*

$$\sigma^2(r) \sim \sum_{m=1}^{\infty} a_m \left[1 - R_m^{(\alpha',\beta')}(\cos r)\right], \qquad r \downarrow 0. \tag{4}$$

Moreover

$$a_m \geq 0, \qquad \sum_{m=1}^{\infty} a_m < \infty.$$

In this case the values of a_m, α', and β' for various spaces are given by formulas (16), (20)*, and* (22)–(27) *below.*

We need an auxiliary result. Put

$$A_m = \sum_{j=m+1}^{\infty} a_j. \tag{5}$$

For all real numbers $\alpha > -1$ and $\beta > -1$ let

$$w_m^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+1)\Gamma(\beta+1)\Gamma(m+\alpha+\beta+1)\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)m!\,\Gamma(m+\beta+1)} \tag{6}$$

where $(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)$ equals 1 for $m=0$ and $\alpha=\beta=-1/2$. Note that the norm of the polynomial $R_m^{(\alpha,\beta)}(\cos\theta)$ in the Hilbert space $L^2([0,\pi],dG)$ is equal to $(w_m^{(\alpha,\beta)})^{-1/2}$ (see [14]). Let the probability measure G be defined by

$$dG(\theta) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sin^{2\alpha+1}(\theta/2)\cos^{2\beta+1}(\theta/2)\,d\theta.$$

Lemma 3. *We have*

$$\sigma^2(r) \sim r^2 \left(\frac{A_0(\alpha'+\beta'+2)}{4(\alpha'+1)} + \sum_{m=1}^{\infty} \frac{c_m w_m^{(\alpha'+1,\beta')}}{m^{2\alpha'+4}} R_m^{(\alpha'+1,\beta')}(\cos r)\right) \tag{7}$$

as $r \downarrow 0$, where

$$c_m = \frac{(2m+\alpha'+\beta'+2)A_m \cdot m^{2\alpha'+4}}{4(\alpha'+1)w_m^{(\alpha'+1,\beta')}}. \tag{8}$$

We also recall some definitions of the theory of regularly varying functions (see [15]). We say that a function $L(t)$ is *slowly varying* at infinity if

$$\lim_{t\to\infty} \frac{L(at)}{L(t)} = 1$$

for all $a > 0$. In what follows we assume that

$$A_m \sim m^{-2\gamma} L(m), \qquad m \to \infty,\ \gamma \in [0,1], \tag{9}$$

for some slowly varying function L. We additionally assume that $L(m) \to 0$ as $m \to \infty$ in the case of $\gamma = 0$, since A_m is the tail of a convergent series.

TABLE 3. Intervals related to types of convergence.

X	Absolute convergence	Conditional convergence	Abel summability
S^1, $\mathbb{R}^2$, $\mathbb{R}\Lambda^2$	$(1/2,1)$	$(0,1/2)$	$\{0\}$
S^2, $\mathbb{R}P^2$, $\mathbb{R}^3$, $\mathbb{R}\Lambda^3$, $\mathbb{C}\Lambda^n$	$(1/4,1)$	$[0,1/4]$	$\varnothing$
S^3, $\mathbb{R}P^3$, $\mathbb{R}^4$, $\mathbb{R}\Lambda^4$	$(0,1)$	$\{0\}$	$\varnothing$
Other spaces	$[0,1)$	$\varnothing$	$\varnothing$

The convergence of the series on the right-hand side of (7) can be understood in various senses depending on the parameter γ and space X. According to [14], we distinguish between the three types of convergence, namely absolute convergence, conditional convergence, and Abel summability. The types of convergence are shown in Table 3.

Table 3 contains the values of the parameter $\gamma \in [0,1)$ for all two-point homogeneous spaces for which the series on the right-hand side of (7) converges for a given type of convergence. For every type of convergence we provide the Abelian theorem and its Tauberian counterpart. We also provide a result for the limit value $\gamma = 1$.

Recall that $\xi(x)$ means a centered square mean continuous homogeneous random field on a space X, while $\eta(x)$ means a centered square mean continuous isotropic random field on a space $\mathbb{R}^n$ with homogeneous increments. These types of random fields are determined by their spectral measure μ. Applying Lemma 1 and equality (5) one can evaluate the coefficients A_m.

Theorem 1. *Let the parameter γ belong to the set of absolute convergence (column 2 of Table 3). Let equality (9) hold. Then the asymptotic behavior of the variance of increments of random fields $\xi(x)$ and $\eta(x)$ is given by*

$$\sigma^2(r) \sim \frac{\Gamma(1-\gamma)\Gamma(\alpha'+1)}{2^{2\gamma}\Gamma(\alpha'+\gamma+1)} r^{2\gamma} L\left(r^{-1}\right), \qquad r \downarrow 0. \tag{10}$$

Theorem 1 claims that the asymptotic relation (10) follows from the asymptotic relation (9) without any additional assumption. Results of this type are called *Abelian theorems*. The Tauberian counterpart of Theorem 1 should claim that, conversely, (9) follows from (10). In most cases (in our case as well) Tauberian theorems hold under extra conditions called *Tauberian conditions*.

We now introduce Tauberian conditions for our case.

Condition 1. There exists a function $\omega(\zeta)$ such that

$$\limsup_{m\to\infty} \max_{m\le j\le \zeta m} \frac{m^{2\gamma}}{L(m)} \left| \left(\frac{m}{j}\right)^{2-2\gamma} A_j - A_m \right| \le \omega(\zeta) < \infty.$$

Moreover

$$\lim_{\zeta\downarrow 1} \omega(\zeta) = 0.$$

We say that a sequence a_m is slowly varying in the Schmidt sense [14] if

$$\liminf_{m\to\infty} \min_{m\le j\le \zeta m} (a_j - a_m) \ge -\omega(\zeta)$$

where

$$0 \le \omega(\zeta) < \infty \quad \text{for } \zeta > 1, \qquad \lim_{\zeta\downarrow 1} \omega(\zeta) = 0.$$

Condition 2. A sequence $m^{2\gamma} A_m / L(m)$, $m \ge 1$, is slowly varying in the Schmidt sense.

Condition 3. Sequence (8) is slowly varying in the Schmidt sense.

Theorem 2. *Assume that the parameter γ belongs to the set of absolute convergence (column 2 of Table 3). Let* (10) *hold as well as any of Conditions 1, 2, or 3. Then equality* (9) *is valid.*

Now we turn to the case of parameters of the set of conditional convergence. A positive function $f(t)$ is called *quasimonotone* [14] if it is of a bounded variation on compact subsets of the interval $[0,\infty)$ and

$$\int_0^x t^\delta |df(t)| = O\left(x^\delta f(x)\right), \qquad x \to \infty,$$

for some $\delta > 0$.

Theorem 3. *Let the parameter γ belong to the set of conditional convergence (column 3 of Table 3). Assume that equality* (9) *holds and the function $L(t)$ is slowly varying and quasimonotone. Then equality* (10) *is satisfied.*

Theorem 4. *Let the parameter γ belong to the set of conditional convergence (column 3 of Table 3). Assume that equality* (10) *and Condition 3 hold. Then equality* (9) *is satisfied.*

The situation becomes simpler if the parameter γ belongs to the set of Abel summability. The set of Tauberian conditions is empty in this case.

Theorem 5. *Let the parameter γ belongs to the set of Abel summability (column 4 of Table 3). Assume that condition* (9) *holds and both sequences $L(m)$ and*

$$m(L(m) - L(m-1))$$

are slowly varying and quasimonotone. Then equality (10) *is satisfied.*

Theorem 6. *Let the parameter γ belong to the set of Abel summability (column 4 of Table 3). Assume that condition* (10) *holds. Then equality* (9) *is satisfied.*

Now let $\gamma = 1$. If the series $\sum_{m=1}^\infty L(m)/m$ converges, then the series on the right-hand side of (7) converges absolutely. In this case

$$\sigma^2(r) \sim \frac{A_0(\alpha' + \beta' + 2)}{4(\alpha' + 1)} r^2.$$

Otherwise the following result holds.

Theorem 7. *Let $A_m \sim m^{-2} L(m)$ $(m \to \infty)$ and*

$$\sum_{m=1}^\infty \frac{L(m)}{m} = \infty.$$

Then

$$\sigma^2(r) \sim \frac{r^2}{2(\alpha' + 1)} \int_0^{r^{-1}} \frac{L(u)}{u}\, du.$$

3. Proofs

Proof of Lemma 1. It is sufficient to prove that all possible correlation functions of centered mean square continuous isotropic random fields with homogeneous increments can be represented in the form (3). Let a field $\eta(x)$, $x \in \mathbb{R}^n$, have homogeneous increments. According to [10], its correlation function is of the form

$$B(x,y) = (Ax, y) + \int_{\mathbb{R}^n} \left(e^{i(p,x)} - 1\right)\left(e^{-i(p,y)} - 1\right) d\nu(p), \tag{11}$$

where A is a positive semidefinite linear operator in the space $\mathbb{R}^n$ and ν is Lévy–Khinchin measure. Assume that the random field $\eta(x)$ is isotropic. Then the operator A commutes with all operators $g \in \mathrm{SO}(n)$. By Schur's lemma [3] the operator A is of the form

$$A = cI. \tag{12}$$

Moreover the measure ν is such that

$$\nu(gC) = \nu(C), \qquad g \in \mathrm{SO}(n),\ C \in \mathfrak{B}(\mathbb{R}^n),$$

where $\mathfrak{B}(\mathbb{R}^n)$ is the σ-algebra of Borel sets in the space $\mathbb{R}^n$. The measure in this case is of the form

$$d\nu(p) = \omega_{n-1}^{-1}(\|p\|)\, dp\, d\mu(\|p\|) \tag{13}$$

where $\omega_{n-1}(\|p\|)$ denotes the Lebesgue measure of a sphere $S^{n-1}(\|p\|)$ of radius $\|p\|$ in the space $\mathbb{R}^n$, while μ denotes the measure on the interval $[0,\infty)$ defined by

$$\mu(C) = \nu(\{p \in \mathbb{R}^n \colon \|p\| \in C\}), \qquad C \in \mathfrak{B}([0,\infty)).$$

Substitute (12) and (13) into (11) and use the following result:

$$\omega_{n-1}^{-1}(\|p\|) \int_{S^{n-1}(\|p\|)} e^{i(q,x)}\, dq = 2^{(n-2)/2}\Gamma(n/2) \frac{J_{(n-2)/2}(\|p\| \cdot \|x\|)}{(\|p\| \cdot \|x\|)^{(n-2)/2}}$$

(see [11]). Then

$$\begin{aligned} B(x,y) = c(x,y) &+ \int_0^\infty \left(1 - 2^{(n-2)/2}\Gamma(n/2)\frac{J_{(n-2)/2}(\lambda\|x\|)}{(\lambda\|x\|)^{(n-2)/2}}\right) d\mu(\lambda) \\ &+ \int_0^\infty \left(1 - 2^{(n-2)/2}\Gamma(n/2)\frac{J_{(n-2)/2}(\lambda\|y\|)}{(\lambda\|y\|)^{(n-2)/2}}\right) d\mu(\lambda) \\ &+ \int_0^\infty \left(1 - 2^{(n-2)/2}\Gamma(n/2)\frac{J_{(n-2)/2}(\lambda\|x-y\|)}{(\lambda\|x-y\|)^{(n-2)/2}}\right) d\mu(\lambda) \end{aligned} \tag{14}$$

where $\lambda = \|p\|$. Substituting (14) into (1) we obtain (3). $\square$

Proof of Lemma 2. First we consider the case of a compact two-point homogeneous space X (Table 1). According to [13], in this case equality (2) takes the following form:

$$B(r) = \sum_{m=0}^{\infty} b_m R_m^{(\alpha,\,\beta)}(\cos r) \tag{15}$$

where the numbers α and β are shown in the corresponding columns in Table 1, and

$$b_m \geq 0, \qquad \sum_{m=1}^{\infty} b_m < \infty.$$

Substituting (15) in (1) we obtain (4) with

$$a_m = 2b_m, \qquad \alpha' = \alpha, \qquad \beta' = \beta. \tag{16}$$

Now we consider the case of a noncompact two-point homogeneous space X (Table 2). In the case of $X = \mathbb{R}^n$ let $M = \mathrm{SO}(n-1)$. In all other cases, the group G is a semisimple Lie group of rank 1. Let $G = KAN$ be its Iwasawa decomposition (see [18]). The group A can be identified with the group $\mathbb{R}$. Let M be the centralizer of the group A in the group K, that is, the set of elements $k \in K$ commuting with all elements $a \in A$. The group M is shown in column 4 of Table 2. We use the following notation in [16]:

$$\mathrm{U}(1)^* = \{(k,k)\colon k \in \mathrm{U}(1)\}, \qquad \mathrm{Sp}(1)^* = \{(k,k)\colon k \in \mathrm{Sp}(1)\}.$$

Denote by $R_\delta(k)$ the zonal spherical function that corresponds to the element $\delta \in \hat{K}_M$. Then

$$T_{00}\left(\lambda; g_2^{-1} g_1\right) = \sum_{\delta \in \hat{K}_M} T_{\delta 0}(\lambda; -a_1) R_\delta\left(k_2^{-1} k_1\right) T_{\delta 0}(\lambda; a_2). \tag{17}$$

Note that the elements $a_j \in \mathbb{R}$ and $k_j \in K/M$ are uniquely determined by the elements g_j, $j = 1, 2$. The symbol $T_{\delta 0}(\lambda; a)$, $\delta \in \hat{K}_M$, denotes the associated spherical function [3]. The homogeneous space K/M is the sphere $S^{N-1} = \{x \in X \colon \rho(x, o) = 1\}$. Relation (17) is proved in [16]. In the special cases $X = \mathbb{R}^n$ and $X = \mathbb{R}\Lambda^n$ it is proved in [3].

Substitute (17) into (2). Thus

$$B(x, y) = \sum_{\delta \in \hat{K}_M} \int_{\hat{G}_K} T_{\delta 0}(\lambda; a_1) T_{\delta 0}(\lambda; a_2)\, d\mu(\lambda) R_\delta\left(k_2^{-1} k_1\right).$$

Substituting the latter result in (1) we obtain

$$\sigma^2(x, y) = 2 \sum_{\delta \in \hat{K}_M} \int_{\hat{G}_K} T_{\delta 0}(\lambda; a_1) T_{\delta 0}(\lambda; a_2)\, d\mu(\lambda) \cdot \left(R_\delta(e) - R_\delta\left(k_2^{-1} k_1\right)\right). \tag{18}$$

If $r = \rho(x, y)$ is sufficiently small, then one can find an element $g \in G$ such that the points gx and gy lie on the sphere S^{N-1}. Since the random field $\xi(x)$ is homogeneous, we have

$$\sigma^2(gx, gy) = \sigma^2(x, y) = \sigma^2(r).$$

This means that one can assume from the beginning that x, $y \in S^{N-1}$. For these points x and y it follows that $a_1 = a_2 = 1$.

Consider the restriction of the random field $\xi(x)$ to the sphere S^{N-1}. This random field is homogeneous with respect to the group K. In the cases of $X = \mathbb{R}^n$ and $X = \mathbb{R}\Lambda^n$ (Table 2, $N = n$) we have $K = \mathrm{SO}(n)$, whence $\xi(x)$, $x \in S^{n-1}$, is an isotropic random field on a sphere. Thus the variance of its increments must be of the following form:

$$\sigma^2(r') = \sum_{m=1}^{\infty} a_m \left[1 - R_m^{((n-3)/2, (n-3)/2)}\left(\cos r'\right)\right] \tag{19}$$

where r' is the distance between points x and y measured along the geodesic line on the sphere S^{n-1}.

It is clear that $r' \sim r$ as $r \downarrow 0$. Comparing (19) and (18) we see that the set $\hat{K}_M$ can be identified with the set $\mathbb{Z}_+$. Moreover, in this case the zonal spherical function R_m depends only on one variable r' and

$$a_m = 2 \int_{\hat{G}_K} |T_{m0}(\lambda; 1)|^2\, d\mu(\lambda), \qquad \alpha' = \beta' = (n-3)/2. \tag{20}$$

Other cases listed in Table 2 are more complicated (see [16]). The group K is a subgroup of $\mathrm{SO}(N-1)$. The set $\hat{K}_M$ can be identified with the set

$$\left\{(k, l) \in \mathbb{Z}_+^2 \colon 0 \le l \le k\right\}.$$

The zonal spherical function R_δ is described in [16] and depends on *two* real variables. The function R_δ is of the following form:

$$R_{kl}(u, \varphi) = R_l^{(\alpha-\beta-1,\, \beta+k-l)}\left(2u^2 - 1\right) r^{k-l} R_{k-l}^{(\beta-1/2,\, \beta-1/2)}(\cos\varphi)$$

where $0 \le u \le 1$ and $-\pi < \varphi \le \pi$ for $X = \mathbb{C}\Lambda^n$, while $0 \le \varphi \le \pi$ for $X = \mathbb{H}\Lambda^n$ or $X = \mathbb{C}\mathrm{a}\Lambda^2$.

Relation (18) can be rewritten as follows:

$$\sigma^2(r) = 2\sum_{k=0}^{\infty}\sum_{l=0}^{k}\int_{\hat{G}_K} T_{kl}(\lambda;-a_1)T_{kl}(\lambda;a_2)\,d\mu(\lambda)\cdot(1-R_{kl}(u,\varphi)) \tag{21}$$

where $r=\rho(x,y)$. It is shown in [16] that

$$\cosh r = \left|\cosh a_1 \cosh a_2 - ue^{i\varphi}\sinh a_1 \sinh a_2\right|.$$

If r is sufficiently small, then there is an element $g \in G$ such that the points gx and gy lie on the surface of the sphere S^{N-1}, whence $a_1 = a_2 = 1$ and $u = 1$. Thus

$$\cosh r = \left|\cosh^2 1 - e^{i\varphi}\sinh^2 1\right|$$

and $r' \sim r$ as $r \downarrow 0$. Hence equality (21) implies an asymptotic relation of the form (4) with

$$a_m = 2\sum_{l=0}^{\infty}\int_{\hat{G}_K} |T_{l+m,l}(\lambda;1)|^2\,d\mu(\lambda), \qquad \alpha' = \beta' = \beta - 1/2. \tag{22}$$

It remains to consider the case of an isotropic random field

$$\eta(x), \qquad x \in \mathbb{R}^n,$$

with homogeneous increments. Let $T_{m0}(\lambda;\|x\|)$ be the associated spherical function of the space $\mathbb{R}^n = \mathrm{ISO}(n)/\mathrm{SO}(n)$ [3]. Then

$$\begin{aligned}
&2^{(n-2)/2}\Gamma(n/2)\frac{J_{(n-2)/2}(\lambda\|x-y\|)}{(\lambda\|x-y\|)^{(n-2)/2}}\\
&\qquad = \sum_{m=0}^{\infty} T_{m0}(\lambda;\|x\|)T_{m0}(\lambda;\|y\|)R_m^{((n-3)/2,(n-3)/2)}(\cos\varphi)
\end{aligned}$$

and

$$1 = \sum_{m=0}^{\infty} T_{m0}(\lambda;\|x\|)T_{m0}(\lambda;\|y\|)$$

where φ denotes the angle between the vectors x and y. Substituting the latter results into (3) and considering the points x and y on the surface of the sphere S^{n-1} we obtain (4). In this case the numbers b_m, α', and β' are defined by (20).

The associated spherical function is given by

$$T_m(\lambda;a) = i^m\sqrt{w_m^{((n-3)/2,(n-3)/2)}}\,2^{(n-2)/2}\Gamma(n/2)\frac{J_{m+(n-2)/2}(\lambda a)}{(\lambda a)^{(n-2)/2}} \tag{23}$$

if $X = \mathbb{R}^n$ (see [3]), or by

$$T_m(\lambda;a) = \sqrt{w_m^{((n-3)/2,(n-3)/2)}}\,\psi_{\lambda,m,0}^{((n-3)/2,(n-3)/2)}(a) \tag{24}$$

if $X = \mathbb{R}\Lambda^n$ (see [16]), or by

$$T_{kl}(\lambda;a) = \sqrt{\pi_{k,l}^{(\alpha-\beta-1,\,\beta-1/2)}}\,\psi_{\lambda,k,l}^{(\alpha,\,\beta)}(a) \tag{25}$$

in all other cases (see [16]), where

$$\begin{aligned}\psi^{(\alpha,\beta)}_{\lambda,k,l}(a) &= \frac{\Gamma((\alpha+\beta+1+i\lambda)/2+k)\Gamma((\alpha-\beta+1+i\lambda)/2+l)\Gamma(\alpha+1)}{\Gamma((\alpha+\beta+1+i\lambda)/2)\Gamma((\alpha-\beta+1+i\lambda)/2)\Gamma(\alpha+k+l+1)}\\ &\quad\times \sinh^{k+l} a \cosh^{k-l} a\\ &\quad\times {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}+k, \frac{\alpha+\beta+1-i\lambda}{2}+k;\right.\\ &\qquad\qquad\left.\alpha+k+l+1; -\sinh^2 a\right)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\pi^{(\alpha,\beta)}_{k,l} &= \frac{(2k-2l+2\beta+1)(k+l+\alpha+\beta+3/2)\Gamma(l+\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\beta+2)\Gamma(\alpha+\beta+5/2)}\\ &\quad\times\frac{\Gamma(k-l+2\beta+1)\Gamma(k+\alpha+\beta+3/2)\Gamma(\beta+3/2)}{l!\,(k-l)!\,\Gamma(k+\beta+3/2)}.\end{aligned} \tag{27}$$

The symbol ${}_2F_1$ stands for the hypergeometric function. The coefficient $\pi^{(\alpha,\beta)}_{k,l}$ equals the dimension in the representation $\delta = (k,l) \in \hat{K}_M$. □

Proof of Lemma 3. We rewrite relation (4) as follows:

$$\sigma^2(r) \sim \sum_{m=1}^{\infty}(A_m - A_{m+1})\left[1 - R^{(\alpha',\beta')}_m(\cos r)\right], \qquad r\downarrow 0.$$

The partial sum of the series on the right-hand side is equal to

$$\begin{aligned}&\sum_{m=1}^{k}(A_m - A_{m+1})\left[1 - R^{(\alpha',\beta')}_m(\cos r)\right]\\ &\quad= A_1\left[1 - R^{(\alpha',\beta')}_1(\cos r)\right] + \sum_{m=1}^{k-1} A_{m+1}\left[R^{(\alpha',\beta')}_m(\cos r) - R^{(\alpha',\beta')}_{m+1}(\cos r)\right]\\ &\qquad - A_{k+1}\left[1 - R^{(\alpha',\beta')}_k(\cos r)\right].\end{aligned} \tag{28}$$

The factor A_{k+1} is the tail of a convergent series, thus it approaches zero as $k\to\infty$. Applying relation (7.32.2) of [9] with $\alpha' \ge \beta' \ge -1/2$ we have

$$\left|R^{(\alpha',\beta')}_m(\cos r)\right| \le R^{(\alpha',\beta')}_m(1) = 1.$$

This implies that the last term on the right-hand side of equality (28) tends to zero as $k\to\infty$. Approaching the limit we get

$$\sigma^2(r) \sim A_1\left[1 - R^{(\alpha',\beta')}_1(\cos r)\right] + \sum_{m=1}^{\infty} A_{m+1}\left[R^{(\alpha',\beta')}_m(\cos r) - R^{(\alpha',\beta')}_{m+1}(\cos r)\right]. \tag{29}$$

According to relation (32) in Section 10.8 in [1] we have

$$R^{(\alpha',\beta')}_m(\cos r) - R^{(\alpha',\beta')}_{m+1}(\cos r) = \frac{(2m+\alpha'+\beta'+2)\sin^2(r/2)}{\alpha'+1} R^{(\alpha'+1,\beta')}_m(\cos r). \tag{30}$$

Substituting (30) into (29) and taking into account that $\sin^2(r/2) \sim r^2/4$ as $r\downarrow 0$ we prove (7) and (8). □

Now we explain how to compute the values shown in Table 3. Assume that relation (9) holds. Applying (6) and the Stirling formula we get

$$c_m \sim \frac{\Gamma(\alpha'+1)\Gamma(\alpha'+\beta'+3)}{4\Gamma(\beta'+1)} m^{2-2\gamma} L(m), \qquad m \to \infty. \tag{31}$$

According to [14], the series on the right-hand side of (31) converges absolutely if

$$0 < 2 - 2\gamma < \alpha' + 3/2.$$

Then $\gamma \in (-\alpha'/2 - 1/4, 1)$. Using Tables 1 and 2 and relations (16), (20), and (22) we obtain the numbers in column 2 of Table 3. The rest of the numbers can be found similarly.

The proofs of all theorems are based on the results in [14]. Theorem 1 follows from (31) and Theorem 1a in [14]. Relation (9) for Theorem 2 in the case of Condition 1 follows from relation (10) and Theorem 2 in [14]. Similarly, relation (9) in the case of Condition 2 follows from (10) by Corollary 1 in [14], while in the case of Condition 3 it follows from (10) by Corollary 2 in [14].

To prove Theorem 3 we use Theorem 3a in [14], whereas to prove Theorem 4 we use Theorem 3b in [14]. Similarly we apply Theorem 4a in [14] to prove Theorem 5, Theorem 4c in [14] to prove Theorem 6, and Theorem 6 in [14] to prove Theorem 7.

4. Concluding remarks

Remark 1. The approach considered in this paper is not applicable for $X = \mathbb{R}^1$. In this case the Abelian and Tauberian theorems can be found in [20].

Remark 2. Let X be one of the spaces listed in Table 2. Let $\eta(x)$, $x \in X$, denote a centered mean square continuous random field such that $\eta(o) = 0$ and whose variance of increments between two points depends only on the distance between these points. Lemma 1 describes all the fields with these properties in the case of $X = \mathbb{R}^n$. Random fields for other cases in Table 2 are partially described in [17]. The Abelian and Tauberian theorems for such fields can be obtained using the methods of this paper.

Remark 3. Condition (9) can be written explicitly in terms of the spectral measure μ in the case of noncompact spaces. For example, if a random field is isotropic and has homogeneous increments on the space $\mathbb{R}^n$, then

$$\frac{2^{n-2}[\Gamma(n/2)]^2}{(n-1)!} \sum_{j=m+1}^{\infty} \frac{(2j+n-2)(j+n-3)!}{j!} \int_0^\infty \frac{[J_{j+(n-2)/2}(\lambda)]^2}{\lambda^{n-2}}\, d\mu(\lambda) \sim m^{-2\gamma} L(m)$$

by (5), (9), (20), and (23).

It would be interesting to obtain equivalent conditions in terms of the asymptotic behavior of the measure μ at infinity.

Bibliography

1. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, vol. II, McGraw–Hill, New York, 1953. MR0058756 (15:419i)
2. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl., Translations of Mathematical Monographs, vol. 188, Amer. Math. Soc., Providence, RI, 2000. MR1743716 (2001g:60089)
3. N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, "Nauka", Moscow, 1991; English transl., Translations of Mathematical Monographs, vol. 22, Amer. Math. Soc., Providence, RI, 1968. MR1177172 (93d:33013)
4. J. A. Wolf, *Spaces of Constant Curvature*, Publish or Perish, Wilmington, DE, 1984. MR0928600 (88k:53002)

5. N. N. Leonenko and A. Ya. Olenko, *Tauberian and Abelian theorems for the correlation function of a homogeneous isotropic random field*, Ukrain. Mat. Zh. **43** (1991), no. 12, 1652–1664; English transl. in Ukrainian Math. J. **43** (1991), 1539–1548. MR1172306 (93f:60069)
6. A. A. Malyarenko, *Local properties of Gaussian random fields on compact symmetric spaces, and Jackson-type and Bernstein-type theorems*, Ukrain. Mat. Zh. **51** (1999), no. 1, 60–68; English transl. in Ukrainian Math. J. **43** (1991), 66–75. MR1712757 (2000j:60053)
7. G. M. Molchan, *Homogeneous random fields on symmetric spaces of rank one*, Teor. Veroyatnost. Matem. Statist. (1979), no. 21, 123–148; English transl. in Theory Probab. Math. Statist. **21** (1980), 143–168. MR0550252 (81f:60077)
8. A. Ya. Olenko, *Tauberian and Abelian theorems for random fields with strong dependence*, Ukrain. Mat. Zh. **48** (1996), no. 3, 368–382; English transl. in Ukrainian Math. J. **48** (1996), 412–427. MR1408658 (97k:60143)
9. G. Szegő, *Orthogonal Polynomials*, Colloquium Publications, vol. XXIII, Amer. Math. Soc., Providence, RI, 1975. MR0372517 (51:8724)
10. A. M. Yaglom, *Certain types of random fields in n-dimensional space similar to stationary stochastic processes*, Teor. Veroyatnost. i Primenen. **2** (1957), 292–338; English transl. in Theory Probab. Appl. MR0094844 (20:1353)
11. M. I. Yadrenko, *Spectral Theory of Random Fields*, "Vyshcha Shkola", Kiev, 1980; English transl., Translation Series in Mathematics and Engineering, Optimization Software, Inc., Publications Division, New York, 1983. MR0590889 (82e:60001)
12. R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, vol. 12, Institute of Mathematical Statistics, Hayward, CA, 1990. MR1088478 (92g:60053)
13. R. Askey and N. H. Bingham, *Gaussian processes on compact symmetric spaces*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **37** (1976), no. 2, 127–143. MR0423000 (54:10984)
14. N. H. Bingham, *Tauberian theorems for Jacobi series*, Proc. London Math. Soc. (3) **36** (1978), no. 2, 285–309. MR0620813 (58:29795)
15. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987. MR0898871 (88i:26004)
16. M. Flensted-Jensen and T. H. Koornwinder, *Positive definite spherical functions on a noncompact, rank one symmetric space*, Analyse Harmonique sur les Groupes de Lie II, Lect. Notes Math., vol. 739, Springer-Verlag, Berlin–Heidelberg–New York, 1979, pp. 249–282. MR0560841 (81j:43015)
17. R. Gangolli, *Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters*, Ann. Inst. H. Poincaré, Sect. B **3** (1967), 121–226. MR0215331 (35:6172)
18. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics, vol. 34, Amer. Math. Soc., Providence, RI, 2001. MR1834454 (2002b:53081)
19. N. N. Leonenko and A. Ya. Olenko, *Tauberian theorems for correlation functions and limit theorems for spherical averages of random fields*, Random Oper. Stochastic Equations **1** (1993), no. 1, 57–67. MR1254176 (95a:60068)
20. E. J. G. Pitman, *On the behavior of the characteristic function of a probability distribution in the neighborhood of the origin*, J. Austral. Math. Soc. **8** (1968), 423–443. MR0231423 (37:6978)
21. A. M. Yaglom, *Second order homogeneous random fields*, Proc. IV Berkeley Symp. Math. Stat. Probab., vol. 2, 1961, pp. 593–622. MR0146880 (26:4399)

International Mathematical Centre, National Academy of Sciences of Ukraine
Current address: Mälardalen University, P. O. Box 883, SE–721 23 Västerås, Sweden
E-mail address: `anatoliy.malyarenko@mdh.se`
URL: `http://www.ima.mdh.se/personal/amo`

Received 3/JAN/2003

Translated by THE AUTHOR