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ON THE WEAK CONVERGENCE OF EXTREMES IN SOME BANACH SPACES

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ABSTRACT. The weak convergence of random elements

$$U_n = b_n(Z_n - a_n\mathfrak{S})$$

is studied for Banach spaces with an unconditional basis, where $Z_n = \max_{1\le k\le n} X_k$ and X_k, $k \ge 1$, are independent copies of a random element X.

1. Introduction

Consider a sequence (ξ_i) of independent identically distributed random variables with distribution function $F(x) = \mathsf{P}(\xi_i < x)$, and let

$$z_n = \max_{1\le k\le n} \xi_k.$$

Assume that

$$b_n(z_n - a_n) \xrightarrow{D} \zeta \tag{1}$$

as $n \to \infty$ for some sequences a_n and $b_n > 0$, and let the distribution of ζ be nondegenerate, $G(x) = \mathsf{P}(\zeta < x)$. We denote by $Y_n \xrightarrow{D} Y$ the weak convergence of random elements.

We say that a distribution function F belongs to the domain of attraction of a law G if relation (1) holds. We write $F \in D(G)$ in this case. According to the well-known theorem on extremal types (see [1]–[3]), the distribution function F belongs to the domain of attraction of one of the following three types of distributions:

$$\begin{aligned} &\text{Type I:} \quad G_1(x) = \exp(-e^{-x}), \quad \text{for } -\infty < x < \infty,\\ &\text{Type II:} \quad G_2(x) = \begin{cases} 0, & \text{for } x \le 0,\\ \exp(-x^{-\alpha}), & \text{for } \alpha > 0,\ x > 0,\end{cases}\\ &\text{Type III:} \quad G_3(x) = \begin{cases} \exp(-(-x)^{\alpha}), & \text{for } \alpha > 0,\ x \le 0,\\ 1, & \text{for } x > 0.\end{cases}\end{aligned} \tag{2}$$

Here we use the classification of extreme value distributions proposed in the book [3].

Let B be a Banach lattice, X a random element assuming values in B, and (X_n) a sequence of independent copies of X. The random elements

$$Z_n = \max_{1\le k\le n} X_k, \qquad n = 1, 2, \ldots,$$

are called extremes.

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It is natural to study the problem of the weak convergence of extremes, or, in other words, a generalization of the asymptotic relation (1) in the infinite-dimensional case.

It is worth mentioning that it is impossible to construct the general theory of the weak convergence in Banach lattices. The main reason for this is, perhaps, that the limit random element for various important spaces (say, for $L_p[0,1]$ or $C[0,1]$) is a process with independent values. Such a process is not well defined in the corresponding function space (moreover, it is nonmeasurable).

Different approaches to the above problem are proposed in different special cases (see [4, 5]).

In this paper, we consider the weak convergence of extremes for Banach spaces with an unconditional basis. We show that the moment conditions used in the paper [4] can be omitted. It turns out that the method we apply in this paper is useful for some other problems concerning the extremes. We also give a new result related to the convergence to the extremal type II distributions.

2. The weak convergence of extreme random elements in Banach spaces with an unconditional basis

Let B be a Banach space with an unconditional basis (e_n). Then there exists a universal constant K such that $\sum b_n e_n \in B$ and

$$\left\| \sum_n b_n e_n \right\| \le K\|x\| \tag{3}$$

if $x = \sum a_n e_n \in B$ and $|b_n| \le |a_n|$ for all n.

A Banach space with an unconditional basis and with $K = 1$ is a Banach lattice. If $K \ne 1$, then the space is a Banach lattice in the equivalent norm

$$|||x||| = \sup\{\|y\| \colon |y| \le |x|\}$$

(see [6]).

Let $\zeta, \zeta_1, \zeta_2, \dots$ be a sequence of independent identically distributed random variables with the distribution function $\mathsf{P}(\zeta < s) = G(s)$, let (η_i) be a sequence of identically distributed random variables with the distribution function $F(s)$, and let (σ_i) be a sequence of real numbers such that $\sigma_i \ge 0$, $i \ge 1$. Assume that the following series converge in the norm of the space B:

$$\mathfrak{S} = \sum_{i \ge 1} \sigma_i e_i, \tag{4}$$

$$X = \sum_{i \ge 1} \eta_i \sigma_i e_i \quad \text{a.s.}, \tag{5}$$

$$Z = \sum_{i \ge 1} \zeta_i \sigma_i e_i \quad \text{a.s.} \tag{6}$$

where “a.s.” stands for “almost surely”.

Consider a sequence (X_n) of independent copies of the random variable X,

$$X_n = \sum_{i \ge 1} \eta_{ni} \sigma_i e_i, \qquad Z_n = \max_{1 \le k \le n} X_k = \sum_{i \ge 1} z_{ni} \sigma_i e_i, \qquad z_{ni} = \max_{1 \le k \le n} \eta_{ki},$$
$$U_n = b_n (Z_n - a_n \mathfrak{S}).$$

Assume that $F \in D(G)$ and that the sequences (a_n) and (b_n) satisfy relation (1). Then the following asymptotic relation is a generalization of (1) to the case of the space B:

$$U_n \xrightarrow{D} Z, \qquad n \to \infty. \tag{7}$$

We are interested in obtaining conditions for the weak convergence of extrems U_n in the case where the components of the random element X are such that

$$\lim_{t\to x(F_1),s\to x(F_2)} \frac{\mathsf{P}(\xi_1 > t, \xi_2 > s)}{\mathsf{P}(\xi_1 > t) + \mathsf{P}(\xi_2 > s)} = 0 \tag{8}$$

where $x(F) = \sup(s\colon F(s) < 1)$, $F_i(s)$ is the distribution function of the random variable ξ_i, $i = 1, 2$ (see [7, 8] for related conditions). The above condition ensures that the extremes constructed from components of the vector are asymptotically independent.

Theorem 1. *Let a Banach space B with an unconditional basis (e_i) be a q-concave Banach lattice for some q, $1 \le q < \infty$. Assume that the series* (4) *and* (5) *converge and*

(i) *$F \in D(G_k)$ and $G_k(x)$ is one of the functions given by* (2)*, and moreover, if $k = 2$, then $\alpha > q$;*
(ii) *for all $i \ne j$ the random vector (η_i, η_j) satisfies equality* (8)*.*

Then the series (6) *converges and asymptotic relation* (7) *holds.*

First we prove some auxiliary results. The first result contains general sufficient conditions for the weak convergence of random elements in the space B.

Lemma 1. *Let $Y = \sum_{i\ge1} \xi_i e_i$, $Y_n = \sum_{i\ge1} \xi_{ni} e_i$, $n \ge 1$, and let random elements in B be such that*

(i) *for all integers m and $i_1, i_2, \dots, i_m$ the sequence of vectors*

$$\bar{\xi_n} = (\xi_{ni_1}, \dots, \xi_{ni_m})$$

converges in distribution to the vector $\bar{\xi} = (\xi_{i_1}, \dots, \xi_{i_m})$ in the space $\mathbf{R}^m$;

(ii) *for all $\delta > 0$,*

$$\lim_{m\to\infty} \sup_{n\ge1} \mathsf{P}\left(\left\|\sum_{i\ge m} \xi_{ni} e_i\right\| > \delta\right) = 0.$$

Then

$$Y_n \xrightarrow{D} Y$$

as $n \to \infty$.

Consider a random vector $\bar{\xi} = (\xi_1, \dots, \xi_k)$. Let $(\bar{\xi}_n = (\xi_{n1}, \dots, \xi_{nk}))$ be a sequence of independent copies of $\bar{\xi}$, $z_{nj} = \max_{1\le i\le n} \xi_{ij}$, $\bar{z}_n = (z_{n1}, \dots, z_{nk})$.

Lemma 2. *If*

$$b_{nj}(z_{nj} - a_{nj}) \xrightarrow{D} \zeta$$

as $n \to \infty$ for all components ξ_j, $j = 1, 2, \dots, k$, of the vector $\bar{\xi}$, and the random vector (ξ_i, ξ_j) satisfies equality (8) *for all $i \ne j$, $1 \le i$, $j \le k$, then*

$$\bar{b}_n(\bar{z}_n - \bar{a}_n) \xrightarrow{D} \bar{\zeta},$$

where $\bar{a}_n = (a_{n1}, \dots, a_{nk})$, $\bar{b}_n = (b_{n1}, \dots, b_{nk})$, $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$, and ζ_i are independent copies of ζ.

Lemma 3. *Let a Banach space B with an unconditional basis (e_i) be a q-concave Banach lattice for some q, $1 \le q < \infty$. Assume that the series*

$$\mathfrak{S} = \sum_{i\ge1} \sigma_i e_i \quad \text{and} \quad Y = \sum_{i\ge1} \xi_i \sigma_i e_i$$

converge almost surely in the norm of B, and

$$\sup_{i\ge1} (\mathsf{E}\,|\xi_i|^q)^{1/q} = C_q < \infty.$$

Then

$$(\mathsf{E}\,\|Y\|^q)^{1/q} \le C_q D_q(B) \|\mathfrak{S}\| \tag{9}$$

where $D_q(B)$ is the constant involved in the definition of a q-concave Banach lattice (see [6, p. 46]*).*

Lemma 4. *Let a Banach space B with an unconditional basis (e_i) be a q-concave Banach lattice for some q, $1 \le q < \infty$. Assume that (τ_i) and (ξ_i) are independent sequences of random variables, the series $\sum_{i\ge 1} \xi_i e_i$ converges almost surely in the norm of B, and*

$$\sup_{i\ge 1} (\mathsf{E}\,|\tau_i|^q)^{1/q} = C_q < \infty.$$

Then

$$\mathsf{P}\left(\left\| \sum_{i\ge 1} \tau_i \xi_i e_i \right\| > t C_q D_q(B) \left\| \sum_{i\ge 1} \xi_i e_i \right\| \right) \le t^{-q} \tag{10}$$

for all $t > 0$, where $D_q(B)$ is the constant on the right-hand side of (9)*.*

Proof of Lemma 4. Without loss of generality we assume that $\Omega = \Omega_1 \times \Omega_2$, $\mathsf{P} = \mathsf{P}_1 \times \mathsf{P}_2$, and the sequence (τ_i) is defined on Ω_1, while the sequence (ξ_i) is defined on Ω_2. By Fubini's theorem, the probability on the left-hand side of (10) can be represented as follows:

$$\int_{\Omega_2} \mathsf{P}_1 \left(\left\| \sum_{i\ge 1} \tau_i(\omega_1) \xi_i(\omega_2) e_i \right\| > t C_q D_q(B) \left\| \sum_{i\ge 1} \xi_i(\omega_2) e_i \right\| \right) \mathsf{P}_2(d\omega_2).$$

Therefore the general case of inequality (10) is reduced to the case where (ξ_i) is a nonrandom sequence. This case is easy, since (10) follows from Markov's inequality and (9). $\square$

Lemma 5. *Let (b_n) be a sequence of positive numbers satisfying equality* (1)*. Then there are constants C and p such that*

$$b_n \le C \cdot n^p \tag{11}$$

for all $n \ge 1$.

Proof of Lemma 5. First we recall a definition of a regularly varying sequence. Let $c(n)$, $n \ge 1$, be a sequence of positive numbers. We say that $(c(n))$ is regularly varying if

$$\lim_{n\to\infty} \frac{c([\lambda n])}{c(n)} = \theta(\lambda) \tag{12}$$

for all $\lambda > 0$, where $0 < \theta(\lambda) < \infty$. In this case, $\theta(\lambda) = \lambda^\rho$ for some $\rho \in (-\infty, +\infty)$ (see [9]).

If a sequence (b_n) satisfies equality (1), then, according to Theorem 2.2.1 of the book [2], the limit

$$\lim_{n\to\infty} \frac{b_{nm}}{b_n} = \theta(m) > 0 \tag{13}$$

exists and is finite for any integer $m \ge 1$. Note that a somewhat different notation is used in [2] for the sequences a_n and b_n in (1). Nevertheless equality (13) remains true.

It can be seen from the proof of Theorem 2.2.1 in [2] that equality (13) for the sequence (b_n) can be strengthened to obtain (12). Thus the sequence (b_n) is regularly varying.

Set $b(x) = b_{[x]}$, $x \ge 1$. Then the function $b(x)$ also is regularly varying at infinity (see [10]). It is well known that

$$b(x) \le Cx^{\rho+\varepsilon} \tag{14}$$

for any function $b(x)$ regularly varying at infinity and having order ρ, for all $\varepsilon > 0$, $x \ge 1$, and some constant $C = C(\varepsilon)$ (see [9]).

Estimate (11) follows explicitly from (14). $\square$

Lemma 1 follows from general results on the weak convergence (see Theorems 3.2 and 3.3 in [11]). Lemma 2 is proved in [5]; estimate (9) follows immediately from a result in [12].

Proof of Theorem 1. We apply Lemma 3 to check the convergence of the series (6). Since the series $\sum_{i\ge 1} \sigma_i e_i$ converges, we have

$$\left(\mathsf{E}\left\|\sum_{i=k}^{m} \zeta_i\sigma_i e_i\right\|^q\right)^{1/q} \le (\mathsf{E}\,|\zeta|^q)^{1/q} D_q(B)\left\|\sum_{i=k}^{m}\sigma_i e_i\right\| \to 0$$

as $k \to \infty$, provided that $m > k$ and

$$\mathsf{E}\,|\zeta|^q < \infty. \tag{15}$$

Note that the latter result follows from the upper bound

$$\mathsf{P}(|\zeta| > x) \le \begin{cases} \exp(-x+1), & k = 1,\\ (e-1)x^{-\alpha}, & k = 2,\\ \exp(-x^\alpha), & k = 3 \end{cases}$$

for $x > 0$ obtained in [4] (if $k = 2$, we additionally assume that $\alpha > q$).

Therefore the series $\sum_{i\ge 1}\zeta_i\sigma_i e_i$ converges in the space $L_q(\Omega, B)$. Since the random variables ζ_i are independent, the series converges almost surely in the norm of B (see [11]).

In order to prove (7), we check conditions of Lemma 1. The components of the vector

$$\bar\xi_n = \big(b_n\sigma_{i_1}(z_{ni_1} - a_n), \dots, b_n\sigma_{i_m}(z_{ni_m} - a_n)\big)$$

are asymptotically independent by assumption (ii) of Theorem 1 and in view of Lemma 2, and moreover

$$\bar\xi_n \xrightarrow{D} \bar\xi$$

where $\bar\xi = (\sigma_{i_1}\zeta_{i_1}, \dots, \sigma_{i_m}\zeta_{i_m})$. Thus it remains to show that condition (ii) of Lemma 1 holds, namely, that for $\delta > 0$

$$\lim_{m\to\infty}\sup_{n\ge 1}\mathsf{P}\left(\left\|\sum_{i\ge m} b_n(z_{ni} - a_n)\sigma_i e_i\right\| > \delta\right) = 0. \tag{16}$$

Put

$$x_+ = \begin{cases} x, & \text{for } x \ge 0,\\ 0, & \text{for } x < 0, \end{cases} \qquad x_- = x_+ - x.$$

Then the triangle inequality implies

$$\mathsf{P}\left(\left\|\sum_{i\ge m} b_n(z_{ni} - a_n)\sigma_i e_i\right\| > \delta\right) \le P_1(n,m) + P_2(n,m)$$

where

$$P_1(n,m) = \mathsf{P}\left(\left\|\sum_{i\ge m} b_n((z_{ni})_+ - a_n)\sigma_i e_i\right\| > \delta/2\right),$$

$$P_2(n,m) = \mathsf{P}\left(\left\|\sum_{i\ge m} b_n(-(z_{ni})_-)\sigma_i e_i\right\| > \delta/2\right).$$

This means that (16) follows from

$$\lim_{m\to\infty}\sup_{n\ge 1} P_1(n,m) = 0, \tag{17}$$

$$\lim_{m\to\infty}\sup_{n\ge 1} P_2(n,m) = 0. \tag{18}$$

First we prove equality (17). Without loss of generality we assume that $x(F) > 0$ (otherwise we consider the random variable $\eta' = \eta + C$, $C > |x(F)|$; it is clear that if sequences $\{a_n\}$ and $\{b_n\}$ are chosen in an appropriate way, then equality (1) holds for the random variable η' if and only if it holds for the random variables η).

If $F_+(x) = \mathsf{P}((\eta_i)_+ < x)$, then condition (i) of Theorem 1 implies that $F_+ \in D(G_k)$. Taking into account that $(z_{ni})_+ = \max_{1\le k\le n}(\eta_{ki})_+$ we deduce that equality (1) holds for random variables $(z_{ni})_+$ and for all $i \ge 1$. Moreover

$$\lim_{n\to\infty} \mathsf{E}\,|b_n((z_{ni})_+ - a_n)|^m = \mathsf{E}\,|\zeta|^m \tag{19}$$

for $i \ge 1$ and $m > 0$ (see [13]).

To estimate $P_1(n,m)$, we apply Markov's inequality and (9):

$$\begin{aligned} P_1(n,m) &\le (2/\delta)^q\,\mathsf{E}\left\|\sum_{i\ge m} b_n((z_{ni})_+ - a_n)\sigma_i e_i\right\|^q \\ &\le (2/\delta)^q (D_q(B))^q\,\mathsf{E}\,|b_n((z_{ni})_+ - a_n)|^q \left\|\sum_{i\ge m}\sigma_i e_i\right\|^q, \end{aligned}$$

whence (17) follows by (15) and (19) becuase the series (4) converges.

It remains to check equality (18). Put

$$I_{ni} = \begin{cases} 1 & \text{if } \min_{2\le k\le n}(\eta_{ki})_- > 0,\\ 0 & \text{otherwise.}\end{cases}$$

Note that

$$-(z_{ni})_- = \max_{1\le k\le n}(-(\eta_{ki})_-)$$

almost surely for all $i \ge 1$, hence

$$(z_{ni})_- \le I_{ni}\cdot|\eta_{1i}|. \tag{20}$$

Moreover

$$\mathsf{E}\,|I_{ni}|^q = p^{n-1} \tag{21}$$

for $p = F(0)$ ($0 \le p < 1$, since we assume that $x(F) > 0$).

Estimates (3) and (20) and equality (21) yield that for $t > 0$

$$
\begin{aligned}
P_2(n,m) &\le \mathsf{P}\left(\left\|\sum_{i\ge m} b_n I_{ni}\eta_{1i}\sigma_i e_i\right\| > \frac{\delta}{2K}\right) \\
&\le \mathsf{P}\left(\left\|\sum_{i\ge m} b_n I_{ni}\eta_{1i}\sigma_i e_i\right\| > t b_n p^{(n-1)/q} D_q(B)\left\|\sum_{i\ge m}\eta_{1i}\sigma_i e_i\right\|\right) \qquad (22)\\
&\quad + \mathsf{P}\left(t b_n p^{(n-1)/q} D_q(B)\left\|\sum_{i\ge m}\eta_{1i}\sigma_i e_i\right\| > \frac{\delta}{2K}\right).
\end{aligned}
$$

The sequences (I_{ni}) and (η_{1i}) are independent, thus one can apply inequality (10):

$$
\mathsf{P}\left(\left\|\sum_{i\ge m} b_n I_{ni}\eta_{1i}\sigma_i e_i\right\| > t b_n p^{(n-1)/q} D_q(B)\left\|\sum_{i\ge m}\eta_{1i}\sigma_i e_i\right\|\right) \le t^{-q}. \tag{23}
$$

Here we used equality (21). Since $0 \le p < 1$, estimate (11) implies that

$$
\sup_{n\ge 1} b_n p^{(n-1)/q} < \infty,
$$

whence

$$
\lim_{m\to\infty}\sup_{n\ge 1}\mathsf{P}\left(t b_n p^{(n-1)/q} D_q(B)\left\|\sum_{i\ge m}\eta_{1i}\sigma_i e_i\right\| > \frac{\delta}{2K}\right) = 0 \tag{24}
$$

for $t > 0$ in view of the convergence of the series (5). Now relations (22)–(24) imply

$$
\lim_{m\to\infty}\sup_{n\ge 1} P_2(n,m) \le t^{-q}.
$$

Since $t > 0$ is arbitrary, we obtain (18). □

We now present two corollaries of Theorem 1. Let l_q be the space of sequences (a_i) such that the series $\sum |a_i|^q$ converges, and let (e_i) be the natural basis in l_q. The space l_q is an example of a q-concave Banach lattice. Using Theorem 1 and Lemma 2 of [4] we obtain the following result.

Corollary 1. *Let X be a random element assuming values in the space l_q and represented in the form of the series* (5). *Assume that conditions (i) and (ii) of Theorem* 1 *hold. Then the series* (6) *converges and* (7) *holds if and only if*

$$
\sum_{i\ge 1} |\sigma_i|^q < \infty.
$$

Let random variables (η_i) have the standard Gaussian distribution function

$$
F(s) = \Phi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} \exp\left(-\frac{y^2}{2}\right) dy.
$$

It is well known that $\Phi(s)$ belongs to the domain of attraction of the distribution of the type I ($\Phi \in D(G_1)$), and

$$
\begin{gathered}
b_n = (2\ln n)^{1/2}, \quad a_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}} \quad \text{for } n > 1, \\
b_n = a_n = 1 \quad \text{for } n = 1
\end{gathered} \tag{25}
$$

(see [3]).

Assume that

$$|\mathsf{E}\,\eta_i\eta_j| < 1 \tag{26}$$

for all $i \neq j$. In the case of the normal distribution, the latter inequality implies (8) (see [7, 8]).

The following result is proved in [14]: *in a Banach space with an unconditional basis* (e_i)*, the convergence of the series* (5) *implies the convergence of the series* (4) *for a random sequence* (η_i) *having the normal distribution.*

Applying the latter result we obtain from Theorem 1 the following corollary.

Corollary 2. *Let a Banach space B with an unconditional basis (e_i) be a q-concave Banach lattice for some q, $1 \leq q < \infty$. Assume that the series* (5) *converges. Let X be a Gaussian random element with values in the space B such that it is represented in the form* (5) *and satisfies condition* (26)*. Then the series* (4) *and* (6) *converge and relation* (7) *holds for $G(x) = G_1(x)$ and sequences $\{a_n\}$ and $\{b_n\}$ defined by* (25).

The following example shows that the condition $\alpha > q$ cannot be improved to $\alpha = q$ in Theorem 1 in the case of $k = 2$ (that is, in the case of the extremal type II distributions).

Example. Let η be a symmetric random variable with the distribution function $F(x)$ such that

$$1 - F(x) = \frac{1}{x^q \ln^2 x} \tag{27}$$

for all $x > 2$ and some $1 \leq q < \infty$.

Denote by (η_i) a sequence of independent copies of η,

$$\sigma_i = \begin{cases} (i \ln^2 i)^{-1/q}, & \text{for } i > 2, \\ 1, & \text{for } i = 1, 2. \end{cases}$$

It is easy to check that

$$\mathsf{E}\,|\eta|^q < \infty$$

if (27) holds. Thus

$$\sum_{i \geq 1} |\eta_i \sigma_i|^q < \infty \quad \text{a.s.},$$

whence $X = \sum_{i \geq 1} \eta_i \sigma_i e_i$ is a random element assuming values in l_q. It is obvious that $\mathfrak{S} = \sum_{i \geq 1} \sigma_i e_i \in l_q$.

Now let ζ be a random variable with the distribution function $G_2(x)$ defined by (2) for $\alpha = q$. Since

$$\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-q},$$

equality (1) holds (see [2, 3]), thus $F \in D(G_2)$.

It is clear that

$$\lim_{x \to \infty} x^q (1 - G_2(x)) = 1. \tag{28}$$

This implies that the almost sure convergence of the series $\sum_{i \geq 1} |\zeta_i \sigma_i|^q$ is equivalent to the convergence of the series $\sum_{i \geq 1} |\sigma_i|^q \ln \sigma_i^{-1}$ (see [15]). On the other hand

$$\sum_{i \geq 1} |\sigma_i|^q \ln \frac{1}{\sigma_i} \geq \sum_{i \geq 2} \frac{1}{i \ln i} = \infty$$

and therefore

$$\sum_{i \geq 1} |\zeta_i \sigma_i|^q = \infty \quad \text{a.s.}$$

This means that the limit random element $Z = \sum_{i \geq 1} \zeta_i \sigma_i e_i$ does not exist in the space l_q.

It turns out that the condition $\alpha > q$ can be dropped in Theorem 1 and relation (7) can be proved for general Banach spaces with an unconditional basis in the case of the extremal type II distributions and under additional conditions on the sequence (η_i).

Assume that $F \in D(G_2)$. Then

$$1 - F(x) = L(x)x^{-\alpha}$$

where $L(x)$ is a slowly varying function at infinity, that is,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1$$

for all $\lambda > 0$ (see [2, 3]).

Every slowly varying function $L(x)$ can be represented in the form

$$L(x) = \exp\left(g(x) + \int_u^x \frac{\varepsilon(t)}{t}\,dt\right) \tag{29}$$

where $x \ge u > 0$, $g(x)$ is a bounded function on $[u,\infty)$ such that $g(x) \to c$ for some $|c| < \infty$, and $\varepsilon(x)$ is a continuous function on $[u,\infty)$ such that $\varepsilon(x) \to 0$ as $x \to \infty$ (see [9]).

Proposition 1. *Let B be a Banach space with an unconditional basis (e_i) and let the series* (4)–(6) *converge. Assume that (η_i) is a sequence of independent identically distributed random variables with the distribution function $F(x)$, $F \in D(G_2)$, and that the function $\varepsilon(x)$ in representation* (29) *is monotone. Then relation* (7) *holds.*

Proof. For the extremal type II distributions, we have

$$a_n = 0, \qquad b_n = \frac{1}{\gamma_n}, \qquad \gamma_n = \inf\left\{x\colon F(x) \ge 1 - \frac{1}{n}\right\}$$

(see [3]). Thus

$$\mathsf{P}(|b_n(z_n - a_n)| > x) = \mathsf{P}\left(\frac{|z_n|}{\gamma_n} > x\right) \le \mathsf{P}(z_n > \gamma_n x) + \mathsf{P}(\eta_1 < -x) \tag{30}$$

and

$$\begin{aligned} \mathsf{P}(z_n > \gamma_n x) &\le n(1 - F(\gamma_n x)) \le \frac{1 - F(\gamma_n x)}{1 - F(\gamma_n)} \\ &\le Cx^{-\alpha} \exp\left(\int_{\gamma_n}^{\gamma_n x} \frac{\varepsilon(t)}{t}\,dt\right) \end{aligned} \tag{31}$$

for sufficiently large n_0 and all $n \ge n_0$ and $x \ge u$.

If $\varepsilon(x)$ increases to 0, we have $\varepsilon(x) \le 0$ on (u,∞). Thus (28) with $q = \alpha$ and (31) yield

$$\mathsf{P}(z_n > \gamma_n x) \le Cx^{-\alpha} \le C_1\,\mathsf{P}(\zeta > x).$$

If $\varepsilon(x)$ is a decreasing function, then

$$\int_{\gamma_n}^{\gamma_n x} \frac{\varepsilon(t)}{t}\,dt \le \int_u^x \frac{\varepsilon(t)}{t}\,dt + C$$

for $n \ge n_0$ and $x \ge u$. This together with (31) implies that

$$\mathsf{P}(z_n > \gamma_n x) \le C_2 x^{-\alpha} L(x) \le C_2\,\mathsf{P}(\eta_1 > x).$$

Thus the following estimate holds in both cases:

$$\mathsf{P}\left(\frac{|z_n|}{\gamma_n} > x\right) \le C\big(\mathsf{P}(\zeta > x) + \mathsf{P}(|\eta_1| > x)\big) \le C_1\,\mathsf{P}\left(|\zeta| + |\eta_1| > x\right). \tag{32}$$

Here we used inequality (30).

Put

$$z'_{ni} = \varepsilon_i \frac{|z_{ni}|}{\gamma_n}, \qquad \xi_i = \varepsilon_i(|\eta_i| + |\zeta_i|)$$

where (ε_i) is a sequence of independent symmetric Bernoulli random variables such that $\mathsf{P}(\varepsilon_i = +1) = \mathsf{P}(\varepsilon_i = -1) = 1/2$. Assume that the sequence (ε_i) does not depend on (η_i) and (ζ_i).

Then it follows from (32) that

$$\mathsf{P}(|z'_{ni}| > x) \le C_1 \, \mathsf{P}(|\xi_i| > x).$$

The random variables z'_{ni} and ξ_i, $i \ge 1$, are independent and symmetric. Thus

$$\mathsf{P}\left(\left\|\sum_{i=m_1}^{m_2} z'_{ni}\sigma_i e_i\right\| > x\right) \le 2C_1 \, \mathsf{P}\left(\left\|\sum_{i=m_1}^{m_2} \xi_i\sigma_i e_i\right\| > x\right) \tag{33}$$

for all $m_2 > m_1$ and $x > 0$ (see also [11]).

Since the series (5) and (6) converge, estimate (3) implies the almost sure convergence of the series $\sum_i \xi_i\sigma_i e_i$ in the norm of the space B. Hence

$$\lim_{m\to\infty} \mathsf{P}\left(\left\|\sum_{i=m}^{\infty} \xi_i\sigma_i e_i\right\| > \delta\right) = 0 \tag{34}$$

for all $\delta > 0$.

Applying (3) once more, we obtain

$$\mathsf{P}\left(\left\|\sum_{i=m}^{\infty} b_n(z_{ni} - a_n)\sigma_i e_i\right\| > x\right) \le \mathsf{P}\left(\left\|\sum_{i=m}^{\infty} z'_{ni}\sigma_i e_i\right\| > K^{-1}x\right). \tag{35}$$

It remains to check conditions (i) and (ii) of Lemma 1 to prove Propositon 1. Condition (i) is obvious under the assumptions of Proposition 1. Now we apply estimates (33)–(35) and prove condition (ii). □

Remark 1. Since the convergence of the series (6) implies condition (4), the latter can be dropped from the set of assumptions of Proposition 1.

Remark 2. In the case of a Gaussian random element X with independent components, relation (7) can be proved for an arbitrary Banach space with an unconditional basis (see [4]).

3. Example of the convergence of extremes to a degenerate law in the space $L_p(T)$

It is worth mentioning that the method applied in the proof of Theorem 1 is useful for some other problems on extremes. For example, when studying the convergence of integral functionals of extremes, this method allows one to drop conditions on the rate of decay of the distribution function $F(x)$ at $-\infty$ (condition b) of Theorem 2 in [5]).

Here we give a result of this type. For brevity we do not provide the proof and restrict ourselves to the case of the space $L_p(T)$, $p \ge 1$.

By $X = \{X(t), t \in T\}$ we denote a stochastic process of the following form:

$$X(t) = \sigma(t)\tilde{X}(t), \qquad t \in T, \tag{36}$$

$$\text{and for all } t \in T: \qquad \mathsf{P}(\tilde{X}(t) < s) = F(s).$$

We assume that the functions $X(t)$, $\tilde{X}(t)$, and $\sigma(t)$ are measurable. For a sequence

$$X_k = \{X_k(t), t \in T\}, \qquad k \ge 1,$$

of independent copies of X, we put

$$\begin{aligned} Z_n &= \left\{ Z_n(t) = \max_{1\le k\le n} X_k(t), t \in T \right\}, \\ U_n &= \{U_n(t) = b_n(Z_n(t) - a_n\sigma(t)), t \in T\}, \\ \mathfrak{S} &= \{\sigma(t), t \in T\}. \end{aligned}$$

Proposition 2. *Let $X = \{X(t), t \in T\}$ be a measurable stochastic process represented in the form* (36). *Assume that $X \in L_p(T)$ almost surely, $\mathfrak{S} \in L_p(T)$, and*

(i) *$F \in D(G_k)$ where $G_k(x)$ is one of the extremal type distributions listed in* (2), *and $\alpha > p$ if $k = 2$;*
(ii) *for almost all $(t, s) \in T \times T$, the random variables $X(t)$ and $X(s)$ satisfy equality* (8).

Then

$$\int_T |U_n(t)|^p \,\mu(dt) \xrightarrow{\mathsf{P}} \mathsf{E}\,|\zeta|^p \int_T |\sigma(t)|^p \,\mu(dt)$$

as $n \to \infty$, where $Y_n \xrightarrow{\mathsf{P}} Y$ means the convergence in probability.

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