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THE ITÔ FORMULA FOR FRACTIONAL BROWNIAN FIELDS

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ABSTRACT. We prove the existence of the stochastic integral of the second kind constructed with respect to Hölder fields, in particular, with respect to fractional Brownian fields, and derive the Itô formula for a linear combination of fractional Brownian fields with different Hurst indices $H_i \in (\frac{1}{2}, 1)$, $i = 1, 2$.

1. Main notation and auxiliary results

Let $\mathbb{R}^2_+$ be the nonnegative quadrant and $t = (t_1, t_2) \in \mathbb{R}^2_+$. Fractional Brownian fields on the plane can be defined in different ways. We consider the fields with the pointwise fractional Brownian property.

Definition 1. A random field $B = \{B_t, t \in \mathbb{R}^2_+\}$ is called a fractional Brownian field with the Hurst indices H_1 and H_2, $H_i \in (0, 1)$, if

(1) B_t is Gaussian such that $B_t = 0$ for $t \in \partial\mathbb{R}^2_+$;
(2) $\mathsf{E}\, B_t = 0$, $\mathsf{E}\, B_t B_s = \frac{1}{4} \prod_{i=1,2} \big(t_i^{2H_i} + s_i^{2H_i} - |t_i - s_i|^{2H_i}\big)$;
(3) the paths of B are almost surely continuous;
(4) the increments $\Delta_s B_t := B_t - B_{s_1 t_2} - B_{t_1 s_2} + B_s$, $s \le t$, are stationary.

Note that for any fixed $t_2 > 0$ the process $B_{\cdot\, t_2}$ is a fractional Brownian motion with the Hurst index H_1, and for any $t_1 > 0$ the process $B_{t_1 \cdot}$ is also a fractional Brownian motion, but with the Hurst index H_2. We consider the case of $H_i \in (\frac{1}{2}, 1)$, $i = 1, 2$.

Consider a rectangle $P = \bigoplus_{i=1,2} [a_i, b_i] \subset \mathbb{R}^2_+$.

Definition 2. Let $0 < \lambda_1 \le 2$ and $0 < \lambda_2 \le 2$. A function $f\colon P \to \mathbb{R}$ belongs to the class $\mathcal{H}^{\lambda_1,\lambda_2}(P)$ (in other words, f is Hölder continuous of orders λ_1 and λ_2 on P) if there exists a constant $C > 0$ such that

$$\begin{gathered} |\Delta_s f(t)| \le C \prod_{i=1,2} (t_i - s_i)^{\lambda_i}, \\ |f(t) - f(s_1, t_2)| \le C (t_1 - s_1)^{\lambda_1}, \\ |f(t) - f(t_1, s_2)| \le C (t_2 - s_2)^{\lambda_2} \end{gathered} \tag{1}$$

for all $s, t \in P$ such that $s \le t$.

We write $f \in \mathcal{H}^{\lambda_1,\lambda_2}(\mathbb{R}^2_+)$ if inequalities (1) hold for all $s, t \in \mathbb{R}^2_+$.

According to the results of [1], the paths of the field B almost surely belong to the class $\mathcal{H}^{H_1-\varepsilon_1, H_2-\varepsilon_2}(\mathbb{R}^2_+)$ for any $0 < \varepsilon_i < H_i$, $i = 1, 2$.

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Definition 3. Let $f\colon P \to \mathbb{R}$. Assume that $\alpha_1, \alpha_2 \in (0,1)$. Then

$$\begin{aligned} D_{a+}^{\alpha_1\alpha_2} f(s) = {} & \big(\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\big)^{-1} \\ & \times \bigg(f(s)(s_1-a_1)^{-\alpha_1}(s_2-a_2)^{-\alpha_2} \\ & + \alpha_1(s_2-a_2)^{-\alpha_2} \int_{a_1}^{s_1} (f(s)-f(u,s_2))(s_1-u)^{-1-\alpha_1}\,du \\ & + \alpha_2(s_1-a_1)^{-\alpha_1} \int_{a_2}^{s_2} (f(s)-f(s_1,v))(s_2-v)^{-1-\alpha_2}\,dv \\ & + \alpha_1\alpha_2 \int_{[a,s]} \Delta_{(u,v)} f(s)(s_1-u)^{-1-\alpha_1}(s_2-v)^{-1-\alpha_2}\,du\,dv \bigg), \\ & s \in P, \end{aligned} \tag{2}$$

and

$$\begin{aligned} D_{b-}^{\alpha_1\alpha_2} f(s) = {} & \big(\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\big)^{-1} \\ & \times \bigg(f(s)(b_1-s_1)^{-\alpha_1}(b_2-s_2)^{-\alpha_2} \\ & + \alpha_1(b_2-s_2)^{-\alpha_2} \int_{s_1}^{b_1} (f(s)-f(u,s_2))(u-s_1)^{-1-\alpha_1}\,du \\ & + \alpha_2(b_1-s_1)^{-\alpha_1} \int_{s_2}^{b_2} (f(s)-f(s_1,v))(v-s_2)^{-1-\alpha_2}\,dv \\ & + \alpha_1\alpha_2 \int_{[s,b]} \Delta_s f(u,v)(u-s_1)^{-1-\alpha_1}(v-s_2)^{-1-\alpha_2}\,du\,dv \bigg), \\ & s \in P, \end{aligned} \tag{3}$$

are called the forward and backward derivatives, respectively, of the function f of orders α_1 and α_2 on the rectangle P if the right-hand sides of (2) and (3) exist for any $s \in P$.

Definition 4. Let $f \in L_1(P)$. Assume that $\alpha_1, \alpha_2 \in (0,1)$. Then

$$I_{a+}^{\alpha_1\alpha_2} f(s) = \big(\Gamma(\alpha_1)\Gamma(\alpha_2)\big)^{-1} \int_{[a,s]} f(u,v)(s_1-u)^{\alpha_1-1}(s_2-v)^{\alpha_2-1}\,du\,dv$$

and

$$I_{b-}^{\alpha_1\alpha_2} f(s) = \big(\Gamma(\alpha_1)\Gamma(\alpha_2)\big)^{-1} \int_{[s,b]} f(u,v)(u-s_1)^{\alpha_1-1}(v-s_2)^{\alpha_2-1}\,du\,dv, \quad s \in P,$$

are called the forward and backward integrals, respectively, of the function f of orders α_1 and α_2 on the rectangle P.

We write

$$f \in I_{a+}^{\alpha_1\alpha_2}(L_p(P)) \quad (\text{respectively, } \; f \in I_{b-}^{\alpha_1\alpha_2}(L_p(P)))$$

for some $p \ge 1$ if f can be represented as

$$f = I_{a+}^{\alpha_1\alpha_2}\varphi \quad (\text{respectively, } \; f = I_{b-}^{\alpha_1\alpha_2}\varphi)$$

for some $\varphi \in L_p(P)$. The corresponding classes are denoted by

$$I_{a_i+}^{\alpha}(L_p[a_i,b_i]) \quad \text{and} \quad I_{b_i-}^{\alpha}(L_p[a_i,b_i]),$$

respectively. (The properties of one-parameter fractional integrals and derivatives can be found in [2].)

Definition 5. Let $f, g\colon P \to \mathbb{R}$ and $g \in C(P)$. Then

$$\begin{aligned}\int_P f\,dg = &\int_P D^{\alpha_1\alpha_2}_{a+} f_{a+}(u,v) D^{1-\alpha_1 1-\alpha_2}_{b-} g_{b-}(u,v)\,du\,dv\\ &+ \int_{a_1}^{b_1} D^{\alpha_1}_{a_1+} f_{a_1+}(u,a_2) D^{1-\alpha_1}_{b_1-} \Delta^1_{(u,a_2)} g_{b_1-}(u,b_2)\,du\\ &+ \int_{a_2}^{b_2} D^{\alpha_2}_{a_2+} f_{a_2+}(a_1,v) D^{1-\alpha_2}_{b_2-} \Delta^2_{(a_1,v)} g_{b_2-}(b_1,v)\,dv + f(a)\Delta_a g(b)\end{aligned} \tag{4}$$

is called the generalized Lebesgue–Stieltjes integral of the function f with respect to the function g on the rectangle P where

$$\begin{aligned} f_{a+}(u,v) &= \Delta_a f(u,v), & g_{b-}(u,v) &= \Delta_{(u,v)} g(b),\\ f_{a_1+}(u,a_2) &= \Delta^1_a f(u,a_2), & f_{a_2+}(a_1,v) &= \Delta^2_a f(a_1,v),\\ g_{b_1-}(u,b_2) &= -\Delta^1_{(u,b_2)} g(b), & g_{b_1-}(u,a_2) &= -\Delta^1_{(u,a_2)} g(b_1,a_2),\\ g_{b_2-}(b_1,v) &= -\Delta^2_{(b_1,v)} g(b), & g_{b_2-}(a_1,v) &= -\Delta^2_{(a_1,v)} g(a_1,b_2),\end{aligned}$$

and the increments Δ^1 and Δ^2 are defined by

$$\Delta^1_s f(t_1,s_2) = f(t_1,s_2) - f(s)$$

and

$$\Delta^2_s f(s_1,t_2) = f(s_1,t_2) - f(s),$$

respectively; $D^{\alpha_1}_{a_1+}$ and $D^{\alpha_2}_{a_2+}$ are forward derivatives, and $D^{1-\alpha_1}_{b_1-}$ and $D^{1-\alpha_2}_{b_2-}$ are backward one-parameter fractional derivatives of the corresponding orders defined according to [2].

In what follows we need the following results (see [3] for the proofs).

(1) The integral defined according to Definition 5 is well defined in the sense that the right-hand side of (4) does not depend on the choice of $\alpha_1, \alpha_2 \in (0,1)$.

(2) The right-hand side of (4) is well defined if

$$\begin{aligned} &f_{a+} \in I^{\alpha_1\alpha_2}_{a+}(L_p(P)), \qquad g_{b-} \in I^{1-\alpha_1,1-\alpha_2}_{b-}(L_q(P)),\\ &f_{a_i+}(\cdot,a_j) \in I^{\alpha_i}_{a_i+}(L_p[a_i,b_i]), \qquad g_{b_i-}(\cdot,b_j) \in I^{1-\alpha_i}_{b_i-}(L_q[a_i,b_i]),\\ &i=1,2, \qquad j = 3-i, \qquad \frac{1}{p}+\frac{1}{q} \le 1, \qquad \alpha_i \in (0,1).\end{aligned}$$

(3) The integral $\int_P f\,dg$ is an additive function of sets P.

(4) If $g \in \mathcal{H}^{\lambda_1,\lambda_2}(P)$, then $g_{b-} \in I^{\varepsilon_1\varepsilon_2}_{b-}(L_q(P))$ for all $q \ge 1$ and $0 < \varepsilon_i < \lambda_i$, $i=1,2$. Moreover, $D^{\varepsilon_1\varepsilon_2}_{b-} g_{b-} \in \mathcal{H}^{\lambda_1-\varepsilon_1,\lambda_2-\varepsilon_2}(P)$. In particular, $D^{\varepsilon_1\varepsilon_2}_{b-} g_{b-} \in C(P)$ and there exists a constant $C>0$ such that

$$\sup_{s\in P} \left| D^{\varepsilon_1\varepsilon_2}_{b-} g_{b-}(s) \right| \le C.$$

(5) Let $\pi_n(P)$, $n \ge 1$, be a sequence of uniform partitions of the rectangle P, that is,

$$\pi_n(P) = \left\{ s^{ik}_n = (s^{in}_1, s^{kn}_2), s^{in}_1 = a_1 + \frac{i(b_1-a_1)}{n},\right.$$
$$\left. s^{kn}_2 = a_2 + \frac{k(b_2-a_2)}{n}, 0 \le i,k \le n \right\}.$$

Assume that $f, g\colon P \to \mathbb{R}$ are nonrandom functions. Consider the sequence of integral sums corresponding to the left endpoints of the intervals of partitions:

$$S_n = \sum_{i,k=0}^{n-1} f(s_n^{ik})\Delta_{s_n^{ik}} g\left(s_n^{i+1,k+1}\right).$$

It is proved in Theorem 7 of [3] that both the generalized Lebesgue–Stieltjes integral $\int_P f\,dg$ and limit $\lim_{n\to\infty} S_n$ exist, and moreover they are equal for functions

$$f \in \mathcal{H}^{\lambda_1,\lambda_2}(P), \qquad g \in \mathcal{H}^{\mu_1,\mu_2}(P), \qquad \lambda_i + \mu_i > 1,\ i = 1,2.$$

In particular, if

$$g(t) = \sum_{l=1}^{m} a_l B_t^{H_1^l, H_2^l}, \qquad H_i^l \in \left(\frac{1}{2}, 1\right),\ a_l \in \mathbb{R},$$

is a linear combination of fractional Brownian fields, the function $F\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous almost surely, and $f(t) = F(g(t))$, then the integral

$$\int_P F(g)\,dg$$

exists almost surely and is equal to the limit of integral sums corresponding to the left endpoints of the partition intervals.

2. The existence of integrals of the second kind with respect to fractional Brownian fields

First we prove the corresponding result for nonrandom functions. Fix a rectangle $P = \bigoplus_{i=1,2}[0, T_i]$ for $T = (T_1, T_2) \in \mathbb{R}_+^2$ and consider a sequence of uniform partitions

$$\widehat{\pi}_n(P) = \pi_{2^n}(P) = \left\{s_n^{ik} = (T_1 i \cdot 2^{-n}, T_2 k \cdot 2^{-n}),\ 0 \le i, k \le 2^n\right\}.$$

Let the functions $f, g\colon P \to \mathbb{R}$ be such that

$$f \in \mathcal{H}^{\lambda_1,\lambda_2}(P), \qquad g \in \mathcal{H}^{\mu_1,\mu_2}(P), \qquad f\big|_{\partial\mathbb{R}_+^2} = f_0 \in \mathbb{R}, \qquad g\big|_{\partial\mathbb{R}_+^2} = g_0 \in \mathbb{R}.$$

Consider the sequence of integral sums of the second kind,

$$\widetilde{S}_n = \sum_{i,k=0}^{2^n-1} f\left(s_n^{ik}\right) \Delta_{ik}^1 g \Delta_{ik}^2 g,$$

where $\Delta_{ik}^1 g = g(s_n^{i+1k}) - g(s_n^{ik})$ and $\Delta_{ik}^2 g = g(s_n^{ik+1}) - g(s_n^{ik})$.

Theorem 1. *Let λ_i, $\mu_i > \frac{1}{2}$, and $\lambda_i + \mu_1 + \mu_2 > 2$, $i = 1,2$. Then the sequence $\{\widetilde{S}_n, n \ge 1\}$ of integral sums of the second kind has the limit $\lim_{n\to\infty} \widetilde{S}_n =: \widetilde{S}$. This limit is called the integral of the second kind with respect to the functions f and g, and is denoted by*

$$\widetilde{S} = \int_P f\,d_1 g\,d_2 g.$$

Proof. Without loss of generality we assume that $T_1 = T_2 = 1$. Let $m > n$. We represent the difference $S_n - S_m$ as

$$S_n - S_m = S_n - S_{n,m}^1 + S_{n,m}^1 - S_m$$

where

$$S_{n,m}^1 = \sum_{i,k=0}^{2^n-1} \sum_{r \in A_i} f(r2^{-m}, k2^{-n})\big(g((r+1)2^{-m}, k2^{-m}) - g(r2^{-m}, k2^{-n})\big) \times \left(g(r2^{-m}, (k+1)2^{-n}) - g(r2^{-m}, k2^{-n})\right)$$

and

$$A_i = \left\{r\colon i2^{m-n} \le r < (i+1)2^{m-n}\right\}.$$

The differences $S_n - S^1_{n,m}$ and $S^1_{n,m} - S_m$ have the same structure and can be estimated in a similar way, hence we estimate only $S_n - S^1_{n,m}$. We have

$$\left|S_n - S^1_{n,m}\right| \le \left|\Delta^1_{n,m} S\right| + \left|\Delta^2_{n,m} S\right|$$

where

$$\Delta^1_{n,m} S = \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} f\left(s^{ik}_n\right) \Delta_{ikr} g \Delta^1_{kr} g,$$

$$\Delta^2_{n,m} S = \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \Delta^1_{ikr} f \Delta^1_{kr} g \Delta^2_{kr} g,$$

and

$$\Delta_{ikr} g = \Delta_{s^{ik}_n} g\left(r2^{-m}, (k+1)2^{-n}\right),$$
$$\Delta^1_{kr} g = \Delta^1_{(r2^{-m},k2^{-n})} g\left((r+1)2^{-m}, k2^{-n}\right),$$
$$\Delta^1_{ikr} f = \Delta^1_{s^{ik}_n} f\left(r2^{-m}, k2^{-n}\right),$$
$$\Delta^2_{kr} g = \Delta^2_{(r2^{-m},k2^{-n})} g\left(r2^{-m}, (k+1)2^{-n}\right).$$

Now we rewrite $\Delta^1_{n,m} S$ as follows:

$$\Delta^1_{n,m} S = \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} f(s^{ik}_n) \Delta_{kr} g \Delta^1_{ikr} g$$

where

$$\Delta_{kr} g = \Delta_{(r2^{-m},k2^{-n})} g\left((r+1)2^{-m}, (k+1)2^{-n}\right),$$
$$\Delta^1_{ikr} g = \Delta^1_{(r2^{-m},k2^{-n})} g\left(s^{i+1k}_n\right).$$

Here the increments $\Delta_{kr} g$ correspond to the rectangles

$$\Delta_{kr} = \left[r2^{-m}, (r+1)2^{-m}\right[\times \left[k2^{-n}, (k+1)2^{-n}\right[$$

such that

$$\Delta_{kr} \cap \Delta_{k'r'} = \varnothing$$

if $(k,r) \ne (k',r')$ and $\bigcup_{k,r} \Delta_{kr} = [0,1[^2$. Therefore the sum $\Delta^1_{n,m} S$ can be represented as the generalized Lebesgue–Stieltjes integral

$$\Delta^1_{n,m} S = \int_P \overline{f}_{m,n}\, dg$$

where $\overline{f}_{m,n}$ is a step function,

$$\overline{f}_{m,n}(s) = \sum_{i,k,r} f(s^{ik}_n) \Delta^1_{ikr} g \cdot I\left\{s \in \Delta_{kr}\right\}.$$

Since f and g are constant on $\partial\mathbb{R}^2_+$, we obtain from (4) that

$$\int_P \overline{f}_{m,n}\, dg = \int_P D^{\alpha_1\alpha_2}_{0+} (\overline{f}_{m,n})_{0+}(s) D^{1-\alpha_1 1-\alpha_2}_{1-} g_{1-}(s)\, ds + \overline{f}_{m,n}(0)\Delta_0 g(1) \tag{5}$$

where $1 = (1,1) \in \mathbb{R}^2_+$, $0 = (0,0) \in \mathbb{R}^2_+$, and $\alpha_i < \lambda_i$, $1-\alpha_i < \mu_i$, $i = 1,2$. As before

$$D^{1-\alpha_1 1-\alpha_2}_{1-} g_{1-} \in \mathcal{H}^{\mu_1+\alpha_1-1,\mu_2+\alpha_2-1}(P)$$

by Theorem 4 in [3] for α_i chosen in the specified way. In particular, there exists a constant $C > 0$ such that

$$\left|D_{1-}^{1-\alpha_1 1-\alpha_2} g_{1-}(s)\right| \le C, \qquad s \in P.$$

Further, $\overline{f}_{m,n}(0) = f(0)(g(2^{-m},0) - g(0)) \to 0$ as $m \to \infty$. Hence (5) implies

$$|\Delta^1_{n,m} S| \to 0$$

if

$$\int_P \left|D_{0+}^{\alpha_1\alpha_2} (\overline{f}_{m,n})_{0+}(s)\right| ds \to 0, \qquad n, m \to \infty.$$

According to equality (2) for fractional derivatives, it is sufficient to prove that

$$\int_P |\varphi_{m,n,k}(s)|\, ds \to 0$$

as $m, n \to \infty$, $k = 1, 2, 3, 4$, where

$$\varphi_{m,n,1}(s) = (\overline{f}_{m,n})_{0+}(s) s_1^{-\alpha_1} s_2^{-\alpha_2},$$

$$\varphi_{m,n,2}(s) = s_2^{-\alpha_2} \int_0^{s_1} \big((\overline{f}_{m,n})_{0+}(s) - (\overline{f}_{m,n})_{0+}(u, s_2)\big)(s_1 - u)^{-1-\alpha_1}\, du,$$

$$\varphi_{m,n,3}(s) = s_1^{-\alpha_1} \int_0^{s_2} \big((\overline{f}_{m,n})_{0+}(s) - (\overline{f}_{m,n})_{0+}(s_1, v)\big)(s_2 - v)^{-1-\alpha_2}\, dv,$$

$$\varphi_{m,n,4}(s) = \int_{[0,s]} \Delta_{(u,v)} \overline{f}_{m,n}(s)(s_1 - u)^{-1-\alpha_1}(s_2 - v)^{-1-\alpha_2}\, du\, dv.$$

Note that the function f is bounded on P, that is, $|f(s_n^{ik})| \le C$ for some constant $C > 0$. Since the functions f and g are Hölder continuous, we get

$$|\varphi_{m,n,1}(s)| \le \left|f(s_n^{ik})\Delta^1_{ikr} g - f(0)\Delta^1_{0k0} g\right| s_1^{-\alpha_1} s_2^{-\alpha_2} \le C 2^{-m\mu_1} s_1^{-\alpha_1} s_2^{-\alpha_2},$$

whence

$$\int_P |\varphi_{m,n,1}(s)|\, ds \to 0 \quad \text{as } m, n \to \infty.$$

Further, if $s \in \Delta_{kr}$, then

$$\begin{aligned}
|\varphi_{m,n,2}(s)| \le{}& s_2^{-\alpha_2} \int_0^{k2^{-n}} \left|\overline{f}_{m,n}(s) - \overline{f}_{m,n}(u, s_2)\right| (s_1 - u)^{-1-\alpha_1}\, du \\
&+ s_2^{-\alpha_2} \int_{k2^{-n}}^{r2^{-m}} \left|\overline{f}_{m,n}(s) - \overline{f}_{m,n}(u, s_2)\right| (s_1 - u)^{-1-\alpha_1}\, du \\
\le{}& s_2^{-\alpha_2} \int_0^{k2^{-n}} \left(|\overline{f}_{m,n}(s)| + |\overline{f}_{m,n}(u, s_2)|\right) (s_1 - u)^{-1-\alpha_1}\, du \\
&+ C s_2^{-\alpha_2} \int_{k2^{-n}}^{r2^{-m}} \left|f(s_n^{ik})\right| (s_1 - u + 2^{-m})^{\mu_1} (s_1 - u)^{-1-\alpha_1}\, du \\
\le{}& C s_2^{-\alpha_2} 2^{-n\mu_1} (s_1 - k2^{-n})^{-\alpha_1} + C s_2^{-\alpha_2} (s_1 - r2^{-m})^{\mu_1 - \alpha_1} \\
&+ C s_2^{-\alpha_2} 2^{-m\mu_1} (s_1 - r2^{-m})^{-\alpha_1}.
\end{aligned}$$

Therefore

$$\begin{aligned}\int_P |\varphi_{m,n,2}(s)|\,ds &\le C\sum_{i,k=0}^{2^n-1}\sum_{r\in A_i}\biggl(2^{-n\mu_1}\int_{\Delta_{kr}} s_2^{-\alpha_2}(s_1-i2^{-n})^{-\alpha_1}\,ds_1\,ds_2\\
&\qquad+\int_{\Delta_{kr}} s_2^{-\alpha_2}(s_1-r2^{-m})^{\mu_1-\alpha_1}\,ds_1\,ds_2\\
&\qquad+2^{-m\mu_1}\int_{\Delta_{kr}} s_2^{-\alpha_2}(s_1-r2^{-m})^{-\alpha_1}\,ds_1\,ds_2\biggr)\\
&\le C\int_0^1 s_2^{-\alpha_2}\,ds_2\biggl(2^{-n(\mu_1-1)}\int_{2^{-n}}^{2^{-n+1}}(s_1-2^{-n})^{-\alpha_1}\,ds_1\\
&\qquad+2^m\int_{2^{-m}}^{2^{-m+1}}(s_1-2^{-m})^{\mu_1-\alpha_1}\,ds_1\\
&\qquad+2^{-m(\mu_1-1)}\int_{2^{-m}}^{2^{-m+1}}(s_1-2^{-m})^{-\alpha_1}\,ds_1\biggr)\\
&\le C\bigl(2^{n(\alpha_1-\mu_1)}+2^{m(\alpha_1-\mu_1)}\bigr)\to 0,\qquad m,n\to\infty,\end{aligned}$$

for α_1 such that

$$\alpha_1<\frac{1}{2}<\mu_1.$$

Further, if $s\in\Delta_{kr}$, then the estimate $|\overline{f}_{m,n}(s)|\le C2^{-n\mu_1}$, $s\in P$, implies that

$$|\varphi_{m,n,3}(s)|\le Cs_1^{-\alpha_1}\int_0^{k2^{-n}}2^{-n\mu_1}(s_2-v)^{-1-\alpha_2}\,dv,$$

whence

$$\begin{aligned}\int_P|\varphi_{m,n,3}(s)|\,ds &\le C\int_0^1 s_1^{-\alpha_1}\,ds_1\cdot\sum_{k=0}^{2^n-1}\int_{k2^{-n}}^{(k+1)2^{-n}}\,ds_2\int_0^{k2^{-n}}(s_2-v)^{-1-\alpha_2}2^{-n\mu_1}\\
&\le C2^{n(1-\mu_1)}\int_{2^{-n}}^{2^{-n+1}}(s_2-2^{-n})^{-\alpha_2}\,ds_2=C2^{n(\alpha_2-\mu_1)}\to 0,\qquad n\to\infty,\end{aligned}$$

if $\alpha_2<\frac{1}{2}<\mu_1$.

It remains to prove that $\int_P|\varphi_{m,n,4}(s)|\,ds\to 0$ as $m,n\to\infty$. We have

$$\begin{aligned}|\varphi_{m,n,4}(s)| &\le C2^{-n\mu_1}\int_0^{k2^{-n}}\int_0^{i2^{-n}}(s_1-u)^{-1-\alpha_1}(s_2-v+2^{-n})^{\mu_2\wedge\lambda_2-1-\alpha_2}\,du\,dv\\
&\quad+C\int_0^{k2^{-n}}\int_{i2^{-n}}^{r2^{-m}}(s_1-u+2^{-m})^{\mu_1-1-\alpha_1}(s_2-v+2^{-n})^{\mu_2\wedge\lambda_2-1-\alpha_2}\,du\,dv\\
&\le C2^{-n\mu_1}(s_1-k2^{-n})^{-\alpha_1}(s_2-(i-1)2^{-n})^{\mu_2\wedge\lambda_2-\alpha_2}\\
&\quad+C(s_1-k2^{-n}+2^{-m})^{\mu_1-\alpha_1}(s_2-r2^{-m}+2^{-n})^{\mu_2\wedge\lambda_2-\alpha_2},\end{aligned}$$

whence

$$\int_P|\varphi_{m,n,4}(s)|\,ds\le C2^{n(\alpha_1+\alpha_2-\mu_1-\mu_2\wedge\lambda_2)}\to 0,\qquad m,n\to\infty.$$

Finally $|\Delta^1_{n,m}S|\to 0$ as $m,n\to\infty$.

Now we prove that $|\Delta^2_{n,m}S|\to 0$ as $m,n\to\infty$.

We rewrite $\Delta^2_{n,m}S$ in the form $\Delta^2_{n,m}S = \sum_{k=0}^{2^n-1} \Delta^{2,k}_{n,m}S$ where

$$\Delta^{2,k}_{n,m}S = \sum_{i=0}^{2^n-1} \sum_{r\in A_i} \Delta^1_{ikr} f \cdot \Delta^2_{kr} g \cdot \Delta^1_{kr} g.$$

The sum $\Delta^{2,k}_{n,m}S$ can be represented as the one-parameter generalized Lebesgue–Stieltjes integral

$$\int_0^1 \varphi_k(u)\, d_1 g(u, k2^{-n})$$

where $\varphi_k(u) = \Delta^1_{ikr} f \cdot \Delta^2_{kr} g$, if $r2^{-m} \le u < (r+1)2^{-m}$, $\varphi(0) = 0$. It follows from relation (22) in [4] that

$$\begin{aligned}\int_0^1 \varphi_k(u)\, d_1 g(u, k2^{-n}) &= \int_0^1 D^\alpha_{0+}(\varphi_k)_{0+}(u) D^{1-\alpha}_{1-} g_{1-}(u, k2^{-n})\, du \\ &= \int_0^1 D^\alpha_{0+}\varphi_k(u) D^{1-\alpha}_{1-} g_{1-}(u, k2^{-n})\, du\end{aligned}$$

where $1-\alpha < \mu_1$. It is evident that $\left|D^{1-\alpha}_{1-} g_{1-}(u, k2^{-n})\right| \le C$, so it is sufficient to establish the relation

$$\sum_{k=0}^{2^n-1} \int_0^1 \left|D^\alpha_{0+}\varphi_k(u)\right| du \to 0, \qquad m,n\to\infty. \tag{6}$$

Note that

$$D^\alpha_{0+}\varphi_k(u) = \big(\Gamma(1-\alpha)\big)^{-1}\left(\varphi_k(u)\cdot u^{-\alpha} + \alpha\int_0^u (\varphi_k(u)-\varphi_k(z))(u-z)^{-\alpha-1}\, dz\right). \tag{7}$$

Moreover, $|\varphi_k(u)| \le C2^{-n(\lambda_1+\mu_2)}$ and

$$\sum_{k=0}^{2^n-1} \int_0^1 |\varphi_k(u)| u^{-\alpha}\, du \le C\int_0^1 u^{-\alpha}\, du \cdot 2^{n(1-\lambda_1-\mu_2)} \to 0, \qquad n\to\infty.$$

If $i2^{-n} \le r2^{-m} \le u < (r+1)2^{-m} \le (i+1)2^{-n}$, then

$$\begin{aligned}\int_0^u (\varphi_k(u)-\varphi_k(z))(u-z)^{-\alpha-1}\, dz &= \int_0^{i2^{-n}} (\varphi_k(u)-\varphi_k(z))(u-z)^{-\alpha-1}\, dz \\ &\quad + \int_{i2^{-n}}^{r2^{-m}} (\varphi_k(u)-\varphi_k(z))(u-z)^{-\alpha-1}\, dz.\end{aligned} \tag{8}$$

Note that $|\varphi_k(u)-\varphi_k(z)| \le |\varphi_k(u)| + |\varphi_k(z)| \le C2^{-n(\lambda_1+\mu_2)}$. Thus

$$\begin{aligned}&\sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \int_{r2^{-m}}^{(r+1)2^{-m}} \left|\int_0^{i2^{-n}} (\varphi_k(u)-\varphi_k(z))(u-z)^{-\alpha-1}\, dz\right| du \\ &\quad\le C\sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} 2^{-n(\lambda_1+\mu_2)} \int_{r2^{-m}}^{(r+1)2^{-m}} (u-i2^{-n})^{-\alpha}\, du \\ &\quad\le C2^{n(1+\alpha-\lambda_1-\mu_2)} \to 0, \qquad n\to\infty,\end{aligned} \tag{9}$$

since one can choose the number $\alpha > 1-\mu_1$ such that $1+\alpha-\lambda_1-\mu_2 < 0$ if $\lambda_1+\mu_1+\mu_2 > 2$.

If $i2^{-n} \le z \le u \le (r+1)2^{-m}$, then $|\varphi_k(u) - \varphi_k(z)| \le 2^{-n\mu_2}(u - z + 2^{-m})^{\lambda_1}$ and therefore

$$
\begin{aligned}
&\sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \int_{r2^{-m}}^{(r+1)2^{-m}} \left| \int_{i2^{-n}}^{r2^{-m}} (\varphi_k(u) - \varphi_k(z))(u-z)^{-\alpha-1}\, dz \right| du \\
&\quad \le \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} 2^{-n\mu_2} \int_{r2^{-m}}^{(r+1)2^{-m}} \int_{i2^{-n}}^{r2^{-m}} (u - r + 2^{-m})^{\lambda_1}(u-z)^{-\alpha-1}\, dz\, du \qquad (10)\\
&\quad \le C2^{n(1-\mu_2)}2^{m(\alpha-\lambda_1)} \le C2^{m(1+\alpha-\lambda_1-\mu_2)} \to 0, \qquad m \to \infty.
\end{aligned}
$$

Relations (7)–(10) imply (6). Theorem 1 is proved. □

Remark 1. If $f(s) = C$ for some constant C, then $\Delta^2_{n,m}S = 0$. It is easily seen from the estimates for $\Delta^1_{n,m}S$ that Theorem 1 remains valid if $\lambda_i > \frac{1}{2}$ and $\mu_i > \frac{1}{2}$, $i = 1, 2$.

Now we improve Theorem 1 for the case of related fields f and g, that is, we assume that there exists a function F such that $f(t) = F(g(t))$ and

$$
F\colon \mathbb{R} \to \mathbb{R}, \qquad F \in C^3(\mathbb{R}), \qquad F''' \text{ is Lipschitz continuous.} \tag{11}
$$

First we prove an auxiliary result.

Theorem 2. *Let assumption* (11) *hold, and moreover* $g \in \mathcal{H}^{\lambda_1,\lambda_2}(\mathbb{R}^2_+)$ *for* $\lambda_i > \frac{1}{2}$, $i = 1, 2$. *Then*

a) $\lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} \Delta^1_{ik} f \Delta_{ik} g = \lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} \Delta^2_{ik} f \Delta_{ik} g = 0$ *where*

$$
\Delta_{ik} g = \Delta_{s_n^{ik}} g\left(s_n^{i+1k+1}\right), \quad \Delta^1_{ik} f = \Delta^1_{s_n^{ik}} f(s_n^{i+1k}), \quad \Delta^2_{ik} f = \Delta^2_{s_n^{ik}} f(s_n^{ik+1});
$$

b) $\lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} f_{ik} \Delta_{ik} g \Delta^1_{ik} g = \lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} f_{ik} \Delta_{ik} g \Delta^2_{ik} g = 0$ *where*

$$
f_{ik} = f\left(s_n^{ik}\right);
$$

c)

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} \Delta^1_{ik} f (\Delta^2_{ik} g)^2 &= \lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} (\Delta^1_{ik} f)^2 \Delta^2_{ik} g \\
&= \lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} F'''(g_{ik}) \Delta^1_{ik} g (\Delta^2_{ik} g)^2 = 0;
\end{aligned}
$$

d) $\lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \Delta^1_{ik} f (\Delta^1_{rk} g)^2 = \lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} f_{ik} \Delta_{rk} g \Delta^1_{rk} g = 0$ *where*

$$
\Delta^1_{rk} g = g((r+1)2^{-m}, k2^{-n}) - g(r2^m, k2^n), \quad \Delta_{rk} g = \Delta^1_{rk+1} g - \Delta^1_{rk} g.
$$

Proof. First we prove a). We compute only the first limit; the second can be treated similarly. As in the proof of Theorem 1 we write

$$
\sum_{i,k=0}^{2^n-1} \Delta_{s_n^{ik}} f\left(s_n^{i+1k}\right) \Delta_{ik} g = \int_P \widetilde{f}_n\, dg
$$

where $\widetilde{f}_n(s) = \Delta^1_{ik} f$ if $s \in \Delta_{ik} = [i2^{-n}, (i+1)2^{-n}[\times [k2^{-n}, (k+1)2^{-n}[$. Further,

$$
\begin{aligned}
\int_P \widetilde{f}_n\, dg &= \int_P D^{\alpha_1\alpha_2}_{0+} (\widetilde{f}_n)_{0+}(s) D^{1-\alpha_1 1-\alpha_2}_{1-} g_{1-}(s)\, ds \\
&= \int_P D^{\alpha_1\alpha_2}_{0+} \widetilde{f}_n(s) D^{1-\alpha_1 1-\alpha_2}_{1-} g_{1-}(s)\, ds
\end{aligned}
$$

and we need to prove the relation

$$\int_P \left|D_{0+}^{\alpha_1\alpha_2}(\widetilde{f}_n)_{0+}(s)\right| ds \to 0, \qquad n\to\infty,$$

which in turn follows from $\int_P |\varphi_{n,i}(s)|\,ds \to 0$ where

$$\begin{aligned}\varphi_{n,1}(s) &= s_1^{-\alpha_1}s_2^{-\alpha_2}\widetilde{f}_n(s),\\ \varphi_{n,2}(s) &= s_2^{-\alpha_2}\int_0^{s_1}\big(\widetilde{f}_n(s)-\widetilde{f}_n(u,s_2)\big)(s_1-u)^{-1-\alpha_1}\,du,\\ \varphi_{n,3}(s) &= s_1^{-\alpha_1}\int_0^{s_2}\big(\widetilde{f}_n(s)-\widetilde{f}_n(s_1,v)\big)(s_2-v)^{-1-\alpha_2}\,dv,\\ \varphi_{n,4}(s) &= \int_{[0,s]}\Delta_{(u,v)}\widetilde{f}_n(s)(s_1-u)^{-1-\alpha_1}(s_2-v)^{-1-\alpha_2}\,du\,dv.\end{aligned}$$

The convergence $\int_P |\varphi_{n,1}(s)|\,ds \to 0$ is obvious.

Next,

$$|\varphi_{n,2}(s)| \le s_2^{-\alpha_2}\int_0^{i2^{-n}}(s_1-u)^{-1-\alpha_1}\,du\cdot 2^{-n\lambda_1}$$

for $i2^{-n}\le s\le (i+1)2^{-n}$, whence

$$\int_P |\varphi_{n,2}(s)|\,ds \le C\int_0^1 s_2^{-\alpha_2}\,ds_2\cdot 2^{n(\alpha_1-\lambda_1)} \to 0, \qquad n\to\infty.$$

The integral $\int_P |\varphi_{n,3}(s)|\,ds$ can be estimated in a similar way.

Finally,

$$\begin{aligned}&\int_P |\varphi_{n,4}(s)|\,ds\\ &\quad\le C2^{-n\lambda_1}\sum_{i,k=0}^{2^n-1}\int_{\Delta_{ik}}\int_0^{i2^{-n}}\int_0^{k2^{-n}}(s_1-u)^{-1-\alpha_1}(s_2-v+2^{-n})^{\lambda_2-\alpha_2-1}\,du\,dv\,ds_1\,ds_2\\ &\quad= C2^{n(\alpha_1+\alpha_2-\lambda_1-\lambda_2)}\to 0\end{aligned}$$

as $n\to\infty$, where

$$\Delta_{ik} = [i2^{-n},(i+1)2^{-n})\times[k2^{-n},(k+1)2^{-n}).$$

Now we prove relation b). The structure of its terms differs from that in the case of relation a), since now they involve f_{ik}. In other words, now we have $\widetilde{f}_n(s) = f_{ik}\Delta_{ik}^1 g$ if $s\in\Delta_{ik}$. Substitute $\widetilde{f}_n(s)$ to the integrals that correspond to the functions $\varphi_{n,i}(s)$, $i=2,3,4$ (as before, $\int_P |\varphi_{n,1}(s)|\,ds\to 0$). Thus

$$\left|\widetilde{f}_n(s)-\widetilde{f}_n(u,s_2)\right| \le \left|\widetilde{f}_n(s)\right| + \left|\widetilde{f}_n(u,s_2)\right| \le C\left|\Delta_{ik}^1 g\right| \le C2^{-n\mu_1}.$$

This means that the estimates for $\varphi_{n,2}$ and $\varphi_{n,3}$ are similar to the preceding estimates. Finally,

$$\left|\Delta_{(u,v)}\widetilde{f}_n(s)\right| \le C2^{-n\mu_1}(s_2-v)^{\mu_2},$$

whence

$$\begin{aligned}
&\int_P |\varphi_{n,4}(s)|\,ds \\
&\quad \le C \sum_{i,k=0}^{2^n-1} \int_{\Delta_{ik}} \int_{[0,i2^{-n}]\times[0,k2^{-n}]} 2^{-n\mu_1}(s_2-v)^{\mu_2-\alpha_2-1}(s_1-u)^{-\alpha_1-1}\,du\,dv\,ds_1\,ds_2 \\
&\quad \le C2^{-n\mu_1} \sum_{i,k=0}^{2^n-1} \int_{\Delta_{ik}} (s_1-i2^{-n})^{-\alpha_1}(s_2-k2^{-n})^{\mu_2-\alpha_2}\,ds_1\,ds_2 \\
&\quad \le C2^{-n(\mu_1-\alpha_1+\mu_2-\alpha_2)} \to 0, \qquad n\to\infty.
\end{aligned}$$

Therefore b) is proved. The second limit in d) can be established similarly.

The second limit in c) and the first in d) are established analogously. It remains to establish the first limit in c). We have

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} f(i2^{-n},1)\left(\Delta^1_{(i2^{-n},1)} g((i+1)2^{-n},1)\right)^2 \le \lim_{n\to\infty} C2^{n-2n\mu_1} = 0.$$

The sum

$$S_n = \sum_{i=0}^{2^n-1} f(i2^{-n},1)\left(\Delta^1_{(i2^{-n},1)} g((i+1)2^{-n},1)\right)^2$$

can be represented as follows:

$$\begin{aligned}
S_n &= \sum_{i,k=0}^{2^n-1} \Big(f_{ik}(\Delta_{ik} g)^2 + 2 f_{ik}\Delta_{ik} g \Delta^1_{ik} g \\
&\qquad\qquad + \Delta^2_{ik} f (\Delta_{ik} g)^2 + \Delta^2_{ik} f (\Delta_{ik} g)^2 + 2\Delta^2_{ik} f \Delta_{ik} g \Delta^1_{ik} g \Big) \\
&= \sum_{l=1,\dots,5} S_n^l
\end{aligned}$$

where

$$S_n^1 \le C2^{-2n(\mu_1+\mu_2-1)} \to 0.$$

Similarly, $S_n^4 \to 0$ and $S_n^5 \to 0$ as $n\to\infty$. Now $S_n^2 \to 0$ in view of b).

This gives $S_n^3 \to 0$ as $n\to\infty$, and the first limit in c) follows. Theorem 2 is proved. □

Remark 2. Analogously to the last limit in c), one can prove that

$$\lim_{n\to\infty} \sum_{i,k=0}^{2^n-1} H\left(g(\theta_n^{ik})\right)\Delta^1_{ik} g \left(\Delta^2_{ik} g\right)^2 = 0$$

for a Lipschitz continuous field H and points $\theta_n^{ik} \in \Delta_{ik}$.

Theorem 3. *Theorem* 1 *holds under the assumptions of Theorem* 2 *and conditions* (11).

Proof. Since the estimates for $\Delta^1_{n,m} S$ hold for $\mu_i, \lambda_i > \frac{1}{2}$, $i=1,2$, we only need to estimate $\Delta^2_{n,m} S$. We have

$$\begin{aligned}
\Delta^2_{n,m} S &= \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \Delta^2_{ik} g \Delta^1_{ikr} f \Delta^1_{kr} g + \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} \left(\Delta^2_{ik} g - \Delta^2_{kr} g\right)\Delta^1_{ikr} f \Delta^1_{kr} g \\
&= \Delta^2_{n,m} S^1 + \Delta^2_{n,m} S^2.
\end{aligned}$$

It is clear that $\Delta^1_{ikr} f = F'(g_{ik})\Delta^1_{ikr} g + \frac{1}{2} F''(\theta_{ikr})(\Delta^1_{ikr} g)^2$, thus

$$\Delta^2_{n,m} S^1 = \Delta^2_{n,m} S^{11} + \Delta^2_{n,m} S^{12}$$

where

$$\Delta^2_{n,m} S^{11} = \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} F'(g_{ik})\Delta^1_{ikr} g \Delta^1_{kr} g \Delta^2_{ik} g,$$

$$\Delta^2_{n,m} S^{12} = \sum_{i,k=0}^{2^n-1} \sum_{r\in A_i} F''(\theta_{ikr})(\Delta^1_{ikr} g)^2 \Delta^1_{kr} g \Delta^2_{ik} g.$$

We get

$$\begin{aligned} \left|\Delta^2_{n,m} S^{11}\right| &\le \left|\sum_{i,k=0}^{2^n-1} F'(g_{ik})\Delta^2_{ik} g \sum_{r\in A_i} \Delta^1_{ikr} g \Delta^1_{kr} g\right| \\ &= \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'(g_{ik})\left(\Delta^1_{ik} g\right)^2 \Delta^2_{ik} g - \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'(g_{ik})\Delta^2_{ik} g \sum_{r\in A_i}\left(\Delta^1_{kr} g\right)^2 \\ &= \frac{1}{2}\sum_{i,k=0}^{2^n-1} \Delta^2_{ik} F(g)\left(\Delta^1_{ik} g\right)^2 - \frac{1}{4}\sum_{i,k=0}^{2^n-1} F''(\theta_{ik})\left(\Delta^1_{ik} g\right)^2\left(\Delta^2_{ik} g\right)^2 \\ &\quad - \frac{1}{2}\sum_{i,k=0}^{2^n-1} \Delta^2_{ik} F(g) \sum_{r\in A_i}\left(\Delta^1_{kr} g\right)^2 + \frac{1}{4}\sum_{i,k=0}^{2^n-1} F''(\theta_{ik})(\Delta^2_{ik} g)^2 \sum_{r\in A_i}(\Delta^1_{kr} g)^2 \to 0, \end{aligned}$$

as $n\to\infty$ due to the relations c) and d) of Theorem 2. Now, in order to estimate $\Delta^2_{n,m} S^{12}$ one follows the argument used to estimate $\Delta^2_{n,m} S$ in Theorem 1. The difference is that, when showing (9) and (10), the estimates for $|\varphi_k(u)-\varphi_k(z)|$ are of the form $C2^{-n(2\mu_1+\mu_2)}$, and thus the right-hand side of (9) is equal to $C2^{n(1+\alpha-2\mu_1-\mu_2)} \to 0$ for $\mu_1,\mu_2 > \frac{1}{2}$ and $\frac{1}{2} > \alpha > 1-\mu_1$. This applies also to the estimates of (10). This completes the proof of Theorem 3. □

3. The Itô formula for a linear combination of fractional Brownian fields

Let $F\colon \mathbb{R}\to\mathbb{R}$ be a function such that $F\in C^3(\mathbb{R})$ and F''' is Lipschitz continuous. Assume that the field $g(t)$ is given by $g(t) = \sum_{l=1}^m a_l B^{H_1^l, H_2^l}$ with $H_i^l > \frac{1}{2}$, $i=1,2$, $l=1,\dots,m$.

Theorem 4. *For every point $T\in\mathbb{R}^2_+$,*

$$F(g(T)) = F(g(0)) + \int_{P_T} F'(g)\,dg + \int_{P_T} F''(g)\,d_1 g\,d_2 g$$

where $P_T = [0,T_1]\times[0,T_2]$.

Proof. According to the one-parameter Itô formula [4],

$$\begin{aligned} F(g(T)) &= F(g(0)) + \int_0^T F'(g(s_1,T_2))\,d_1 g(s_1,T_2) \\ &= F(g(0)) + \lim_{n\to\infty}\sum_{i=0}^{2^n-1} F'(g(T_1 i 2^{-n},T_2))\Delta^1_{(T_1 i 2^{-n},T_2)} g(T_1(i+1)2^{-n},T_2) \end{aligned}$$

almost surely. The sum under the limit sign can be rewritten as follows:

$$\begin{aligned}
&\sum_{i=0}^{2^n-1} F'(g(T_1 i 2^{-n}, T_2))\Delta^1_{(T_1 i2^{-n}, T_2)} g(T_1(i+1)2^{-n}, T_2) \\
&\quad = \sum_{i,k=0}^{2^n-1} F'\left(g(s_n^{ik})\right)\Delta_{ik} g + \sum_{i,k=0}^{2^n-1} F''\left(g(s_n^{ik})\right)\Delta^1_{ik} g \Delta^2_{ik} g \\
&\quad\quad + \sum_{i,k=0}^{2^n-1} F''\left(g(s_n^{ik})\right)\Delta_{ik} g \Delta^2_{ik} g + \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''\left(g(\theta_n^{ik})\right)\left(\Delta^2_{ik} g\right)^2 \Delta^1_{ik} g \\
&\quad\quad + \frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''\left(g(\theta_n^{ik})\right)\left(\Delta^2_{ik} g\right)^2 \Delta_{ik} g
\end{aligned} \tag{12}$$

where

$$\Delta_{ik} g = \Delta_{s_n^{ik}} g\left(s_n^{i+1k+1}\right).$$

Other increments are defined in a similar way for

$$s_n^{ik} = (T_1 i 2^{-n}, T_2 k 2^{-n}), \qquad \theta_n^{ik} \in \Delta_{ik}.$$

According to Theorem 8 in [3] we have

$$\sum_{i,k=0}^{2^n-1} F'\left(g(s_n^{ik})\right)\Delta_{ik} g \to \int_{P_T} F'(g)\, dg.$$

It follows from Theorems 2 and 3 and Remark 2 that

$$\sum_{i,k=0}^{2^n-1} F''\left(g(s_n^{ik})\right)\Delta^1_{ik} g \Delta^2_{ik} g \to \int_{P_T} F''(g)\, d_1 g\, d_2 g,$$

$$\sum_{i,k=0}^{2^n-1} F''\left(g(s_n^{ik})\right)\Delta_{ik} g \Delta^2_{ik} g \to 0,$$

$$\frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''(g(\theta_n^{ik}))\left(\Delta^2_{ik} g\right)^2 \Delta_{ik} g \to 0,$$

$$\frac{1}{2}\sum_{i,k=0}^{2^n-1} F'''\left(g(\theta_n^{ik})\right)\left(\Delta^2_{ik} g\right)^2 \Delta^1_{ik} g \to 0$$

almost surely. Theorem 4 is proved. □

Remark 3. Theorem 4 holds for a function $F \in C^2(\mathbb{R})$ such that the derivative F'' is a Lipschitz function. To prove this result we rewrite the second and fourth terms on the right-hand side of (12) as follows:

$$\sum_{i,k=0}^{2^n-1} F''\left(g(\theta_n^{ik})\right)\Delta^1_{ik} g \Delta^2_{ik} g.$$

The existence of the limit of this sum can be proved similarly to Theorems 1 and 3. The third and fifth terms are represented as follows:

$$\sum_{i,k=0}^{2^n-1} F''\left(g(s_n^{ik})\right)\Delta_{ik} g \Delta^2_{ik} g.$$

It is easy to prove that this sum equals 0.

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