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EXPLICIT EXTRAPOLATION FORMULAS FOR CORRELATION MODELS OF HOMOGENEOUS ISOTROPIC RANDOM FIELDS

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Abstract. For some correlation models of homogeneous isotropic random fields, we obtain explicit formulas for linear extrapolation of a random field to the center of a sphere from observations on the sphere.

1. Homogeneous isotropic random fields on $\mathbf{R}^n$

A random function $\xi(x)$ $(x \in \mathbf{R}^n)$ is called a homogeneous isotropic random field if $\mathsf{E}\,\xi(x) = \text{const}$ (in what follows we assume that $\mathsf{E}\,\xi(x) = 0$), $\mathsf{E}\,\xi^2(x) < \infty$, and $\mathsf{E}\,\xi(x)\xi(y) = \varphi(r)$ depends only on the distance $r = |x-y|$ between points x and y.

The correlation function $\varphi(r)$ of a homogeneous isotropic random field admits the following representation (see, for example, [1, 2]):

$$\varphi(r) = 2^{(n-2)/2}\Gamma\left(\frac{n}{2}\right)\int_0^\infty \frac{J_{(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}}\,dF(\lambda) \tag{1}$$

where F is a nondecreasing bounded function on $[0,+\infty]$ and J is the Bessel function of a real argument. Relation (1) is called the *spectral representation* of the correlation function of the homogeneous isotropic random field, while the function $F(\lambda)$ is called the *spectral function* of the field. If $F(\lambda)$ is absolutely continuous, that is, there exists a function $f(\lambda)$ such that

$$F(\lambda) = \int_0^\lambda f(u)\,du,$$

then $f(\lambda)$ is called the *spectral density* of the homogeneous and isotropic random field. Relation (1) in the case of $n = 1$ is reduced to the well-known expansion

$$\varphi(r) = \int_0^\infty \cos(\lambda r)\,dF(\lambda)$$

of the correlational function of a real stationary process.

The spectral density $f(\lambda)$ can be expressed in terms of the correlational function $\varphi(r)$, namely

$$f(\lambda) = \int_0^\infty \sqrt{\lambda r}J_{(n-2)/2}(\lambda r)\frac{(\lambda r)^{(n-1)/2}}{2^{(n-2)/2}\Gamma(n/2)}\varphi(r)\,dr. \tag{2}$$

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The spectral function $F(\lambda)$ can also be expressed in terms of $\varphi(r)$ as follows:

$$F(\lambda) = \frac{1}{2^{(n-2)/2}\Gamma(n/2)} \int_0^\infty \sqrt{\lambda r} J_{(n-2)/2}(\lambda r)(\lambda r)^{n/2} \frac{\varphi(r)}{r} \, dr \tag{3}$$

(see [1]). Note that relation (1) implies that $\varphi(0) = F(+\infty)$.

Let Φ_n be the class of correlation functions of mean square continuos, homogeneous isotropic random fields defined on the Euclidean space $\mathbf{R}^n$. The classes Φ_n are nonincreasing in n,

$$\Phi_1 \supset \Phi_2 \supset \Phi_3 \supset \cdots \supset \Phi_n \supset \cdots . \tag{4}$$

The class

$$\Phi_\infty = \bigcap_{n \geq 1} \Phi_n$$

consists of functions $\varphi(r)$ of the following form:

$$\varphi(r) = \int_0^\infty e^{-\lambda^2 r^2} \, dF(\lambda) \tag{5}$$

(see [1]).

Assume that a homogeneous isotropic random field $\xi(x)$ is observed on a sphere and the problem is to find the linear extrapolation formula for $\xi(0)$ that is the best estimator for $\xi(0)$ in the sense of the minimum mean square error.

The following result is obtained in [1].

Theorem 1. *The extrapolation to the center O of the sphere*

$$S_r = \{x \colon |x| = r\}$$

of a mean square continuous and homogeneous random field $\xi(x)$ defined on the n-dimensional Euclidean space and observed on the sphere S_r is given by

$$\eta(0) = \frac{r^{(n-2)/2}}{2\pi^{n/2}} \cdot \frac{\int_0^\infty \lambda^{(2-n)/2} J_{(n-2)/2}(\lambda r) \, dF(\lambda)}{\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \, dF(\lambda)} \int_{S_n} \xi(x) \, dm_n \tag{6}$$

where $m_n(\cdot)$ is Lebesgue measure on the unit sphere in $\mathbf{R}^n$. The mean square error of extrapolation

$$\sigma^2(0, r) = \mathsf{E}[\xi(0)]^2 - \mathsf{E}[\eta(0)]^2$$

is given by

$$\sigma^2(0, r) = F(+\infty) - \frac{\left[\int_0^\infty \lambda^{(2-n)/2} J_{(n-2)/2}(\lambda r) \, dF(\lambda)\right]^2}{\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \, dF(\lambda)}. \tag{7}$$

Using (1) we rewrite $\eta(0)$ and $\sigma^2(0, r)$ as follows:

$$\eta(0) = \frac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)} \cdot \frac{r^{n-2} B(r)}{\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \, dF(\lambda)} \int_{S_n} \xi(x) \, dm_n, \tag{8}$$

$$\sigma^2(0, r) = F(+\infty) - \frac{1}{2^{n-2}\Gamma^2(n/2)} \cdot \frac{r^{n-2} B^2(r)}{\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \, dF(\lambda)}. \tag{9}$$

Note that one must calculate the integral

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \, dF(\lambda) \tag{10}$$

to evaluate $\eta(0)$ and $\sigma^2(0, r)$ according to (8) and (9), respectively, in the case of a known correlation function.

TABLE 1. Buell's list of correlation functions.

Index	Model $\varphi(r)$	Membership in Φ_n
1	e^{-r^2}	belongs to Φ_∞
2	$(1+r^2)^{-\nu}$	belongs to Φ_n if $\nu \ge 0$
3	$(1+r+\kappa r^2)e^{-r}$	belongs to Φ_n if $-\frac{1}{n} \le \kappa \le \frac{1}{3}$
3a	$(1+r)e^{-r}$	belongs to Φ_∞
3b	$(1+r+r^2/3)e^{-r}$	belongs to Φ_∞
3c	$(1+r-r^2/2)e^{-r}$	belongs to Φ_2
4	$(ae^{-br}-be^{-ar})/(a-b)$	belongs to Φ_n if $b \ne a = 0$ or $a \ne b = 0$ or $a > b > 0$ or $b > a > 0$
5	$\sin r/r$	belongs to Φ_3
6	$\frac{2^{1/3}}{\Gamma(2/3)} r^{2/3} K_{2/3}(r)$	belongs to Φ_∞

2. ON MODELS OF CORRELATION FUNCTIONS OF HOMOGENEOUS AND ISOTROPIC RANDOM FIELDS

Isotropic random fields play an important role in modeling various spatial phenomena in natural science and engineering. Isotropic correlation models are used in various branches of science, for example, in geodesy, geology, geomorphology, hydrology, meteorology, agronomy, and geophysics. Buell's list of "empirical correlation functions" ([3, p. 53]) became a standard source of correlation models in meteorology. Consider the Whittle–Mattern class of correlation functions

$$\varphi_\nu(r) = \frac{2^{1-\nu}}{\Gamma(\nu)}(r)^\nu K_\nu(r) \tag{11}$$

where K is the modified Bessel function, $\nu > 0$. If

$$\nu = k + \frac{1}{2}$$

and k is a positive integer, then (11) can be written as the product of the exponent e^{-r} and a polynomial of order k with respect to r (see [4, 8.468]). For example, the correlation functions

$$\varphi_{1/2}(r) = e^{-r}$$

and

$$\varphi_{3/2}(r) = (1+r)e^{-r}$$

can be obtained in this way for $\nu = 1/2$ and $\nu = 3/2$, respectively.

In this paper we consider the Whittle–Mattern and Buell correlation models and, for all of them, find spectral densities and extrapolation formulas for the center of a sphere from observations of the field on the sphere.

Note that the Whittle–Mattern class contains the correlation functions of homogeneous isotropic random fields that are solutions of the following stochastic equation:

$$\left(\Delta - c^2\right)^m \xi(x) = w'(x) \tag{12}$$

where Δ is the Laplace operator on $\mathbf{R}^n$ and $w'(x)$ is the white noise on $\mathbf{R}^n$ ([1, 5]).

Buell [3] proposed to apply the correlation models shown in Table 1 in meteorology.

Now we turn to the Whittle–Mattern family of correlation functions (11). If $\nu = 2/3$, $3/2$, or $5/2$, then formula (11) coincides with the Buell model of indices 6, 3a, or 3b, respectively (see [4, 8.468]). Note that the Buell model of index 3 can be written as a linear combination of the Whittle–Mattern models for $\nu = 3/2$ and $5/2$, namely

$$\left(1 + r + \kappa r^2\right) e^{-r} = (1 - 3\kappa)\varphi_{3/2}(r) + 3\kappa\varphi_{5/2}(r). \tag{13}$$

3. Explicit extrapolation formulas for some correlation models

First we prove the following auxiliary result.

Lemma 1. *If $n > \nu > 0$, then*

$$\begin{aligned}
&\int_0^\infty J^2_{(n-2)/2}(\lambda r)\frac{\lambda}{(\lambda^2+a^2)^{n/2+\nu}}\,d\lambda \\
&\quad= \frac{r^{n-2}B(1/2,(n-1)/2+\nu)}{2\pi} \\
&\qquad\times\Bigg(\frac{B((n-1)/2,\nu)}{a^{2\nu}\Gamma(n-1)}\cdot {}_1F_2\left((n-1)/2;1-\nu,n-1;a^2r^2\right) \\
&\qquad\quad+\frac{r^{2\nu}\Gamma(-\nu)}{\Gamma(n+\nu-1)}\cdot {}_1F_2\left((n-1)/2+\nu;n+\nu-1\nu+1;a^2r^2\right)\Bigg),
\end{aligned} \tag{14}$$

where ${}_pF_q$ is the generalized hypergeometric series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{k=0}^\infty \frac{(a_1)_k\cdots(a_p)}{(b_1)_k\cdots(b_q)}\cdot\frac{z^k}{k!}.$$

Proof. We use the following integral representation for the product of Bessel functions:

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi}\int_0^{\pi/2} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\,d\theta,$$

and a representation of an integral of a Bessel function in terms of a sum of hypergeometric functions

$$\begin{aligned}
\int_0^\infty J^2_{(n-2)/2}(\lambda r)\frac{\lambda\,d\lambda}{(\lambda^2+a^2)^{n/2+\nu}} &= \frac{2}{\pi}\int_0^{\pi/2}\int_0^\infty J_{n-2}(2\lambda r\cos\theta)\frac{\lambda\,d\lambda\,d\theta}{(\lambda^2+a^2)^{n/2+\nu}} \\
= \frac{2}{\pi}\int_0^{\pi/2}\Bigg(&\frac{r^{n-2}B(n/2,\nu)}{2a^{2\nu}\Gamma(n-1)}\cos^{n-2}\theta\cdot{}_1F_2\left(n/2;1-\nu,n-1;a^2r^2\cos^2\theta\right) \\
&+\frac{r^{n-2+2\nu}\Gamma(-\nu)}{2\Gamma(n+\nu-1)}\cos^{n-2+2\nu}\theta \\
&\times{}_1F_2\left(n/2+\nu;n+\nu-1,\nu+1;a^2r^2\cos^2\theta\right)\Bigg)d\theta
\end{aligned}$$

(see [6]). We consider separately the integrals

$$\int_0^{\pi/2}\cos^{n-2}\theta\cdot{}_1F_2\left(n/2;1-\nu,n-1;a^2r^2\cos^2\theta\right)\,d\theta$$

and

$$\int_0^{\pi/2}\cos^{n-2+2\nu}\theta\cdot{}_1F_2\left(n/2+\nu;n+\nu-1,\nu+1;a^2r^2\cos^2\theta\right)\,d\theta.$$

Using the definition of the generalized hypergeometric series ${}_pF_q$ we expand the integrand into the series, integrate the series term by term (see [4]), and then evaluate its sum:

$$\begin{aligned}
&\int_0^{\pi/2} \cos^{n-2}\theta \cdot {}_1F_2\left(n/2; 1-\nu, n-1; a^2r^2\cos^2\theta\right)\, d\theta \\
&\quad = \sum_{k=0}^{\infty} \frac{(n/2)_k}{(1-\nu)_k(n-1)_k}\frac{\left(a^2r^2\right)^k}{k!}\int_0^{\pi/2}\cos^{n-2+2k}\theta\, d\theta \\
&\quad = \sum_{k=0}^{\infty} \frac{(n/2)_k}{(1-\nu)_k(n-1)_k}\frac{\left(a^2r^2\right)^k}{k!}\cdot\frac{1}{2}B(1/2,(n-1)/2+k) \\
&\quad = \frac{1}{2}B(1/2,(n-1)/2)\cdot {}_1F_2\left((n-1)/2; 1-\nu, n-1; a^2r^2\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\pi/2} \cos^{n-2+2\nu}\theta \cdot {}_1F_2\left(n/2+\nu; n+\nu-1, \nu+1; a^2r^2\cos^2\theta\right)\, d\theta \\
&\quad = \sum_{k=0}^{\infty} \frac{(n/2+\nu)_k}{(n+\nu-1)_k(\nu+1)_k}\frac{\left(a^2r^2\right)^k}{k!}\int_0^{\pi/2}\cos^{n-2+2\nu+2k}\theta\, d\theta \\
&\quad = \sum_{k=0}^{\infty} \frac{(n/2+\nu)_k}{(n+\nu-1)_k(\nu+1)_k}\frac{\left(a^2r^2\right)^k}{k!}\cdot\frac{1}{2}B(1/2,(n-1)/2+\nu+k) \\
&\quad = \frac{1}{2}B(1/2,(n-1)/2+\nu)\cdot {}_1F_2\left((n-1)/2+\nu; n+\nu-1, \nu+1; a^2r^2\right).
\end{aligned}$$

Now relation (14) follows easily. □

Model 1. Consider a homogeneous isotropic random field whose correlation function is given by (11). The spectral density for this model is of the form

$$f_\nu(\lambda) = \frac{2\lambda^{n-1}}{B(\nu,n/2)(\lambda^2+1)^{n/2+\nu}}, \qquad n>\nu. \tag{15}$$

Using (14) we obtain for $a=1$ that

$$\begin{aligned}
&\int_0^{\infty} \lambda^{2-n}J^2_{(n-2)/2}(\lambda r)\, dF(\lambda) \\
&\quad = \frac{B(1/2,(n-1)/2+\nu)r^{n-2}}{\pi B(\nu,n/2)} \\
&\qquad \times\left(\frac{B((n-1)/2,\nu)}{\Gamma(n-1)}\cdot {}_1F_2\left((n-1)/2; 1-\nu, n-1; r^2\right)\right. \\
&\qquad\qquad \left. + \frac{r^{2\nu}\Gamma(-\nu)}{\Gamma(n+\nu-1)}\cdot {}_1F_2\left((n-1)/2+\nu; n+\nu-1, \nu+1; r^2\right)\right).
\end{aligned} \tag{16}$$

Let

$$\begin{aligned}
G_1(r) &= \frac{B((n-1)/2,\nu)}{\Gamma(n-1)}\cdot {}_1F_2\left((n-1)/2; 1-\nu, n-1; r^2\right), \\
G_2(r) &= \frac{\Gamma(-\nu)}{\Gamma(n+\nu-1)}\cdot {}_1F_2\left((n-1)/2+\nu; n+\nu-1, \nu+1; r^2\right).
\end{aligned}$$

According to (8) the extrapolation formula for the center of a sphere is of the following form:

$$\eta(0) = \frac{2^{2-n-\nu}\pi^{(2-n)/2}}{\Gamma\left(n/2+\nu\right)B\left(1/2,(n-1)/2+\nu\right)}\cdot\frac{K_\nu(r)}{r^{-\nu}G_1(r)+r^\nu G_2(r)}\int_{S_n}\xi(x)\, dm_n. \tag{17}$$

As was mentioned before, $F(+\infty) = 1$. According to (9), the mean square error of the extrapolation is

$$\sigma^2(0,r) = 1 - \frac{\pi 2^{4-n-2\nu}\Gamma(n/2+\nu)}{\Gamma(n/2)\Gamma(\nu)B(1/2,(n-1)/2+\nu)} \cdot \frac{K_\nu^2(r)}{r^{-2\nu}G_1(r)+G_2(r)}. \tag{18}$$

Model 2. Let $\varphi(r) = \exp\{-r^2\}$. Using relation (2) and formula (10) of Section 8.6 in [7] we get the spectral density

$$f(\lambda) = \frac{\lambda^{n-1}\exp\{-\lambda^2/4\}}{2^{n-1}\Gamma(n/2)}.$$

Taking into account relation (39) of Section 4.14 in [7], we get

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f(\lambda)\,d\lambda = \frac{e^{-2r^2} I_{(n-2)/2}(2r^2)}{2^{n-2}\Gamma(n/2)}.$$

After simple calculations we obtain from (8) the following extrapolation formula for the center of the sphere:

$$\eta(0) = \frac{1}{2\pi^{n/2}} \cdot \frac{r^{n-2}e^{r^2}}{I_{(n-2)/2}(2r^2)} \cdot \int_{S_n} \xi(x)\,dm_n.$$

In view of (9) the mean square error of the extrapolation is given by

$$\sigma^2(0,r) = 1 - \frac{1}{\Gamma(n/2)} \cdot \frac{r^{n-2}}{I_{(n-2)/2}(2r^2)}.$$

Model 3. Let

$$\varphi(r) = (1+r^2)^{-\nu}, \qquad \nu > 0.$$

We apply (2) to find the spectral density. Using relation (20) of Section 8.5 in [8] we get

$$f(\lambda) = \frac{\lambda^{(n-2)/2+\nu} K_{n/2-\nu}(\lambda)}{2^{(n-2)/2+\nu-1}\Gamma(n/2)\Gamma(\nu)}.$$

According to equality (25) of Section 10.3 in [8],

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f(\lambda)\,d\lambda = \frac{2^{2-n}r^{n-2}}{\Gamma^2(n/2)} F\left(\frac{n-1}{2}, \nu; n-1; -4r^2\right)$$

where F is the Gauss hypergeometric series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z).$$

After a simple calculation we obtain from (8) the following extrapolation formula for the center of the sphere

$$\eta(0) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \cdot \frac{(1+r^2)^{-\nu}}{F((n-1)/2, \nu; n-1; -4r^2)} \int_{S_n} \xi(x)\,dm_n.$$

In view of (9) the mean square error of the extrapolation is given by

$$\sigma^2(0,r) = 1 - \frac{(1+r^2)^{-2\nu}}{F((n-1)/2, \nu; n-1; -4r^2)}.$$

Model 4. Let $\varphi(r) = (1+r)e^{-r}$. As mentioned before, this correlation model corresponds to the Whittle–Mattern model $\varphi_{3/2}(r)$. Thus we use (15) and obtain

$$f_{3/2}(\lambda) = \frac{2\lambda^{n-1}}{B(3/2, n/2)\,(\lambda^2+1)^{(n+3)/2}}, \qquad n > 3/2.$$

It follows from (16) that

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r)\, dF(\lambda) = \frac{nr^{n-2}}{\pi}\left(G_1(r) + r^3 G_2(r)\right)$$

where

$$G_1(r) = \frac{B((n-1)/2, 3/2)}{\Gamma(n-1)} \cdot {}_1F_2\left(\frac{n-1}{2}; -\frac{1}{2}, n-1; r^2\right),$$
$$G_2(r) = \frac{4\sqrt{\pi}}{3\Gamma(n+1/2)} \cdot {}_1F_2\left(\frac{n}{2}+1; n+1/2, \frac{5}{2}; r^2\right).$$

Since $K_{3/2} = (\pi/(2r^3))^{1/2}(1+r)e^{-r}$, we obtain from (17) the linear extrapolation formula for the center of the sphere:

$$\eta(0) = \frac{\pi^{(2-n)/2}}{2^n\Gamma(n/2+1)} \frac{(1+r)e^{-r}}{(G_1(r)+r^3G_2(r))} \int_{S_n} \xi(x)\, dm_n.$$

The mean square error of the extrapolation is equal to

$$\sigma^2(0,r) = 1 - \frac{\pi 2^{1-n}}{\Gamma(n/2)\Gamma\left(n/2+1\right)} \cdot \frac{(1+r)^2 e^{-2r}}{(G_1(r)+r^3G_2(r))}$$

(see (18)).

Model 5. Let $\varphi(r) = (1+r+r^2/3)e^{-r}$. This correlation model corresponds to the Whittle–Mattern model $\varphi_{5/2}(r)$. It follows from (15) that

$$f_{5/2}(\lambda) = \frac{2\lambda^{n-1}}{B(5/2, n/2)(\lambda^2+1)^{(n+5)/2}}, \qquad n > 5/2.$$

Relation (16) implies that

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r)\, dF(\lambda) = \frac{n(n+2)r^{n-2}}{3\pi}\left(g_1(r) - r^5 g_2(r)\right)$$

where

$$g_1(r) = \frac{B((n-1)/2, 5/2)}{\Gamma(n-1)} \cdot {}_1F_2\left(\frac{(n-1)}{2}; -\frac{3}{2}, n-1; r^2\right),$$
$$g_2(r) = \frac{8\sqrt{\pi}}{15\Gamma(n+3/2)} \cdot {}_1F_2\left(\frac{n}{2}+2; n+\frac{3}{2}, \frac{7}{2}; r^2\right).$$

According to (17) the linear extrapolation formula for the center of the sphere is given by

$$\eta(0) = \frac{3\pi^{(2-n)/2}}{2^{n+1}\Gamma(n/2+2)} \frac{\left(1+r+r^2/3\right)e^{-r}}{(g_1(r)-r^5g_2(r))} \int_{S_n} \xi(x)\, dm_n,$$

since

$$K_{5/2} = \left(\pi/\left(8r^5\right)\right)^{1/2}\left(1+r+\frac{r^2}{3}\right)e^{-r}.$$

The mean square error of the extrapolation is equal to

$$\sigma^2(0,r) = 1 - \frac{3\pi 2^{-n}}{\Gamma(n/2)\Gamma(n/2+2)} \cdot \frac{\left(1+r+r^2/3\right)^2 e^{-2r}}{\left(g_1(r)-r^5g_2(r)\right)}$$

(see (18)).

Model 6. Consider the following correlation model:

$$\varphi(r) = \left(1 + r + \kappa r^2\right) e^{-r}.$$

As mentioned before, this model is a linear combination of the models $\varphi_{3/2}(r)$ and $\varphi_{5/2}(r)$, namely

$$\left(1 + r + \kappa r^2\right) e^{-r} = (1 - 3\kappa)\varphi_{3/2}(r) + 3\kappa\varphi_{5/2}(r).$$

Thus the spectral density $f(\lambda)$ of this correlation model is the linear combination of $\varphi_{3/2}(r)$ and $\varphi_{5/2}(r)$ with the same coefficients. Thus

$$f(\lambda) = \frac{2(n+1)\lambda^{n-1}}{B(1/2, n/2)(1+\lambda^2)^{(n+5)/2}} \left((1 - 3\kappa)\lambda^2 + n\kappa + 1\right)$$

by the results obtained for the two preceding models. The same arguments yield

$$\begin{aligned} &\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r)\, dF(\lambda) \\ &\quad = \frac{n}{\pi} r^{n-2} \left((1 - 3\kappa)\left(G_1(r) + r^3 G_2(r)\right) + \kappa(n+2)\left(g_1(r) - r^5 g_2(r)\right)\right) \end{aligned} \tag{19}$$

where

$$\begin{aligned} G_1(r) &= \frac{B((n-1)/2, 3/2)}{\Gamma(n-1)} \cdot {}_1F_2\left(\frac{(n-1)}{2}; -\frac{1}{2}, n-1; r^2\right), \\ G_2(r) &= \frac{4\sqrt{\pi}}{3\Gamma(n+1/2)} \cdot {}_1F_2\left(\frac{n}{2} + 1; n + \frac{1}{2}, \frac{5}{2}; r^2\right); \\ g_1(r) &= \frac{B((n-1)/2, 5/2)}{\Gamma(n-1)} \cdot {}_1F_2\left(\frac{(n-1)}{2}; -\frac{3}{2}, n-1; r^2\right), \\ g_2(r) &= \frac{8\sqrt{\pi}}{15\Gamma(n+3/2)} \cdot {}_1F_2\left(n/2 + 2; n + \frac{3}{2}, \frac{7}{2}; r^2\right). \end{aligned}$$

Now we obtain from (8) the extrapolation formula for the center of the sphere:

$$\begin{aligned} \eta(0) = &\frac{2^{-n}\pi^{(2-n)/2}}{\Gamma(n/2+1)} \\ &\times \frac{(1 + r + \kappa r^2)e^{-r}}{\left((1-3\kappa)(G_1(r) + r^3 G_2(r)) + \kappa(n+2)(g_1(r) - r^5 g_2(r))\right)} \int_{S_n} \xi(x)\, dm_n. \end{aligned} \tag{20}$$

The mean square error of the extrapolation is given by

$$\begin{aligned} \sigma^2(0, r) = 1 - &\frac{\pi 2^{1-n}}{\Gamma(n/2)\Gamma(n/2+1)} \\ &\times \frac{\left(1 + r + \kappa r^2\right)^2 e^{-2r}}{\left((1-3\kappa)(G_1(r) + r^3 G_2(r)) + \kappa(n+2)(g_1(r) - r^5 g_2(r))\right)} \end{aligned} \tag{21}$$

(see (9)).

Model 7. Consider the following correlation model:

$$\varphi(r) = \left(1 + r - r^2/2\right) e^{-r}, \qquad n = 2.$$

This is the preceding model with $\kappa = -1/2$, and we may use the preceding results:

$$f(\lambda) = \frac{15}{2}\lambda^3 \left(1 + \lambda^2\right)^{-7/2}.$$

Similarly we obtain from (19) that

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r)\, dF(\lambda) = \frac{2}{\pi}\left(\frac{5}{2}\left(G_1(r) + r^3 G_2(r)\right) + 2\left(r^5 g_2(r) - g_1(r)\right)\right)$$

where

$$G_1(r) = \frac{\pi}{2} \cdot {}_1F_2\left(\frac{1}{2}; -\frac{1}{2}, 1; r^2\right),$$
$$G_2(r) = \frac{2^5}{5 \cdot 3^2} \cdot {}_1F_2\left(2; \frac{5}{2}, \frac{5}{2}; r^2\right);$$
$$g_1(r) = \frac{3\pi}{4} \cdot {}_1F_2\left(\frac{1}{2}; -\frac{3}{2}, 1; r^2\right),$$
$$g_2(r) = \frac{2^6}{5^2 \cdot 3^2} \cdot {}_1F_2\left(3; \frac{7}{2}, \frac{7}{2}; r^2\right).$$

The extrapolation formula for the center of the sphere is found from (20):

$$\eta(0) = \frac{(1+r-r^2/2)e^{-r}}{\big(5(G_1(r)+r^3G_2(r))+4(r^5g_2(r)-g_1(r))\big)} \int_{S_n} \xi(x)\, dm_n.$$

The mean square error of the extrapolation is given by (21) as follows:

$$\sigma^2(0,r) = 1 - \frac{\pi}{2} \cdot \frac{\big(1+r-r^2/2\big)^2 e^{-2r}}{\big(5/2(G_1(r)+r^3G_2(r))+2(r^5g_2(r)-g_1(r))\big)}.$$

Model 8. Consider the following correlation model:

$$\varphi(r) = \left(ae^{-br} - be^{-ar}\right)/(a-b).$$

Let $\varphi_a(r) = e^{-ar}$ and $\varphi_b(r) = e^{-br}$, and let the respective spectral densities be $f_a(\lambda)$ and $f_b(\lambda)$. The correlation function $\varphi(r)$ is a linear combination of the functions $\varphi_a(r)$ and $\varphi_b(r)$. Thus the spectral density $f(\lambda)$ corresponding to the correlation function $\varphi(r)$ is a linear combination of the functions $f_a(\lambda)$ and $f_b(\lambda)$. Using (2) and relation (4) of Section 8.6 in [8] we obtain

$$f_a(\lambda) = \frac{2a\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \cdot \frac{\lambda^{n-1}}{(\lambda^2+a^2)^{(n+1)/2}}.$$

We use equality (14) to evaluate the integral

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f_a(\lambda)\, d\lambda.$$

We have

$$\begin{aligned}\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) \cdot \frac{a\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \cdot \frac{\lambda^{n-1}\, d\lambda}{(a^2+\lambda^2)^{(n+1)/2}} \\ = \frac{ar^{n-2}}{\pi}\Bigg(\frac{B((n-1)/2,1/2)}{a\Gamma(n-1)} {}_1F_2\left(\frac{n-1}{2}; \frac{1}{2}, n-1; a^2r^2\right) \\ - \frac{2\sqrt{\pi}}{\Gamma(n-1/2)} r {}_1F_2\left(\frac{n}{2}; \frac{3}{2}, n-\frac{1}{2}; a^2r^2\right)\Bigg).\end{aligned}$$

Let

$$\begin{aligned}L(a,r) = \Bigg(\frac{B((n-1)/2,1/2)}{a\Gamma(n-1)} {}_1F_2\left(\frac{n-1}{2}; \frac{1}{2}, n-1; a^2r^2\right) \\ - \frac{2\sqrt{\pi}}{\Gamma(n-1/2)} r {}_1F_2\left(\frac{n}{2}; \frac{3}{2}, n-\frac{1}{2}; a^2r^2\right)\Bigg).\end{aligned}$$

Then

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f_a(\lambda)\, d\lambda = \frac{ar^{n-2}}{\pi} L(a,r).$$

Similarly

$$f_b(\lambda) = \frac{2b\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \cdot \frac{\lambda^{n-1}}{(\lambda^2+b^2)^{(n+1)/2}}$$

and

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f_b(\lambda)\, d\lambda = \frac{br^{n-2}}{\pi} L(b,r)$$

where

$$L(b,r) = \left(\frac{B((n-1)/2, 1/2)}{b\Gamma(n-1)} {}_1F_2\left(\frac{n-1}{2}; \frac{1}{2}, n-1; b^2r^2\right) \right.$$
$$\left. - \frac{2\sqrt{\pi}}{\Gamma(n-1/2)} r\, {}_1F_2\left(\frac{n}{2}; \frac{3}{2}, n-\frac{1}{2}; b^2r^2\right)\right).$$

Thus the spectral density of the correlation model under consideration is of the following form:

$$f(\lambda) = \frac{2ab}{a-b}\frac{\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)}\left(\frac{\lambda^{n-1}}{(\lambda^2+b^2)^{(n+1)/2}} - \frac{\lambda^{n-1}}{(\lambda^2+a^2)^{(n+1)/2}}\right).$$

Moreover

$$\int_0^\infty \lambda^{2-n} J^2_{(n-2)/2}(\lambda r) f(\lambda)\, d\lambda = \frac{abr^{n-2}(L(b,r)-L(a,r))}{\pi(a-b)}.$$

The linear extrapolation formula (8) takes the form

$$\eta(0) = \frac{2^{1-n}\pi^{(2-n)/2}}{\Gamma(n/2)} \cdot \frac{ae^{-br} - be^{-ar}}{abr^{n-2}(L(b,r)-L(a,r))} \int_{S_n} \xi(x)\, dm_n.$$

The mean square error of extrapolation is given by (9) as follows:

$$\sigma^2(0,r) = 1 - \frac{\pi 2^{2-n}}{(a-b)\Gamma^2(n/2)} \cdot \frac{(ae^{-br}-be^{-ar})^2}{ab(L(b,r)-L(a,r))}.$$

Model 9. Let $\varphi(r) = \sin(r)/r$, $n = 3$. This is the so-called Markov type field. The problem of extrapolation for the Markov type field in $\mathbf{R}^n$ (and, in particular, in $\mathbf{R}^3$) is considered in [1].

Model 10. Let $\varphi(r) = 2^{1/3} r^{2/3} K_{2/3}(r)/\Gamma(2/3)$. This correlation model corresponds to the Whittle–Mattern model $\varphi_{2/3}(r)$. Thus (15) implies that

$$f_{2/3}(\lambda) = \frac{2\lambda^{n-1}}{B(2/3, n/2)(\lambda^2+1)^{n/2+2/3}}, \qquad n > 2/3.$$

Let

$$G_1(r) = \frac{B((n-1)/2, 2/3)}{\Gamma(n-1)} \cdot {}_1F_2\left((n-1)/2; 1/3, n-1; r^2\right),$$
$$G_2(r) = \frac{\Gamma(-2/3)}{\Gamma(n-1/3)} \cdot {}_1F_2\left(n/2+1/6; n-1/3, 5/3; r^2\right).$$

The extrapolation formula is given by

$$\eta(0) = \frac{2^{1/3-n}\pi^{(2-n)/2}}{\Gamma(n/2+2/3)B(1/2, n/2+1/6)} \cdot \frac{K_{2/3}(r)}{r^{-2/3}G_1(r) + r^{2/3}G_2(r)} \int_{S_n} \xi(x)\, dm_n.$$

The mean square error of the extrapolation is given by (18) as follows:

$$\sigma^2(0,r) = 1 - \frac{\pi 2^{8/3-n}\Gamma(n/2+2/3)}{\Gamma(n/2)\Gamma(2/3)B(1/2, n/2+1/6)} \cdot \frac{K^2_{2/3}(r)}{r^{-4/3}G_1(r) + G_2(r)}.$$

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