ON THE JOINT DISTRIBUTION OF THE SUPREMUM, INFIMUM, 
AND THE VALUE OF A SEMICONTINUOUS PROCESS WITH 
INDEPENDENT INCREMENTS

UDC 519.21

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Abstract. The joint distribution of the supremum, infimum, and the value of a 
homogeneous lower semicontinuous process with independent increments is found in 
this paper.

The weak convergence of the boundary distribution to the corresponding distri-
bution of the Wiener process is proved in the case of $E\xi(1) = 0$ and $E\xi^2(1) < \infty$.

Exact and asymptotic relations are obtained for this distribution.

Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, be a homogeneous lower semicontinuous process with independent 
increments $[1]$ and let $k(p)$ be its cumulant:

$$\xi(0) = 0, \quad E e^{\xi(t)} = e^{tk(p)}, \quad \Re p = 0.$$ 

The aim of this paper is to determine the joint distribution

$$Q^t(-y, \alpha, \beta, x) = P \left[ -y \leq \inf_{u \leq t} \xi(u), \xi(t) \in (\alpha, \beta), \sup_{u \leq t} \xi(u) \leq x \right]$$

where

$$x, y > 0, \quad -y \leq \alpha < \beta \leq x.$$ 

This problem is solved in $[2]$ for homogeneous processes with independent increments. 
The problem for semicontinuous processes with independent increments can be solved in 
the closed form in terms of the resolvent

$$R^s(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{px}}{k(p) - s} dp, \quad \gamma > c(s)$$

(see $[2]$–$[6]$) where $c(s) > 0$ for $s > 0$ is a unique in the half-plane $\Re p > 0$ positive root 
of the equation

$$k(p) - s = 0$$

(see $[2]$).

Now we state the main results of the paper.

Theorem 1. Let $\xi(t)$, $t \geq 0$, be a homogeneous lower semicontinuous process, $\nu_s$ an 
exponential random variable with parameter $s > 0$, and let

$$Q^s(-y, \alpha, \beta, x) = \int_0^\infty e^{-st} P \left[ -y \leq \inf_{u \leq t} \xi(u), \xi(t) \in (\alpha, \beta), \sup_{u \leq t} \xi(u) \leq x \right] dt,$$

$$Q^s_{\nu_s}(-y, x) = \int_{-y}^x e^{-pu_s} P \left[ -y \leq \inf_{u \leq \nu_s} \xi(u), \xi(\nu_s) \in du, \sup_{u \leq \nu_s} \xi(u) \leq x \right] du$$

2000 Mathematics Subject Classification. Primary 60J25, 60J75.
be the integral transforms of the joint distribution (1).

Then

\[ Q_y^p(-y, x) = U_y^p(x) - e^{pg} \frac{R^p(x)}{R^p(B)} U_y^p(B), \quad B = x + y, \]

\[ \tilde{Q}(-y, \alpha, \beta, x) = \frac{R^p(x)}{R^p(B)} \int_0^\beta R^p(y + u) \, du - \int_{\max\{0, \beta\}}^{\max\{0, \alpha\}} R^p(u) \, du \]

where

\[ U^p_n(x) = E \left[ e^{-p \xi_{\nu_s}}; \xi^+(\nu_s) \leq x \right] = \frac{c(s)}{c(s) - p} E \left[ e^{-p \xi^+(\nu_s)}; \xi^+(\nu_s) \leq x \right], \]

\[ \xi^+(t) = \sup_{u \leq t} \xi(u), \quad \xi^-(t) = \inf_{u \leq t} \xi(u). \]

**Corollary 1.** Let \( w(t), t \geq 0 \), be the Wiener process with cumulant \( k(p) = \frac{1}{2} \sigma^2 p^2 \) and let

\[ \chi = \inf\{t > 0: w(t) \notin (-y, x)\} \]

be the first exit time of the process \( w(t), t \geq 0 \), from the interval \((-y, x)\).

Then

1) the following equalities hold:

\[ P \left[ -y \leq \inf_{u \leq t} w(u), w(t) \in (\alpha, \beta), \sup_{u \leq t} w(u) \leq x \right] \overset{\text{def}}{=} Q^l(-y, \alpha, \beta, x) = \frac{4}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} e^{-\nu(\pi B)^2/2} \sin \left( \frac{B}{\nu} \right) \sin \left( \frac{\beta - \alpha}{2B} \right), \]

\[ P[\chi > t] = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} e^{-\nu(\pi(2\nu + 1)B)^2/2} \sin \left( \frac{B}{\nu} (2\nu + 1) \right); \]

2) the first two moments of the random variable \( \chi \) are given by

\[ E \chi = \frac{1}{\sigma^2} xy, \quad E \chi^2 = \frac{1}{3\sigma^4} xy \left( x^2 + 3xy + y^2 \right), \]

\[ \text{Var} \chi = \frac{1}{3\sigma^4} xy \left( x^2 + y^2 \right). \]

Moreover, if \( x = y \), then

\[ E \chi^n = \frac{1}{(2n - 1)!!} \left( \frac{\chi}{\sigma} \right)^{2n} E_n, \quad n > 0, \]

where \( E_n, n > 0 \), are the Euler numbers;

3) the probability \( Q^l(-y, \alpha, \beta, x) \) is such that

\[ \tilde{Q}(-y, \alpha, \beta, x) = \frac{1}{\sigma \sqrt{2\pi t}} \int_0^\beta \left( \sum_{k=-\infty}^{\infty} e^{-(2Bk+u)^2/2\sigma^2 t} - \sum_{k=-\infty}^{\infty} e^{-(2Bk+2x-u)^2/2\sigma^2 t} \right) \, du \]

(see [2]).

**Theorem 2.** Let \( E \xi(1) = 0, E \xi^2(1) = \sigma^2 < \infty \), and

\[ x, y > 0, \quad x + y = 1, \quad -y \leq \alpha < \beta \leq x. \]
Then the joint distribution

\( P \left[ -yB \leq \xi^- (tB^2), \xi(2tB^2) \in (\alpha B, \beta B), \xi^+ (tB^2) \leq xB \right] \)

weakly converges as \( B \to \infty \) to the joint distribution

\[
P \left[ -y \leq \inf_{u \leq t} w(u), \ w(t) \in (\alpha, \beta), \ \sup_{u \leq t} w(u) \leq x \right]
\]

of the supremum, infimum, and the value of the symmetric Wiener process \( w(t), t \geq 0 \), with the cumulant

\[
k(p) = \frac{1}{2} \sigma^2 p^2.
\]

Moreover,

\[
\lim_{B \to \infty} Q^t(-y, \alpha, \beta, x, B) = \frac{4}{\pi} \sum_{\nu=1}^{\infty} \frac{1}{\nu} e^{-t(\pi \nu)^2/2} \sin \left( \frac{2x - \alpha - \beta}{2} \pi \nu \right) \sin \left( \frac{\beta - \alpha}{2} \pi \nu \right).
\]

The limit distribution (11) is such that

\[
\lim_{B \to \infty} Q^t(-y, \alpha, \beta, x, B) = \frac{1}{\sigma \sqrt{2 \pi}} \int_0^\beta \left( \sum_{k=-\infty}^{\infty} e^{-(2k+u)^2/2\sigma^2 t} - \sum_{k=-\infty}^{\infty} e^{-(2k+2u-x)^2/2\sigma^2 t} \right) du.
\]

Corollary 2. Let \( E \xi(1) = 0, \ E \xi^2(1) = \sigma^2 < \infty, \ x, y > 0, \ x + y = 1 \), and let

\[
\chi(B) = \inf\{t > 0: \xi(t) \notin (-yB, xB)\}
\]

be the first exit time of the process \( \xi(t), t \geq 0 \), from the interval \((-yB, xB)\).

Then the random variable \( \chi(B) \) weakly converges as \( B \to \infty \) to the first exit time

\[
\chi = \inf\{t > 0: w(t) \notin (-y, x)\}
\]

of the Wiener process from the interval \((-y, x)\). Moreover,

\[
\lim_{B \to \infty} P \left[ \frac{\chi(B)}{B^2} > t \right] = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} e^{-t(\pi(2\nu+1)\sigma)^2/2} \sin(x(2\nu + 1)\pi)
\]

\[
= P[\chi > t].
\]

Remark. The right-hand sides of equalities (6), (7), (11), and (12) can be used to determine the asymptotic expansions for the probabilities on the corresponding left-hand sides. For example, considering only the first terms in expansions (6) and (7) we get

\[
P \left[ -y \leq \inf_{u \leq t} w(u), \ w(t) \in (0x), \ \sup_{u \leq t} w(u) \leq x \right] = \frac{4}{\pi} e^{-t(\pi \sigma/B)^2/2} \sin \left( \frac{x}{B} \pi \right) \sin \left( \frac{x}{2B} \pi \right) + o \left( e^{-t(\pi \sigma/B)^2/2} \right),
\]

\[
P[\chi > t] = \frac{4}{\pi} e^{-t(\pi \sigma/B)^2/2} \sin \left( \frac{x}{B} \pi \right) + o \left( e^{-t(\pi \sigma/B)^2/2} \right)
\]

as \( t \to \infty \).

Proof of Theorem 1. Let

\[
\chi = \inf\{t > 0: \xi(t) \notin (-y, x)\}
\]
be the first exit time of the process \( \xi(t), \ t \geq 0, \) from the interval \((-y, x)\). According to the total probability formula we have

\[
E \left[ e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq x \right] = E \left[ e^{-p\xi(\nu_s)}; -y \leq \xi^-(\nu_s), \xi^+(\nu_s) \leq x \right] + E \left[ e^{-s\chi e^{-p\xi(\chi)}}; \xi(\chi) = -y \right] E \left[ e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq B \right],
\]

(13)

since the event that the process \( \xi(t) \) does not exceed the upper level \( x \) on the interval \([0, \nu_s]\) (the left-hand side of the equality) occurs if either \( \xi(t) \) does not cross the lower level \(-y\) (the first term on the right-hand side of (13)) or it crosses the lower level \(-y\) but then its increments do not exceed the upper level \( x + y = B \) (the second term on the right-hand side of (13)). According to results of the paper [7],

\[
E \left[ e^{-s\chi e^{-p\xi(\chi)}}; \xi(\chi) = -y \right] = e^{py} \frac{R^s(x)}{R^s(B)},
\]

whence

\[
E \left[ e^{-p\xi(\nu_s)}; -y \leq \xi^-(\nu_s), \xi^+(\nu_s) \leq x \right] = Q^s_p(-y, x).
\]

Thus (13) implies (3). Now we prove (5).

Let \( \xi(t), \ t \geq 0, \) be a homogeneous process with independent increments such that \( \xi(0) = 0 \). For \( x > 0 \) put

\[
\tau_x = \inf \{ t > 0; \xi(t) > x \}.
\]

Applying the total probability formula we obtain

\[
E e^{-p\xi(\nu_s)} = E \left[ e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \leq x \right] + E \left[ e^{-s\tau_x e^{-p\xi(\tau_x)}}; \xi^+(\nu_s) \right],
\]

(14)

since the behavior of the process \( \xi(t), \ t \geq 0, \) on the interval \([0, \nu_s]\) (the left-hand side of the equality) is such that either \( \xi \) does not exceed the upper level \( x \) (the first term on the right-hand side of (14)) or it exceeds the level \( x \) but then it varies in an exponentially distributed time \([0, \nu_s]\) (the second term on the right-hand side of (14)).

Using the Spitzer–Rogozin equality

\[
E e^{-p\xi(\nu_s)} = E \left[ e^{-p\xi^+(\nu_s)} \right] E \left[ e^{-p\xi^-(\nu_s)} \right], \quad \text{Re} \ p = 0,
\]

we rewrite (14) as follows:

\[
\left( E e^{-p\xi^-(\nu_s)} \right)^{-1} E \left[ e^{-p(\xi(\nu_s) - x)}; \xi^+(\nu_s) \leq x \right] - E \left[ e^{-p(\xi^+(\nu_s) - x)}; \xi^+(\nu_s) \leq x \right] = E \left[ e^{-p(\xi^+(\nu_s) - x)}; \xi^+(\nu_s) > x \right] - E e^{-p\xi^+(\nu_s)} E \left[ e^{-s\tau_x e^{-p(\xi(\tau_x) - x)}} \right],
\]

\[
\text{Re} \ p = 0.
\]

Following the standard reasoning based on the factorization (see [8]) we obtain from the latter equality that

\[
E \left[ e^{-s\tau_x e^{-p\xi(\tau_x)}} \right] = \left( E e^{-p\xi^+(\nu_s)} \right)^{-1} E \left[ e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) > x \right],
\]

\[
\text{Re} \ p \geq 0;
\]

(15)

\[
E \left[ e^{-p\xi(\nu_s)}; \xi^+(\nu_s) \leq x \right] = E e^{-p\xi^+(\nu_s)} E \left[ e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \leq x \right],
\]

\[
\text{Re} \ p \leq 0.
\]

Note that the above approach is only one of several methods that can be used to get the distributions of \( (\xi(\cdot), \xi^+(\cdot)) \) and \( (\tau_x, \xi(\tau_x)) \). The joint distribution of a homogeneous process with independent increments and its maximum is studied in [9].
Since
\[ E e^{-p\xi^-(\nu_s)} = \frac{c(s)}{c(s) - p}, \quad \text{Re} \, p \leq 0, \]
\[ E e^{-p\xi^+(\nu_s)} = \frac{s \cdot p - c(s)}{c(s) \cdot k(p) - s}, \quad \text{Re} \, p \geq 0, \]
for lower semicontinuous processes, relation (15) implies (5).

Now we find the resolvent representation of the right-hand side of (15). Multiplying both sides of this equality by \( e^{-\lambda x} \) and integrating with respect to \( x > 0 \), we get
\[ \int_0^\infty e^{-\lambda x} E \left[ e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \leq x \right] \, dx = \frac{s}{c(s) - p} R^s(x) - s \int_0^x e^{-pu} R^s(u) \, du. \]

Using definition (2) of the resolvent, we obtain from the latter equality that
\[ E \left[ e^{-p\xi^+(\nu_s)}; \xi^+(\nu_s) \leq x \right] = \frac{s}{c(s) - p} e^{-px} \, R^s(x) - s \int_0^x e^{-pu} R^s(u) \, du. \]

Now we find the resolvent representation for the function \( Q^s_{\rho}(-y, x) \).

Substituting (16) into (3) we get
\[ Q^s_{\rho}(-y, x) = s \frac{R^s(x)}{R^s(B)} \int_0^B e^{-pu} R^s(u) \, du - s \int_0^x e^{-pu} R^s(u) \, du \]
or
\[ \int_{-y}^x e^{-pu} P \left[ -y \leq \xi^-(\nu_s), \xi(\nu_s) \in du, \xi^+(\nu_s) \leq x \right] \]
\[ = \int_{-y}^x e^{-pu} \left( s \frac{R^s(x)}{R^s(B)} R^s(y + u) - s R^s(u) \right) \, du, \]

whence
\[ P \left[ -y \leq \xi^-(\nu_s), \xi(\nu_s) \in du, \xi^+(\nu_s) \leq x \right] = s \left\{ \frac{R^s(x)}{R^s(B)} R^s(y + u) - s R^s(u) \right\}, \]
\[ u \in [-y, x]. \]

Integrating the latter equality in the interval \((\alpha, \beta)\) and taking into account that
\[ R^s(u) = 0 \]
for \( u \leq 0 \) we prove that
\[ s \hat{Q}^s(-y, \alpha, \beta, x) = s \frac{R^s(x)}{R^s(B)} \int_{\alpha}^{\beta} R^s(y + u) \, du - s \int_{\max(0, \alpha)}^{\max(0, \beta)} R^s(u) \, du. \]

This result coincides with (4). Thus Theorem 1 is proved. Another method to obtain equality (17) for a Poisson process with two-sided reflection is described in [10].

**Proof of Corollary 1.** If \( w(t), t \geq 0 \), is a Wiener process with cumulant \( \frac{1}{2} \beta^2 \sigma^2 \), then
\[ R^s(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{px} \frac{dp}{\frac{1}{2} \beta^2 \sigma^2 - s} = \frac{2}{\sigma \sqrt{2s}} \text{sh} \left( \frac{x}{\sigma \sqrt{2s}} \right), \quad \gamma > \frac{1}{\sigma \sqrt{2s}}, \]
where
\[ \text{sh} \, u = \frac{1}{2} (e^u - e^{-u}). \]
(19) yields
\[\hat{Q}^∗(-y, \alpha, \beta, x) = \frac{1}{s} \frac{\sqrt{2s}}{\sqrt{2s}} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \]
(19
\[= \frac{1}{s} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \]
(19
\[+ \frac{1}{s} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \]
where
\[\alpha^{+} = \max\{0, \alpha\}, \quad \beta^{+} = \max\{0, \beta\},\]
\[\text{ch} u = \frac{1}{2} (e^u + e^{-u}).\]

Considering the cases \(0 \leq \alpha < \beta, \alpha < 0 < \beta, \text{and} \alpha < \beta \leq 0\) separately we obtain from (19) that
\[\hat{Q}^∗(-y, \alpha, \beta, x) = \frac{2}{s} \frac{\sqrt{2s}}{\sqrt{2s}} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \left( \frac{2x - \alpha - \beta}{2\sigma} \right) \left( \frac{\beta - \alpha}{2\sigma} \sqrt{2s} \right),\]
(20
\[0 \leq \alpha < \beta,\]
(20
\[\hat{Q}^∗(-y, \alpha, \beta, x) = \frac{2}{s} \frac{\sqrt{2s}}{\sqrt{2s}} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \left( \frac{2x - \beta}{2\sigma} \sqrt{2s} \right) \left( \frac{\beta - \alpha}{2\sigma} \sqrt{2s} \right),\]
(20
\[\alpha < 0 < \beta,\]
(20
\[\hat{Q}^∗(-y, \alpha, \beta, x) = \frac{2}{s} \frac{\sqrt{2s}}{\sqrt{2s}} \left[ \frac{\theta^{+}}{\sqrt{2s}} \right] \left( \frac{2x + \alpha + \beta}{2\sigma} \sqrt{2s} \right) \left( \frac{\beta - \alpha}{2\sigma} \sqrt{2s} \right),\]
(20
\[\alpha < \beta \leq 0.\]

Now we turn to equality (20). According to the inversion formula
\[Q^t(-y, \alpha, \beta, x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \hat{Q}^∗(-y, \alpha, \beta, x) ds, \quad \gamma > 0\]
(see [3]). The integrand is an analytic function everywhere in the plane except for the points
\[s_\nu = -\frac{1}{2} \left( \frac{\nu}{B} \right)^2, \quad \nu \in \mathbb{N}^+ = \{1, 2, \ldots\}\]
where it has simple poles. Considering appropriate contours for integration (see [3]) and performing necessary transforms for the probability \(Q^t(-y, \alpha, \beta, x)\) we obtain
\[Q^t(-y, \alpha, \beta, x) = \sum_{\nu=1}^{\infty} \text{Re} s_{s-s_\nu} \left( e^{st} \hat{Q}^∗(-y, \alpha, \beta, x) \right)\]
(21
\[= \frac{4}{\pi} \sum_{\nu=1}^{\infty} e^{-\frac{t}{\nu}} \frac{1}{\sin (\frac{x}{B}) \sin (\frac{2x - \alpha - \beta}{2B}) \sin (\frac{\beta - \alpha}{2B})} \right.\]
Applying the same method to equalities (20*) and (20**) we prove that the joint distribution \(F^t(-y, \alpha, \beta, x)\) is given by (21) for all cases under consideration. Thus equality (6) is proved.

Since
\[P[\chi > t] = P \left[ -y \leq \inf_{u \leq t} w(u), \sup_{u \leq t} w(u) \leq x \right],\]
we put $\alpha = -y$ and $\beta = x$ in (21) and obtain

\begin{equation}
P[\chi > t] = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} e^{-t[(2\nu + 1)\sigma / B]^{2/3}} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right)
\end{equation}

which in fact coincides with equality (7). Putting $t = 0$ in (22) we evaluate the series

\begin{equation}
\sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right) = \frac{\pi}{4}, \quad x, y > 0, \quad B = x + y.
\end{equation}

Now we evaluate the first two moments of the random variable $\chi$.

It follows from (20*) for $\beta = x$ and $\alpha = -y$ that

\begin{equation}
E e^{-s\chi} = \text{ch} \left( \frac{x - y}{2\sigma} \sqrt{2s} \right) / \text{ch} \left( \frac{x + y}{2\sigma} \sqrt{2s} \right).
\end{equation}

Expanding the right-hand side of the latter equality into a series in powers of $s$, we find

\begin{align*}
E \chi &= \frac{1}{\sigma^2} xy, \\
E \chi^2 &= \frac{1}{3\sigma^4} xy \left( x^2 + 3xy + y^2 \right), \\
\text{Var} \chi &= \frac{1}{3\sigma^4} xy \left( x^2 + y^2 \right),
\end{align*}

$x, y > 0$.

On the other hand, integrating equality (22) we obtain

\begin{align*}
E \chi &= \frac{8B^2}{\pi^3\sigma^2} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu + 1)^3} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right), \\
E \chi^2 &= \frac{32B^4}{3\pi^5\sigma^4} \sum_{\nu=0}^{\infty} \frac{1}{(2\nu + 1)^5} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right).
\end{align*}

The term-by-term integration is justified by the uniform convergence of series (22) for $t \geq 0$, which in turn follows from (23). Comparing the latter equalities with the preceding ones we get for $x, y > 0$ and $B = x + y$ that

\begin{equation}
\sum_{\nu=0}^{\infty} \frac{1}{(2\nu + 1)^3} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right) = \frac{\pi^3}{8B^3} xy,
\end{equation}

\begin{equation}
\sum_{\nu=0}^{\infty} \frac{1}{(2\nu + 1)^5} \sin \left( \frac{x}{B} (2\nu + 1)\pi \right) = \frac{\pi^5}{96B^5} xy \left( x^2 + 3xy + y^2 \right).
\end{equation}

In particular, we put $x = y > 0$ in (23) and (25) and obtain

\begin{align*}
\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu + 1} &= \frac{\pi}{4}, \\
\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu + 1)^3} &= \frac{\pi^3}{32}, \\
\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu + 1)^5} &= \frac{5\pi^5}{1536}.
\end{align*}

Using (22) and (24) one can evaluate higher moments of the random variable $\chi$.

Further, relation (24) for $x = y$ implies that

\begin{equation}
E e^{-s\chi} = \frac{1}{\text{ch} \left( \frac{x}{\sigma} \sqrt{2s} \right)} = \text{sech} \left( \frac{x}{\sigma} \sqrt{2s} \right).
\end{equation}

Using the Taylor expansion of the function sech$(\cdot)$ we get

\begin{equation}
\sum_{n=0}^{\infty} (-1)^n \frac{s^n}{n!} E \chi^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \frac{x}{\sigma} \sqrt{2s} \right)^n E_n
\end{equation}

where $E_0 = 1$ and $E_n, n > 0$, are Euler numbers.

Comparing the coefficients of the powers of $s$ we conclude that

\begin{equation}
E \chi^n = \frac{1}{(2n - 1)!!} \left( \frac{x}{\sigma} \right)^{2n} E_n, \quad n > 0.
\end{equation}
Now we derive representation (8) for the probability \( \tilde{Q}^t(-y, \alpha, \beta, x) \). Equalities (17) and (18) imply

\[
\int_0^\infty e^{-st} P \left[ -y \leq \inf_{u \leq t} w(u), \ w(u) \in du, \sup_{u \leq t} w(u) \leq x \right] \overset{\text{def}}{=} \tilde{q}^t(-y, u, x) du \tag{26}
\]

It follows from (26) that

\[
\tilde{q}^t(-y, u, x) = \frac{1}{\sigma \sqrt{2s}} \frac{1}{\text{sh}(\frac{B}{\sigma} \sqrt{2s})} \left[ \text{ch} \left( \frac{B - \sigma |u| \sqrt{2s}}{2} \right) - \text{ch} \left( \frac{x - y - u \sqrt{2s}}{\sigma} \right) \right], \quad u \in [-y, x].
\]

To invert the Laplace transform on the right-hand side of (27) we use the following expansion:

\[
\frac{1}{2 \text{sh}(B \sqrt{2s}/\sigma)} = \sum_{k=0}^{\infty} e^{-B(2k+1)\sqrt{2s}/\sigma}
\]

(see [2]) and the equality

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \frac{1}{\sqrt{s}} e^{-a \sqrt{s}} ds = \frac{1}{\sqrt{\pi t}} e^{-a^2/4t}, \quad \gamma > 0, \ a > 0.
\]

Then

\[
P \left[ -y \leq \inf_{u \leq t} w(u), \ w(t) \in du, \sup_{u \leq t} w(u) \leq x \right] = \frac{1}{\sigma \sqrt{2\pi t}} \left\{ \sum_{k=\infty}^{\infty} e^{-(2k+1)\sigma^2 t} - \sum_{k=\infty}^{\infty} e^{-(2k+2)\sigma^2 t} \right\} du,
\]

\[
\quad u \in [-y, x].
\]

Integrating this equality in the interval \((\alpha, \beta)\) we prove equality (8). This completes the proof of Corollary 1.

**Proof of Theorem 2.** Suppose the assumptions of Theorem 2 hold. Since

\[
\frac{s}{B^2} \tilde{Q}^{s/B^2}(-yB, \alpha B, \beta B, xB)
\]

\[
= \frac{s}{B^2} \int_0^\infty e^{-us/B^2} P[-yB \leq \xi(u) \leq \xi^+(u) \in (\alpha B, \beta B), \xi^-(u) \leq xB] du
\]

\[
= \frac{s}{B^2} \int_0^\infty e^{-st} P[-yB \leq \xi^-(tB^2) \leq \xi^+(tB^2) \in (\alpha B, \beta B)] dt
\]

\[
= \frac{s}{B^2} \int_0^\infty e^{-st} Q^t(-y, \alpha, \beta, x, B) dt,
\]

we get

\[
\lim_{B \to \infty} \frac{s}{B^2} \int_0^\infty e^{-st} Q^t(-y, \alpha, \beta, x, B) dt = \lim_{B \to \infty} \frac{s}{B^2} \tilde{Q}^{s/B^2}(-yB, \alpha B, \beta B, xB, B)
\]

\[
= \lim_{B \to \infty} \frac{s}{B^2} \left( \frac{R^s/B^2(xB)}{R^s/B^2(B)} \int_{B(y+B)}^{B(y+\beta)} R^s/B^2(u) du - \int_{B\max\{0,\alpha\}}^{B\max\{0,\beta\}} R^s/B^2(u) du \right)
\]

by relation (4) and the preceding chain of equalities.
Asymptotic properties of the resolvent and potential of a semicontinuous process with independent increments are studied in [5, 6]. In particular, it is proved in [5, 6] that
\[
\lim_{B \to \infty} \frac{1}{B} R^x/B^2(xB) = \frac{1}{\sigma} \sqrt{\frac{2}{s}} \text{sh} \left( \frac{x}{\sigma} \sqrt{2s} \right),
\]

\[
\lim_{B \to \infty} \frac{8}{B^2} \int_0^{xB} R^x/B^2(u) \, du = \text{ch} \left( \frac{x}{\sigma} \sqrt{2s} \right) - 1
\]

under the assumptions of the theorem. Using the latter equalities and evaluating the limits on the right-hand side of (28) we deduce
\[
\lim_{B \to \infty} \int_0^\infty e^{-st} Q^t(-y, \alpha, \beta, x, B) \, dt = \frac{1}{s} \sqrt{\frac{2}{s}} \text{sh} \left( \frac{y + \beta}{\sigma} \sqrt{2s} \right) - \text{ch} \left( \frac{y + \alpha}{\sigma} \sqrt{2s} \right) - \frac{1}{s} \left[ \text{ch} \left( \frac{\beta^+}{\sigma} \sqrt{2s} \right) - \text{ch} \left( \frac{\alpha^+}{\sigma} \sqrt{2s} \right) \right],
\]

\[
\alpha^+ = \max \{0, \alpha\}, \quad \beta^+ = \max \{0, \beta\}.
\]

The right-hand side of this equality coincides with the right-hand side of equality (19) for \( B = 1 \). Thus the left-hand sides of these equalities coincide as well. Therefore
\[
\lim_{B \to \infty} \int_0^\infty e^{-st} P \left[ -yB \leq \xi - (tB^2), \xi(tB^2) \in (\alpha B, \beta B), \xi^+(tB^2) \leq xB \right] \, dt
\]

\[
= \int_0^\infty e^{-st} P \left[ -y \leq \inf_{u \leq t} w(u), w(u) \in du, \sup_{u \leq t} w(u) \leq x \right] \, dt,
\]

\[
x, y > 0, \quad x + y = 1,
\]

and the weak convergence as \( B \to \infty \) of the joint distribution \( Q^t(-y, \alpha, \beta, x, B) \) to the corresponding joint distribution of the Wiener process is proved. Equality (11) follows from (6) for \( B = 1 \).

To prove Corollary 2 we note that
\[
P \left[ \frac{1}{B^2} \chi(B) > t \right] = P \left[ \chi(B) > tB^2 \right] = Q^t(-y, -y, x, x, B),
\]

whence
\[
\lim_{B \to \infty} P \left[ \frac{1}{B^2} \chi(B) > t \right] = \lim_{B \to \infty} Q^t(-y, -y, x, x, B)
\]

\[
= P \left[ -y \leq \inf_{u \leq t} w(u), \sup_{u \leq t} w(u) \leq x \right] \, dt = P[\chi > t]
\]

\[
= \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} e^{-t(\pi(2\nu+1)\sigma/B)^2/2} \sin(x(2\nu + 1)\pi).
\]

Thus Theorem 2 and Corollary 2 are proved. \( \square \)

**Bibliography**


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Received 21/MAR/2003

Translated by V. SEMENOV