Теорія Ймовір. та Матем. Статист.
Вип. 70, 2004

Theor. Probability and Math. Statist.
No. 70, 2005, Pages 71–82
S 0094-9000(05)00632-0
Article electronically published on August 26, 2005

DIFFUSION APPROXIMATION OF EVOLUTIONARY SYSTEMS WITH EQUILIBRIUM IN ASYMPTOTIC SPLIT PHASE SPACE

UDC 519.21

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Abstract. In this paper we consider an additive functional of a Markov process with locally independent increments switched by a Markov process. For this functional, we obtain nonhomogeneous diffusion approximation results without balance condition on the drift parameter. A more general diffusion approximation result is obtained in the case of an asymptotic split phase space of the switching Markov process.

1. Introduction

Additive functionals of Markov processes are of great interest in theory (compensator of certain potentials, change of time in processes, local time problems, etc.) and applications (reward function in stochastic system theory, etc.); see [22, 12, 20]. In studying stochastic systems, such as additive functionals, two problems usually arise: the first concerns the complexity of the phase space, and the second concerns the fact that the local characteristics of the systems are not fixed but depend upon random factors. For the first problem, in order to obtain analytical or numerical tractable models, we have to reduce (simplify) the phase space. This is possible when some subsets are both connected with each other by small transition probabilities, and asymptotically connected. For the second problem, we describe the random changes of local characteristics by a stochastic process, called a switching process; see [1, 2, 3, 4].

In order to simplify the study of complex and nontractable systems by the classical analytical-numerical methods, diffusion approximation offers a real possibility to do this in particular in problems of optimization and control. Nowadays, diffusion approximation of birth-and-death processes is common practice in real problems. The most effective application of diffusion approximation algorithms seems to be for stochastic systems which describe queueing systems and networks as well as storage and transport processes widely used in problems of communication, insurance, various networks (computers, transport, biological, industrial, etc.) and more recently in reliability and maintenance problems [4, 5, 10, 11, 16]. Another technique to simplify the study of complex perturbed systems is to consider asymptotic split of phase space of perturbing process. In this paper we will consider both techniques for asymptotic study of additive functionals. Skorokhod [9, 23] and Kushner [19] considered weak convergence under the singular perturbation conditions for averaging problems, without diffusion approximation part.

2000 *Mathematics Subject Classification.* Primary 60J55, 60J75, 60F17.

Key words and phrases. Diffusion approximation, additive functional, asymptotic split phase space, Markov process with locally independent increments, nonhomogeneous diffusion.

This work is partially supported by INTAS project # 9900016.

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We will consider here additive functionals of the following form:

$$\xi(t) = \int_0^t \eta(ds; x(s)), \tag{1}$$

where $x(t)$, $t \geq 0$, is a switching Markov process and $\eta(t;x)$, $t \geq 0$, $x \in E$, is a switched $\mathbf{R}^d$-valued Markov process with locally independent increments. Such processes are also called "weakly differentiable" [9], or "locally infinitely divisible" [8], or "piecewise deterministic" [6] Markov processes.

We will consider this stochastic system in two cases. In the first case, $x(t)$, $t \geq 0$, is supposed to be ergodic and fixed. In the second case, the phase space is supposed to be asymptotically split, which is more general and gives a reduced phase space [12].

In our previous works [16, 17], we have discussed averaging and diffusion approximations of additive functionals of the form (1), for a reducible Markov process $x(t)$, $t \geq 0$, on the fluctuations with the balance condition

$$\int_E \pi(dx) a(u;x) = 0, \tag{2}$$

where $\pi(dx)$ is the stationary distribution of $x(t)$, $t \geq 0$, and $a(u;x)$ is the drift velocity of $\eta(t;x)$.

Another diffusion approximation can be obtained by considering fluctuations with respect to the average process $\hat{\xi}(t)$, $t \geq 0$, defined as the limit process in the following weak convergence scheme [16, 17]:

$$\xi^\varepsilon(t) = \int_0^t \eta(ds; x(s/\varepsilon)) \Longrightarrow \hat{\xi}(t) \quad \text{as } \varepsilon \to 0 \tag{3}$$

(see [17], Theorem 1).

In that case, we obtain a nonhomogeneous diffusion process $\hat{\zeta}(t)$, $t \geq 0$, in the following normalized scheme:

$$\varepsilon^{-1} \left[\int_0^t \eta\big(ds; x(s/\varepsilon^2)\big) - \hat{\xi}(t) \right] \Longrightarrow \hat{\zeta}(t) \quad \text{as } \varepsilon \to 0. \tag{4}$$

For such a diffusion approximation scheme, i.e., without balance condition (2), results concerning random evolution with semi-Markov switching were obtained in [18], Chapter 7, and in [2, 3] using other techniques.

In Section 2, we give the general setting for processes and examples. In Section 3, we give a diffusion approximation result without splitting. In Section 4, we give a diffusion approximation result for an asymptotic split phase space of the switching Markov process. And finally, in Section 5, we give the proofs of these results.

2. Preliminaries and examples

Let us consider a family of time-homogeneous cadlag Markov processes $\eta^\varepsilon(t;x)$, $t \geq 0$, $x \in E$, with locally independent increments in the series scheme, with a small series parameter $\varepsilon > 0$, depending on the phase state $x \in E$. They take values in the Euclidean space $\mathbf{R}^d$, $d \geq 1$, and their generators are given by

$$\mathbb{G}_\varepsilon(x)\varphi(u) = a_\varepsilon(u;x)\varphi'(u) + \varepsilon^{-1} \int_{\mathbf{R}^d} [\varphi(u + \varepsilon v) - \varphi(u) - \varepsilon v \varphi'(u)]\, \Gamma(u, dv; x). \tag{5}$$

The drift velocity $a_\varepsilon(u;x)$ and the measure of the random jumps $\Gamma(u, dv; x)$ depend on the phase state $x \in E$. A complete characterization of the above generator is given in [6]. It is worth noticing that the drift velocity of $a_\varepsilon(u;x)$ in (5) contains an initial drift and the drift due to the jumps. Note also that $\eta^\varepsilon(\cdot,\cdot)$ contains no diffusion part (see, e.g., [6, 8, 9]).

The time-homogeneous cadlag Markov jump process $x(t)$, $t \geq 0$, taking values in a measurable compact state space $(E, \mathcal{E})$, is defined by its generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)]. \tag{6}$$

The stochastic evolutionary system with Markov switching in series scheme is represented as follows:

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon)). \tag{7}$$

The regular Markov jump process can be defined by the Markov renewal process $(x_n, \theta_n, n \geq 0)$ given by the semi-Markov kernel

$$Q(x, B, t) = \mathsf{P}(x_{n+1} \in B, \theta_{n+1} \leq t \mid x_n = x) = P(x, B)\left(1 - e^{-q(x)t}\right). \tag{8}$$

We introduce the counting process

$$\nu(t) := \max\{n \colon \tau_n \leq t\}, \tag{9}$$

where the renewal moments and the auxiliary processes are

$$\tau_n = \sum_{k=1}^n \theta_k, \qquad n \geq 1,\ \tau_0 = 0,$$
$$\tau(t) = \tau_{\nu(t)}, \qquad \theta(t) = t - \tau(t).$$

The evolutionary system (7) can be represented in the following form:

$$\xi^\varepsilon(t) := \xi_0^\varepsilon + \sum_{k=0}^{\nu(t/\varepsilon)-1} \eta^\varepsilon(\varepsilon\theta_{k+1}; x_k) + \eta^\varepsilon(\varepsilon\theta(t); x(t/\varepsilon)). \tag{10}$$

As an illustration, we give here four typical evolutionary systems [16, 17].

1. A stochastic integral functional is determined by

$$\alpha^\varepsilon(t) := \int_0^t a_\varepsilon(x(s/\varepsilon))\, ds. \tag{11}$$

The corresponding generators (5) have the following form:

$$\mathbb{G}_\varepsilon(x)\varphi(u) = a_\varepsilon(x)\varphi'(u). \tag{12}$$

2. A dynamical system with Markov switching is determined by a solution of the evolutionary equation

$$\frac{d}{dt} U^\varepsilon(t) := a_\varepsilon(U^\varepsilon(t); x(t/\varepsilon)). \tag{13}$$

The respective generators (5) have the following form:

$$\mathbb{G}_\varepsilon(x)\varphi(u) = a_\varepsilon(u; x)\varphi'(u). \tag{14}$$

3. The storage jump process with Markov switching is determined by the generators

$$\mathbb{G}_\varepsilon(x)\varphi(u) = \varepsilon^{-1} \int_{\mathbf{R}^d} [\varphi(u + \varepsilon v) - \varphi(u)]\, \Gamma_\varepsilon(u, dv; x). \tag{15}$$

4. A compound Poisson process with Markov switching is determined by the generators

$$\mathbb{G}_\varepsilon(x)\varphi(u) = \varepsilon^{-1} \int_{\mathbf{R}^d} [\varphi(u + \varepsilon v) - \varphi(u)]\, \Gamma_\varepsilon(dv; x). \tag{16}$$

3. Diffusion approximation without the balance condition

The stochastic evolutionary system with Markov switching in the diffusion approximation scheme considered here is rescaled as follows:

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon\big(ds; x(s/\varepsilon^2)\big). \tag{17}$$

Consider the following centered stochastic system:

$$\zeta^\varepsilon(t) = \varepsilon^{-1}\big[\xi^\varepsilon(t) - \hat{\xi}(t)\big], \tag{18}$$

with the deterministic process $\hat{\xi}(t)$, $t \geq 0$, defined by the evolutionary equation

$$\frac{d}{dt}\hat{\xi}(t) = \hat{a}\big(\hat{\xi}(t)\big), \quad \hat{\xi}(0) = \hat{\xi}_0, \tag{19}$$

where $\hat{a}(\cdot)$ is the averaged drift coefficient defined below.

We will give a diffusion approximation result concerning this system without considering the balance condition (2).

Theorem 1 (Diffusion approximation without the balance condition)**.** *Let the stochastic evolutionary system $\xi^\varepsilon(t)$, $t \geq 0$, be defined by relations* (5) *and* (17). *Let the following conditions be fulfilled:*

C0: *The switching Markov process $x(t)$, $t \geq 0$, is uniformly ergodic with stationary distribution $\pi(dx)$ on the compact phase space E.*

C1: *The drift velocity has the following representation:*

$$a_\varepsilon(u;x) = a(u;x) + \varepsilon a_1(u;x).$$

C2: *The second moments of the jumps,*

$$\int_{\mathbf{R}^d} zz^*\,\Gamma(u, dz; x) = C(u,x),$$

are bounded.

C3: *The following asymptotic expansions hold:*

$$C(v+\varepsilon u, x) = \int_{\mathbf{R}^d} zz^*\,\Gamma(v+\varepsilon u, dz; x) = C(v,x) + \theta_\varepsilon(v,u,x),$$

$$a(v+\varepsilon u, x) = a(v,x) + \varepsilon u a'_v(v,x) + \theta_\varepsilon(v,u,x),$$

$$a_1(v+\varepsilon u, x) = a_1(v,x) + \theta_\varepsilon(v,u,x),$$

where negligible terms satisfy the following condition: for any $R > 0$,

$$\sup_{\substack{|u|<R\\ x\in E}} |\theta_\varepsilon(v,u,x)| \to 0 \quad \text{as } \varepsilon \to 0.$$

C4: *The initial values satisfy $\xi^\varepsilon(0) - \hat{\xi}_0 = \varepsilon\zeta_0^\varepsilon$, where*

$$\zeta_0^\varepsilon \Longrightarrow \hat{\zeta} \quad \text{and} \quad \sup_{\varepsilon>0} \mathsf{E}\,|\zeta_0^\varepsilon| \leq c < +\infty.$$

Then the weak convergence

$$\zeta^\varepsilon(t) \Longrightarrow \hat{\zeta}(t) \quad \text{as } \varepsilon \to 0,$$

takes place.

The limit diffusion process $\hat{\zeta}(t)$, $t \geq 0$, is determined by the generator of the coupled process $\hat{\zeta}(t)$, $\hat{\xi}(t)$, $t \geq 0$, which is

$$\hat{\mathbb{L}}\varphi(u,v) = a(u,v)\varphi'_u(u,v) + \frac{1}{2}B(v)\varphi''_{uu}(u,v) + \hat{a}(v)\varphi'_v(u,v),$$

where $a(u,v) = \hat{a}_1(v) + u\hat{a}'(v)$.

That means the limit diffusion process $\hat{\zeta}(t)$, $t \ge 0$, is nonhomogeneous in time and is defined by the generator

$$\hat{\mathbb{L}}_t\varphi(u) = \big[\hat{a}_1\big(\hat{\xi}(t)\big) + u\hat{a}'(\hat{\xi}(t))\big]\varphi'(u) + \frac{1}{2}\hat{B}\big(\hat{\xi}(t)\big)\varphi''(u).$$

The covariance function $\hat{B}(v)$ is defined by

$$\hat{B}(v) = \hat{A}(v) + \hat{C}(v), \tag{20}$$

where

$$\hat{A}(v) = 2\int_E \pi(dx)\tilde{a}(v;x)R_0\tilde{a}(v;x),$$

$$\hat{C}(v) = \int_E \pi(dx)C(v;x), \qquad C(v;x) = \frac{1}{2}\int_{\mathbf{R}^d} zz^*\mathbb{G}(v,dz;x),$$

$$\tilde{a}(v;x) := a(v;x) - \hat{a}(v), \qquad \hat{a}(v) := \int_E \pi(dx)a(v;x), \qquad \hat{a}_1(v) := \int_E \pi(dx)a_1(v;x),$$

and R_0 is the potential operator of Q (see [12]*)*

$$QR_0 = R_0Q = \Pi - I.$$

Remark 3.1. The stationary regime in the averaged process (19) is realized when the velocity has an equilibrium point ρ, i.e., $\hat{a}(\rho) = 0$. Then the limit diffusion process $\hat{\zeta}(t)$, $t \ge 0$, is of the Ornstein–Uhlenbeck type with the generator

$$\hat{\mathbb{L}}^0\varphi(u) = b(u)\varphi'(u) + \frac{1}{2}B\varphi''(u),$$

where

$$b(u) = b_0 + ub_1,$$

$$b_0 = \hat{a}_1(\rho), \qquad b_1 = \hat{a}'(\rho), \qquad B = \hat{B}(\rho).$$

4. Diffusion approximation with asymptotic split phase space

Let us suppose here that the phase space $(E, \mathcal{E})$, of the family of Markov jump processes $x^\varepsilon(t)$, $t \ge 0$, $\varepsilon > 0$, is split in the following way:

$$E = \bigcup_{k=1}^{N} E_k, \qquad E_k \cap E_{k'} = \varnothing, \quad k \ne k'.$$

These processes are defined by the generators

$$Q^\varepsilon\varphi(x) = q(x)\int_E P^\varepsilon(x,dy)[\varphi(y) - \varphi(x)].$$

The stochastic kernel $P^\varepsilon(x,dy)$ is represented by

$$P^\varepsilon(x,dy) = P(x,dy) + \varepsilon^2 P_1(x,dy),$$

where the stochastic kernel $P(x,dy)$ defines the embedded Markov chain x_n, $n \ge 0$, uniformly ergodic in every class E_k, $1 \le k \le N$, with stationary distributions $\rho_k(dx)$, $1 \le k \le N$. The Markov process $x(t)$, $t \ge 0$, defined by the generator

$$Q\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)],$$

is also uniformly ergodic in each class with the stationary distributions $\pi_k(dx)$, $1 \le k \le N$, which are represented as follows:

$$\pi_k(dx)q(x) = q_k\rho_k(dx), \qquad q_k = \int_{E_k} \pi_k(dx)q(x).$$

The perturbing kernel $P_1(x, dy)$ is supposed to satisfy the merging conditions [12].

Let us now introduce the following process:

$$\hat{x}^\varepsilon(t) := m\big(x^\varepsilon(t/\varepsilon^2)\big), \tag{21}$$

where the merging function m is defined by

$$m(x) = k \quad \text{if } x \in E_k,$$

and the limit merged Markov jump process $\hat{x}(t)$, $t \geq 0$, is defined on the merged phase space $\hat{E} = \{1, \dots, N\}$ by the intensity matrix

$$\hat{Q} = [q_{k\ell}; 1 \leq k, \ell \leq N],$$
$$q_{k\ell} = q_k p_{k\ell}, \quad k \neq \ell, \qquad p_{k\ell} = \int_{E_k} \pi_k(dx) P_1(x, E_\ell), \qquad q_{kk} = q_k p_{kk}.$$

The rescaled process that we will consider here is

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon\big(ds; x^\varepsilon(s/\varepsilon^2)\big), \tag{22}$$

and

$$\zeta^\varepsilon(t) = \varepsilon^{-1}\big[\xi^\varepsilon(t) - \hat{\xi}^\varepsilon(t)\big], \tag{23}$$

where the process $\hat{\xi}^\varepsilon(t)$, $t \geq 0$, is defined by the following evolutionary equation:

$$\frac{d}{dt}\hat{\xi}^\varepsilon(t) = \hat{a}\big(\hat{\xi}^\varepsilon(t); \hat{x}^\varepsilon(t)\big), \qquad \hat{\xi}^\varepsilon(0) = \hat{\xi}_0, \tag{24}$$

with $\hat{a}(v, y) = \int_{E_y} a(v; x)\, \pi_y(dx)$, $y \in \hat{E}$.

Theorem 2 (Diffusion approximation in split phase space)**.** *Let the stochastic evolutionary system $\xi^\varepsilon(t)$, $t \geq 0$, be defined by relation* (22) *and the asymptotic split of phase space as given above. Let conditions* C1–C4 *of Theorem* 1*, be fulfilled.*

Then the weak convergence

$$\zeta^\varepsilon(t) \Longrightarrow \hat{\zeta}(t) \quad \text{as } \varepsilon \to 0$$

takes place.

The limit conditional perturbed diffusion process $\hat{\zeta}(t)$, $t \geq 0$, is determined by the generator of the Markov process $\hat{\zeta}(t)$, $\hat{\xi}(t)$, $\hat{x}(t)$, $t \geq 0$, which is

$$\mathbb{L}\varphi(u, v, y) = \hat{\mathbb{L}}_t \varphi(u, \cdot, \cdot) + \hat{\mathbb{L}}(y)\varphi(\cdot, v, y),$$

where $\hat{\mathbb{L}}(y)$ is the generator of the Markov process $\hat{\xi}(t)$, $\hat{x}(t)$, $t \geq 0$,

$$\hat{\mathbb{L}}(y)\varphi(v, y) = \hat{Q}\varphi(\cdot, y) + \hat{a}(v, y)\varphi_v'(v, y),$$

and the generator of $\hat{\zeta}(t)$, $t \geq 0$, is

$$\hat{\mathbb{L}}_t \varphi(u) = \Big[\hat{a}_1\big(\hat{\xi}(t); \hat{x}(t)\big) + u\hat{a}_v'\big(\hat{\xi}(t); \hat{x}(t)\big)\Big]\varphi'(u) + \frac{1}{2}\hat{B}\big(\hat{\xi}(t); \hat{x}(t)\big)\varphi''(u),$$

where the drift and the diffusion coefficients are

$$\hat{B}(v;y) = \hat{C}(v;y) + \hat{A}(v;y),$$
$$\hat{C}(v;y) = \int_{E_y} \pi(dx) C(v;x),$$
$$\hat{A}(v;y) = \int_{E_y} \pi_y(dx)\tilde{a}(v;x)R_0\tilde{a}(v;x),$$
$$\hat{a}_1(v;y) = \int_{E_y} \pi_y(dx)a_1(v;x), \qquad \hat{a}'_v(v;y) = \int_{E_y} \pi_y(dx)a'_v(v;x).$$

The process $\hat{\xi}(t)$, $t \geq 0$, is defined by a solution of the following equation:

$$\frac{d}{dt}\hat{\xi}(t) = \hat{a}\big(\hat{\xi}(t);\hat{x}(t)\big), \qquad \hat{\xi}^\varepsilon(0) = \hat{\xi}_0. \tag{25}$$

Remark 4.1. In terms of stochastic differential equations, the limit process

$$\hat{\zeta}(t), \qquad t \geq 0,$$

is defined as follows:

$$d\hat{\zeta}(t) = \hat{A}\big(\hat{\xi}(t);\hat{x}(t)\big)\,dt + \hat{B}\big(\hat{\xi}(t);\hat{x}(t)\big)\,dw(t),$$
$$d\hat{\xi}(t) = \hat{a}\big(\hat{\xi}(t),\hat{x}(t)\big)\,dt,$$

where $w(t)$, $t \geq 0$, is the standard Wiener process.

5. Proof of theorems

Let us start by giving the proof of Theorem 2, which is the most general. The proof of Theorem 1 is a special case for which we will give the additional elements. Let $C_0^2(\mathbf{R}^d)$ be the space of real-valued functions defined on $\mathbf{R}^d$ with compact support.

Proof of Theorem 2. Consider the $\mathbf{R} \times E \times \mathbf{R} \times \hat{E}$-valued family of processes

$$\zeta^\varepsilon(t), \qquad x^\varepsilon\big(t/\varepsilon^2\big), \qquad \hat{\xi}^\varepsilon(t), \qquad \hat{x}^\varepsilon(t), \qquad t \geq 0, \qquad \varepsilon > 0. \tag{26}$$

We will denote by (u,x,v,y) the generic element of the state space $\mathbf{R} \times E \times \mathbf{R} \times \hat{E}$, with $y := m(x)$.

Lemma 1. *The generator of the quadruple Markov process* (26)*, under conditions* C1–C6 *of Theorem* 1*, is*

$$\mathbb{L}^\varepsilon = \varepsilon^{-2}Q^\varepsilon + \varepsilon^{-1}\tilde{\mathbb{A}}(v,x) + \tilde{\mathbb{G}}_\varepsilon(v,x) + \hat{\mathbb{L}}(y(x)), \tag{27}$$

where

$$\tilde{\mathbb{A}}(v,y(x),x) = \mathbb{A}(v,x) - \hat{\mathbb{A}}(v,y(x)),$$
$$\mathbb{A}(v,x)\varphi(u) = a(v;x)\varphi'(u),$$
$$\hat{\mathbb{A}}(v,y(x))\varphi(u) = \hat{a}(v,y(x))\varphi'(u),$$
$$\tilde{\mathbb{G}}_\varepsilon(v,x) = \mathbb{L}_0(v,x) + \gamma_\varepsilon(v;x),$$
$$\mathbb{L}_0(v,x)\varphi(u) = \big[a_1(v;x) + ua'_v(v;x)\big]\varphi'(u) + \frac{1}{2}C(v;x)\varphi''(u).$$

The negligible operator $\gamma_\varepsilon(v;x)$ is such that, for all $\varphi \in C_0^2(\mathbf{R}^d)$,

$$\|\gamma_\varepsilon(v;x)\varphi\| \to 0 \quad \textit{as } \varepsilon \to 0,$$

and

$$\hat{\mathbb{L}}(y)\varphi(v,y) = \hat{a}(v;y)\varphi'(v;\cdot) + \hat{Q}_1\varphi(\cdot,y)$$

is the generator of the coupled Markov process $\hat{\xi}(t)$, $\hat{x}(t)$, $t \geq 0$.

Proof. We compute the part of the generator corresponding to the first two components, i.e., $\zeta^\varepsilon(t)$, $x_t^\varepsilon := x^\varepsilon(t/\varepsilon^2)$. We get

$$
\begin{aligned}
&\mathsf{E}\left[\varphi\big(u+\Delta\zeta^\varepsilon(t), x^\varepsilon_{t+\Delta t}\big) - \varphi(u,x) \mid \zeta^\varepsilon(t) = u, x_t^\varepsilon = x, \hat{\xi}^\varepsilon(t) = v, \hat{x}^\varepsilon(t) = y\right] \\
&\quad = \varepsilon^{-2}\Delta t Q^\varepsilon \varphi(\cdot, x) \\
&\qquad + \mathsf{E}\left[\varphi(u+\Delta\zeta^\varepsilon(t), x) - \varphi(u,x) \mid \zeta^\varepsilon(t) = u, x_t^\varepsilon = x, \hat{\xi}^\varepsilon(t) = v, \hat{x}^\varepsilon(t) = y\right] + o(\Delta t) \\
&\quad = \varepsilon^{-2}\Delta t Q^\varepsilon \varphi(\cdot, x) \\
&\qquad + \mathsf{E}\Big[\varphi\left(u + \varepsilon^{-1}\big(\Delta\xi^\varepsilon(t) - \Delta\hat{\xi}^\varepsilon(t)\big), x\right) - \varphi(u,x) \;\Big| \\
&\qquad\qquad\qquad\qquad\qquad \xi^\varepsilon(t) = v + \varepsilon u, x_t^\varepsilon = x, \hat{\xi}^\varepsilon(t) = v, \hat{x}^\varepsilon(t) = y\Big] \\
&\qquad + o(\Delta t) \\
&\quad = \varepsilon^{-2}\Delta t Q^\varepsilon \varphi(\cdot, x) \\
&\qquad + \mathsf{E}\left[\varphi(u+\varepsilon^{-1}\Delta\xi^\varepsilon(t), x) - \varphi(u,x) \mid \xi^\varepsilon(t) = v + \varepsilon u, x_t^\varepsilon = x, \hat{\xi}^\varepsilon(t) = v, \hat{x}^\varepsilon(t) = y\right] \\
&\qquad - \varepsilon^{-1}\,\mathsf{E}\left[\Delta\hat{\xi}^\varepsilon(t) \mid x_t^\varepsilon = x, \hat{\xi}^\varepsilon(t) = v, \hat{x}^\varepsilon(t) = y\right] \varphi_u'(u,x) + o(\Delta t) \\
&\quad = \Delta t \varepsilon^{-2} Q^\varepsilon \varphi(\cdot, x) + \Delta t \varepsilon^{-1}\mathbb{G}_\varepsilon(v,x)\varphi(u,x) - \Delta t\varepsilon^{-1}\hat{\mathbb{A}}(v,y)\varphi(u,x) + o(\Delta t),
\end{aligned}
$$

where

$$
\mathbb{G}_\varepsilon(v,x)\varphi(u) = a_\varepsilon(v+\varepsilon u; x)\varphi'(u) + \varepsilon^{-1}\int_{\mathbf{R}^d}[\varphi(u+\varepsilon z) - \varphi(u) - \varepsilon z\varphi'(u)]\,\Gamma(v+\varepsilon u, dz; x).
$$

Now, the generator of the Markov process $\hat{\xi}^\varepsilon(t)$, $\hat{x}^\varepsilon(t)$ is obtained by a straightforward calculus, and thus we get the global generator of the above quadruple Markov process (26). $\square$

We will consider the test functions

$$
\varphi^\varepsilon(u,v;x,y) = \varphi(u,v) + \varepsilon\varphi_1(u,v;x,y) + \varepsilon^2\varphi_2(u,v;x,y).
$$

The singular perturbation problem is formulated as follows:

$$
\mathbb{L}^\varepsilon\varphi^\varepsilon = \mathbb{L}\varphi + \gamma_\varepsilon\varphi. \tag{28}
$$

Now, as in Lemma 3.3 in [12], we get the following result (see also [4]).

Lemma 2. *The perturbed limit conditional diffusion process $\hat{\zeta}(t)$, $t \geq 0$, is determined by the generator of the triple Markov process $\hat{\zeta}(t)$, $\hat{\xi}(t)$, $\hat{x}(t)$, $t \geq 0$,*

$$
\hat{\mathbb{L}} = \hat{\mathbb{L}}_0(v,y) + \hat{\mathbb{L}}(y),
$$

where

$$
\hat{\mathbb{L}}_0(v,y)\varphi(u) = [\hat{a}_1(v;y) + u\hat{a}_v'(v;y)]\varphi'(u) + \frac{1}{2}\hat{B}(v;y)\varphi''(u).
$$

Proof. From the asymptotic representation (28), we get

$$
\begin{gathered}
Q\varphi(u,v) = 0, \\
Q\varphi_1 + \tilde{\mathbb{A}}(x)\varphi = 0, \\
Q\varphi_2 + \tilde{\mathbb{A}}(v,x)\varphi_1 + \mathbb{L}_0(v,x)\varphi + \hat{\mathbb{L}}(y)\varphi = \mathbb{L}.
\end{gathered}
$$

So, $Q\varphi(u,v) = 0$ means that $\varphi \in N_Q$, the null-space of the operator Q. We have $\Pi\tilde{\mathbb{A}}(v,x)\Pi\varphi = 0$, which means that the second equation above satisfies the solvability condition, and we get $\varphi_1 = R_0\tilde{\mathbb{A}}(u,v)\varphi$. The solvability condition of the third equation gives

$$\hat{\mathbb{L}} = \widehat{\tilde{\mathbb{A}}(v,x)R_0\tilde{\mathbb{A}}}(v,x) + \widehat{\mathbb{L}_0(v,x)} + \hat{\mathbb{L}}(y),$$

from which the claimed result follows immediately. □

For the compactness of the process (26), we need to prove only the compactness of the family $\zeta^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$. For this, we need the following theorem, which is a compilation for our conditions of Theorem 9.4, p. 145, and Corollary 8.6, p. 231, of Ethier and Kurtz [7]. See also Theorems A and B in [17].

Theorem C. *Consider the family of coupled processes*

$$\zeta^\varepsilon(t), \qquad x^\varepsilon\left(t/\varepsilon^2\right), \qquad t \geq 0, \qquad \varepsilon > 0, \tag{29}$$

a Markov process $\zeta(t)$, $t \geq 0$, of generator $\mathbb{L}$ with domain $\mathcal{D}(\mathbb{L})$, and an algebra

$$C_a \subset \overline{\mathcal{D}(\mathbb{L})}$$

that separates points. Consider also the test functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x), \qquad \varphi \in C_a.$$

Suppose that the following conditions are fulfilled:

(C1) *The compact containment condition for the family* (29) *holds.*
(C2) *For every $T \in \mathbf{R}_+$ we have*

$$\lim_{\varepsilon\to 0} \mathsf{E}\left[\sup_{0\leq t\leq T}\left|\varphi^\varepsilon\big(\zeta^\varepsilon(t),\hat{\xi}(t)\big) - \varphi\big(\zeta^\varepsilon(t)\big)\right|\right] = 0. \tag{30}$$

(C3) *For every $T \in \mathbf{R}_+$ we have*

$$\sup_{\varepsilon>0} \mathsf{E}\left[\|\mathbb{L}^\varepsilon\varphi^\varepsilon\|_{\infty,T}\right] < +\infty, \tag{31}$$

where $\|\varphi\|_{\infty,T} = \sup_{0\leq t\leq T}|\varphi(\zeta(t))|$.
(C4) *The convergence in probability of the initial values holds, i.e.,*

$$\zeta^\varepsilon(0) \xrightarrow{\mathsf{P}} \zeta(0) \quad \text{as } \varepsilon \to 0,$$

with uniformly bounded expectation

$$\sup_{\varepsilon>0} \mathsf{E}\,|\zeta^\varepsilon(0)| \leq c < +\infty.$$

Then

$$\zeta^\varepsilon(t) \Longrightarrow \zeta(t) \quad \text{as } \varepsilon \to 0.$$

Let us first prove the compactness containment condition for the processes (26).

Lemma 3. *If $\sup_{\varepsilon>0}\mathsf{E}\,|\zeta^\varepsilon(0)| \leq c < +\infty$, then the family of stochastic processes $\zeta^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$, satisfies the compact containment condition*

$$\lim_{\ell\to\infty}\sup_{\varepsilon>0} \mathsf{P}^\varepsilon\left(\sup_{0\leq t\leq T}|\zeta^\varepsilon(t)| \geq \ell\right) = 0. \tag{32}$$

Proof. Let us consider the test functions

$$\varphi_0^\varepsilon(u,x) = \varphi_0(u) + \varepsilon\varphi_1(u,x),$$

where $\varphi_0(u) = \sqrt{1+u^2}$.

From the asymptotic representation

$$\mathbb{L}^\varepsilon\varphi_0^\varepsilon(u,x) = \mathbb{L}\varphi_0 + \gamma_\varepsilon\varphi_0$$

and the definition of the operator $\mathbb{A}^0(v,x) := -R_0\tilde{\mathbb{A}}(v,x)$, we get

$$\varphi_1 = -R_0\mathbb{G}(x)\varphi_0 = \mathbb{G}^0(x)\varphi_0.$$

Hence

$$\varphi_0^\varepsilon(u,x) = \varphi_0(u) + \varepsilon\varphi_1(u,x) = [1+\varepsilon\mathbb{A}^0(v,x)]\varphi_0(u).$$

Then the proof follows as in Lemma 6 in [17]. □

The other conditions of Theorem C are as follows. The separating points algebra C_a considered here is $C_0^2(\mathbf{R}^d \times \hat{E})$. Conditions (C2) and (C3) are as follows:

$$\begin{aligned}\mathsf{E}\left[\sup_{0\le t\le T}\left|\varphi^\varepsilon(\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2)) - \varphi(\xi^\varepsilon(t))\right|\right]&\\ = \varepsilon\,\mathsf{E}\sup_{0\le t\le T}\left|\varphi_1(\xi^\varepsilon(t), x^\varepsilon(t/\varepsilon^2))\right| \to 0 \quad \text{as } \varepsilon\to 0,&\end{aligned} \tag{33}$$

and

$$\sup_{0<\varepsilon\le\varepsilon_0}\mathsf{E}\left[\|\mathbb{L}^\varepsilon\varphi^\varepsilon\|_{\infty,T}\right] \le \sup_{0<\varepsilon\le\varepsilon_0}\mathsf{E}\left[\|\mathbb{L}\varphi\|_{\infty,T}\right] + \sup_{0<\varepsilon\le\varepsilon_0}\mathsf{E}\left[\|\theta^\varepsilon\|_{\infty,T}\right] < +\infty. \tag{34}$$

The proof of Theorem 2 is now achieved by considering Theorem C. □

Proof of Theorem 1. Consider the following $\mathbf{R}\times\mathbf{R}\times E$-valued family of processes:

$$\zeta^\varepsilon(t), \qquad \hat{\xi}(t), \qquad x\big(t/\varepsilon^2\big), \qquad t\ge 0, \qquad \varepsilon > 0, \tag{35}$$

where $\zeta^\varepsilon(t)$ is given by (18), and where $\hat{\xi}(t)$, $t\ge 0$, is the averaged system defined by the differential equation (19).

Denote by (u,v,x) the generic element of the space $\mathbf{R}\times\mathbf{R}\times E$. Now, by an easy calculus we get the following result.

Lemma 4. *The generator of the above Markov process* (35) *is*

$$\mathbb{L}^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}\tilde{\mathbb{A}}(v;x) + \mathbb{L}_0(v;x) + \hat{\mathbb{A}} + \gamma_\varepsilon(v;x), \tag{36}$$

where

$$\mathbb{L}_0(v;x)\varphi(u) = [ua_v'(v;x) + a_1(v;x)]\varphi'(u) + \frac{1}{2}C(v;x)\varphi''(u), \tag{37}$$

and

$$\begin{gathered}\tilde{\mathbb{A}}(v;x) = \mathbb{A}(v;x) - \hat{\mathbb{A}}(v),\\ \mathbb{A}(v;x)\varphi(u) = a(v;x)\varphi'(u),\\ \hat{\mathbb{A}}(v)\varphi(u) = \hat{a}(v)\varphi'(u),\\ \hat{\mathbb{A}}\varphi(v) = \hat{a}(v)\varphi'(v),\\ \hat{a}(v) := \int_E \pi(dx)a(v;x).\end{gathered}$$

Proof. This generator is a particular case of the generator (27), where $\hat{E} = \{1\}$. □

As previously, we obtain the limit operator $\hat{\mathbb{L}}(v)$ of the operators $\mathbb{L}^\varepsilon$, as $\varepsilon\to 0$, namely

$$\hat{\mathbb{L}}(v) = \hat{\mathbb{L}}_0(v) + \hat{\mathbb{A}}, \tag{38}$$

where

$$\hat{\mathbb{L}}_0(v)\varphi(u) = [u\hat{a}'(v) + \hat{a}_1(v)]\,\varphi'(u) + \frac{1}{2}\hat{B}(v)\varphi''(u), \tag{39}$$

and

$$\hat{B}(v) = \hat{C}(v) + \hat{\mathbb{A}}(v). \tag{40}$$

$$\hat{\mathbb{A}}(v) = 2 \int \pi(dx)\tilde{a}(v;x)R_0\tilde{a}(v;x). \tag{41}$$

The limit process $\hat{\zeta}(t)$, $t \geq 0$, is a Markov nonhomogeneous process, with generator

$$\mathbb{L}_t\varphi(u) = \hat{\mathbb{L}}_0\big(\hat{\xi}(t)\big)\varphi(u). \tag{42}$$

The compactness condition is a particular case of those of Theorem 2, hence we omit it here. □

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Received 20/JAN/2004
Originally published in English