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ON THE PRODUCT OF A RANDOM AND A REAL MEASURE

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Abstract. The product of a random measure X and a real measure Y is defined as a random measure on $X \times Y$. We obtain conditions under which the integral of a real function with respect to the product measure equals the iterated integrals of this function.

Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be measurable spaces, $Z = X \times Y$, and $\mathcal{B}_Z = \mathcal{B}_X \otimes \mathcal{B}_Y$. By $L_0 = L_0(\Omega, \mathcal{F}, \mathsf{P})$ we denote the set of all random variables defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ (to be more specific, L_0 is the set of classes of equivalent random variables). The convergence in L_0 is the convergence in probability.

Definition 1. Any σ-additive mapping $\mu\colon \mathcal{B}_X \to L_0$ is called a random measure on $\mathcal{B}_X$.

Note that we do not assume that μ is nonnegative and we do not pose any moment condition.

Here are some examples. If $X(t)$, $0 \le t \le T$, is a continuous square-integrable martingale, then $\mu(A) = \int_0^T I_A(t)\, dX(t)$ is a random measure on Borel sets of $[0, T]$. A fractional Brownian motion $B^H(t)$ for $H > \frac{1}{2}$ defines a random measure in a similar way (this follows from inequality (3.11) in [1]). Other examples as well as conditions for increments of a stochastic process to generate a random measure can be found in Chapters 7 and 8 of [2].

Further let μ be a random measure on $\mathcal{B}_X$, and m a finite nonnegative measure on $\mathcal{B}_Y$. A set $A \in \mathcal{B}_X$ is called μ-negligible if

$$\mu(B) = 0 \quad \text{a.s.}$$

for all $B \in \mathcal{B}_X$ such that $B \subset A$. Let ξ be a random variable and put

$$\|\xi\| = \sup\{\delta\colon\ \mathsf{P}\{|\xi| > \delta\} > \delta\}.$$

The integral $\int_A f\, d\mu$ is defined and studied in [3] where $f\colon X \to \mathbb{R}$ is a real measurable function and $A \in \mathcal{B}_X$. When constructing this integral one starts with simple functions and proceeds similarly to [2, Chapter 7] (see also [4]). In particular, any measurable bounded function f is integrable with respect to any measure μ.

In this paper, we define the product of a random and a real measure and prove analogs of Fubini's theorem for integrals of real functions.

Theorem 1. *There exists a unique random measure η on $\mathcal{B}_Z$ such that*

$$\eta(A_1 \times A_2) = \mu(A_1) m(A_2)$$

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for all $A_1 \in \mathcal{B}_X$ and $A_2 \in \mathcal{B}_Y$. If $f\colon Z \to \mathbb{R}$ is integrable on Z with respect to η, then for all fixed $x \in X$, except for a μ-negligible set, the function $f(x,\cdot)\colon Y \to \mathbb{R}$ is integrable on Y with respect to m and $\int_Y f(x,y)\,dm(y)$ is integrable on X with respect to μ. Moreover

$$\int_Z f(x,y)\,d\eta = \int_X d\mu(x)\int_Y f(x,y)\,dm(y). \tag{1}$$

Proof. Let $A \in \mathcal{B}_Z$ and put

$$\eta(A) = \int_X d\mu(x)\int_Y I_A(x,y)\,dm(y). \tag{2}$$

The latter integral exists, since the inner integral does not exceed $m(Y)$ and any bounded function of x is integrable with respect to μ. Corollary 1.2 of [3] implies that η is σ-additive in probability. The equality $\eta(A_1\times A_2) = \mu(A_1)m(A_2)$ is obvious; the uniqueness of η can be proved in a standard way.

Now we prove (1). If $f = I_B$, $B \in \mathcal{B}_Z$, then (1) follows from (2). Thus (1) holds for simple functions on Z. Let $f\colon Z \to \mathbb{R}$ be a measurable bounded function and let f_n, $n \ge 1$, be a sequence of simple functions such that $f_n \to f$ and $|f_n| \le |f|$. Equality (1) for f follows from the same equality for functions f_n by passing to the limit with the help of Corollary 1.2 in [3]. Since m is finite, the functions f, $f(x,\cdot)$, and $\int_Y f(x,y)\,dm(y)$ are bounded and integrable. The latter result is an analog of the Lebesgue dominated convergence theorem.

Let $f\colon Z \to \mathbb{R}$ be an arbitrary function integrable with respect to η. Let D be the set of points $x \in X$ for which $f(x,\cdot)$ is nonintegrable with respect to m. For $x \in D$ we have

$$\int_Y |f(x,y)|\,dm(y) = +\infty$$

and thus the classical Fubini theorem implies that $D \in \mathcal{B}_X$. Assume that D is not a μ-negligible set. Then for some $\varepsilon_0 > 0$ and all $k \ge 1$ there are $n(k) > k$ and $D_1 \subset D$, $D_1 \in \mathcal{B}_X$, such that $\|\mu(D_1)\| > \varepsilon_0$ and

$$\int_Y |f(x,y)| I_{\{k<|f|\le n(k)\}}\,dm(y) > 1$$

for $x \in D_1$. Theorem 1.3 of [3] with $h(x) = 1$ and $A = D_1$ implies that

$$\|\mu(D_1)\| \le 16 \sup_{B\subset D_1}\left\|\int_B d\mu(x)\int_Y |f(x,y)| I_{\{k<|f|\le n(k)\}}\,dm(y)\right\|.$$

Equality (1) is already proved for the bounded function $|f(x,y)|I_{\{k<|f|\le n(k)\}}$. Now Corollary 1.2 of [3] implies that

$$\sup_{B\in\mathcal{B}_X}\left\|\int_{B\times Y} |f(x,y)| I_{\{k<|f|\le n(k)\}}\,d\eta\right\| \to 0, \qquad k\to\infty,$$

since f is integrable with respect to η. This result contradicts the condition$\|\mu(D_1)\| > \varepsilon_0$. Thus the set of points x where $f(x,\cdot)$ is nonintegrable is μ-negligible. In what follows we assume that this set is empty (in fact, we change the values of f on a η-negligible set).

Now we prove that the function $g(x) = \int_Y f(x,y)\,dm(y)$ is integrable with respect to μ. Consider the functions $g_n(x) = \int_Y f(x,y)I_{\{|f|\le n\}}\,dm(y)$, $n \ge 1$, and

$$h(x) = \int_Y |f(x,y)|\,dm(y).$$

Equality (1) is already proved for the bounded functions $fI_{\{|f|\le n\}}$; we also have that $g_n(x) \to g(x)$, $x \in X$. For all $c > 0$

$$\{x\colon |g_n(x)| > c\} \subset \{x\colon |h(x)| > c\},$$

thus

$$\sup_{n,A\in\mathcal{B}_X}\left\|\int_{A\cap\{|g_n|>c\}} g_n\,d\mu\right\| = \sup_{n,A\in\mathcal{B}_X}\left\|\int_{(A\cap\{|g_n|>c\})\times Y} fI_{\{|f|\le n\}}\,d\eta\right\|$$
$$\le \sup_{B\in\mathcal{B}_Z}\left\|\int_{B\cap(\{|h|>c\}\times Y)} f\,d\eta\right\|.$$

The set $\{x\colon |h(x)|>c\}$ approaches the empty set as $c\to\infty$. Since f is integrable with respect to η, Corollary 1.2 of [3] yields that conditions for the uniform integrability hold in Theorem 1.7 of [3] (see also the theorem in [5]). Thus Theorem 1.7 of [3] implies that g is integrable. Now equality (1) can be proved by passing to the limit along the sequence g_n. □

Remark 1. The existence of a random measure η defined by (2) and the equality between the integrals on the left- and right-hand sides of (1) is stated without proof in Example 10.1.2 of [2].

The product of a random measure with independent values with itself is constructed in [2, Chapter 10]. If m is Lebesgue measure and μ is generated by increments of fractional Brownian motion, then a result on the product of m and μ is obtained in [6].

The iterated integrals coincide only under some additional assumptions. First we prove some auxiliary results.

Lemma 1. *If $a_k\in\mathbb{R}$, $a_k>0$, and $A_k\in\mathcal{B}_X$, $k\ge 1$, are such that*

$$\sup_{x\in\mathbb{R}}\sum_{k=1}^{\infty} a_k I_{A_k}(x)<\infty,$$

then

$$\sum_{k=1}^{\infty} a_k^2\mu^2(A_k)<\infty \quad a.s. \tag{3}$$

Proof. If inequality (3) does not hold, then for some $\varepsilon_0>0$ and all $c>0$ there exists $n\ge 1$ such that $\mathsf{P}(\Omega_1)\ge\varepsilon_0$ for $\Omega_1=\left\{\omega\in\Omega:\sum_{k=1}^n a_k^2\mu^2(A_k)\ge c\right\}$.

Consider independent Bernoulli random variables ε_k, $1\le k\le n$, defined on another probability space $(\Omega',\mathcal{F}',\mathsf{P}')$, that is, $\mathsf{P}'(\varepsilon_k=1)=\mathsf{P}'(\varepsilon_k=-1)=\frac12$. Lemma V.4.3 (a) of [7] yields that

$$\mathsf{P}'\left[\left(\sum_{k=1}^n\lambda_k\varepsilon_k\right)^2\ge\frac14\sum_{k=1}^n\lambda_k^2\right]\ge\frac18,\qquad \lambda_k\in\mathbb{R}.$$

Thus

$$\mathsf{P}'\left[\omega'\colon\ \left(\sum_{k=1}^n\varepsilon_k(\omega')a_k\mu(A_k,\omega)\right)^2\ge\frac{c}{4}\right]\ge\frac18$$

for all $\omega\in\Omega_1$. Integrating over the set Ω_1 we get

$$\mathsf{P}\times\mathsf{P}'\left[(\omega,\omega')\colon\ \left(\sum_{k=1}^n\varepsilon_k(\omega')a_k\mu(A_k,\omega)\right)^2\ge\frac{c}{4}\right]\ge\frac{\varepsilon_0}{8}.$$

Hence there exists $\omega_0'\in\Omega'$ such that

$$\mathsf{P}\left[\omega:\left(\sum_{k=1}^n\varepsilon_k(\omega_0')a_k\mu(A_k,\ \omega)\right)^2\ge\frac{c}{4}\right]\ge\frac{\varepsilon_0}{8}.$$

Since $\varepsilon_k(\omega_0') = \pm 1$, there exists a simple function $f\colon X \to \mathbb{R}$ such that

$$\mathsf{P}\left[\left|\int_X f(x)\,d\mu(x)\right| \ge \frac{\sqrt{c}}{2}\right] \ge \frac{\varepsilon_0}{8}, \qquad |f(x)| \le \sup_{x\in\mathbb{R}} \sum_{k=1}^{\infty} a_k I_{A_k}(x).$$

Recall that $\varepsilon_0 > 0$ is fixed, while c is arbitrary. The latter inequality contradicts the boundedness in probability of the set of values of integrals of simple functions $\int_X f\,d\mu$ such that $|f(x)| \le 1$ (see Theorem 1.1 in [3] or Theorem 2 in [4]). Therefore the lemma is proved. $\square$

In what follows, $X = (a, b] \subset \mathbb{R}$ and $\mathcal{B}_X$ is the Borel σ-algebra. Let

$$\Delta_{kn} = \left(a + (k-1)2^{-n}(b-a), a + k2^{-n}(b-a)\right], \qquad n \ge 0,\ 1 \le k \le 2^n.$$

Lemma 2. *For all* $\alpha > \frac{1}{2}$

$$\sum_{n=0}^{\infty} 2^{-n\alpha} \sum_{k=1}^{2^n} |\mu(\Delta_{kn})| < \infty \quad a.s. \tag{4}$$

Proof. Let $\alpha = \frac{1}{2} + \beta$. Using the Cauchy–Bunyakovskiĭ inequality we obtain

$$\begin{aligned}\left(\sum_{n=0}^{\infty} 2^{-n\alpha} \sum_{k=1}^{2^n} |\mu(\Delta_{kn})|\right)^2 &\le \left(\sum_{n=0}^{\infty} 2^{-n\beta}\right)\left(\sum_{n=0}^{\infty} 2^{-n(1+\beta)}\left(\sum_{k=1}^{2^n} |\mu(\Delta_{kn})|\right)^2\right)\\ &\le \left(\sum_{n=0}^{\infty} 2^{-n\beta}\right)\left(\sum_{n=0}^{\infty} 2^{-n\beta}\sum_{k=1}^{2^n} \mu^2(\Delta_{kn})\right).\end{aligned}$$

It remains to apply Lemma 1 to the second factor. $\square$

In the sequel the integrals of random functions $\xi(y) = \xi(y, \omega)$, $y \in Y$, with respect to a real measure m are defined according to Definition 5.2 in [3] (see also [8]) (an equivalent condition is given in Theorem 3.8 of [3]). A random function $\xi(y, \omega)$ is integrable with respect to m if it is measurable with respect to the pair of arguments (y, ω) and $\mathsf{P}\left\{\sup_{y\in Y} |\xi(y)| < \infty\right\} = 1$. The integral $\int_Y \xi(y)\,dm(y)$ can be defined for any fixed ω as the limit of the Lebesgue integrals of simple functions.

Theorem 2. *Let* $X = (a, b] \subset \mathbb{R}$, *and let* $\mathcal{B}_X$ *be the Borel* σ*-algebra. Let* $f\colon Z \to \mathbb{R}$ *be a bounded and measurable function. Assume that there exist numbers* $\alpha > \frac{1}{2}$ *and* $L > 0$ *such that*

$$|f(x_1, y) - f(x_2, y)| \le L|x_1 - x_2|^{\alpha}$$

for all $x_1, x_2 \in X$ *and* $y \in Y$. *Then*

$$\int_Z f(x, y)\,d\eta = \int_Y dm(y) \int_X f(x, y)\,d\mu(x). \tag{5}$$

Proof. The left-hand side of equality (5) is well defined, since f is a bounded function. Now we show that the right-hand side of (5) is well defined, too. Let $x_{kn} \in \Delta_{kn}$ be arbitrary numbers and

$$S_n(y) = \sum_{k=1}^{2^n} f(x_{kn}, y)\mu(\Delta_{kn}).$$

The Hölder condition and Corollary 1.2 of [3] imply that

$$S_n(y) \xrightarrow{\mathsf{P}} \int_X f(x, y)\,d\mu(x) \quad \text{as } n \to \infty$$

for all $y \in Y$. It follows from the Hölder condition that

$$\begin{aligned}\sum_{n=1}^{\infty} &|S_n(y) - S_{n-1}(y)| \\ &= \sum_{n=1}^{\infty} \Bigg| \sum_{k=1}^{2^{n-1}} \big((f(x_{(2k-1)n}, y) - f(x_{k(n-1)}, y))\mu(\Delta_{(2k-1)n}) \\ &\qquad\qquad + (f(x_{(2k)n}, y) - f(x_{k(n-1)}, y))\mu(\Delta_{(2k)n})\big)\Bigg| \\ &\le L(b-a)^{\alpha} \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} 2^{-(n-1)\alpha} (|\mu(\Delta_{(2k-1)n})| + |\mu(\Delta_{(2k)n})|) < \infty \quad \text{a.s.}\end{aligned}$$

in view of $\Delta_{k(n-1)} = \Delta_{(2k-1)n} \cup \Delta_{(2k)n}$.

Since f is a bounded function, we obtain $\sup_{y\in Y} |S_0(y)| < \infty$ a.s. Thus the set of random variables $\sup_{y\in Y} |S_n(y)|$, $n \ge 1$, is bounded in probability. Theorem 3.9 of [3] (see also Theorem 2 in [8]) implies that the random function $S(y) = \int_X f(x,y)\, d\mu(x)$ is integrable with respect to m and

$$\int_Y S_n(y)\, dm(y) \xrightarrow{\mathsf{P}} \int_Y S(y)\, dm(y), \qquad n \to \infty.$$

To check (5) we consider the functions

$$f_n(x,y) = \sum_{k=1}^{2^n} f(x_{kn}, y) I_{\Delta_{kn}}(x), \qquad n \ge 1.$$

Then $S_n(y) = \int_X f_n(x,y)\, d\mu(x)$ and the Hölder condition implies that $f_n(x,y) \to f(x,y)$, $n \to \infty$, for all x and y. According to Theorem 1

$$\int_Z f_n(x,y)\, d\eta = \sum_{k=1}^{2^n} \mu(\Delta_{kn}) \int_Y f(x_{kn}, y)\, dm(y) = \int_Y S_n(y)\, dm(y).$$

Now we pass to the limit as $n \to \infty$ and apply Corollary 1.2 of [3] to the left-hand side of the latter relation, while for its right-hand side, we take into account the convergence of integrals of $S_n(y)$ proved above. □

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