

ON THE PRODUCT OF A RANDOM AND A REAL MEASURE

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V. M. RADCHENKO

ABSTRACT. The product of a random measure X and a real measure Y is defined as a random measure on $X \times Y$. We obtain conditions under which the integral of a real function with respect to the product measure equals the iterated integrals of this function.

Let (X, \mathcal{B}_X) and (Y, \mathcal{B}_Y) be measurable spaces, $Z = X \times Y$, and $\mathcal{B}_Z = \mathcal{B}_X \otimes \mathcal{B}_Y$. By $L_0 = L_0(\Omega, \mathcal{F}, \mathbb{P})$ we denote the set of all random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (to be more specific, L_0 is the set of classes of equivalent random variables). The convergence in L_0 is the convergence in probability.

Definition 1. Any σ -additive mapping $\mu: \mathcal{B}_X \rightarrow L_0$ is called a random measure on \mathcal{B}_X .

Note that we do not assume that μ is nonnegative and we do not pose any moment condition.

Here are some examples. If $X(t)$, $0 \leq t \leq T$, is a continuous square-integrable martingale, then $\mu(A) = \int_0^T I_A(t) dX(t)$ is a random measure on Borel sets of $[0, T]$. A fractional Brownian motion $B^H(t)$ for $H > \frac{1}{2}$ defines a random measure in a similar way (this follows from inequality (3.11) in [1]). Other examples as well as conditions for increments of a stochastic process to generate a random measure can be found in Chapters 7 and 8 of [2].

Further let μ be a random measure on \mathcal{B}_X , and m a finite nonnegative measure on \mathcal{B}_Y . A set $A \in \mathcal{B}_X$ is called μ -negligible if

$$\mu(B) = 0 \quad \text{a.s.}$$

for all $B \in \mathcal{B}_X$ such that $B \subset A$. Let ξ be a random variable and put

$$\|\xi\| = \sup\{\delta: \mathbb{P}\{|\xi| > \delta\} > \delta\}.$$

The integral $\int_A f d\mu$ is defined and studied in [3] where $f: X \rightarrow \mathbb{R}$ is a real measurable function and $A \in \mathcal{B}_X$. When constructing this integral one starts with simple functions and proceeds similarly to [2, Chapter 7] (see also [4]). In particular, any measurable bounded function f is integrable with respect to any measure μ .

In this paper, we define the product of a random and a real measure and prove analogs of Fubini's theorem for integrals of real functions.

Theorem 1. *There exists a unique random measure η on \mathcal{B}_Z such that*

$$\eta(A_1 \times A_2) = \mu(A_1)m(A_2)$$

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for all $A_1 \in \mathcal{B}_X$ and $A_2 \in \mathcal{B}_Y$. If $f: Z \rightarrow \mathbb{R}$ is integrable on Z with respect to η , then for all fixed $x \in X$, except for a μ -negligible set, the function $f(x, \cdot): Y \rightarrow \mathbb{R}$ is integrable on Y with respect to m and $\int_Y f(x, y) dm(y)$ is integrable on X with respect to μ . Moreover

$$(1) \quad \int_Z f(x, y) d\eta = \int_X d\mu(x) \int_Y f(x, y) dm(y).$$

Proof. Let $A \in \mathcal{B}_Z$ and put

$$(2) \quad \eta(A) = \int_X d\mu(x) \int_Y I_A(x, y) dm(y).$$

The latter integral exists, since the inner integral does not exceed $m(Y)$ and any bounded function of x is integrable with respect to μ . Corollary 1.2 of [3] implies that η is σ -additive in probability. The equality $\eta(A_1 \times A_2) = \mu(A_1)m(A_2)$ is obvious; the uniqueness of η can be proved in a standard way.

Now we prove (1). If $f = I_B$, $B \in \mathcal{B}_Z$, then (1) follows from (2). Thus (1) holds for simple functions on Z . Let $f: Z \rightarrow \mathbb{R}$ be a measurable bounded function and let f_n , $n \geq 1$, be a sequence of simple functions such that $f_n \rightarrow f$ and $|f_n| \leq |f|$. Equality (1) for f follows from the same equality for functions f_n by passing to the limit with the help of Corollary 1.2 in [3]. Since m is finite, the functions f , $f(x, \cdot)$, and $\int_Y f(x, y) dm(y)$ are bounded and integrable. The latter result is an analog of the Lebesgue dominated convergence theorem.

Let $f: Z \rightarrow \mathbb{R}$ be an arbitrary function integrable with respect to η . Let D be the set of points $x \in X$ for which $f(x, \cdot)$ is nonintegrable with respect to m . For $x \in D$ we have

$$\int_Y |f(x, y)| dm(y) = +\infty$$

and thus the classical Fubini theorem implies that $D \in \mathcal{B}_X$. Assume that D is not a μ -negligible set. Then for some $\varepsilon_0 > 0$ and all $k \geq 1$ there are $n(k) > k$ and $D_1 \subset D$, $D_1 \in \mathcal{B}_X$, such that $\|\mu(D_1)\| > \varepsilon_0$ and

$$\int_Y |f(x, y)| I_{\{k < |f| \leq n(k)\}} dm(y) > 1$$

for $x \in D_1$. Theorem 1.3 of [3] with $h(x) = 1$ and $A = D_1$ implies that

$$\|\mu(D_1)\| \leq 16 \sup_{B \subset D_1} \left\| \int_B d\mu(x) \int_Y |f(x, y)| I_{\{k < |f| \leq n(k)\}} dm(y) \right\|.$$

Equality (1) is already proved for the bounded function $|f(x, y)| I_{\{k < |f| \leq n(k)\}}$. Now Corollary 1.2 of [3] implies that

$$\sup_{B \in \mathcal{B}_X} \left\| \int_{B \times Y} |f(x, y)| I_{\{k < |f| \leq n(k)\}} d\eta \right\| \rightarrow 0, \quad k \rightarrow \infty,$$

since f is integrable with respect to η . This result contradicts the condition $\|\mu(D_1)\| > \varepsilon_0$. Thus the set of points x where $f(x, \cdot)$ is nonintegrable is μ -negligible. In what follows we assume that this set is empty (in fact, we change the values of f on a η -negligible set).

Now we prove that the function $g(x) = \int_Y f(x, y) dm(y)$ is integrable with respect to μ . Consider the functions $g_n(x) = \int_Y f(x, y) I_{\{|f| \leq n\}} dm(y)$, $n \geq 1$, and

$$h(x) = \int_Y |f(x, y)| dm(y).$$

Equality (1) is already proved for the bounded functions $f I_{\{|f| \leq n\}}$; we also have that $g_n(x) \rightarrow g(x)$, $x \in X$. For all $c > 0$

$$\{x: |g_n(x)| > c\} \subset \{x: |h(x)| > c\},$$

thus

$$\begin{aligned} \sup_{n,A \in \mathcal{B}_X} \left\| \int_{A \cap \{|g_n| > c\}} g_n d\mu \right\| &= \sup_{n,A \in \mathcal{B}_X} \left\| \int_{(A \cap \{|g_n| > c\}) \times Y} f I_{\{|f| \leq n\}} d\eta \right\| \\ &\leq \sup_{B \in \mathcal{B}_Z} \left\| \int_{B \cap (\{|h| > c\} \times Y)} f d\eta \right\|. \end{aligned}$$

The set $\{x: |h(x)| > c\}$ approaches the empty set as $c \rightarrow \infty$. Since f is integrable with respect to η , Corollary 1.2 of [3] yields that conditions for the uniform integrability hold in Theorem 1.7 of [3] (see also the theorem in [5]). Thus Theorem 1.7 of [3] implies that g is integrable. Now equality (1) can be proved by passing to the limit along the sequence g_n . \square

Remark 1. The existence of a random measure η defined by (2) and the equality between the integrals on the left- and right-hand sides of (1) is stated without proof in Example 10.1.2 of [2].

The product of a random measure with independent values with itself is constructed in [2, Chapter 10]. If m is Lebesgue measure and μ is generated by increments of fractional Brownian motion, then a result on the product of m and μ is obtained in [6].

The iterated integrals coincide only under some additional assumptions. First we prove some auxiliary results.

Lemma 1. *If $a_k \in \mathbb{R}$, $a_k > 0$, and $A_k \in \mathcal{B}_X$, $k \geq 1$, are such that*

$$\sup_{x \in \mathbb{R}} \sum_{k=1}^{\infty} a_k I_{A_k}(x) < \infty,$$

then

$$(3) \quad \sum_{k=1}^{\infty} a_k^2 \mu^2(A_k) < \infty \quad a.s.$$

Proof. If inequality (3) does not hold, then for some $\varepsilon_0 > 0$ and all $c > 0$ there exists $n \geq 1$ such that $\mathbb{P}(\Omega_1) \geq \varepsilon_0$ for $\Omega_1 = \{\omega \in \Omega : \sum_{k=1}^n a_k^2 \mu^2(A_k) \geq c\}$.

Consider independent Bernoulli random variables ε_k , $1 \leq k \leq n$, defined on another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, that is, $\mathbb{P}'(\varepsilon_k = 1) = \mathbb{P}'(\varepsilon_k = -1) = \frac{1}{2}$. Lemma V.4.3 (a) of [7] yields that

$$\mathbb{P}' \left[\left(\sum_{k=1}^n \lambda_k \varepsilon_k \right)^2 \geq \frac{1}{4} \sum_{k=1}^n \lambda_k^2 \right] \geq \frac{1}{8}, \quad \lambda_k \in \mathbb{R}.$$

Thus

$$\mathbb{P}' \left[\omega' : \left(\sum_{k=1}^n \varepsilon_k(\omega') a_k \mu(A_k, \omega) \right)^2 \geq \frac{c}{4} \right] \geq \frac{1}{8}$$

for all $\omega \in \Omega_1$. Integrating over the set Ω_1 we get

$$\mathbb{P} \times \mathbb{P}' \left[(\omega, \omega') : \left(\sum_{k=1}^n \varepsilon_k(\omega') a_k \mu(A_k, \omega) \right)^2 \geq \frac{c}{4} \right] \geq \frac{\varepsilon_0}{8}.$$

Hence there exists $\omega'_0 \in \Omega'$ such that

$$\mathbb{P} \left[\omega : \left(\sum_{k=1}^n \varepsilon_k(\omega'_0) a_k \mu(A_k, \omega) \right)^2 \geq \frac{c}{4} \right] \geq \frac{\varepsilon_0}{8}.$$

Since $\varepsilon_k(\omega'_0) = \pm 1$, there exists a simple function $f: X \rightarrow \mathbb{R}$ such that

$$\mathbb{P} \left[\left| \int_X f(x) d\mu(x) \right| \geq \frac{\sqrt{c}}{2} \right] \geq \frac{\varepsilon_0}{8}, \quad |f(x)| \leq \sup_{x \in \mathbb{R}} \sum_{k=1}^{\infty} a_k I_{A_k}(x).$$

Recall that $\varepsilon_0 > 0$ is fixed, while c is arbitrary. The latter inequality contradicts the boundedness in probability of the set of values of integrals of simple functions $\int_X f d\mu$ such that $|f(x)| \leq 1$ (see Theorem 1.1 in [3] or Theorem 2 in [4]). Therefore the lemma is proved. \square

In what follows, $X = (a, b] \subset \mathbb{R}$ and \mathcal{B}_X is the Borel σ -algebra. Let

$$\Delta_{kn} = (a + (k-1)2^{-n}(b-a), a + k2^{-n}(b-a)], \quad n \geq 0, \quad 1 \leq k \leq 2^n.$$

Lemma 2. For all $\alpha > \frac{1}{2}$

$$(4) \quad \sum_{n=0}^{\infty} 2^{-n\alpha} \sum_{k=1}^{2^n} |\mu(\Delta_{kn})| < \infty \quad a.s.$$

Proof. Let $\alpha = \frac{1}{2} + \beta$. Using the Cauchy–Bunyakovskiĭ inequality we obtain

$$\begin{aligned} \left(\sum_{n=0}^{\infty} 2^{-n\alpha} \sum_{k=1}^{2^n} |\mu(\Delta_{kn})| \right)^2 &\leq \left(\sum_{n=0}^{\infty} 2^{-n\beta} \right) \left(\sum_{n=0}^{\infty} 2^{-n(1+\beta)} \left(\sum_{k=1}^{2^n} |\mu(\Delta_{kn})| \right)^2 \right) \\ &\leq \left(\sum_{n=0}^{\infty} 2^{-n\beta} \right) \left(\sum_{n=0}^{\infty} 2^{-n\beta} \sum_{k=1}^{2^n} \mu^2(\Delta_{kn}) \right). \end{aligned}$$

It remains to apply Lemma 1 to the second factor. \square

In the sequel the integrals of random functions $\xi(y) = \xi(y, \omega)$, $y \in Y$, with respect to a real measure m are defined according to Definition 5.2 in [3] (see also [8]) (an equivalent condition is given in Theorem 3.8 of [3]). A random function $\xi(y, \omega)$ is integrable with respect to m if it is measurable with respect to the pair of arguments (y, ω) and $\mathbb{P} \{ \sup_{y \in Y} |\xi(y)| < \infty \} = 1$. The integral $\int_Y \xi(y) dm(y)$ can be defined for any fixed ω as the limit of the Lebesgue integrals of simple functions.

Theorem 2. Let $X = (a, b] \subset \mathbb{R}$, and let \mathcal{B}_X be the Borel σ -algebra. Let $f: Z \rightarrow \mathbb{R}$ be a bounded and measurable function. Assume that there exist numbers $\alpha > \frac{1}{2}$ and $L > 0$ such that

$$|f(x_1, y) - f(x_2, y)| \leq L|x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in X$ and $y \in Y$. Then

$$(5) \quad \int_Z f(x, y) d\eta = \int_Y dm(y) \int_X f(x, y) d\mu(x).$$

Proof. The left-hand side of equality (5) is well defined, since f is a bounded function. Now we show that the right-hand side of (5) is well defined, too. Let $x_{kn} \in \Delta_{kn}$ be arbitrary numbers and

$$S_n(y) = \sum_{k=1}^{2^n} f(x_{kn}, y) \mu(\Delta_{kn}).$$

The Hölder condition and Corollary 1.2 of [3] imply that

$$S_n(y) \xrightarrow{\mathbb{P}} \int_X f(x, y) d\mu(x) \quad \text{as } n \rightarrow \infty$$

for all $y \in Y$. It follows from the Hölder condition that

$$\begin{aligned} & \sum_{n=1}^{\infty} |S_n(y) - S_{n-1}(y)| \\ &= \sum_{n=1}^{\infty} \left| \sum_{k=1}^{2^{n-1}} ((f(x_{(2k-1)n}, y) - f(x_{k(n-1)}, y))\mu(\Delta_{(2k-1)n}) \right. \\ & \qquad \qquad \qquad \left. + (f(x_{2kn}, y) - f(x_{k(n-1)}, y))\mu(\Delta_{(2k)n})) \right| \\ & \leq L(b-a)^\alpha \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} 2^{-(n-1)\alpha} (|\mu(\Delta_{(2k-1)n})| + |\mu(\Delta_{(2k)n})|) < \infty \quad \text{a.s.} \end{aligned}$$

in view of $\Delta_{k(n-1)} = \Delta_{(2k-1)n} \cup \Delta_{(2k)n}$.

Since f is a bounded function, we obtain $\sup_{y \in Y} |S_0(y)| < \infty$ a.s. Thus the set of random variables $\sup_{y \in Y} |S_n(y)|$, $n \geq 1$, is bounded in probability. Theorem 3.9 of [3] (see also Theorem 2 in [8]) implies that the random function $S(y) = \int_X f(x, y) d\mu(x)$ is integrable with respect to m and

$$\int_Y S_n(y) dm(y) \xrightarrow{P} \int_Y S(y) dm(y), \quad n \rightarrow \infty.$$

To check (5) we consider the functions

$$f_n(x, y) = \sum_{k=1}^{2^n} f(x_{kn}, y) I_{\Delta_{kn}}(x), \quad n \geq 1.$$

Then $S_n(y) = \int_X f_n(x, y) d\mu(x)$ and the Hölder condition implies that $f_n(x, y) \rightarrow f(x, y)$, $n \rightarrow \infty$, for all x and y . According to Theorem 1

$$\int_Z f_n(x, y) d\eta = \sum_{k=1}^{2^n} \mu(\Delta_{kn}) \int_Y f(x_{kn}, y) dm(y) = \int_Y S_n(y) dm(y).$$

Now we pass to the limit as $n \rightarrow \infty$ and apply Corollary 1.2 of [3] to the left-hand side of the latter relation, while for its right-hand side, we take into account the convergence of integrals of $S_n(y)$ proved above. \square

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MATHEMATICAL INSTITUTE, UNIVERSITY OF JENA, 07740 JENA, GERMANY
E-mail address: vradchenko@univ.kiev.ua

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