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SOME PROPERTIES OF ASYMPTOTIC QUASI-INVERSE FUNCTIONS AND THEIR APPLICATIONS. II

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Abstract. We continue to study properties of functions which are asymptotic (quasi-)inverse for PRV and POV functions. The equivalence of all quasi-inverses for POV functions is proved. Under appropriate conditions, we derive the limiting behaviour of the ratio of asymptotic quasi-inverse functions from the corresponding asymptotics of their original versions. Several applications of these general results to the asymptotic stability of a Cauchy problem, to the asymptotics of the solution of a stochastic differential equation, and to the limiting behavior of generalized renewal processes are also presented.

This is a continuation of Buldygin et al. [7]. For the sake of consistency we continue the numbering of sections, results, and formulas.

8. Main properties and characterizations of POV functions and their asymptotic quasi-inverses

The main results of this section (Theorems 8.1–8.4) show that the class of POV functions is similar to the class of RV functions with positive index.

Proposition 8.1. *Let $f(\cdot) \in \mathbb{F}^{(\infty)}$ be a WPRV function and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. Then,*

1) *if there exists a nondecreasing function $h(\cdot)$ such that $f(\cdot) \sim h(\cdot)$, then condition (2.2) for the function $\tilde{f}^{(-1)}(\cdot)$ holds, that is*

$$(8.1)\qquad \left(\tilde{f}^{(-1)}\right)_*(c) = \liminf_{t\to\infty} \frac{\tilde{f}^{(-1)}(ct)}{\tilde{f}^{(-1)}(t)} > 1 \quad \text{for all } c > 1;$$

2) *if there exists a nondecreasing function $h(\cdot)$ such that $\tilde{f}^{(-1)}(\cdot) \sim h(\cdot)$, then condition (8.1) holds;*
3) *if $f(\cdot)$ is a POV function, then $\tilde{f}^{(-1)}(\cdot)$ is a WPRV function;*
4) *if $f(\cdot)$ is a POV function, then $\tilde{f}^{(-1)}(\cdot)$ is a WPOV function.*

Proof of Proposition 8.1. 1) Assume that condition (8.1) does not hold. Then there exist a number $c_0 > 1$ and a sequence of positive numbers $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$, and

$$(8.2)\qquad \lim_{n\to\infty} \frac{\tilde{f}^{(-1)}(t_n)}{\tilde{f}^{(-1)}(c_0 t_n)} = \beta \in [1, \infty].$$

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Therefore, by Theorem 2.1, if $\beta = 1$, we have that

$$1 > \frac{1}{c_0} = \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(t_n)\big)}{f\big(\tilde{f}^{(-1)}(c_0 t_n)\big)} = 1, \tag{8.3}$$

and, by the monotonicity of $h(\cdot)$, if $\beta \in (1, \infty]$, then

$$1 > \frac{1}{c_0} = \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(t_n)\big)}{f\big(\tilde{f}^{(-1)}(c_0 t_n)\big)} = \lim_{n\to\infty} \frac{h\big(\tilde{f}^{(-1)}(t_n)\big)}{h\big(\tilde{f}^{(-1)}(c_0 t_n)\big)} \geq 1,$$

since $f(\cdot) \sim h(\cdot)$. These contradictions prove assertion 1).

2) Assume that condition (8.1) does not hold. In view of assertion (8.2) and by the monotonicity of $h(\cdot)$, we have

$$1 \geq \lim_{n\to\infty} \frac{h(t_n)}{h(c_0 t_n)} = \lim_{n\to\infty} \frac{\tilde{f}^{(-1)}(t_n)}{\tilde{f}^{(-1)}(c_0 t_n)} = \beta \geq 1,$$

since $f(\cdot) \sim h(\cdot)$. Thus $\beta = 1$ and (8.3) follows. This contradiction proves assertion 2).

3) Now, assume that $f(\cdot)$ is a POV function, but condition (7.1) of Lemma 7.1 does not hold. Then, there exist a number $\gamma \in (1, \infty]$ and sequences of positive numbers $\{a_n\}$ and $\{u_n\}$ such that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} u_n = \infty$, and

$$\lim_{n\to\infty} \frac{\tilde{f}^{(-1)}(a_n u_n)}{\tilde{f}^{(-1)}(u_n)} = \gamma.$$

Consequently, by Proposition 3.1, we have

$$1 = \lim_{n\to\infty} \frac{a_n u_n}{u_n} = \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(a_n u_n)\big)}{f\big(\tilde{f}^{(-1)}(u_n)\big)} > 1.$$

This contradiction proves assertion 3).

Assertion 4) follows from 1), 3) and Theorem 4.1, which completes the proof of Proposition 8.1. □

Corollary 8.1. *Let $f(\cdot) \in \mathbb{F}^\infty$ and $\tilde{f}^{-1}(\cdot)$ be an asymptotic inverse function for $f(\cdot)$. Assume that $\tilde{f}^{-1}(\cdot)$ is a WPRV function. Then*

1) *if there exists a nondecreasing function $h(\cdot)$ which is asymptotically equivalent either to $f(\cdot)$ or to $\tilde{f}^{(-1)}(\cdot)$, then condition (2.2) holds;*
2) *if $\tilde{f}^{-1}(\cdot)$ is a POV function, then $f(\cdot)$ is a WPOV function.*

Applying Proposition 8.1, Corollary 8.1 and Theorem 2.1, we get the following result.

Theorem 8.1. *Let both $f(\cdot) \in \mathbb{F}^\infty$ and its asymptotic inverse function $\tilde{f}^{-1}(\cdot)$ be measurable functions. Assume that there exists a nondecreasing function $h(\cdot)$ which is asymptotically equivalent either to $f(\cdot)$ or to $\tilde{f}^{-1}(\cdot)$. Then, the following four conditions are equivalent:*

(a) *$f(\cdot)$ is POV;*
(b) *$\tilde{f}^{-1}(\cdot)$ is POV;*
(c) *both $f(\cdot)$ and $\tilde{f}^{-1}(\cdot)$ are PRV;*
(d) *both $f(\cdot)$ and $\tilde{f}^{-1}(\cdot)$ preserve the equivalence of functions and sequences.*

With the help of Theorem 8.1, it is possible to prove a more precise version of Theorem 6.2.

Theorem 8.2. *Let $f(\cdot)$ be a POV function and let the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ be as defined in Lemma 6.1. Then*

1) *$\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotically equivalent and are nondecreasing asymptotic inverse functions for $f(\cdot)$. Moreover, each of these functions possesses the POV property.*
2) *If a function $q(\cdot) \in \mathbb{F}_+$ is such that $\varphi_1(s) \le q(s) \le \psi_1(s)$ for all large s, then $q(\cdot)$ is a WPOV function, being an asymptotic inverse function for $f(\cdot)$, and it is asymptotically equivalent both to $\varphi_1(\cdot)$ and to $\psi_1(\cdot)$.*

Theorem 8.2 follows from Theorems 6.2 and 8.1, since every nondecreasing function is measurable.

The next two theorems demonstrate that any function, which is an asymptotic quasi-inverse function either for a POV function $f(\cdot)$ or for a function which is asymptotically equivalent to $f(\cdot)$, is also an asymptotic inverse function for $f(\cdot)$ and a WPOV function, and it is uniquely determined up to asymptotic equivalence. If this function is measurable, then it is a POV function.

Theorem 8.3. *Let $f(\cdot)$ be a POV function and let $\tilde{f}^{-1}(\cdot)$ be an asymptotic inverse function for $f(\cdot)$. If $q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$, then*

1) *$q(\cdot) \sim \tilde{f}^{-1}(\cdot)$ and $q(\cdot)$ is WPOV;*
2) *$q(\cdot)$ is an asymptotic inverse function for $f(\cdot)$.*

Proof of Theorem 8.3. First, recall that, by Remark 3.2, $f(\cdot) \in \mathbb{F}^\infty$ and, by Theorem 8.2, there exists a nondecreasing asymptotic inverse function for $f(\cdot)$ which is a POV function.

We have $f(q(x)) \sim x$ as $x \to \infty$, since $q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$. By Proposition 8.1 and Theorem 2.1, $\tilde{f}^{-1}(\cdot)$ is a WPOV function and thus preserves the equivalence of functions. By definition, $q(x) \to \infty$ as $x \to \infty$, so

$$q(x) \sim \tilde{f}^{-1}(f(q(x))) \sim \tilde{f}^{-1}(x) \quad \text{as } x \to \infty,$$

that is, $q(\cdot) \sim \tilde{f}^{-1}(\cdot)$. Moreover, either from above or from Proposition 8.1, we have that $q(\cdot)$ is WPOV. Therefore assertion 1) is proved.

By assertion 1), $q(f(x)) \sim \tilde{f}^{-1}(f(x)) \sim x$ as $x \to \infty$. This proves assertion 2). □

The next result complements Theorem 8.3.

Theorem 8.4. *Let $f(\cdot)$ be a POV function. Then*

1) *there exists a continuous POV function $f_0(\cdot)$, asymptotically equivalent to $f(\cdot)$ and strictly increasing to ∞, for which the inverse function $f_0^{-1}(\cdot)$ is a continuous POV function, strictly increasing to ∞;*
2) *$f_0^{-1}(\cdot)$ is an asymptotic inverse function for $f(\cdot)$;*
3) *if $\tilde{f}^{(-1)}(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$ and $\tilde{f}^{(-1)}(\cdot) \sim q(\cdot)$, then $q(\cdot)$ is an asymptotic inverse function for $f(\cdot)$, $q(\cdot)$ is a WPOV function and $q(\cdot) \sim f_0^{-1}(\cdot)$;*
4) *if $h(\cdot) \sim f(\cdot)$, then any asymptotic quasi-inverse function $q(\cdot)$ for $h(\cdot)$ is asymptotically equivalent to any asymptotic quasi-inverse function for $f(\cdot)$, $q(\cdot)$ is an asymptotic inverse function both for $f(\cdot)$ and for $h(\cdot)$, and $q(\cdot)$ is a WPOV function.*

Proof of Theorem 8.4. Assertion 1) follows from Theorems 4.1 and 8.1.

Now we show that $f_0^{-1}(\cdot)$ is an asymptotic inverse function for $f(\cdot)$. Indeed,

$$f\big(f_0^{-1}(x)\big) \sim f_0\big(f_0^{-1}(x)\big) \sim x \quad \text{as } x \to \infty,$$

since $f(\cdot) \sim f_0(\cdot)$. By assertion 1) and Theorem 8.1, the function $f_0^{-1}(\cdot)$ preserves the equivalence of functions. Hence

$$f_0^{-1}(f(x)) \sim f_0^{-1}(f_0(x)) \sim x \quad \text{as } x \to \infty.$$

Thus, assertion 2) is proved.

To prove assertion 3) we recall that any POV function preserves the equivalence of functions (see Theorem 2.1). Hence $f(q(x)) \sim f(\tilde{f}^{(-1)}(x)) \sim x$ as $x \to \infty$, and $q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$. Thus assertion 3) follows from Theorem 8.3 together with assertion 2).

Now, let $h(\cdot) \sim f(\cdot)$ and let $q(\cdot)$ be an asymptotic quasi-inverse function for $h(\cdot)$. Then $f(q(x)) \sim h(q(x)) \sim x$ as $x \to \infty$; that is, $q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$. By assertion 3), we get that $q(\cdot)$ is an asymptotic inverse function for $f(\cdot)$, $q(\cdot) \sim f_0^{-1}(\cdot)$ and $q(\cdot)$ is WPOV. By Theorem 2.1, $q(\cdot)$ preserves the equivalence of functions. Hence $q(h(x)) \sim q(f(x)) \sim x$ as $x \to \infty$, and $q(\cdot)$ is an asymptotic inverse function for $h(\cdot)$. □

9. Limiting behaviour of the ratio of asymptotic quasi-inverse functions

The following theorem extends Theorems 8.3 and 8.4 to the case of WPMPV functions, that is, functions satisfying condition (2.2). It demonstrates that every function, which is an asymptotic quasi-inverse function either for $f(\cdot)$ or for any asymptotically equivalent version of $f(\cdot)$, is an asymptotic inverse function for $f(\cdot)$, and that the asymptotic inverse function for $f(\cdot)$ is uniquely determined up to asymptotic equivalence.

Theorem 9.1. *Let $f(\cdot)$ be a WPMPV function. If $f(\cdot) \sim f_0(\cdot) \in \mathbb{F}^{\infty}_{\mathrm{ndec}}$, and $\tilde{f}_0^{-1}(\cdot)$ is an asymptotic inverse function for $f_0(\cdot)$, then*

1) *$\tilde{f}_0^{-1}(\cdot)$ is an asymptotic inverse function for $f(\cdot)$ and $\tilde{f}_0^{-1}(\cdot)$ is WPRV;*
2) *if $h(\cdot) \sim f(\cdot)$, then any asymptotic quasi-inverse function $q(\cdot)$ for $h(\cdot)$ is asymptotically equivalent to any asymptotic quasi-inverse function for $f(\cdot)$, $q(\cdot)$ is an asymptotic inverse function both for $f(\cdot)$ and for $h(\cdot)$, and $q(\cdot)$ is WPRV.*

Proof of Theorem 9.1. First we show that $\tilde{f}_0^{-1}(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$. Indeed,

$$f\big(\tilde{f}_0^{-1}(x)\big) \sim f_0\big(\tilde{f}_0^{-1}(x)\big) \sim x \quad \text{as } x \to \infty,$$

since $f(\cdot) \sim f_0(\cdot)$. Moreover, condition (2.2) also holds for $f_0(\cdot)$, and, by Proposition 7.1, $\tilde{f}_0^{-1}(\cdot)$ is a WPRV function, i.e. it preserves the equivalence of functions. Hence

$$\tilde{f}_0^{-1}(f(x)) \sim \tilde{f}_0^{-1}(f_0(x)) \sim x \quad \text{as } x \to \infty,$$

which proves the first assertion of the theorem.

Now to prove assertion 2) we assume that $h(\cdot) \sim f(\cdot)$ and let $q(\cdot)$ be an asymptotic quasi-inverse function for $h(\cdot)$. Then, $f(q(x)) \sim h(q(x)) \sim x$ as $x \to \infty$. Hence $q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$. By assertion 1) and Theorem 2.1, we have that $\tilde{f}_0^{-1}(\cdot)$ is WPRV, thus preserves the equivalence of functions, and

$$q(x) \sim \tilde{f}_0^{-1}(f_0(q(x))) \sim \tilde{f}_0^{-1}(f(q(x))) \sim \tilde{f}_0^{-1}(x) \quad \text{as } x \to \infty.$$

Hence $q(\cdot) \sim \tilde{f}_0^{-1}(\cdot)$, which immediately implies $q(f(x)) \sim \tilde{f}_0^{-1}(f(x)) \sim x$ as $x \to \infty$. Moreover, $q(h(x)) \sim q(f(x)) \sim x$ as $x \to \infty$, since $q(\cdot)$ is WPRV and, by Theorem 2.1, preserves the equivalence of functions. □

The following results describe the relationship between the limiting behaviour of the ratio of asymptotic quasi-inverse functions and that of their original functions.

Corollary 9.1. *Let $f(\cdot)$ be a WPMPV function. Assume that $f(\cdot) \sim f_0(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$, and that $\tilde{f}_0^{-1}(\cdot)$ is an asymptotic inverse function for $f_0(\cdot)$. If, for some function $x(\cdot) \in \mathbb{F}^{\infty}$,*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \quad \text{for some } a \in (0,\infty), \tag{9.1}$$

then, for any asymptotic quasi-inverse function $\tilde{x}^{(-1)}(\cdot)$ of $x(\cdot)$ and for any asymptotic quasi-inverse function $\tilde{f}^{(-1)}(\cdot)$ of $f(\cdot)$, we have

$$\lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s/a)} = \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}_0^{-1}(s/a)} = 1. \tag{9.2}$$

Corollary 9.2 (Buldygin et al. [5]). *Let $f(\cdot) \in \mathbb{C}^{\infty}_{\text{ndec}}$ and let $f(\cdot)$ satisfy condition (2.2). If, for some function $x(\cdot) \in \mathbb{F}^{\infty}$,*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \quad \text{for some } a \in (0,\infty), \tag{9.3}$$

then, for any quasi-inverse function $x^{(-1)}(\cdot)$ of $x(\cdot)$, we have

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{f^{-1}(s/a)} = 1. \tag{9.4}$$

Corollary 9.1 and Examples 6.1 and 6.2 imply the following result.

Corollary 9.3. *Assume $f(\cdot) \sim f_0(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$, let $\tilde{f}_0^{-1}(\cdot)$ be an asymptotic inverse function for $f_0(\cdot)$ and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f_0(\cdot)$. If $f(\cdot)$ is WPMPV and relation (9.1) holds for some function $x(\cdot) \in \mathbb{C}^{\infty}$, then we have:*

1) *(9.2) follows both for $\tilde{x}_1^{(-1)}(s) = \inf\{t \ge 0\colon x(t) \ge s\}$ and for*
$$\tilde{x}_2^{(-1)}(s) = \sup\{t \ge 0\colon x(t) \le s\};$$

2)
$$\lim_{s\to\infty} \frac{z(s)}{\tilde{f}^{(-1)}(s/a)} = \lim_{s\to\infty} \frac{z(s)}{\tilde{f}_0^{-1}(s/a)} = 1$$
for any function $z(\cdot)$ satisfying $\tilde{x}_1^{(-1)}(s) \le z(s) \le \tilde{x}_2^{(-1)}(s)$ for all large s.

The case of POV functions. For POV functions we have more complete results compared to those of Corollaries 9.1–9.3.

Theorem 9.2. *Let $f(\cdot)$ be a POV function and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. Assume that $x(\cdot) \in \mathbb{F}^{(\infty)}$ and $\tilde{x}^{(-1)}(\cdot)$ is an asymptotic quasi-inverse function for $x(\cdot)$. Then, we have*

1) *for $a \in (0,\infty)$,*
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s/a)} = 1;$$

2) *if $\tilde{x}^{-1}(\cdot)$ is an asymptotic inverse function for $x(\cdot)$, then, for $a \in (0,\infty)$,*
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{-1}(s)}{\tilde{f}^{(-1)}(s/a)} = 1;$$

3) *if $x(\cdot)$ is a WPRV function, then, for $a \in (0,\infty)$,*
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s/a)} = 1;$$

4) *if* (9.1) *holds and* $x(\cdot)$ *is measurable, then* $x(\cdot)$ *is a POV function and, as an asymptotic quasi-inverse function* $\tilde{x}^{(-1)}(\cdot)$ *for* $x(\cdot)$*, we can take in 1) the function* $\tilde{x}_1^{(-1)}(s) = \inf\{t \ge 0 \colon x(t) \ge s\}$ *as well as* $\tilde{x}_2^{(-1)}(s) = \sup\{t \ge 0 \colon x(t) \le s\}$*;*
5) *if* $x(\cdot)$ *is measurable, then the relation* (9.1) *implies the relation*

$$\lim_{s\to\infty} \frac{z(s)}{\tilde{f}^{(-1)}(s/a)} = 1,$$

for any function $z(\cdot)$ *satisfying* $\tilde{x}_1^{(-1)}(s) \le z(s) \le \tilde{x}_2^{(-1)}(s)$ *for all large* s*.*

The proof of Theorem 9.2 is an immediate consequence of Theorems 8.2, 8.4 and 2.1.

Zero and infinite limits of ratios. The following results discuss relationships between the limiting behaviour of the ratio of asymptotic quasi-inverse functions in case the limit of the ratio of the original functions equals 0 or ∞. In this situation, Corollary 9.1 can be retained for zero and infinite limits, but with the additional condition that

$$\liminf_{s\to\infty} \frac{\tilde{f}^{-1}(c_0 s)}{\tilde{f}^{-1}(s)} > 1 \quad \text{for some } c_0 > 1. \tag{9.5}$$

Proposition 9.1. *Let* $f(\cdot) \sim h(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$*, let* $\tilde{f}^{-1}(\cdot)$ *be an asymptotic inverse function for* $f(\cdot)$*, and let* $\tilde{f}^{-1}(\cdot) \sim q(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$*. Assume that* $x(\cdot) \in \mathbb{F}^{(\infty)}$*, and let* $\tilde{x}^{(-1)}(\cdot)$ *be an asymptotic quasi-inverse function for* $x(\cdot)$*. If both conditions* (9.5) *and (2.2) hold, then the following relations follow:*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = \infty \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = 0; \tag{9.6}$$

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = 0 \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = \infty. \tag{9.7}$$

Proof of Proposition 9.1*.* First we prove that

$$\lim_{c\to\infty} l(c) = \infty \quad \text{and} \quad \lim_{c\to 0} r(c) = 0, \tag{9.8}$$

where

$$l(c) = \liminf_{s\to\infty} \frac{\tilde{f}^{-1}(cs)}{\tilde{f}^{-1}(s)}, \quad r(c) = \limsup_{s\to\infty} \frac{\tilde{f}^{-1}(cs)}{\tilde{f}^{-1}(s)}.$$

Indeed,

$$l_\infty = \liminf_{c\to\infty} l(c) = \liminf_{c\to\infty} l\left(c^2\right) \ge \left(\liminf_{c\to\infty} l(c)\right)^2 = l_\infty^2$$

and

$$l(c) = \liminf_{s\to\infty} \frac{\tilde{f}^{-1}(cs)}{\tilde{f}^{-1}(s)} = \liminf_{s\to\infty} \frac{q(cs)}{q(s)}.$$

The first relation in (9.8) follows from condition (9.5), since $l(\cdot)$ is nondecreasing in view of $q(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$. The second relation is a consequence of the relation $r(c) = 1/l(1/c)$.

By Proposition 7.1, the function $\tilde{f}^{-1}(\cdot)$ preserves the equivalence of functions. Hence we have

$$1 = \lim_{s\to\infty} \frac{\tilde{f}^{-1}\big(x\big(\tilde{x}^{(-1)}(s)\big)\big)}{\tilde{f}^{-1}(s)} = \lim_{s\to\infty} \frac{\tilde{f}^{-1}\big(x\big(\tilde{x}^{(-1)}(s)\big)\big)}{\tilde{x}^{(-1)}(s)} \cdot \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)}, \tag{9.9}$$

since $x(\tilde{x}^{(-1)}(s)) \sim s$ as $s \to \infty$.

Put $a(t) = x(t)/f(t)$ for $t > 0$. It follows from the left-hand side of (9.6) that $a(t) \to \infty$ as $t \to \infty$, and hence

$$\begin{aligned}
\liminf_{s\to\infty} \frac{\tilde{f}^{-1}\big(x\big(\tilde{x}^{(-1)}(s)\big)\big)}{\tilde{x}^{(-1)}(s)} &\geq \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(x(t))}{t} = \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(a(t)f(t))}{t} \\
&= \liminf_{t\to\infty} \frac{q(a(t)f(t))}{t} \geq \liminf_{c\to\infty} \liminf_{t\to\infty} \frac{q(cf(t))}{t} \\
&= \liminf_{c\to\infty} \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(cf(t))}{t} = \liminf_{c\to\infty} \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(cf(t))}{\tilde{f}^{-1}(f(t))} \\
&\geq \liminf_{c\to\infty} \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(ct)}{\tilde{f}^{-1}(t)} = \lim_{c\to\infty} l(c).
\end{aligned}$$

By (9.8),

$$\liminf_{s\to\infty} \frac{\tilde{f}^{-1}\big(x\big(\tilde{x}^{(-1)}(s)\big)\big)}{\tilde{x}^{(-1)}(s)} = \infty,$$

and, in view of (9.9),

$$\limsup_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{f^{-1}(s)} = 0.$$

This proves (9.6).

Relation (9.7) can be proved via a similar reasoning. $\square$

For POV functions the latter result reads as follows.

Proposition 9.2. *Let $f(\cdot)$ be a POV function, and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. Assume that $x(\cdot) \in \mathbb{F}^{(\infty)}$ and $\tilde{x}^{(-1)}(\cdot)$ is an asymptotic quasi-inverse function for $x(\cdot)$. Then the following relations follow:*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = \infty \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s)} = 0;$$
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = 0 \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s)} = \infty.$$

If $\tilde{x}^{-1}(\cdot)$ is an asymptotic inverse function for $x(\cdot)$, then

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = \infty \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{-1}(s)}{\tilde{f}^{(-1)}(s)} = 0;$$
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = 0 \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{-1}(s)}{\tilde{f}^{(-1)}(s)} = \infty.$$

Proposition 9.2 follows from Proposition 9.1, Theorem 8.4 and Theorem 8.1.

Remark 9.1. Proposition 9.2 extends Theorem 9.2 and allows for considering $a = 0$ or ∞ in the limiting relations (9.1) and (9.2).

Limiting behaviour of the ratio of asymptotic quasi-inverse functions for RV functions. For RV functions, Theorem 9.2 and Proposition 9.2 read as follows.

Corollary 9.4. *Let $f(\cdot)$ be an RV function with positive index α, and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. Assume that $x(\cdot) \in \mathbb{F}^{(\infty)}$ and $\tilde{x}^{(-1)}(\cdot)$ is an asymptotic quasi-inverse function for $x(\cdot)$. Then, we have*

1) *if*

$$\lim_{t\to\infty}\frac{x(t)}{f(t)} = a \in [0,\infty], \tag{9.10}$$

then

$$\lim_{s\to\infty}\frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \in [0,\infty]. \tag{9.11}$$

Here and in the sequel, it is assumed that $(1/\infty)=0$ *and* $(1/0)=\infty$.

2) *If* $\tilde{x}^{-1}(s)$ *is an asymptotic inverse function for* $x(\cdot)$*, then*

$$\lim_{t\to\infty}\frac{x(t)}{f(t)} = a \in [0,\infty] \quad\Longleftrightarrow\quad \lim_{s\to\infty}\frac{\tilde{x}^{-1}(s)}{\tilde{f}^{(-1)}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \in [0,\infty].$$

3) *If* $x(\cdot)$ *is a WPRV function, then relations* (9.10) *and* (9.11) *are equivalent.*
4) *If* (9.10) *holds and* $x(\cdot)$ *is measurable then, as a quasi-inverse function* $\tilde{x}^{(-1)}(\cdot)$ *for* $x(\cdot)$*, we can choose in* (9.11) *the function* $\tilde{x}_1^{(-1)}(s) = \inf\{t\ge 0\colon x(t)\ge s\}$ *as well as* $\tilde{x}_2^{(-1)}(s) = \sup\{t\ge 0\colon x(t)\le s\}$.
5) *Moreover, if* $x(\cdot)$ *is measurable, then* (9.10) *implies the relation*

$$\lim_{s\to\infty}\frac{z(s)}{\tilde{f}^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \in [0,\infty],$$

for any function $z(\cdot)$ *such that* $\tilde{x}_1^{(-1)}(s)\le z(s)\le \tilde{x}_2^{(-1)}(s)$ *for all large* s.

10. Piecewise linear interpolations and generalized renewal sample functions

In view of Section 9, some results of Buldygin et al. [5, Section 7] can be improved. Before doing so, we briefly recall some necessary facts from [5].

Piecewise linear interpolations of sequences and functions. The continuous function

$$\hat{x}(t) = ([t]+1-t)x_{[t]} + (t-[t])x_{[t]+1}, \qquad t\ge 0,$$

is called the *piecewise linear interpolation of the sequence* $\{x_n\} = \{x_n, n\ge 0\}$. The sequence $\{x_n\}$ is called PRV (POV) if its piecewise linear interpolation is PRV (POV). The continuous function

$$\hat{f}(t) = ([t]+1-t)f([t]) + (t-[t])f([t]+1), \qquad t\ge 0,$$

is called the *piecewise linear interpolation of the function* $f(\cdot)$. Observe that $\widehat{f}(\cdot)$ is a piecewise linear interpolation of the sequence $\{f(n)\}$.

Lemma 10.1 (Buldygin et al. [5])**.** (A) *If a function* $f(\cdot)$ *is PRV, then* $\widehat{f}(\cdot)$ *is also PRV, and* $f(\cdot)\sim\widehat{f}(\cdot)$.
(B) *If a function* $f(\cdot)$ *is POV, then* $\widehat{f}(\cdot)$ *is also POV, and* $f(\cdot)\sim\widehat{f}(\cdot)$.

Lemma 10.2 (Buldygin et al. [5])**.** *Let* $\{x_n\}$ *and* $\{c_n\}$ *be two sequences with* $c_n>0$ *for large* n*. Then for all* $a\in[0,\infty]$,

$$\lim_{n\to\infty}\frac{x_n}{c_n} = a \quad\Longleftrightarrow\quad \lim_{t\to\infty}\frac{\widehat{x}(t)}{\widehat{c}(t)} = a.$$

Generalized renewal sample functions. For a given function $x(\cdot) \in \mathbb{F}^{(\infty)}$, two *generalized renewal sample functions* $f_{x(\cdot)}(\cdot)$ and $l_{x(\cdot)}(\cdot)$ are defined as follows:

$$f_{x(\cdot)}(s) = \inf\{t \geq 0 \colon x(t) \geq s\}, \qquad l_{x(\cdot)}(s) = \sup\{t \geq 0 \colon x(t) \leq s\}$$

for $s \geq \max\{0, x(0)\}$; if $0 \leq s < x(0)$, then we put $f_{x(\cdot)}(s) = l_{x(\cdot)}(s) = 0$. If the function $x(\cdot)$ is measurable, then one more *generalized renewal sample function* $\tau_{x(\cdot)}(\cdot)$ for the function $x(\cdot)$ is defined as follows:

$$\tau_{x(\cdot)}(s) = \operatorname{meas}(\{t \geq 0 \colon x(t) \leq s\}) = \int_0^\infty I(x(t) \leq s)\, dt$$

for $s \geq \max\{0, x(0)\}$; if $0 \leq s < x(0)$, then we put $\tau_{x(\cdot)}(s) = 0$. Here $I(A)$ is the indicator function of a set A.

These three generalized renewal sample functions have a natural "physical" interpretation. The number $f_{x(\cdot)}(s)$ can be interpreted as "*an instant of first exit*" of the values of the function $x(\cdot)$ from the set $(-\infty, s)$, or as "*an instant of first visit*" of the values of the function $x(\cdot)$ to the set $[s, \infty)$. Similarly, the number $l_{x(\cdot)}(s)$ can be interpreted as "*an instant of last visit*" of the values of the function $x(\cdot)$ to the set $(-\infty, s]$. The number $\tau_{x(\cdot)}(\cdot)$ can be interpreted as "*the total time spent*" by the values of the function $x(\cdot)$ in the set $(-\infty, s]$. There are other possibilities for defining generalized renewal sample functions, but we will focus our attention only on the functions $f_{x(\cdot)}(\cdot)$, $\tau_{x(\cdot)}(\cdot)$ and $l_{x(\cdot)}(\cdot)$ introduced above.

Note that the latter functions are nondecreasing and that, for $s \geq 0$,

$$f_{x(\cdot)}(s) \leq \tau_{x(\cdot)}(s) \leq l_{x(\cdot)}(s).$$

If $x(\cdot) \in \mathbb{C}^{(\infty)}$, then $f_{x(\cdot)}(s) = \min\{t \geq 0 \colon x(t) = s\}$, $s \geq x(0)$, and the function $f_{x(\cdot)}(\cdot)$ is a quasi-inverse function for $x(\cdot)$. If $x(\cdot) \in \mathbb{C}^{\infty}$, then $l_{x(\cdot)}(s) = \max\{t \geq 0 \colon x(t) = s\}$, $s \geq x(0)$, and the function $l_{x(\cdot)}(\cdot)$ is also a quasi-inverse function for $x(\cdot)$. If $x(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{ndec}}$, then the functions $f_{x(\cdot)}(\cdot)$, $\tau_{x(\cdot)}(\cdot)$ and $l_{x(\cdot)}(\cdot)$ coincide. Otherwise they are different.

If $x(\cdot)$ is a PRV function and $x(\cdot) \in \mathbb{F}^{(\infty)}$, then, by Lemma 6.1, the function $f_{x(\cdot)}(\cdot)$ is an asymptotic quasi-inverse for $x(\cdot)$. If $x(\cdot)$ is a PRV function and $x(\cdot) \in \mathbb{F}^{\infty}$, then, similarly, both the function $f_{x(\cdot)}(\cdot)$ and the function $l_{x(\cdot)}(\cdot)$ are asymptotic quasi-inverse functions for $x(\cdot)$. If $x(\cdot)$ is a PRV function and $x(\cdot) \sim h(\cdot) \in \mathbb{F}^{\infty}_{\mathrm{ndec}}$, then, by Corollary 6.1, all three functions $f_{x(\cdot)}(\cdot)$, $l_{x(\cdot)}(\cdot)$ and $\tau_{x(\cdot)}(\cdot)$ are asymptotic quasi-inverse functions for $x(\cdot)$. Finally, if $x(\cdot)$ is a POV function, then, by Theorem 6.1, all three functions are asymptotically equivalent and are asymptotic inverse functions for $x(\cdot)$. These facts enable us to apply the results of the above sections to study the asymptotic behaviour of generalized renewal sample functions.

10.1. **Generalized renewal sample functions for sequences.** Consider a real-valued sequence $\{x_n\} = \{x_n, n \geq 0\}$ with $x(0) = 0$. The *generalized renewal sample functions for the sequence* $\{x_n\}$ are defined as follows:

$$f_{\{x_n\}}(s) = \min\{n \geq 0 \colon x_n \geq s\}, \qquad l_{\{x_n\}}(s) = \max\{n \geq 0 \colon x_n \leq s\},$$

$$\tau_{\{x_n\}}(s) = \sum_{n=1}^{\infty} I(x_n \leq s),$$

for $s \geq \max\{0, x(0)\}$; if $0 \leq s < x(0)$, then we put $f_{\{x_n\}}(s) = l_{\{x_n\}}(s) = \tau_{\{x_n\}}(s) = 0$. Sometimes the notation $r_{\{x_n\}}(\cdot)$ will be used for any of the generalized renewal sample functions defined above.

If $\lim_{n\to\infty} x_n = \infty$, then all its generalized renewal sample functions are well defined for $s > 0$. If the sequence $\{x_n\}$ is strictly increasing, then these three functions coincide.

Otherwise they are different and, for all $s > 0$,

$$f_{\{x_n\}}(s) \le \tau_{\{x_n\}}(s) \le l_{\{x_n\}}(s). \tag{10.1}$$

Along with the generalized renewal sample functions for a sequence $\{x_n\}$ we will consider generalized renewal sample functions for its piecewise linear interpolation. Assume $\lim_{n\to\infty} x_n = \infty$ and let $\widehat{x}(\cdot)$ be the piecewise linear interpolation for $\{x_n\}$. It is clear that, for all $s > 0$,

$$f_{\{x_n\}}(s) - 1 \le f_{\hat{x}(\cdot)}(s) \le f_{\{x_n\}}(s) \quad \text{and} \quad l_{\{x_n\}}(s) \le l_{\hat{x}(\cdot)}(s) \le l_{\{x_n\}}(s) + 1. \tag{10.2}$$

From Lemmas 10.1–10.2, inequalities (10.1)–(10.2), Theorem 9.2 and Proposition 9.2, we now get the main result of Section 10.

Theorem 10.1. *Let $p(\cdot)$ be a POV function, let $\tilde{p}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $p(\cdot)$, and let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = \infty$. Then, for any generalized renewal sample function $r_{\{x_n\}}(\cdot)$ and any $a \in (0,\infty)$, the following implications hold:*

$$\lim_{n\to\infty} \frac{x_n}{p(n)} = a \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{r_{\{x_n\}}(s)}{\tilde{p}^{(-1)}(s/a)} = 1;$$

$$\lim_{n\to\infty} \frac{x_n}{p(n)} = \infty \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{r_{\{x_n\}}(s)}{\tilde{p}^{(-1)}(s)} = 0;$$

$$\lim_{n\to\infty} \frac{x_n}{p(n)} = 0 \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{r_{\{x_n\}}(s)}{\tilde{p}^{(-1)}(s)} = \infty.$$

For RV functions, Theorem 10.1 reads as follows:

Corollary 10.1. *Let $p(\cdot)$ be an RV function with positive index α, let $\tilde{p}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $p(\cdot)$, and let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = \infty$. Then, for any generalized renewal sample function $r_{\{x_n\}}(\cdot)$, the following implication holds:*

$$\lim_{n\to\infty} \frac{x_n}{p(n)} = a \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{r_{\{x_n\}}(s)}{\tilde{p}^{(-1)}(s)} = \left(\frac{1}{a}\right)^{1/\alpha},$$

where $a \in [0,\infty]$ with $(1/\infty) = 0$ and $(1/0) = \infty$.

11. Applications

In this section, we discuss several applications of the general results above. Most of the examples are taken from probability theory; however, we start with one taken from the theory of differential equations.

11.1. **Asymptotic stability of a Cauchy problem.** Consider the Cauchy problem

$$d\mu(t) = g(\mu(t))\,dt, \qquad \mu(0) = b > 0, \tag{11.1}$$

$t \ge 0$, where $g(u), u > 0$, is a positive continuous function such that the problem (11.1) has a unique solution for all fixed $b > 0$. We say that problem (11.1) is *asymptotically stable* with respect to the initial condition if

$$\lim_{t\to\infty} \frac{\mu_{b_1}(t)}{\mu_{b_2}(t)} = 1, \tag{11.2}$$

for all positive b_1 and b_2, where $\mu_b(\cdot)$ is a solution of problem (11.1) with initial condition b.

Note, for example, that problem (11.1) is not asymptotically stable with respect to the initial condition, if $g(u) = u$, $u > 0$, while it is asymptotically stable for $g(u) = u^r$, $u > 0$, with $r < 1$. Observe also, that a solution reaches infinity in finite time, if $g(u) = u^r$, $u > 0$, with $r > 1$, so that we do not discuss this case.

Conditions for the asymptotic stability of problem (11.1) can easily be obtained from the results of Section 9. Indeed, given $b > 0$, consider the function

$$G_b(s) = \int_b^s \frac{du}{g(u)}, \qquad s \geq b, \tag{11.3}$$

and note that $G_b(\cdot)$ is a strictly increasing and continuous function, and it is the inverse for $\mu_b(\cdot)$. Put $G(\cdot) = G_1(\cdot)$.

Proposition 11.1. *Let $g(\cdot)$ be such that, for any given $b > 0$,*

$$\int_b^\infty \frac{du}{g(u)} = \infty. \tag{11.4}$$

If $G(\cdot)$ satisfies condition (2.2), i.e.

$$G_*(s) = \liminf_{s\to\infty} \left(\int_1^{cs} \frac{du}{g(u)} \bigg/ \int_1^s \frac{du}{g(u)} \right) > 1 \quad \textit{for all } c > 1, \tag{11.5}$$

then problem (11.1) *is asymptotically stable with respect to the initial condition.*

Proof of Proposition 11.1. By condition (11.4), one has that $\lim_{s\to\infty} G_b(s) = \infty$ for any $b > 0$. Hence $G_{b_1}(\cdot) \sim G_{b_2}(\cdot)$ for all $b_1, b_2 > 0$. Moreover, by conditions (11.5) and (11.4), the continuous and strictly increasing to infinity function $G_b(\cdot)$ is WPMPV for any $b > 0$. Thus, by Theorem 9.1, (11.2) holds, since the function $\mu_b(\cdot)$ is the inverse function for $G_b(\cdot)$. □

Condition (11.4) excludes the possibility of explosions (that is, the solution does not reach infinity in finite time under this condition). Note that the function $g(u) = u, u > 0$, satisfies condition (11.4), but does not satisfy condition (11.5). Observe also, that, by Corollary 7.1, under condition (11.4) the function $\mu_b(\cdot)$ is PRV if and only if condition (11.5) holds.

Remark 11.1. Below are three sufficient conditions for $g(\cdot)$ to satisfy the conditions of Proposition 11.1, namely either

(1) $0 < \inf_{u>0} g(u)$, $\sup_{u>0} g(u) < \infty$; or
(2) $g^*(c) < c$ for all $c > 1$, and condition (11.4) holds; or
(3) $g(\cdot)$ is an RV function with index $\alpha \in (-\infty, 1)$.

Asymptotic behavior of the solution of a stochastic differential equation. Now, consider a stochastic differential equation

$$dX(t) = g(X(t))\,dt + \sigma(X(t))\,dW(t), \qquad X(0) = 1, \tag{11.6}$$

$t \geq 0$, where $W(\cdot)$ is a standard Wiener process and both functions $g(\cdot)$ and $\sigma(\cdot)$ are positive and continuously differentiable. Theorem 1 in Keller et al. [14] states that, under certain conditions on $g(\cdot)$ and $\sigma(\cdot)$ (denoted (A1)–(A4) there), one has

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.s. on } \left\{ \lim_{t\to\infty} X(t) = \infty \right\}, \tag{11.7}$$

where $G(\cdot) = G_1(\cdot)$ is as defined in (11.3). Moreover, Theorem 2 in Keller et al. [14] states that, under some extra conditions,

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = 1 \quad \text{in probability on } \left\{ \lim_{t\to\infty} X(t) = \infty \right\}, \tag{11.8}$$

where $\mu(\cdot)$ is the solution of the deterministic problem (11.1) with initial condition

$$\mu(0) = 1.$$

Observe that condition (11.4) is contained in conditions (A1)–(A4) of Keller et al. [14]. Moreover, under conditions (A1)–(A4), $X(\cdot)$ is almost surely sample continuous.

As already mentioned, $\mu(\cdot)$ is the inverse function for $G(\cdot)$, and therefore relation (11.7) and Proposition 7.1 imply the following extension of (11.8) to the case of almost sure convergence.

Proposition 11.2. *Let $g(\cdot)$ be such that conditions (A1)–(A4) of Keller et al.* [14] *and condition* (11.5) *hold. Then,*

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = 1 \quad \textit{a.s. on } \Big\{ \lim_{t\to\infty} X(t) = \infty \Big\}.$$

Consider *generalized renewal processes* for the stochastic processes $X(\cdot)$ which are constructed according to the generalized renewal sample functions for functions (see Section 10). Let

$$F_{X(\cdot)}(s) = \inf\{t \ge 0\colon X(t) \ge s\}, \qquad s > 1,$$

the *first* time the stochastic process $X(\cdot)$ crosses the level s, let

$$L_{X(\cdot)}(s) = \sup\{t \ge 0\colon X(t) \le s\}, \qquad s > 1,$$

the *last* time the process $X(\cdot)$ visits the set $(-\infty, s]$, and let

$$T_{X(\cdot)}(s) = \operatorname{meas}(\{t \ge 0\colon x(t) \le s\}) = \int_0^\infty I(X(t) \le s)\, dt, \qquad s > 1,$$

the *total time* spent by the process $X(\cdot)$ in $(-\infty, s]$.

Proposition 11.3. *Let $g(\cdot)$ be such that conditions (A1)–(A4) of Keller et al.* [14] *hold. Then,*

$$\lim_{s\to\infty} \frac{R_{X(\cdot)}(s)}{G(s)} = 1 \quad \textit{a.s. on } \Big\{ \lim_{t\to\infty} X(t) = \infty \Big\}, \tag{11.9}$$

for any generalized renewal process $R_{X(\cdot)}(\cdot) \in \{F_{X(\cdot)}(\cdot), T_{X(\cdot)}(\cdot), L_{X(\cdot)}(\cdot)\}$.

Proof of Proposition 11.3. By Examples 6.1–6.2, the processes $F_{X(\cdot)}(\cdot)$ and $L_{X(\cdot)}(\cdot)$ are almost surely quasi-inverse for $X(\cdot)$ on $\{\lim_{t\to\infty} X(t) = \infty\}$, since $X(\cdot)$ has almost surely continuous sample paths. Then, by Proposition (11.2), we have

$$1 = \lim_{t\to\infty} \frac{t}{G(X(t))} = \lim_{t\to\infty} \frac{F_{X(\cdot)}(t)}{G\big(X\big(F_{X(\cdot)}(t)\big)\big)} = \lim_{t\to\infty} \frac{F_{X(\cdot)}(t)}{G(t)}$$

almost surely on $\{\lim_{t\to\infty} X(t) = \infty\}$. Thus

$$\lim_{t\to\infty} \frac{F_{X(\cdot)}(t)}{G(t)} = 1 \quad \text{a.s. on } \Big\{ \lim_{t\to\infty} X(t) = \infty \Big\}.$$

Similarly one can prove that

$$\lim_{s\to\infty} \frac{L_{X(\cdot)}(s)}{G(s)} = 1 \quad \text{a.s. on } \Big\{ \lim_{t\to\infty} X(t) = \infty \Big\}.$$

To complete the proof, we note that almost surely on $\{\lim_{t\to\infty} X(t) = \infty\}$ we have $F_{X(\cdot)}(t) \le T_{X(\cdot)}(t) \le L_{X(\cdot)}(t)$, for all $t > 1$. Thus (11.9) is proved. $\square$

11.2. **Sojourn times for general counting processes.** Let $(\nu(t), t \geq 0)$ be an integer-valued stochastic process such that a.s. its trajectories are step functions, continuous from the right at every point $t \geq 0$. Assume that $\nu(\cdot) \in \mathbb{F}^\infty$ a.s. and

$$\lim_{t\to\infty} \frac{\nu(t)}{f(t)} = 1 \quad \text{a.s.},$$

where $f(\cdot) \in \mathbb{F}^\infty$. Consider the compound stochastic process $X(\cdot)$, generated by $\nu(\cdot)$ and defined as

$$X(t) = \sum_{i=1}^{\nu(t)} X_i, \qquad t > 0,$$

where $\{X_i, i \geq 1\}$ is a sequence of independent, identically distributed random variables. If the mean $\mu = \mathsf{E}X_1$ exists and is positive then, by the strong law of large numbers,

$$\lim_{t\to\infty} \frac{X(t)}{f(t)} = \lim_{t\to\infty} \frac{X(t)}{\nu(t)} \cdot \lim_{t\to\infty} \frac{\nu(t)}{f(t)} = \mu \quad \text{a.s.} \tag{11.10}$$

Observe that a.s. the trajectories of the process $X(\cdot)$ are also step functions, continuous from the right at every point $t \geq 0$. Hence, these trajectories are measurable. Moreover, by (11.10), $X(\cdot) \in \mathbb{F}^\infty$ a.s.

Consider the three generalized renewal processes $F_{X(\cdot)}(\cdot)$, $T_{X(\cdot)}(\cdot)$ and $L_{X(\cdot)}(\cdot)$ as given in Proposition 11.3. From relation (11.10) and Theorem 9.2, we get the following result on the asymptotic behaviour of these generalized renewal processes.

Proposition 11.4. *Let $f(\cdot)$ be a POV function and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. Then, for any generalized renewal process*

$$R_{X(\cdot)}(\cdot) \in \big\{F_{X(\cdot)}(\cdot), T_{X(\cdot)}(\cdot), L_{X(\cdot)}(\cdot)\big\},$$

we have

$$\lim_{s\to\infty} \frac{R_{X(\cdot)}(s)}{\tilde{f}^{(-1)}(s/\mu)} = 1 \quad \textit{a.s.} \tag{11.11}$$

Example 11.1. An example of a process $\nu(\cdot)$ satisfying the above properties is given by a (not necessarily homogeneous) Poisson process with a nonnegative intensity function $\lambda(t)$, $t \geq 0$, such that

$$\int_0^\alpha \lambda(t)\,dt < \infty \quad \text{for all } \alpha > 0, \quad \text{and} \quad \int_0^\infty \lambda(t)\,dt = \infty. \tag{11.12}$$

Put $f(t) = \int_0^t \lambda(s)\,ds$, $t > 0$.

Below are three sufficient conditions on $\lambda(\cdot)$ ensuring the POV property for $f(\cdot)$, namely either

(1) $0 < \inf_{s\geq 0} \lambda(s)$, $\sup_{s\geq 0} \lambda(s) < \infty$; or
(2) condition (11.12) holds, $c\lambda_*(c) > 1$ for all $c > 1$, and $\liminf_{c\downarrow 1} \lambda^*(c) \leq 1$; or
(3) $\lambda(\cdot)$ is an RV function with index $\alpha \in (-1, \infty)$.

Sojourn times for stationary sequences. Consider a random sequence $\{X_n, n \geq 0\}$, $X_0 = 0$, and define generalized renewal processes as follows:

$$F_{\{X_n\}}(s) = \min\{n \geq 0\colon X_n \geq s\}, \qquad s > 0,$$
$$L_{\{X_n\}}(s) = \sup\{n \geq 0\colon X_n \leq s\}, \qquad s > 0,$$
$$T_{\{X_n\}}(s) = \sum_{n=1}^\infty I(X_n \leq s), \qquad s > 0.$$

Any of the above generalized renewal processes will be written as

$$R_{\{X_n\}}(\cdot) = \left(R_{\{X_n\}}(s), s > 0\right).$$

Let $\{\xi_n, n \in \mathbf{Z}\}$ be a strongly stationary, real-valued random sequence such that $\mathsf{E}|\xi_0| < \infty$ and $\mathsf{E}(\xi_0|S_\infty) = \eta$ a.s., where S_∞ is the σ-algebra of shift invariant events. Then

$$\lim_{n\to\infty} \frac{X_n}{n} = \eta \quad \text{a.s.},$$

where $X_0 = 0$, and $X_n = \sum_{k=1}^n \xi_k$, $n \geq 1$.

Hence, by Corollary 10.1, the following proposition holds.

Proposition 11.5. *Let $\{\xi_n, n \in \mathbf{Z}\}$ be a strongly stationary sequence satisfying the above conditions, and let $X_n = \sum_{k=1}^n \xi_k$, $n \geq 1$. Then the following asymptotics hold:*

1) *for any generalized renewal process $R_{\{X_n\}}(\cdot)$, one has*

$$\lim_{s\to\infty} \frac{R_{\{X_n\}}(s)}{s} = \frac{1}{\eta} \quad \text{a.s. on } \{\eta > 0\}.$$

In particular, if $\mathsf{P}\{\eta > 0\} > 0$, then

$$\lim_{s\to\infty} \mathsf{P}\left\{\frac{R_{\{X_n\}}(s)}{s} < u \;\middle|\; \eta > 0\right\} = \frac{\mathsf{P}\{\eta > \frac{1}{u}\}}{\mathsf{P}\{\eta > 0\}},$$

for any $u > 0$ such that $1/u$ is a point of continuity of the distribution function of the random variable η.

2) *for any generalized renewal process $R_{\{|X_n|\}}(\cdot)$, one has*

$$\lim_{s\to\infty} \frac{R_{\{|X_n|\}}(s)}{s} = \frac{1}{|\eta|} \quad \text{a.s. on } \{|\eta| > 0\}, \tag{11.13}$$

In particular, if $\mathsf{P}\{|\eta| > 0\} > 0$, then

$$\lim_{s\to\infty} \mathsf{P}\left\{\frac{R_{\{|X_n|\}}(s)}{s} < u \;\middle|\; |\eta| > 0\right\} = \frac{\mathsf{P}\{|\eta| > \frac{1}{u}\}}{\mathsf{P}\{|\eta| > 0\}},$$

for any $u > 0$ such that $1/u$ is a point of continuity of the distribution function of the random variable $|\eta|$.

3) *if $\limsup_{n\to\infty} |X_n| = \infty$ a.s. on $\{\eta = 0\}$, then*

$$\lim_{s\to\infty} \frac{F_{\{X_n\}}(s)}{s} = \infty \quad \text{a.s. on } \{\eta = 0\}.$$

Example 11.2. Let $\{\xi_n, n \in \mathbf{Z}\}$, be a stationary Gaussian random sequence with

$$\mathsf{E}\xi_0 = \mu \in \mathbf{R}$$

and spectral function $G(\lambda)$, $\lambda \in [-\pi, \pi]$, continuous for all $\lambda \neq 0$. Assume that

$$\sigma_0^2 = \lim_{\lambda\downarrow 0}(G(\lambda) - G(-\lambda)) > 0.$$

Then $\eta = \mathsf{E}(\xi(0)|S_\infty)$ is a Gaussian random variable with $\mathsf{E}\eta = \mu$ and $\operatorname{var}(\eta) = \sigma_0^2$. Therefore, $1/|\eta|$ is an almost surely positive random variable with density

$$q(u) = \frac{1}{\sqrt{2\pi}\sigma_0 u^2}\left(\exp\left\{\frac{-(1-u\mu)^2}{2\sigma_0^2 u^2}\right\} + \exp\left\{\frac{-(1+u\mu)^2}{2\sigma_0^2 u^2}\right\}\right), \qquad u > 0.$$

Hence, by statement 2) of Proposition 11.5, for any generalized renewal process $R_{\{|X_n|\}}(\cdot)$, relation (11.13) holds and

$$\lim_{s\to\infty} \mathsf{P}\left\{\frac{R_{\{|X_n|\}}(s)}{s} < u\right\} = \mathsf{P}\left\{|\eta| > \frac{1}{u}\right\} = \int_{1/u}^{\infty} q(u)\,du,$$

for any $u > 0$.

11.3. **Sojourn times for Gaussian Markov sequences.** Let $\{X_n, n \geq 1\}$ be a zero-mean, real-valued Gaussian Markov random sequence with $\sigma_n^2 = \mathsf{E}X_n^2 > 0$, $n \geq 1$, and $r_{n,n+1} \neq 0$, $n \geq 1$. Here $r_{k,n}$ denotes the correlation coefficient between X_k and X_n, that is, $r_{k,n} = \mathsf{E}(X_k X_n / \sigma_k \sigma_n)$. Gaussian Markov sequences are characterized by the relations

$$r_{k,n} = r_{k,m} r_{m,n}, \qquad 1 \leq k \leq m \leq n;$$

see Feller [8]. By these formulas, one has

$$r_{1,n} = \prod_{m=1}^{n-1} r_{m,m+1} \neq 0, \qquad n \geq 1;$$

thus the limit $|r|_{1,\infty} = \lim_{n\to\infty} |r_{1,n}|$ always exists with $|r|_{1,\infty} \in [0,1]$.

Moreover, every Gaussian Markov sequence $\{X_n, n \geq 1\}$ satisfies the following recurrence relations:

$$X_1 = \sigma_1 w_1, \qquad X_n = a_n X_{n-1} + b_n w_n, \quad n \geq 2,$$

where $a_n = (\sigma_n/\sigma_{n-1}) r_{n-1,n}$, $b_n^2 = \sigma_n^2(1 - r_{n-1,n}^2)$, $n \geq 2$, and $\{w_n, n \geq 1\}$ is a sequence of independent, identically distributed, zero-mean normal random variables with $\mathsf{E}w_n^2 = 1$.

It is clear that the sequence

$$Y_1 = X_1, \qquad Y_n = \frac{X_n}{a_2 \cdots a_n} = \frac{\sigma_1 X_n}{\sigma_n r_{1,n}}, \quad n \geq 2,$$

is a Gaussian martingale.

Assume that $|r|_{1,\infty} \neq 0$. Then, by Doob's theorem, the martingale $\{Y_n\}$ converges almost surely, since

$$\sup_{n\geq 2} \mathsf{E}|Y_n|^2 = \frac{\sigma_1^2}{|r|_{1,\infty}^2} < \infty.$$

Therefore,

$$\lim_{n\to\infty} \frac{X_n}{\sigma_n} = \eta \quad \text{a.s.},$$

and

$$\lim_{n\to\infty} \frac{|X_n|}{\sigma_n} = |\eta| \quad \text{a.s.}, \tag{11.14}$$

where η is a standard normal random variable. Hence, by (11.14) and Theorem 10.1 we have:

Proposition 11.6. *Let $\{X_n, n \geq 1\}$ be a zero-mean Gaussian Markov sequence satisfying the conditions above. Let $p(\cdot)$ be a POV function and let $\tilde{p}^{-1}(\cdot)$ be an asymptotic inverse function for $p(\cdot)$. Assume that $\sigma_n = p(n)$, $n \geq 1$. Then, for any generalized renewal process $R_{\{|X_n|\}}(\cdot)$ (see Proposition 11.5), one has*

$$\lim_{s\to\infty} \frac{R_{\{|X_n|\}}(s)}{\tilde{p}^{-1}(s/|\eta|)} = 1 \quad \textit{a.s.}$$

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