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ON A MULTIVARIATE STORAGE PROCESS

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Abstract. A multivariate storage process that satisfies the Langevin equation is studied in the paper.

1. Introduction

Let a process $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ satisfy the Langevin equation

$$dx(t) = Ax(t)\,dt + dz(t), \tag{1}$$

where $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathbb{R}^n$ is a generalized Poisson process with parameter λ and jumps $\eta^1, \eta^2, \dots, \eta^j, \dots$; $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and $\|a_{ij}\|_{i,j=1}^n$ is the matrix of its representation in some basis of $\mathbb{R}^n$.

Equation (1) with initial data $x(0) = x_0$ has a unique solution in the class of measurable processes. This solution can be written in the following form:

$$x(t) = \exp\{At\}x_0 + \int_0^t \exp\{A(t-u)\}\,dz(u). \tag{2}$$

It is shown in [1] that the process $x(t)$ has the limit distribution as $t \to \infty$ and this distribution does not depend on the initial data x_0 if and only if

a) the eigenvalues of A belong to the left semiplane,
b) $\mathsf{E}(\ln|\eta^1|; |\eta^1| > 1) < \infty$.

It is also proved in [1] that the limit distribution is a unique stationary distribution of the process $x(t)$ if both of the above conditions hold. The characteristic function of the limit distribution is given by

$$\psi(s) = \exp\left\{-\lambda \int_0^\infty \left(1 - \varphi\left(\exp\left\{A^T u\right\} s\right)\right)\,du\right\} \tag{3}$$

where $\varphi(s) = \mathsf{E}\{\exp i(s, \eta^1)\}$.

As is seen from equality (2), the stationary distribution of $x(t)$ coincides with the distribution of the vector

$$\xi = \int_0^\infty \exp\{Au\}\,dz(u). \tag{4}$$

Moreover, equality (3) implies that the characteristic function of the stationary distribution of $x(t)$ is of the form $\psi(s) = \exp\{\lambda K(s)\}$ where $K(s)$ does not depend on λ. In the stationary regime, $x(\cdot, \lambda)$ can be viewed as values of a stochastically continuous homogeneous process with independent increments at the moment λ.

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2. Setting of the problem

The limit behavior of the distribution of $x(\cdot,\lambda)$ as $\lambda \to 0$ is studied in [1] for the case of $A = U\Lambda U^{-1}$ where $\Lambda = \|\delta_{ij}\lambda_i\|_{i,j=1}^n$, λ_i $(i = 1,\dots,n)$ are real eigenvalues of the matrix A such that $\lambda_i < 0$ for all i, and $U = \|u_{ij}\|_{i,j=1}^n$ is a nonsingular matrix.

The limit behavior as $\lambda \to 0$ of the distribution of $x(\cdot,\lambda)$ is obtained in [5] for the case of $A = UJU^{-1}$ where J is a Jordan matrix,[1] $A = \|a_{ij}\|_{i,j=1}^2$, and $U = \|u_{ij}\|_{i,j=1}^2$.

In this paper, we consider the general case of $A = UJU^{-1}$ where J is a Jordan matrix, $U = \|u_{ij}\|_{i,j=1}^n$ is a nonsingular matrix, and $A = \|a_{ij}\|_{i,j=1}^n$. We study the limit behavior as $\lambda \to 0$ of the vector

$$\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_n)^T = U^{-1}x(\cdot,\lambda)$$

under the assumption that the distribution of $x(\cdot,\lambda)$ is stationary.

Below we show that the components of the vector $\tilde{x}$ are completely determined by the form of the Jordan blocks. Thus we obtain the limit behavior, as $\lambda \to 0$, of the part of the vector $\tilde{x}$ that corresponds to a Jordan block J_i. In doing so, we consider separately the cases of real and complex eigenvalues λ_i of the matrix A.

3. Auxiliary results and notation

The process $z(t)$ is completely determined by the heights of the jumps $\eta^1,\eta^2,\dots$ and by the lengths of the intervals $\lambda^{-1}\tau_1,\lambda^{-1}\tau_2,\dots$ between the jumps. All the random variables η^j, $j = 1,2,\dots$, and τ_i, $i = 1,2,\dots$, are independent and $\mathsf{P}\{\tau_i > t\} = \exp\{-t\}$ for $t \ge 0$. Thus we obtain from (4) that

$$\begin{aligned}\xi = \exp\left\{\lambda^{-1}\tau_1 A\right\}\eta^1 + \exp\left\{\lambda^{-1}(\tau_1+\tau_2)A\right\}\eta^2 + \dots \\ + \exp\left\{\lambda^{-1}A\sum_{k=1}^j \tau_k\right\}\eta^j + \dots .\end{aligned} \tag{5}$$

Below we use the following notation: $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_n)^T = U^{-1}x(\cdot,\lambda)$;

$$\tilde{\eta}^j = \left(\tilde{\eta}_1^j,\dots,\tilde{\eta}_n^j\right)^T = U^{-1}\eta^j, \qquad j = 1,2,\dots;$$

$$\tilde{\xi} = (\tilde{\xi}_1,\dots,\tilde{\xi}_n)^T = U^{-1}\xi; p_r = \mathsf{P}\left\{\tilde{\eta}_r^j = 0\right\}; p_r^+ = \mathsf{P}\{\tilde{\eta}_r^j > 0\};$$

$$\operatorname{sgn} z = (\operatorname{sgn} z_1,\dots,\operatorname{sgn} z_n)^T \quad \text{for } z = (z_1,\dots,z_n)^T \in \mathbb{R}^n;$$

$J = \{J_1,\dots,J_m\}$ where J_i is the Jordan block of order k_i corresponding to the eigenvalue λ_i, $i = 1,\dots,m$, of the matrix A (there could be equal numbers among the λ_i, $i = 1,\dots,m$); $\sum_{i=1}^m k_i = l_m$, $m = 1,\dots,n$; $l_n = n$; $\nu_i = \lambda\lambda_i^{-1}$ for real λ_i and $\kappa_i = \lambda a_i^{-1}$ for complex $\lambda_i = a_i + ib_i$.

Recall that the matrix $f(A)$ is well defined if $f(t)$ is an analytic function. Since $A = UJU^{-1}$, the matrix $f(J)$ is well defined and, moreover, $f(A) = Uf(J)U^{-1}$. Thus relation (5) can be rewritten in the following form:

$$\begin{aligned}\xi = U\exp\left\{\lambda^{-1}\tau_1 J\right\} \\ \times U^{-1}\left(\eta^1 + U\exp\left\{\lambda^{-1}\tau_2 J\right\}U^{-1}\eta^2 + \dots + U\exp\left\{\lambda^{-1}J\sum_{k=2}^j \tau_k\right\}U^{-1}\eta^j + \dots\right)\end{aligned}$$

or

$$\xi = U\exp\left\{\lambda^{-1}\tau_1 J\right\}U^{-1}\left(\eta^1 + \xi^1\right),$$

where the random variables τ_1, ξ^1, and η^1 are independent and the distributions of ξ and ξ^1 are identical.

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[1] I.e., a matrix in the Jordan normal form.

Therefore

$$\tilde{\xi} = \exp\left\{\lambda^{-1}\tau_1 J\right\}\left(\tilde{\eta}^1 + \exp\left\{\lambda^{-1}\tau_2 J\right\}\tilde{\eta}^2 + \dots + \exp\left\{\lambda^{-1} J \sum_{k=2}^{j}\tau_k\right\}\tilde{\eta}^j + \dots\right) \tag{6}$$

or

$$\tilde{\xi} = \exp\left\{\lambda^{-1}\tau_1 J\right\}\left(\tilde{\eta}^1 + \tilde{\xi}^1\right), \tag{7}$$

where $\tilde{\xi}^1 = (\tilde{\xi}^1_1, \dots, \tilde{\xi}^1_n) = U^{-1}\xi^1$, the distributions of $\tilde{x}$, $\tilde{\xi}$, and $\tilde{\xi}^1$ are identical, and

$$\exp\left\{\lambda^{-1}\tau_1 J\right\} = \left\{\exp\{\lambda^{-1}\tau_1 J_1\}, \dots, \exp\{\lambda^{-1}\tau_1 J_m\}\right\},$$

$$\exp\left\{\lambda^{-1}\tau_1 J_i\right\} = \exp\left\{\lambda^{-1}\tau_1 \lambda_i\right\}\begin{pmatrix} 1 & \frac{\lambda^{-1}\tau_1}{1!} & \dots & \frac{(\lambda^{-1}\tau_1)^{k_i-1}}{(k_i-1)!} \\ 0 & 1 & \dots & \frac{(\lambda^{-1}\tau_1)^{k_i-2}}{(k_i-2)!} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{8}$$

It is seen from (6) and (7) that the components of the vector $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_n)^T$ as well as those of the vector $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ are determined by the Jordan blocks. Thus, without loss of generality, we restrict our consideration below to the investigation of the part of the vector $\tilde{x}$ that corresponds to the Jordan block J_i of order k_i related to the eigenvector λ_i.

Denote by $(\tilde{x}_{l_{i-1}+1}, \tilde{x}_{l_{i-1}+2}, \dots, \tilde{x}_{l_i})^T$ the part of the vector $\tilde{x}$ that corresponds to the Jordan block J_i and let $(\tilde{\eta}^j_{l_{i-1}+1}, \tilde{\eta}^j_{l_{i-1}+2}, \dots, \tilde{\eta}^j_{l_i})^T$, $j = 1, 2 \dots$, be the part of the vector $\tilde{\eta}^j$ related to the Jordan block J_i.

We introduce the random events $A_1 = \{\tilde{\eta}^1_{l_i} \neq 0\}$,

$$A_j = \left\{\tilde{\eta}^1_{l_i} = 0, \dots, \tilde{\eta}^{j-1}_{l_i} = 0, \tilde{\eta}^j_{l_i} \neq 0\right\},$$

$B_1 = \{\tilde{\eta}^1_{l_i-1} \neq 0\}$, $B_j = \{\tilde{\eta}^1_{l_i-1} = 0, \dots, \tilde{\eta}^{j-1}_{l_i-1} = 0, \tilde{\eta}^j_{l_i-1} \neq 0\}$, $j = 2, 3 \dots$, and denote the indicators of events A_j and B_j by $1(A_j)$ and $1(B_j)$, respectively. Let $\mathsf{P}\{A_1\} = p$ and $\mathsf{P}\{B_1\} = q$. In what follows we assume that all stochastic processes and random variables are defined on the same probability space.

4. Main results

We distinguish between the following two cases.

I. An eigenvalue $\lambda_i < 0$ of the matrix A is real ($\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i}$ are real in this case).

II. An eigenvalue $\lambda_i < 0$ of the matrix A is complex; that is,

$$\lambda_i = a_i + i b_i, \qquad a_i < 0,\ b_i \neq 0.$$

In this case, $\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i}$ are complex. We represent these numbers as follows:

$$\tilde{x}_{l_{i-1}+1} = |\tilde{x}_{l_{i-1}+1}| \exp\{i\varphi_{l_{i-1}+1}\}, \qquad \dots, \qquad \tilde{x}_{l_i} = |\tilde{x}_{l_i}| \exp\{i\varphi_{l_i}\},$$

where $\varphi_{l_{i-1}+1} = \arg \tilde{x}_{l_{i-1}+1}, \dots, \varphi_{l_i} = \arg \tilde{x}_{l_i}$, $\varphi_{l_{i-1}+1}, \dots, \varphi_{l_i} \in (0, 2\pi)$.

4.1. Case I.

Theorem 1. *If $p_{l_i} = 0$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i}, \dots, \left|\tilde{x}_{l_i}\right|^{-\nu_i}, \operatorname{sgn}(\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i})\right)$$

converges weakly as $\lambda \to 0$ to the distribution of $(\alpha, \dots, \alpha, \operatorname{sgn}(\tilde{\eta}^1_{l_i}, \dots, \tilde{\eta}^1_{l_i}))$, where α has the uniform distribution on the interval $(0, 1)$ and does not depend on $\tilde{\eta}^1_{l_i}$.

Proof. Since $p_{l_i} = 0$ and the distributions of the vectors $\tilde{x}$ and $\tilde{\xi}$ are identical, we use relations (7) and (8) and obtain

$$\begin{aligned}(9)\qquad \left(\tilde{\xi}_{l_{i-1}+1},\dots,\tilde{\xi}_{l_i}\right)^T &= \exp\left\{\lambda^{-1}\tau_1 J_i\right\}\left(\tilde{\xi}^1_{l_{i-1}+1}+\tilde{\eta}^1_{l_{i-1}+1},\dots,\tilde{\xi}^1_{l_i}+\tilde{\eta}^1_{l_i}\right)^T \\ &= \exp\left\{\lambda^{-1}\tau_1\lambda_i\right\}\left(\tilde{\zeta}^1_{l_{i-1}+1},\tilde{\zeta}^1_{l_{i-1}+2},\dots,\tilde{\zeta}^1_{l_i}\right)^T,\end{aligned}$$

where

$$\tilde{\zeta}^1_{l_{i-1}+1} = \sum_{m=0}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\left(\tilde{\xi}^1_{l_{i-1}+m+1}+\tilde{\eta}^1_{l_{i-1}+m+1}\right),$$

$$\tilde{\zeta}^1_{l_{i-1}+2} = \sum_{m=0}^{k_i-2}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\left(\tilde{\xi}^1_{l_{i-1}+m+2}+\tilde{\eta}^1_{l_{i-1}+m+2}\right),\qquad \dots,\qquad \tilde{\zeta}^1_{l_i} = \tilde{\xi}^1_{l_i}+\tilde{\eta}^1_{l_i}.$$

According to Lemma 6.4 in [1], $\tilde{\xi}^1 \xrightarrow{\mathsf{P}} \bar{0}$ as $\lambda \to 0$ and

$$\begin{aligned}&\mathsf{P}\left\{\tilde{\eta}^1_{l_{i-1}+1}+\sum_{m=1}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\tilde{\eta}^1_{l_{i-1}+m+1}=0\right\}\\ &\qquad = \mathsf{P}\left\{\sum_{m=1}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\tilde{\eta}^1_{l_{i-1}+m+1}=-\tilde{\eta}^1_{l_{i-1}+1}\right\}=0,\end{aligned}$$

since the random variables τ_1 and $\tilde{\eta}^1_r$ $(r = l_{i-1}+1,\dots,l_i)$ are independent and

$$\sum_{m=1}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\tilde{\eta}^1_{l_{i-1}+m+1}$$

has an absolutely continuous distribution. Thus

$$\begin{aligned}&\lim_{\lambda\to 0}\ln\left|\sum_{m=0}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\left(\tilde{\xi}^1_{l_{i-1}+m+1}+\tilde{\eta}^1_{l_{i-1}+m+1}\right)\right|^{-\nu_i}\\ &\qquad = -\lambda_i^{-1}\lim_{\lambda\to 0}\frac{\tau_1^{k_i-1}\tilde{\eta}^1_{l_i}}{\tau_1^{k_i-2}\tilde{\eta}^1_{l_i-1}+\lambda^{-1}\tau_1^{k_i-1}\tilde{\eta}^1_{l_i}}=0.\end{aligned}$$

Therefore relation (9) implies that

$$\left|\tilde{\xi}_{l_{i-1}+1}\right|^{-\nu_i} = \exp\{-\tau_1\}\left|\sum_{m=0}^{k_i-1}\frac{\left(\lambda^{-1}\tau_1\right)^m}{m!}\left(\tilde{\xi}^1_{l_{i-1}+m+1}+\tilde{\eta}^1_{l_{i-1}+m+1}\right)\right|^{-\nu_i}\xrightarrow{\mathsf{P}}\exp\{-\tau_1\},$$

$$\left|\tilde{\xi}_{l_{i-1}+2}\right|^{-\nu_i}\xrightarrow{\mathsf{P}}\exp\{-\tau_1\},\qquad\dots,\qquad\left|\tilde{\xi}_{l_i}\right|^{-\nu_i}\xrightarrow{\mathsf{P}}\exp\{-\tau_1\},$$

$$\operatorname{sgn}\tilde{\xi}_{l_{i-1}+1}\xrightarrow{\mathsf{P}}\operatorname{sgn}\tilde{\eta}^1_{l_i},\qquad\operatorname{sgn}\tilde{\xi}_{l_{i-1}+2}\xrightarrow{\mathsf{P}}\operatorname{sgn}\tilde{\eta}^1_{l_i},\qquad\dots,\qquad\operatorname{sgn}\tilde{\xi}_{l_i}\xrightarrow{\mathsf{P}}\operatorname{sgn}\tilde{\eta}^1_{l_i}$$

as $\lambda \to 0$. Moreover the random variable $\exp\{-\tau_1\} = \alpha$ has the uniform distribution on the interval $(0, 1)$ and does not depend on $\tilde{\eta}^1_{l_i}$.

Now Theorem 1 follows, since the convergence in probability of the corresponding coordinates of the vectors implies the weak convergence of the distributions of the vectors. □

Theorem 2. *If* $0 < p_{l_i} < 1$ *and the distribution of* $x(\cdot, \lambda)$ *is stationary, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i},\dots,\left|\tilde{x}_{l_i}\right|^{-\nu_i},\operatorname{sgn}\tilde{x}_{l_{i-1}+1},\dots,\operatorname{sgn}\tilde{x}_{l_i}\right)$$

weakly converges as $\lambda \to 0$ *to the distribution of* $(\alpha^{1/p}, \dots, \alpha^{1/p}, \gamma, \dots, \gamma)$, *where the random variable* α *has the uniform distribution on the interval* $(0, 1)$, *while the random variable* γ *assumes the values* 1 *and* -1 *with the probabilities* $p_{l_i}^{+}$ *and* $(1 - p_{l_i}^{+})$, *respectively, and does not depend on* α.

Proof. If $0 < p_{l_i} < 1$, then we get from relations (6) and (8) that

$$\left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}\right)^T = \exp\left\{\lambda^{-1}\tau_1\lambda_i\right\}\left(\tilde{\zeta}_{l_{i-1}+1}, \tilde{\zeta}_{l_{i-1}+2}, \dots, \tilde{\zeta}_{l_i}\right)^T,$$

where

$$\tilde{\zeta}_{l_{i-1}+1} = \sum_{m=1}^{k_i}\left(\frac{\left(\lambda^{-1}\tau_1\right)^{k_i-m}}{(k_i-m)!}\tilde{\eta}^1_{l_i-m+1} + \sum_{j=2}^{\infty}\exp\left\{\lambda^{-1}\lambda_i\sum_{k=2}^{j}\tau_k\right\}\frac{\left(\lambda^{-1}\sum_{k=1}^{j}\tau_k\right)^{k_i-m}}{(k_i-m)!}\tilde{\eta}^j_{l_i-m+1}\right),$$

$$\tilde{\zeta}_{l_{i-1}+2} = \sum_{m=2}^{k_i}\left(\frac{\left(\lambda^{-1}\tau_1\right)^{k_i-m}}{(k_i-m)!}\tilde{\eta}^1_{l_i-m+2} + \sum_{j=2}^{\infty}\exp\left\{\lambda^{-1}\lambda_i\sum_{k=2}^{j}\tau_k\right\}\frac{\left(\lambda^{-1}\sum_{k=1}^{j}\tau_k\right)^{k_i-m}}{(k_i-m)!}\tilde{\eta}^j_{l_i-m+2}\right), \qquad \dots,$$

$$\tilde{\zeta}_{l_i} = \tilde{\eta}^1_{l_i} + \sum_{j=2}^{\infty}\exp\left\{\lambda^{-1}\lambda_i\sum_{k=2}^{j}\tau_k\right\}\tilde{\eta}^j_{l_i}.$$

Consider the random events

$$A_1 = \left\{\tilde{\eta}^1_{l_i} \neq 0\right\}, \qquad A_2 = \left\{\tilde{\eta}^1_{l_i} = 0, \tilde{\eta}^2_{l_i} \neq 0\right\}, \qquad \dots,$$
$$A_j = \left\{\tilde{\eta}^1_{l_i} = 0, \dots, \tilde{\eta}^{j-1}_{l_i} = 0, \tilde{\eta}^j_{l_i} \neq 0\right\}, \qquad \dots.$$

One can treat $\Omega = \{A_1, A_2, \dots, A_j, \dots\}$ as the space of elementary events. Moreover $\mathsf{P}\{A_j\} = p(1-p)^{j-1}$, where $\mathsf{P}\{A_1\} = p$.

The restriction of the random variable $\tilde{\xi}_r$, $r = l_{i-1}+1, \dots, l_i$, on the elementary event A_j, $j = 1, 2, \dots$, is denoted by $\tilde{\xi}_r\big|_{A_j}$. Then

$$\begin{aligned}
&\tilde{\xi}_{l_{i-1}+1}\Big|_{A_1} \equiv \exp\left\{\lambda^{-1}\tau_1\lambda_i\right\}\frac{\left(\lambda^{-1}\tau_1\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^1_{l_i},\\
&\tilde{\xi}_{l_{i-1}+1}\Big|_{A_2} \equiv \exp\left\{\lambda^{-1}(\tau_1+\tau_2)\lambda_i\right\}\frac{\left(\lambda^{-1}(\tau_1+\tau_2)\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^2_{l_i}, \qquad \dots,\\
&\tilde{\xi}_{l_{i-1}+1}\Big|_{A_j} \equiv \exp\left\{\lambda^{-1}\lambda_i\sum_{k=1}^{j}\tau_k\right\}\frac{\left(\lambda^{-1}\sum_{k=1}^{j}\tau_k\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^j_{l_i}, \qquad \dots.
\end{aligned} \tag{10}$$

Thus we have

$$\left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_1}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \exp\{-\tau_1\}, \qquad \left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_2}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \exp\{-(\tau_1+\tau_2)\}, \qquad \dots,$$
$$\left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_j}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \exp\left\{-\sum_{k=1}^{j}\tau_k\right\}, \qquad \dots,$$

as $\lambda \to 0$; that is,

$$\left|\tilde{\xi}_{l_{i-1}+1}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\,(A_j) = \chi.$$

A similar reasoning for $\tilde{\xi}_{l_{i-1}+2}, \ldots, \tilde{\xi}_{l_i}$ shows that

$$\left|\tilde{\xi}_{l_{i-1}+2}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\,(A_j), \qquad \ldots,$$

$$\left|\tilde{\xi}_{l_i}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\,(A_j).$$

Analogously we obtain that

$$\operatorname{sgn} \tilde{\xi}_{l_{i-1}+1} = \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^{j}\, 1\,(A_j) = \gamma,$$

$$\operatorname{sgn} \tilde{\xi}_{l_{i-1}+2} = \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^{j}\, 1\,(A_j), \qquad \ldots, \qquad \operatorname{sgn} \tilde{\xi}_{l_i} = \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^{j}\, 1\,(A_j)$$

as $\lambda \to 0$.

Since the random variables

$$\exp\{-\tau_j\} = \alpha_j, \qquad j = 1, 2, \ldots,$$

have the uniform distribution on the interval (0,1), the Laplace transform of χ is given by

$$\begin{aligned}
\mathsf{E} \exp\{-s\chi\} &= \mathsf{E} \exp\left\{-s \sum_{j=1}^{\infty} \alpha_1 \ldots \alpha_j 1\,(A_j)\right\} \\
&= \sum_{k=1}^{\infty} \mathsf{P}(A_k) \mathsf{E}_{A_k} \exp\left\{-s \sum_{j=1}^{\infty} \alpha_1 \ldots \alpha_j 1(A_j)\right\} \\
&= \sum_{k=1}^{\infty} \mathsf{P}(A_k) \mathsf{E} \exp\left\{-s\alpha_1 \ldots \alpha_k\right\},
\end{aligned}$$

where

$$\mathsf{E} \exp\{-s\alpha_1 \ldots \alpha_k\} = \int_0^1 \ldots \int_0^1 \exp\{-s x_1 \ldots x_k\}\, dx_1 \ldots\, dx_k.$$

The series

$$\exp\{-s x_1 \ldots x_k\} = \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} x_1^j \ldots x_k^j$$

converges by the d'Alembert criterion for any $x^0 = x_1^0 \ldots x_k^0$; thus it converges on $(0,1)$. By the Weierstrass criterion this series is uniformly convergent on $(0,1)$. Since the terms of this series are continuous functions on $(0,1)$, we get

$$\mathsf{E} \exp\left\{-s\alpha_1 \ldots \alpha_k\right\} = \sum_{j=0}^{\infty} \int_0^1 \ldots \int_0^1 \frac{(-s)^j}{j!} x_1^j \ldots x_k^j\, dx_1 \ldots dx_k = \sum_{j=0}^{\infty} \frac{(-s)^j}{j!\,(j+1)^k}.$$

Hence

$$\begin{aligned}\mathsf{E}\exp\{-s\chi\} &= \sum_{k=1}^{\infty} p(1-p)^{k-1} \sum_{j=0}^{\infty} \frac{(-s)^j}{j!\,(j+1)^k} = p\sum_{j=0}^{\infty}\frac{(-s)^j}{j!}\sum_{k=1}^{\infty}\frac{(1-p)^{k-1}}{(j+1)^k}\\ &= p\sum_{j=0}^{\infty}\frac{(-s)^j}{j!}\frac{1}{(j+1)}\sum_{k=1}^{\infty}\left(\frac{1-p}{j+1}\right)^{k-1} = p\sum_{j=0}^{\infty}\frac{(-s)^j}{j!}\frac{1}{j+p} = \phi(s)\end{aligned}$$

or

$$\begin{aligned}\phi(s)(-1)^p s^p &= p\sum_{j=0}^{\infty}\frac{(-s)^{j+p}}{j!\,(j+p)} = \int_0^s d_u\left(\phi(u)(-u)^p\right) = -p\int_0^s \sum_{j=0}^{\infty}\frac{(-u)^{j+p-1}}{j!}\,du\\ &= -p\int_0^s(-u)^{p-1}\sum_{j=0}^{\infty}\frac{(-u)^j}{j!}\,du = p(-1)^p\int_0^s u^{p-1}\exp\{-u\}\,du\\ &= p(-1)^p\gamma(p,s),\end{aligned}$$

where

$$(-1)^p = \exp\{(2n+1)p\pi i\} = \cos(2n+1)p\pi + i\sin(2n+1)p\pi, \qquad n = 0, \pm1, \pm2, \dots,$$

and $\gamma(p,s)$ is the incomplete gamma function.

Therefore

$$\phi(s) = ps^{-p}\gamma\,(p,s).$$

Using corresponding relations for the inversion of the Laplace–Carson transform [3], we evaluate the density of χ:

$$g(t) = \begin{cases} pt^{p-1}, & 0 < t < 1,\\ 0, & t > 1,\end{cases}$$

and the corresponding distribution function

$$G(t) = \begin{cases} t^{p}, & 0 < t < 1,\\ 0, & t > 1.\end{cases}$$

If α has the uniform distribution on $(0,1)$, then $\chi = \alpha^{1/p}$. The theorem is proved. $\square$

Theorem 3. *If $p_{l_i} = 1$, $p_{l_i-1} = 0$, and the distribution of $x(\cdot,\lambda)$ is stationary, then the distribution of $\big(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i-1}|^{-\nu_i}, |\tilde{x}_{l_i}|, \operatorname{sgn}\tilde{x}_{l_{i-1}+1}, \dots, \operatorname{sgn}\tilde{x}_{l_i-1}, \operatorname{sgn}\tilde{x}_{l_i}\big)$ weakly converges as $\lambda \to 0$ to the distribution of $\big(\alpha, \dots, \alpha, 0, \operatorname{sgn}\tilde{\eta}^1_{l_i-1}, \dots, \operatorname{sgn}\tilde{\eta}^1_{l_i-1}, 0\big)$, where α has the uniform distribution on $(0,1)$ and does not depend on $\tilde{\eta}^1_{l_i-1}$.*

Proof. The proof of Theorem 3 is similar to that of Theorem 1. $\square$

Theorem 4. *If $p_{l_i} = 1$, $0 < p_{l_i-1} < 1$, and the distribution of $x(\cdot,\lambda)$ is stationary, then the distribution of*

$$\big(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i-1}|^{-\nu_i}, |\tilde{x}_{l_i}|, \operatorname{sgn}\tilde{x}_{l_{i-1}+1}, \dots, \operatorname{sgn}\tilde{x}_{l_i-1}, \operatorname{sgn}\tilde{x}_{l_i}\big)$$

weakly converges as $\lambda \to 0$ to the distribution of $\big(\alpha^{1/q}, \dots, \alpha^{1/q}, 0, \gamma, \dots, \gamma, 0\big)$, where α has the uniform distribution on $(0,1)$, while the random variable γ assumes the values 1 and -1 with probabilities $p^+_{l_i-1}$ and $1 - p^+_{l_i-1}$, respectively, and does not depend on α.

Proof. The proof of Theorem 4 is analogous to that of Theorem 2. Note however that the random events $B_1 = \{\tilde{\eta}^1_{l_i-1} \neq 0\}$, $B_j = \{\tilde{\eta}^1_{l_i-1} = 0, \dots, \tilde{\eta}^{j-1}_{l_i-1} = 0, \tilde{\eta}^j_{l_i-1} \neq 0\}$, $j = 2, 3 \dots$, should be substituted for the random events A_j, $j \geq 1$, in the proof of Theorem 4. $\square$

4.2. **Case II.**

Theorem 5. *If* $p_{l_i} = 0$ *and the distribution of* $x(\cdot, \lambda)$ *is stationary, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\kappa_i}, \dots, \left|\tilde{x}_{l_i}\right|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \dots, \varphi_{l_i}\right)$$

weakly converges as $\lambda \to 0$ *to the distribution of* $(\alpha, \dots, \alpha, \beta, \dots, \beta)$, *where the distribution of the random variable* α *is uniform on the interval* $(0,1)$, *the distribution of the random variable* β *is uniform on the interval* $(0, 2\pi)$, *and* β *does not depend on* α.

Proof. Since $p_{l_i} = 0$, $\lambda_i = a_i + ib_i$, and the matrix U is complex, the vector

$$\left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}\right)^T$$

is of the form

$$\begin{aligned} &\left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}\right)^T \\ &\qquad = \exp\left\{\lambda^{-1}\tau_1 a_i\right\} \exp\left\{i\lambda^{-1}\tau_1 b_i\right\} \left(\zeta_{l_{i-1}+1}, \zeta_{l_{i-1}+2}, \dots, \zeta_{l_i}\right)^T \end{aligned} \tag{11}$$

in view of relations (7) and (8), where

$$\zeta_r = |\zeta_r| \exp\left\{i\gamma_r\right\}, \qquad \gamma_r = \arg \zeta_r, \qquad \gamma_r \in (0, 2\pi), \qquad r = l_{i-1}+1, \dots, l_i.$$

Therefore (11) implies that

$$\begin{gathered} \left|\tilde{\xi}_{l_{i-1}+1}\right|^{-\kappa_i} = \exp\left\{-\tau_1\right\} \left|\exp\left\{i\left(\gamma_{l_{i-1}+1} + \lambda^{-1}\tau_1 b_i\right)\right\}\right|^{-\kappa_i} \xrightarrow{\mathsf{P}} \exp\left\{-\tau_1\right\}, \\ \left|\tilde{\xi}_{l_{i-1}+2}\right|^{-\kappa_i} \xrightarrow{\mathsf{P}} \exp\left\{-\tau_1\right\}, \qquad \dots, \qquad \left|\tilde{\xi}_{l_i}\right|^{-\kappa_i} \xrightarrow{\mathsf{P}} \exp\left\{-\tau_1\right\} \end{gathered}$$

as $\lambda \to 0$. Moreover the distribution of the random variable $\exp\left\{-\tau_1\right\} = \alpha$ is uniform on the interval $(0,1)$, and

$$\begin{aligned} \arg \tilde{\xi}_{l_{i-1}+1} &\equiv \left(\gamma_{l_{i-1}+1} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi}, \\ \arg \tilde{\xi}_{l_{i-1}+2} &\equiv \left(\gamma_{l_{i-1}+2} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi}, \qquad \dots, \\ \arg \tilde{\xi}_{l_i} &\equiv \left(\gamma_{l_i} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi} \end{aligned}$$

as $\lambda \to 0$.

Let $f_{l_{i-1}+1}(t), f_{l_{i-1}+2}(t), \dots, f_{l_i}(t)$, $t \in (0, 2\pi)$, be the densities of the random variables

$$\begin{gathered} \left(\gamma_{l_{i-1}+1} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi}, \\ \left(\gamma_{l_{i-1}+2} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi}, \qquad \dots, \\ \left(\gamma_{l_i} + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi}, \end{gathered}$$

respectively, defined on a circle of length 2π, while $f(t)$ is the density of the random variable $\lambda^{-1}\tau_1 b_i$. The random variable $\lambda^{-1}\tau_1 b_i \pmod{2\pi}$ is defined on a circle, and its density $\tilde{f}$ is given by

$$\tilde{f}(t) = \sum_{k=0}^{\infty} f\left(t + 2\pi k\right) = \lambda b_i^{-1} \sum_{k=0}^{\infty} \exp\left\{-\lambda b_i^{-1}\left(t + 2\pi k\right)\right\} = \frac{\lambda \exp\left\{-\lambda b_i^{-1} t\right\}}{b_i\left(1 - \exp\left\{-\lambda b_i^{-1} 2\pi\right\}\right)},$$

$$\lim_{\lambda \to 0} \tilde{f}(t) = \lim_{\lambda \to 0} \frac{\exp\left\{-\lambda b_i^{-1} t\right\} - \lambda b_i^{-1} t \exp\left\{-\lambda b_i^{-1} t\right\}}{2\pi \exp\left\{-\lambda b_i^{-1} 2\pi\right\}} = \frac{1}{2\pi}.$$

It is known [2] that the convolution of the uniform density on a circle with an arbitrary density on a circle is the density of the uniform distribution. Thus

$$\lim_{\lambda \to 0} f_{l_{i-1}+1}(t) = \frac{1}{2\pi}, \qquad \lim_{\lambda \to 0} f_{l_{i-1}+2}(t) = \frac{1}{2\pi}, \qquad \dots, \qquad \lim_{\lambda \to 0} f_{l_i}(t) = \frac{1}{2\pi}$$

as $\lambda \to 0$. Therefore the distributions of the random variables

$$(\gamma_{l_{i-1}+1} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi},$$
$$(\gamma_{l_{i-1}+2} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi}, \qquad \dots, \qquad (\gamma_{l_i} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi}$$

as $\lambda \to 0$ coincide with the uniform distribution on the interval $(0, 2\pi)$, and these random variables do not depend on α.

Now Theorem 5 follows from the properties of the weak convergence and convergence in probability. □

Theorem 6. *If $0 < p_{l_i} < 1$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$\left(|\tilde{x}_{l_{i-1}+1}|^{-\kappa_i}, \dots, |\tilde{x}_{l_i}|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \dots, \varphi_{l_i}\right)$$

weakly converges as $\lambda \to 0$ to the distribution of $(\alpha^{1/p}, \dots, \alpha^{1/p}, \beta, \dots, \beta)$ where the random variable α has the uniform distribution on the interval $(0,1)$, the random variable β has the uniform distribution on the interval $(0, 2\pi)$, and β does not depend on α.

Proof. The proof of Theorem 6 is analogous to that of Theorem 2.

Since $\lambda_i = a_i + ib_i$, relation (10) can be rewritten as follows:

$$\tilde{\xi}_{l_{i-1}+1}\Big|_{A_1} \equiv \exp\left\{\lambda^{-1}\tau_1 a_i\right\} \exp\left\{i\lambda^{-1}\tau_1 b_i\right\} \frac{\left(\lambda^{-1}\tau_1\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^1_{l_i},$$
$$\tilde{\xi}_{l_{i-1}+1}\Big|_{A_2} \equiv \exp\left\{\lambda^{-1}\left(\tau_1+\tau_2\right)a_i\right\} \exp\left\{i\lambda^{-1}\left(\tau_1+\tau_2\right)b_i\right\} \frac{\left(\lambda^{-1}\left(\tau_1+\tau_2\right)\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^2_{l_i}, \ \dots,$$
$$\tilde{\xi}_{l_{i-1}+1}\Big|_{A_j} \equiv \exp\left\{\lambda^{-1}a_i\sum_{k=1}^{j}\tau_k\right\} \exp\left\{i\lambda^{-1}b_i\sum_{k=1}^{j}\tau_k\right\} \frac{\left(\lambda^{-1}\sum_{k=1}^{j}\tau_k\right)^{k_i-1}}{(k_i-1)!}\tilde{\eta}^j_{l_i}, \ \dots,$$

where

$$\tilde{\eta}^j_{l_i} = \left|\tilde{\eta}^j_{l_i}\right| \exp\{i\phi_j\}, \qquad \phi_j = \arg\tilde{\eta}^j_{l_i}, \qquad j = 1, 2, \dots, \qquad \phi_j \in (0, 2\pi),$$

and

$$\arg\tilde{\xi}_{l_{i-1}+1}\Big|_{A_1} \equiv \left(\phi_1 + \lambda^{-1}\tau_1 b_i\right) \pmod{2\pi},$$
$$\arg\tilde{\xi}_{l_{i-1}+1}\Big|_{A_2} \equiv \left(\phi_2 + \lambda^{-1}\tau_1 b_i + \lambda^{-1}\tau_2 b_i\right) \pmod{2\pi}, \qquad \dots,$$
$$\arg\tilde{\xi}_{l_{i-1}+1}\Big|_{A_j} \equiv \left(\phi_j + \lambda^{-1}\tau_1 b_i + \lambda^{-1}b_i\sum_{k=2}^{j}\tau_k\right) \pmod{2\pi}, \qquad \dots.$$

Then

$$\arg\tilde{\xi}_{l_{i-1}+1} = \sum_{j=1}^{\infty}\left(\phi_j + \lambda^{-1}\tau_1 b_i + \lambda^{-1}b_i\sum_{k=2}^{j}\tau_k\right) \pmod{2\pi} 1\left(A_j\right).$$

Therefore

$$\left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_1}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp\left\{-\tau_1\right\}, \qquad \left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_2}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp\left\{-\left(\tau_1+\tau_2\right)\right\}, \qquad \dots,$$
$$\left|\tilde{\xi}_{l_{i-1}+1}\Big|_{A_j}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp\left\{-\sum_{k=1}^{j}\tau_k\right\}, \qquad \dots,$$

as $\lambda \to 0$; that is,

$$\left|\tilde{\xi}_{l_{i-1}+1}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\left(A_j\right) = \chi.$$

As in the proof of Theorem 2, $\chi = \alpha^{1/p}$ where the random variable α has the uniform distribution on the interval $(0, 1)$.

A similar reasoning for $\tilde{\xi}_{l_{i-1}+2}, \ldots, \tilde{\xi}_{l_i}$ yields

$$\left|\tilde{\xi}_{l_{i-1}+2}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\left(A_j\right), \qquad \ldots,$$

$$\left|\tilde{\xi}_{l_i}\right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp\left\{-\sum_{k=1}^{j} \tau_k\right\} 1\left(A_j\right),$$

$$\arg \tilde{\xi}_{l_{i-1}+2} = \sum_{j=1}^{\infty} \left(\phi_j + \lambda^{-1}\tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^{j} \tau_k\right) \pmod{2\pi} 1\left(A_j\right), \qquad \ldots,$$

$$\arg \tilde{\xi}_{l_i} = \sum_{j=1}^{\infty} \left(\phi_j + \lambda^{-1}\tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^{j} \tau_k\right) \pmod{2\pi} 1\left(A_j\right).$$

Since the random variable $\lambda^{-1}\tau_1 b_i \pmod{2\pi}$ has the uniform density on $(0, 2\pi)$ as $\lambda \to 0$ (see Theorem 5) and the convolution of the uniform density on a circle with an arbitrary density on a circle is again uniform, the densities of the random variables

$$\gamma_j = \left(\phi_j + \lambda^{-1}\tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^{j} \tau_k\right) \pmod{2\pi}, \qquad j = 1, 2, \ldots,$$

are again uniform on $(0, 2\pi)$ as $\lambda \to 0$. Thus

$$\arg \tilde{\xi}_{l_{i-1}+1} = \sum_{j=1}^{\infty} \gamma_j 1\left(A_j\right) = \beta,$$

$$\arg \tilde{\xi}_{l_{i-1}+2} = \sum_{j=1}^{\infty} \gamma_j 1\left(A_j\right), \qquad \ldots, \qquad \arg \tilde{\xi}_{l_i} = \sum_{j=1}^{\infty} \gamma_j 1\left(A_j\right)$$

as $\lambda \to 0$.

The Laplace transform of the random variable β is given by

$$\mathsf{E} \exp\{-s\beta\} = \mathsf{E} \exp\left\{-s \sum_{j=1}^{\infty} \gamma_j 1\left(A_j\right)\right\} = \sum_{k=1}^{\infty} \mathsf{P}\left(A_k\right) \mathsf{E} \exp\{-s\gamma_k\},$$

where

$$\mathsf{E} \exp\{-s\gamma_k\} = \int_0^{2\pi} \exp\{-s x_k\}\, d\left(\frac{x_k}{2\pi}\right) = \frac{1 - \exp\{-2\pi s\}}{2\pi s}.$$

Therefore

$$\mathsf{E} \exp\{-s\beta\} = \frac{1}{2\pi} \frac{1 - \exp\{-2\pi s\}}{s} \sum_{k=1}^{\infty} p\left(1-p\right)^{k-1} = \frac{1}{2\pi} \frac{1 - \exp\{-2\pi s\}}{s}$$

and thus the random variable β has the uniform distribution on $(0, 2\pi)$ and does not depend on α.

Theorem 6 is proved. □

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