

## LINEAR EQUATIONS AND STOCHASTIC EXPONENTS IN A HILBERT SPACE

UDC 519.21

YULIYA MISHURA AND GEORGIY SHEVCHENKO

ABSTRACT. We consider linear stochastic differential equations in a Hilbert space and obtain general limit theorems. As a corollary, we get a result on the convergence of finite-dimensional approximations of solutions of such equations.

### 0. INTRODUCTION

The linear stochastic differential equation

$$(0.1) \quad X(t) = X_0 + \int_0^t AX(s) ds + \int_0^t BX(s) dW(s)$$

is a classical model in financial mathematics. It also appears in some problems of mathematical physics. The solution of the above equation in the one-dimensional case is called the stochastic exponent. The stochastic exponent is given by

$$X(t) = X_0 \exp \left\{ BW(t) + \left( A - \frac{1}{2}B^2 \right) t \right\}.$$

For higher dimensions this formula holds (now with matrix exponent) only if the operator  $A$  and the coefficients for different coordinates of the Wiener process are pairwise commuting. An exact formula for the solution of equation (0.1) is known for some cases in higher dimensions. However, if the space is infinite-dimensional, the form of the exact solution of equation (0.1) is unknown in general.

Our idea used in this paper is to approximate the solution of equation (0.1) in an infinite-dimensional space by solutions of linear finite-dimensional equations that can be solved explicitly.

The first section below contains general limit theorems and convergence results for finite-dimensional approximations of the linear homogeneous equation (0.1). In the second section, analogous results for the linear nonhomogeneous equation are obtained. The third section is devoted to some examples where the solution of the finite-dimensional equation (0.1) can be written in an explicit form.

### 1. GENERAL LIMIT THEOREMS

Let  $\mathcal{X}$  be a separable Hilbert space,  $\mathcal{L}(\mathcal{X})$  be the space of linear continuous operators on  $\mathcal{X}$ , and let  $\mathcal{L}_2(\mathcal{X})$  be the space of the Hilbert–Schmidt operators. In what follows we use the abbreviations  $\mathcal{L}$  and  $\mathcal{L}_2$  for  $\mathcal{L}(\mathcal{X})$  and  $\mathcal{L}_2(\mathcal{X})$ , respectively. All unessential constants are denoted by  $C$ .

---

2000 *Mathematics Subject Classification.* Primary 60H10; Secondary 34G10, 47A50, 47D06.

*Key words and phrases.* Linear stochastic differential equation, stochastic exponent.

The second author is partially supported by INTAS grant YSF 03-55-2447.

Consider a linear stochastic differential equation in  $\mathcal{X}$ :

$$(1.1) \quad X(t) = X_0 + \int_0^t (AX(s) ds + BX(s) dW(s)), \quad t \in [0, T],$$

where  $A$  is a closed linear operator with domain  $D$ , that is,  $A$  is the generator of the  $C_0$ -semigroup  $\{U(t), t \geq 0\}$ ,  $B$  is a linear operator acting from  $\mathcal{X}$  to  $\mathcal{L}_2$ ,  $W(t)$  is an cylindrical Wiener process in  $\mathcal{X}$ , and  $X_0$  is a  $\mathcal{F}_0$ -measurable square integrable random variable. It is known that equation (1.1) has a unique pathwise continuous solution if

$$(1.2) \quad \int_0^T \|AU(t)B\|_{\mathcal{L}(\mathcal{X}, \mathcal{L}_2)}^2 dt < \infty.$$

This solution is also a “mild” solution, i.e., it satisfies the equation

$$(1.3) \quad X(t) = U(t)X_0 + \int_0^t U(t-s)BX(s) dW(s)$$

and moreover  $\sup_{t \leq T} \mathbf{E} \|X(t)\|^2 < \infty$  (see, for example, Theorems 2.2.1, 2.2.2, 2.3.1, and 2.3.2 in [1]).

In some cases it is impossible to write the solution of the equation (1.1) in an explicit form. In such a case, one has to use an approximation by solutions of the equations

$$(1.4) \quad X_n(t) = X_0^n + \int_0^t (A_n X_n(s) ds + B_n X_n(s) dW(s))$$

whose coefficients satisfy the same conditions as the coefficients of equation (1.1).

More precisely, assume that  $A_n$  is a closed linear operator with domain  $D(A_n) \supset D$  and that  $A_n$  generates the  $C_0$ -semigroup  $\{U_n(t), t \geq 0\}$  such that

$$(1.5) \quad \int_0^T \|A_n U_n(t) B_n\|_{\mathcal{L}(\mathcal{X}, \mathcal{L}_2)}^2 dt < \infty.$$

Then, as we have already mentioned, equation (1.4) has a unique solution that satisfies the equation

$$(1.6) \quad X_n(t) = U_n(t)X_0^n + \int_0^t U_n(t-s)B_n X_n(s) dW(s).$$

Furthermore, we assume that

$$(1.7a) \quad A_n x \rightarrow Ax, \quad n \rightarrow \infty, \quad x \in D,$$

$$(1.7b) \quad \|Bx - B_n x\|_{\mathcal{L}_2} \rightarrow 0, \quad n \rightarrow \infty, \quad x \in \mathcal{X}.$$

In view of the Banach–Steinhaus theorem, it follows from the latter assumptions that the norms  $\|B_n\|_{\mathcal{L}_2}$  are bounded in  $n$ .

Finally, assume that

$$(1.8) \quad \|U_n(t)\|_{\mathcal{L}} \leq C, \quad t \in [0, T].$$

We get from the Gronwall–Bellman lemma that  $\sup_{t \leq T} \mathbf{E} \|X_n(t)\|^2 < \infty$  uniformly in  $n$ . It follows from (1.7a) and (1.8) that

$$U_n(t)x \rightarrow U(t)x, \quad n \rightarrow \infty,$$

for  $x \in \mathcal{X}$  and uniformly on the segment  $[0, T]$  (cf. [5]). In turn, this implies

$$(1.9) \quad \sup_{t \in [0, T]} \|(U(t) - U_n(t))Bx\|_{\mathcal{L}_2} \rightarrow 0, \quad n \rightarrow \infty.$$

To justify the latter relation we consider finite-dimensional projections  $B^N x$  of the operator  $Bx$ . It is clear that

$$\|Bx - B^N x\|_{\mathcal{L}_2} \rightarrow 0, \quad N \rightarrow \infty,$$

and

$$(1.10) \quad \sup_{t \in [0, T]} \|(U(t) - U_n(t))Bx\|_{\mathcal{L}_2}$$

$$(1.11) \quad \leq \sup_{t \in [0, T]} \|(U(t) - U_n(t))B^N x\|_{\mathcal{L}_2} + \sup_{t \in [0, T]} \|U(t)(Bx - B^N x)\|_{\mathcal{L}_2}$$

$$(1.12) \quad + \sup_{t \in [0, T]} \|U_n(t)(Bx - B^N x)\|_{\mathcal{L}_2}$$

$$(1.13) \quad \leq \sup_{t \in [0, T]} \|(U(t) - U_n(t))B^N x\|_{\mathcal{L}_2} + C \|Bx - B^N x\|_{\mathcal{L}_2}.$$

The convergence to zero of the latter expression can be proved in a standard way: namely, we choose  $N$  such that the second term is sufficiently small; then we choose  $n$  such that the first term is small.

**Theorem 1.1.** *If conditions (1.5), (1.7), and (1.8) hold and*

$$\mathbf{E} \|X_0 - X_0^n\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

then

$$\mathbf{E} \|X(t) - X_n(t)\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

uniformly on the segment  $[0, T]$ .

*Proof.* Put

$$Z_n(t) = \mathbf{E} \|X(t) - X_n(t)\|^2.$$

We have  $Z_n(t) \leq C(A_1 + A_2 + A_3 + A_4)$ , where

$$\begin{aligned} A_1 &= \mathbf{E} \|U(t)X_0 - U_n(t)X_0^n\|^2 \leq C \left( \mathbf{E} \|U_n(t)\|_{\mathcal{L}} \|X_0 - X_0^n\|^2 + \mathbf{E} \|(U(t) - U_n(t))X_0\|^2 \right) \\ &\leq C \left( \mathbf{E} \|X_0 - X_0^n\|^2 + \mathbf{E} \|(U(t) - U_n(t))X_0\|^2 \right), \end{aligned}$$

$$\begin{aligned} A_2 &= \mathbf{E} \left\| \int_0^t U_n(t-s)(B_n X(s) - B_n X_n(s)) dW(s) \right\|^2 \\ &\leq \mathbf{E} \int_0^t \|U_n(t-s)\|_{\mathcal{L}}^2 \|B_n\|_{\mathcal{L}(X, \mathcal{L}_2)}^2 \|X(s) - X_n(s)\|^2 ds \leq C \int_0^t Z_n(s) ds, \end{aligned}$$

$$\begin{aligned} A_3 &= \mathbf{E} \left\| \int_0^t U_n(t-s)(B - B_n)X(s) dW(s) \right\|^2 \\ &\leq \mathbf{E} \int_0^t \|U_n(t-s)\|_{\mathcal{L}}^2 \|(B - B_n)X(s)\|_{\mathcal{L}_2}^2 ds \leq C \mathbf{E} \int_0^t \|(B - B_n)X(s)\|_{\mathcal{L}_2}^2 ds, \end{aligned}$$

$$\begin{aligned} A_4 &= \mathbf{E} \left\| \int_0^t (U(t-s) - U_n(t-s))BX(s) dW(s) \right\|^2 \\ &\leq \mathbf{E} \int_0^t \|(U(t-s) - U_n(t-s))BX(s)\|_{\mathcal{L}_2}^2 ds. \end{aligned}$$

Applying the Gronwall–Bellman lemma, we get from the latter bounds that

$$\begin{aligned} Z_n(t) \leq C & \left( \mathbb{E} \|X_0 - X_0^n\|^2 + \mathbb{E} \|(U(t) - U_n(t))X_0\|^2 \right. \\ & + CE \int_0^t \|(B - B_n)X(s)\|_{\mathcal{L}_2}^2 ds \\ & \left. + \int_0^t \sup_{u \in [0, T]} \mathbb{E} \|(U(u) - U_n(u))BX(s)\|_{\mathcal{L}_2}^2 ds \right) e^{Ct}. \end{aligned}$$

The integrands vanish as  $n \rightarrow \infty$  and have an integrable majorant  $C(1 + \|X(s)\|^2)$ ; therefore  $Z_n(t) \leq C_n e^{Ct}$  and

$$C_n \rightarrow 0, \quad n \rightarrow \infty. \quad \square$$

To prove the uniform convergence in probability, we need a stronger assumption. Instead of condition (1.8), we assume that the semigroups  $\{U_n(t)\}$  are of the uniformly contracting type. This means that there exist a number  $\beta$  and, for every  $n$ , an equivalent norm  $\|\cdot\|^{\sim, n}$  on  $\mathcal{X}$  such that

$$\|U_n(t)\|_{\mathcal{L}}^{\sim, n} \leq e^{\beta t}, \quad t \in [0, T].$$

According to the Phillips–Lumer theorem (cf. [4]) the latter condition is equivalent to

$$(1.14) \quad (A_n x, x) \leq \beta \|x\|^2, \quad x \in D(A_n).$$

**Theorem 1.2.** *If conditions (1.5), (1.7), and (1.14) hold and also*

$$\mathbb{E} \|X_0 - X_0^n\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

*then the uniform convergence in probability holds; that is, for all  $\delta > 0$ ,*

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|X(t) - X_n(t)\| > \delta \right) \rightarrow 0, \quad n \rightarrow \infty.$$

*Proof.* We check the conditions of the Kotelenez theorem [2]. First,

$$\begin{aligned} & \sup_{t \in [0, T]} \|U(t)Ax - U_n(t)A_n x\| \\ & \leq \sup_{t \in [0, T]} \|U_n(t)\| \cdot \|(A - A_n)x\| + \sup_{t \in [0, T]} \|(U(t) - U_n(t))Ax\| \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

for  $x \in D$ . Second,

$$\begin{aligned} & \mathbb{E} \left\| \int_0^t BX(s) dW(s) - \int_0^t B_n X_n(s) dW(s) \right\|^2 \\ & \leq 2 \int_0^t \mathbb{E} \|(B - B_n)X(s)\|_{\mathcal{L}_2}^2 ds + 2 \int_0^t \mathbb{E} \|B\|_{\mathcal{L}(\mathcal{X}, \mathcal{L}_2)}^2 \|X_n(s) - X(s)\|^2 ds \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$  by what we proved above. From the Kotelenez theorem we obtain the uniform convergence in probability:

$$\int_0^t U_n(t-s)B_n X_n(s) ds \rightarrow \int_0^t U(t-s)BX(s) ds, \quad n \rightarrow \infty.$$

Moreover

$$\begin{aligned} & \mathbb{P} \left( \sup_{t \in [0, T]} \|U(t)X_0 - U_n(t)X_0^n\| > \delta \right) \\ & \leq \mathbb{P} \left( \sup_{t \in [0, T]} \|U(t)X_0 - U_n(t)X_0\| > \delta/2 \right) \\ & \quad + \mathbb{P} \left( \sup_{t \in [0, T]} \|U_n(t)X_0 - U_n(t)X_0^n\| > \delta/2 \right) \\ & \leq \mathbb{P} \left( \sup_{t \in [0, T]} \|U(t)X_0 - U_n(t)X_0\| > \delta/2 \right) + \mathbb{P} \left( \sup_{t \in [0, T]} \|X_0 - X_0^n\| > \delta/2K \right) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$  by the assumption of the theorem (here  $K = \sup_{n \geq 1, t \leq T} \|U_n(t)\|$ ). Combining all the results above we get

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|X(t) - X_n(t)\| > \delta \right) \rightarrow 0$$

as  $n \rightarrow \infty$ . □

**1.1. Finite-dimensional approximations.** Let  $\{e_i, i \geq 1\}$  be an orthonormal basis in  $\mathcal{X}$ , and let

$$E_n = \text{span}\{e_i, i \leq n\}, \quad n \geq 1.$$

In this section,  $A_n$  and  $B_n$  are finite-dimensional approximations of the operators  $A$  and  $B$ , respectively. We consider the case of

$$(1.15) \quad E_n \subset D(A).$$

*Remark 1.1.* Condition (1.15) holds if, for example,  $\mathcal{X} = \mathcal{L}^2(\mathcal{O})$ ,  $O \subset \mathbb{R}^d$ ,  $A$  is a differential operator, and  $e_n$  are orthogonal polynomials.

At first glance, condition (1.15) seems to contradict the condition  $D(A_n) \supset D(A)$  used above. This, however, is not the case, since the domain of  $A_n$  is the entire space  $\mathcal{X}$  (not  $E_n$  as in the above case).

Now we may put  $A_n = P_n A P_n$ ,  $X_n^0 = P_n X_0$ , and  $B_n x = P_n (Bx) P_n$ , where  $P_n$  is the orthogonal projector on  $E_n$ . Equation (1.4) becomes of the form

$$(1.16) \quad X_n(t) = P_n X_0 + P_n \int_0^t (A X_n(s) ds + B X_n(s) P_n dW(s)).$$

Conditions (1.5) and (1.7b) evidently hold in this case. If the semigroup generated by  $A$  is of the contracting type, that is

$$(1.17) \quad (Ax, x) \leq \beta \|x\|^2,$$

then the semigroups generated by  $A_n$  are of the uniform contracting type. Indeed,

$$(A_n x, x) = (P_n A P_n x, x) = (A P_n x, P_n x) \leq \beta \|P_n x\|^2 \leq \beta \|x\|^2.$$

It is obvious that the convergence  $A_n x \rightarrow Ax$ ,  $n \rightarrow \infty$ , holds for  $x \in \bigcup_{k \geq 1} E_k$ .

Sufficient conditions for the convergence for all  $x \in D$  are not obvious in general. Nevertheless  $U_n(t) \xrightarrow{s} U(t)$  if the convergence  $A_n \rightarrow Ax$  holds on a dense set  $D_1$  such that the set

$$(A - \lambda)D_1$$

is also dense in  $\mathcal{X}$  for sufficiently large  $\lambda$ . In other words, it is sufficient to assume that

$$(1.18) \quad \text{the set } (A - \lambda)E \text{ is dense in } \mathcal{X} \text{ for all sufficiently large } \lambda$$

to prove the convergence, where  $E = \bigcup_{k=1}^{\infty} E_k$ . Thus we have proved the following result.

**Theorem 1.3.** *If conditions (1.15), (1.17), and (1.18) hold, then the finite-dimensional approximations (being solutions of equation (1.16)) uniformly with respect to  $t \in [0, T]$  converge in the mean-square sense to the solution of equation (1.1); that is,*

$$\mathbb{E} \|X(t) - X_n(t)\|^2 \rightarrow 0, \quad n \rightarrow \infty.$$

Moreover, the uniform convergence in probability also holds:

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|X(t) - X_n(t)\| > \delta \right) \rightarrow 0$$

as  $n \rightarrow \infty$ .

## 2. THE CONVERGENCE FOR NONHOMOGENEOUS EQUATIONS

In this section, we generalize the above results to the case where coefficients  $A$  and  $B$  depend on “time”  $t$ . Consider the following linear nonhomogeneous equations on the segment  $[0, T]$ :

$$(2.1a) \quad X(t) = X_0 + \int_0^t A(s)X(s) ds + \int_0^t B(s)X(s) dW(s),$$

$$(2.1b) \quad X_n(t) = X_0^n + \int_0^t A_n(s)X_n(s) ds + \int_0^t B_n(s)X_n(s) dW(s),$$

where  $A(t)$  and  $A_n(t)$  are linear densely defined closed operators in  $\mathcal{X}$ , and  $B(t)$  and  $B_n(t)$  are linear continuous operators acting from  $\mathcal{X}$  to  $\mathcal{L}_2(\mathcal{X})$ . For convenience, we set  $A_0(t) = A(t)$  and  $B_0(t) = B(t)$  in the below conditions. Operators  $A(t)$  and  $A_n(t)$  are assumed to satisfy the usual Kato–Tanabe conditions [6] uniformly in  $n$ ; that is,

- (i) for all  $n \geq 0$  and  $t \in [0, T]$ , the operators  $A_n(t)$  are the generators of some semigroups;
- (ii) the families  $A_n(t)$  are uniformly *stable*, i.e., there exist constants  $M$  and  $\beta$  such that

$$\|(A_n(t_k) - \lambda)^{-1}(A_n(t_{k-1}) - \lambda)^{-1} \cdots (A_n(t_1) - \lambda)^{-1}\| \leq M(\lambda - \beta)^{-k}$$

for all  $n \geq 0$ ,  $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ , and  $\lambda > \beta$ ;

- (iii) there exists a dense subspace  $D \subset \bigcap D(A_n(t))$  equipped with a norm  $\|\cdot\|_D$  such that the embedding of  $D$  in  $\mathcal{X}$  is continuous and  $D$  is  $A_n(t)$ -*admissible*, i.e.,  $e^{sA_n(t)}D \subset D$  and  $\{e^{sA_n(t)}|_D, s \geq 0\}$  are semigroups;
- (iv) for all  $n \geq 0$  and  $t \in [0, T]$ , the operator  $A_n(t)$  maps continuously  $(D, \|\cdot\|_D)$  to the space  $\mathcal{X}$ .

Then for any  $n \geq 0$ , the operators  $A_n(t)$  generate an *evolution family*

$$\{U_n(t, s), 0 \leq s \leq t \leq T\}$$

of linear continuous operators that possesses the following properties:

- 1)  $U_n(t, t) = I$  is the identity operator,
- 2)  $U_n(t, r)U_n(r, s) = U_n(t, s)$  for  $0 \leq s \leq r \leq t \leq T$ ,
- 3)  $U_n(t, s)$  is strongly continuous in  $s$  and  $t$ ,
- 4) for  $x \in D(A_n(s))$ ,

$$\frac{\partial}{\partial t} U_n(t, s)x = A_n(t)U_n(t, s)x.$$

As in the preceding section, assume that

$$(2.2) \quad \int_0^T \int_0^t \|A(t)U(t,s)B(s)\|_{\mathcal{L}(\mathcal{X},\mathcal{L}_2)}^2 ds dt < \infty$$

in order to ensure the existence of solutions of equations (2.1).

The conditions

$$(2.3) \quad A_n(t)x \rightarrow A(t)x, \quad n \rightarrow \infty, \quad \text{for all } x \in D,$$

$$(2.4) \quad \lim_{\lambda(E) \rightarrow 0} \sup_{n \geq 1} \int_E \|A_n(t)\|_{\mathcal{L}(D,\mathcal{X})} dt = 0 \quad (\lambda \text{ is the Lebesgue measure}),$$

together with (i)–(iv) guarantee that

$$U_n(t,s)x \rightarrow U(t,s)x, \quad n \rightarrow \infty,$$

for all  $x \in \mathcal{X}$  and uniformly with respect to  $s \leq t \leq T$  (cf. [5]). Assume further that

$$(2.5) \quad \|B_n(t)x - B(t)x\|_{\mathcal{L}_2} \rightarrow 0, \quad n \rightarrow \infty,$$

for all  $x$  and uniformly in  $t \in [0, T]$ . Analogously to the preceding section, we prove the following result.

**Theorem 2.1.** *If conditions (i)–(iv) and (2.2)–(2.5) hold and*

$$\mathbb{E} \|X_0 - X_0^n\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

*then solutions of equations (2.1b) converge in the mean-square sense to the solution of equation (2.1a); that is,*

$$\mathbb{E} \|X(t) - X_n(t)\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

*uniformly on the segment  $[0, T]$ .*

To prove the uniform convergence in probability, one should assume that the families  $U_n(t,s)$  are of the uniform contracting type; that is, there is  $\beta > 0$  and, for all  $n \geq 0$ , there exists an equivalent norm  $\|\cdot\|^{\sim,n}$  such that

$$(2.6) \quad \|U_n(t,s)\|^{\sim,n} \leq e^{\beta(t-s)}, \quad 0 \leq s \leq t \leq T.$$

The Kotelenetz theorem holds also for the nonhomogeneous case; hence we are able to prove the following theorem.

**Theorem 2.2.** *If conditions (i)–(iv) and (2.2)–(2.6) hold and*

$$\mathbb{E} \|X_0 - X_0^n\|^2 \rightarrow 0, \quad n \rightarrow \infty,$$

*then solutions of equations (2.1b) converge to the solution of equation (2.1a) uniformly on the segment  $[0, T]$ ; that is, for all  $\delta > 0$ ,*

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|X(t) - X_n(t)\| > \delta \right) \rightarrow 0, \quad n \rightarrow \infty.$$

**2.1. Finite-dimensional approximations for nonhomogeneous equations.** Consider finite-dimensional approximations for equation (2.1a). We use the notation introduced for the homogeneous equations. As in the previous section, we assume that

$$E_n \subset D(A(t))$$

and moreover that  $E_n \subset D$  ( $D$  is the space involved in condition (iii)). Then we put  $A_n(t) = P_n A(t) P_n$  and equation (2.1b) becomes of the form

$$(2.7) \quad X_n(t) = P_n X_0 + P_n \int_0^t (A(s)X_n(s) ds + B(s)X_n(s)P_n dW(s)).$$

We further assume that the semigroups generated by  $A(t)$  are of the contracting type; that is,

$$(2.8) \quad (A(t)x, x) \leq \beta \|x\|^2, \quad x \in D(A(t)).$$

Note that the condition on the stability is equivalent to

$$\left\| e^{s_k A(t_k)} e^{s_{k-1} A(t_{k-1})} \dots e^{s_1 A(t_1)} \right\| \leq M e^{\beta(s_1 + \dots + s_k)}$$

for all

$$t_1 \leq \dots \leq t_k \leq T, \quad s_1, \dots, s_n > 0.$$

Then, conditions (i) and (ii) are satisfied for the operators  $A(t)$ , and the semigroups generated by the operators  $A_n(t)$  are uniformly contracting. Moreover, conditions (i) and (ii) hold for  $A_n(t)$  with constants  $M$  and  $\beta$  that are independent of  $n$ . Condition (iii) holds for  $A_n(t)$ , since  $E_n \subset D$ . If condition (iv) holds for  $A(t)$ , then this condition holds for  $A_n(t)$ , too. Moreover,  $\|A_n(t)\|_{\mathcal{L}(D, \mathcal{X})} \leq \|A(t)\|_{\mathcal{L}(D, \mathcal{X})}$  and (2.4) follows from

$$(2.9) \quad \int_0^T \|A(t)\|_{\mathcal{L}(D, \mathcal{X})} dt < \infty.$$

Convergence (2.3) holds for  $x \in E = \bigcup_n E_n$ . This convergence holds for  $x \in D$ , too, if, for example,

$$(2.10) \quad E \text{ is dense in } (D, \|\cdot\|_D).$$

Finally, (2.6) holds if

$$(2.11) \quad B(t)x \in C([0, T], \mathcal{L}_2)$$

for all  $x \in \mathcal{X}$ . (At every point  $t \in [0, T]$ , the sequence  $\|B(t)x - B_n(t)x\|$  is monotone in  $n$  and tends to zero as  $n \rightarrow \infty$ ; thus the convergence is uniform on  $[0, T]$  by the Dini theorem.)

**Theorem 2.3.** *Let the operators  $A(t)$  satisfy conditions (iii), (iv), and (2.2). If conditions (2.8)–(2.11) hold, then the solutions of equations (2.7) converge in the mean-square sense and uniformly in probability to the solution of equation (2.1a); that is,*

$$\begin{aligned} \mathbb{E} \|X(t) - X_n(t)\|^2 &\rightarrow 0, \quad n \rightarrow \infty, \\ \mathbb{P} \left( \sup_{t \in [0, T]} \|X(t) - X_n(t)\| > \delta \right) &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

### 3. REPRESENTATION OF SOLUTIONS OF FINITE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

It is well known that if  $a \in L_1[0, T]$  and  $b \in L_2[0, T]$ , then the one-dimensional stochastic differential equation

$$(3.1) \quad x(t) = x_0 + \int_0^t a(s)x(s) ds + \int_0^t b(s)x(s) dw(s), \quad t \in [0, T],$$

has a unique solution given by

$$(3.2) \quad x(t) = x_0 \exp \left\{ \int_0^t b(s) dw(s) + \int_0^t \left( a(s) - \frac{1}{2} b^2(s) \right) ds \right\}, \quad t \in [0, T].$$

This solution is called the stochastic exponent constructed from the functions  $a$  and  $b$ . We refer to the solution of equation (1.1) as the stochastic exponent constructed from the operators  $A$  and  $B$ . Denote this solution by

$$X(t) = \mathcal{E}_t(A, B).$$



It is impossible to express  $X(t)$  in terms of  $A$ ,  $B$ , and  $W(t)$  in an explicit form similar to formula (3.2). On the other hand, there are certain conditions under which the stochastic exponent can be represented in a closed form in the case of finite-dimensional equations. Below we provide such conditions.

Taking into account that  $\mathcal{E}_t(A, B)$  is a mean-square limit (and a uniform limit in probability, as well) of the finite-dimensional stochastic exponents  $\mathcal{E}_t(A_n, B_n)$  if the assumptions of Theorem 1.3 hold, we propose to represent  $\mathcal{E}_t(A, B)$  as a limit of finite-dimensional stochastic exponents.

Let  $n > 1$ ,  $A \in \mathcal{L}(R^n)$ ,  $B \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ , and let  $W(t)$  be a  $d$ -dimensional Wiener process. Consider the linear stochastic differential equation

$$(3.3) \quad X(t) = X_0 + \int_0^t AX(s) ds + \int_0^t BX(s) dW(s).$$

One can rewrite this equation in the coordinate form as follows:

$$x_i(t) = x_i(0) + \sum_{j=1}^n a_{ij} \int_0^t x_j(s) ds + \sum_{k=1}^d \sum_{j=1}^n b_{ij}^k \int_0^t x_j(s) dw_k(s),$$

where  $a_{i,j} = (Ae_j, e_i)$  and  $b_{ij}^k = ((Be_j)e_k, e_i)$ .

If the operators  $A$  and  $B^k = (b_{ij}^k)_{i,j=1}^n$  are pairwise commuting, that is, if

$$AB^k = B^kA, \quad B^k B^l = B^l B^k, \quad k, l = 1, \dots, d,$$

then one can represent the solution of equation (3.3) in a closed form. We have in this case

$$X(t) = \exp \left\{ \int_0^t \left( A - \frac{1}{2} \sum_{k=1}^d (B^k)^2 \right) ds + \sum_{k=1}^d B^k dw_k \right\} X_0.$$

This formula remains true also in the case where  $X_0$  is a common eigenvector of operators  $A$  and  $B^k$ ,  $k = 1, \dots, d$ , since the equation can be viewed in this case as the one-dimensional one on the line determined by this eigenvector.

The case where the matrices of the operators  $A$  and  $B^k$  are of the upper-triangular form, that is if

$$a_{ij} = b_{ij}^k = 0 \quad \text{for } i > j,$$

is the most interesting one. The solution of (3.3) can be represented in this case as follows:

$$x_n(t) = M_n(t)^{-1} x_n(0),$$

$$x_i(t) = M_i(t)^{-1} \left( x_i(0) + \int_0^t M_i(s) d\eta_i(s) \right), \quad 1 \leq i \leq n-1,$$

where

$$\eta_i(t) = \sum_{j=i+1}^n \int_0^t x_j(s) d\xi_j^i(s),$$

$$\xi_j^i(t) = \sum_{k=1}^d \left[ b_{ij}^k w_k(t) + \left( a_{ij} - \frac{1}{2} (b_{ij}^k)^2 \right) t \right],$$

$$M_i(t) = \exp \{ -\xi_i^i(t) \}.$$

Kunita [3] obtained the following result for homogeneous-in-time stochastic differential equations on smooth manifolds whose coefficients are infinitely differentiable vector fields: if the Lie algebra generated by the coefficients of the equation is solvable, then the solution

of the equation can be represented in a certain closed form. Moreover, the Lie algebra generated by commuting matrices and the Lie algebra generated by upper-triangular matrices are solvable. Thus both cases above follow from the Kunita result. In order to rewrite the Kunita formula for the solution in the case of upper-triangular operators, it is convenient to set

$$B^0 := A - \frac{1}{2} \sum_{k=1}^d (B^k)^2, \quad w_0(t) := t,$$

and to consider equation (3.3) in the Stratonovich form

$$dX(t) = \sum_{k=0}^d B^k X(s) \circ dw_k(s).$$

Let

$$D^k = (\delta_{ij} b_{ij}^k)_{i,j=1}^n$$

be the diagonal part of  $B^k$ , and let  $N^k = B^k - D^k$  be the nilpotent part of  $B^k$ , where  $\delta_{ij}$  is the Kronecker delta. The Kunita formula for the solution becomes of the following form:

$$\begin{aligned} X(t) &= e^{M(t)} e^{V(t)} X_0, \\ M(t) &= \sum_{k=0}^d w_k(t) D^k, \\ (3.4) \quad V(t) &= \sum_{k=0}^d \int_0^t e^{-M(s)} N^k e^{M(s)} \circ dw_k(s) \\ &\quad + \frac{1}{2} \sum_{i < j} \iint_{0 < s < u < t} \left[ e^{-M(s)} N^i e^{M(s)}, e^{-M(u)} N^j e^{M(u)} \right] \\ &\quad \circ (dw_i(s) dw_j(u) - dw_j(s) dw_i(u)) + \dots, \end{aligned}$$

where  $[\cdot, \cdot]$  denotes the Lie brackets. The second sum is followed by a finite number of terms, and each of them is a sum of multiple integrals similar to the above multiple commutators of  $e^{-M(s)} N^k e^{M(s)}$ . Moreover the coefficients for the terms coincide with the corresponding coefficients in the classical Campbell–Hausdorff formula

$$\begin{aligned} e^X e^Y &= e^Z, \\ Z &= X + Y - \frac{1}{2} [X, Y] + \dots \end{aligned}$$

The case of upper-triangular operators is essentially all that is given by the Kunita theorem for linear equations considered above. Indeed, the Lie theorem gives the following solvability criterion of a Lie algebra generated by complex matrices: a Lie algebra is solvable if and only if all the matrices are of upper-triangular form in a certain basis. We deal with real numbers; nevertheless we may apply the Lie criterion, although this can be done at the cost of complexification, that is, by doubling the dimension of the phase space (but not the dimension of noise). More precisely, assuming that the Lie algebra generated by  $B^k$  is solvable, define matrices  $\tilde{B}^k$  of dimension  $2n \times 2n$  composed of  $2 \times 2$  blocks

$$\tilde{B}_{ij}^k = \begin{pmatrix} b_{ij}^k & 0 \\ 0 & b_{ij}^k \end{pmatrix}.$$

Then there exists a basis for which the matrices are of “almost” upper-triangular form. This means that the matrices  $\tilde{B}^k$  are composed of the blocks  $\tilde{B}_{ij}^k$  that equal zero for  $i > j$

and are of the form

$$\widehat{B}_{ij}^k = \begin{pmatrix} \alpha_{ij}^k & -\beta_{ij}^k \\ \beta_{ij}^k & \alpha_{ij}^k \end{pmatrix}$$

for  $i \leq j$ . Taking into account that  $\widehat{B}_{ij}^k$  are pairwise commuting, we prove that equality (3.4) remains true for

$$D^k = \begin{pmatrix} \widehat{B}_{11}^k & 0 & 0 & \dots & 0 \\ 0 & \widehat{B}_{22}^k & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \widehat{B}_{nn}^k \end{pmatrix}, \quad N^k = \widehat{B}^k - D^k.$$

#### BIBLIOGRAPHY

1. W. Greksch and C. Tudor, *Stochastic Evolution Equations: A Hilbert Space Approach*, Mathematical Research, vol. 85, Akademie Verlag, 1995. MR1353910 (96m:60130)
2. P. Kotelenez, *A submartingale type inequality with application to stochastic evolution equations*, Stochastics **8** (1982), no. 2, 139–152. MR0686575 (84h:60115)
3. H. Kunita, *On the Representation of Solutions of SDEs*, Séminaire de Prob. XIV, Lect. Notes Math., Springer-Verlag, Berlin, 1980, 282–304. MR0580134 (82e:58028)
4. K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin–New York, 1971. MR0350358 (50:2851)
5. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer Verlag, Berlin–Heidelberg–New York, 1983. MR0710486 (85g:47061)
6. H. Tanabe, *Equations of Evolution*, Pitman, London, 1979. MR0533824 (82g:47032)

CHAIR OF PROBABILITY THEORY AND MATHEMATICAL STATISTICS, DEPARTMENT OF MECHANICS AND MATHEMATICS, KYIV NATIONAL TARAS SHEVCHENKO UNIVERSITY, GLUSHKOV PR. 6, KYIV 03127, UKRAINE

*E-mail address:* myus@univ.kiev.ua

CHAIR OF PROBABILITY THEORY AND MATHEMATICAL STATISTICS, DEPARTMENT OF MECHANICS AND MATHEMATICS, KYIV NATIONAL TARAS SHEVCHENKO UNIVERSITY, GLUSHKOV PR. 6, KYIV 03127, UKRAINE

*E-mail address:* zhora@univ.kiev.ua

Received 18/DEC/2002

Translated by G. SHEVCHENKO