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THRESHOLD STRUCTURE OF OPTIMAL STOPPING STRATEGIES FOR AMERICAN TYPE OPTION. I

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Abstract. The paper presents results of theoretical studies of optimal stopping domains of American type options in discrete time. Sufficient conditions on the payoff functions and the price process for the optimal stopping domains to have one-threshold structure are given. We consider monotone, convex and inhomogeneous-in-time payoff functions. The underlying asset's price is modelled by an inhomogeneous discrete time Markov process.

1. Introduction

The holder of an American type option can exercise it at any time up to and including the expiration date. The problem for the holder is to determine when it is optimal to exercise. We define the optimal moment to be the moment when the holder maximizes the expected discounted exercise profit. The optimal moment to exercise the options is an optimal stopping time and to compute the optimal moment of exercise we need to know the structure of the optimal stopping domains.

The optimal stopping domain is defined for each moment as the set of all prices of the underlying asset for which the payoff of exercising the option is greater than the expected future payoff. The complement to the stopping domain is the continuation domain.

The present paper studies the structure of the optimal stopping boundary and the optimal stopping domains for American type options with monotone, convex, inhomogeneous-in-time payoff functions written on a single underlying asset. The price of the underlying asset follows a general inhomogeneous discrete time Markov process.

We present general sufficient conditions on the price process and the payoff functions such that the stopping boundary and the optimal stopping domains have the same structure as for the standard American options.

The structure of the optimal stopping boundary and the optimal stopping domains for the standard American option is well known; see, e.g., van Moerbeke (1976), Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992), Broadie and Detemple (1999).

The optimal stopping domains for the standard put option (with homogeneous payoff function) are intervals such that if x' is in the optimal stopping domain at time t, then every x, $0 \leq x \leq x'$, is in the optimal stopping domain at time t. The boundary of the optimal stopping domains is nondecreasing in time; i.e., if x' belongs to the optimal

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stopping domain at time t', then x' belongs to the optimal stopping domain for all t, $t' \leq t \leq T$, where T is the expiration date.

The standard call option (with homogeneous payoff function) has optimal stopping domains that are semi-infinite intervals; i.e., if x' is in the optimal stopping domain at time t, then every $x \geq x'$ is in the optimal stopping domain at time t. The boundary of the optimal stopping domains is nonincreasing in time; i.e., if x' is in the stopping domain at time t', then x' is in the stopping domain at time t, for all $t' \leq t \leq T$.

In Shiryaev *et al.* (1994) the structure of the optimal stopping domains for American type call options with inhomogeneous-in-time payoff function,

$$g_n(x) = a^n[x - K]^+,$$

where $0 < a \leq 1$, is studied under the assumption that the value of the underlying asset follows a binomial random walk. It is shown that, for $0 < a < 1$, the structure of the optimal stopping domains is equivalent with the structure of the optimal stopping domains for the standard American call option; i.e., the up-connectedness property and the right-connectedness property hold.

In Kukush and Silvestrov (2004) the structure of the optimal stopping domains for the holder of an American type call option is studied for a price process described by a two-component inhomogeneous-in-time Markov process. Under general monotonicity and convexity conditions imposed on the transition dynamical function of the price process, it is shown that the structure of the optimal stopping domains is of one-threshold type if the payoff function is an inhomogeneous-in-time analogue of the payoff function for the standard American option. But the structure of the optimal stopping domains can be of multi-threshold structure, i.e. union of disjoint intervals, if the payoff function is a convex piecewise linear function.

The results presented in Kukush and Silvestrov (2004) were confirmed by experimental studies presented in Jönsson (2001). In Jönsson (2001) optimal stopping domains for American type call options with standard, piecewise linear, stepwise, quadratic and logarithmic payoff functions are presented.

In Jönsson, Kukush, and Silvestrov (2002), both theoretical and experimental studies of the structure of the optimal stopping domains for American type call options are presented. Sufficient local conditions, connecting the payoff and the expected payoff from two sequential time steps, for the optimal stopping domains to be semi-infinite intervals for American type call options are given. In the paper, the price process is a discrete time Markov process with multiplicative increments. The payoff functions are nondecreasing and convex. A sufficient condition for the existence of a semi-infinite interval structure for a piecewise linear payoff function is given explicitly. It is shown, by experimental results, that the single interval structure of the stopping domain can switch to a more complex structure, a union of two disjoint intervals, when the sufficient condition is violated.

The present paper is a continuation of the research presented in Kukush and Silvestrov (2000), Jönsson (2001) and Jönsson, Kukush, and Silvestrov (2002). We consider both put and call American type options with nonincreasing and nondecreasing payoff functions, respectively.

The knowledge of the structure of the optimal stopping domains and the optimal stopping boundaries can be used in Monte Carlo algorithms to estimate the risk-neutral price of the option and different characteristics of the option, such as, expectation and variance of the profit of holding the option, the mean exercise time and the probability of early exercise.

See Boyle, Broadie, and Detemple (2001) for a recent review of Monte Carlo methods in option pricing. See also Basso, Nardon, and Pianca (2002) and the references therein.

Examples of Monte Carlo algorithms using the knowledge of the structure of the optimal stopping domains can also be found in Silvestrov, Galochkin, and Sibirtsev (1999) and Silvestrov, Galochkin, and Malyarenko (2001).

The paper is organized as follows. In Section 2 the general optimal stopping problem in discrete time is formulated. Section 3 presents sufficient conditions on the price process and the payoff functions, such that the stopping domains have a one-threshold structure. To model the payoff of a general put option we use nonincreasing and convex functions. In Section 4 we give examples of put type payoff functions and corresponding sufficient conditions for a one-threshold structure. In Section 5 we give equivalent results for American type call options. The call options payoff functions are modelled as nondecreasing and convex functions. The paper is divided into two parts: Sections 1–3 and Sections 4–5, respectively. The first part is presented here. The second is presented in the next issue of the journal.

2. Optimal stopping in discrete time

We consider a general inhomogeneous discrete time stochastic Markov process to model the underlying asset's price,

$$S_n = A_n(S_{n-1}, Y_n), \qquad n = 1, \dots, N, \tag{1}$$

where $A_n(x, y)$, $n = 1, 2, \dots, N$, is a measurable function acting on $\mathbb{R}^+ \times \mathbb{Y}$ to $\mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, and $\mathbb{Y}$ is a measurable space.

Here S_n is the stock price at moment n and Y_n is a sequence of independent random variables that take values in the space $\mathbb{Y}$. The initial value, $S_0 > 0$, of the price process is nonrandom.

We assume the following condition for the price process:

A1: For every $n = 1, \dots, N$ and $y \in \mathbb{Y}$,

$$A_n(0, y) = 0.$$

Condition **A1** is a natural assumption. It assures that if $S_n = 0$, then $S_m = 0$ for all $n \le m \le N$.

We assume also the following condition:

A2: For every $n = 1, \dots, N$ and $y \in \mathbb{Y}$, $A_n(x, y)$ is a nondecreasing and concave function in x.

The concavity of the function $A_n(\cdot, y)$ reflects the effect that the larger S_{n-1} is the slower S_n will increase. This is a natural assumption for the behavior of the price of the underlying asset.

For example, the classical multiplicative model, $A_n(x, Y_n) = xY_n$, $n = 1, 2, \dots, N$, satisfies **A1** and **A2**.

Let $n = 0, \dots, N$ be possible moments of execution of the option, where $N < \infty$ is the expiration date.

Denote by $\mathbf{G}$ the class of nonincreasing and convex functions acting from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Note that functions from the class $\mathbf{G}$ are bounded and continuous.

Let $g'_{(-)}(x)$ denote the left derivative of a function $g(x) \in \mathbf{G}$. The left derivative $g'_{(-)}(x)$ exists for all $x > 0$. Moreover, the ordinary derivative $g'(x)$ exists for all $x \ge 0$ except for at most a countable set, and both derivatives coincide at the points where the function $g(x)$ is differentiable. Note also that $g'_{(-)}(x)$ is a nonpositive and nondecreasing function.

Let $g_n(x)$ be the payoff function at moment $n = 0, 1, \dots, N$. We impose the following conditions on the payoff functions:

G1: For every $n = 0, 1, \dots, N$, the payoff function $g_n(x)$ belongs to the class $\mathbf{G}$.

Condition **G1** actually means that we analyze put type options.

Note that condition **G1** holds for the standard payoff function of the American type put option $g(x) = [K - x]_+$, as well as for an inhomogeneous-in-time version

$$g_n(x) = a_n[K_n - x]_+, \qquad a_n > 0.$$

Let $r_n > 0$ be a risk free interest rate valid between moments $n-1$ and n for $n = 1, \dots, N$, and $r_0 = 0$. Also let $R_n = r_0 + \cdots + r_n$ for $n = 0, \dots, N$.

Denote by $\mathbf{M}_N$ the class of Markov moments $\tau \in \{0, 1, \dots, N\}$ for the random sequence $S_0, S_1, \dots, S_N$.

The problem of finding the structure of the stopping boundaries is formulated as an optimal stopping problem, i.e., a maximization problem where we maximize over all Markov moments $\tau \in \mathbf{M}_N$ the functional

$$\Phi_g(\tau) = \mathsf{E} e^{-R_\tau} g_\tau(S_\tau). \tag{2}$$

The optimal Markov moment τ_{opt} maximizing (2) is given by

$$\Phi_g(\tau_{\text{opt}}) = \sup_{\tau \in \mathbf{M}_N} \mathsf{E} e^{-R_\tau} g_\tau(S_\tau). \tag{3}$$

We introduce the operators T_n, $n = 1, \dots, N$, acting on nonnegative, nonincreasing functions $f(x)$ defined on $\mathbb{R}^+$ by the formula

$$T_n f(x) = \mathsf{E}\{f(A_n(S_{n-1}, Y_n)) \mid S_{n-1} = x\} = \mathsf{E} f(A_n(x, Y_n)). \tag{4}$$

Let us formulate without the proof the following simple lemma.

Lemma 1. *If the function $g(x) \in \mathbf{G}$ and condition **A2** holds, then functions $T_n g(x) \in \mathbf{G}$, $n = 1, \dots, N$.*

The following lemma will also be useful.

Lemma 2. *If condition **A2** holds and the function $g(x) \in \mathbf{G}$, then for every $n = 1, \dots, N$ and $x \in \mathbb{R}^+$ there exist*

$$\begin{aligned} (T_n g)'_{(-)}(x) &= \mathsf{E}[g(A_n(x, Y_n))]'_{(-)} \\ &= \mathsf{E} g'_{(-)}(A_n(x, Y_n)) A'_{n,(-)}(x, Y_n). \end{aligned} \tag{5}$$

Proof. By the definition of the left derivative,

$$\begin{aligned} g'_{(-)}(A_n(x, Y_n)) &= \lim_{h \to 0+} \frac{g(A_n(x, Y_n)) - g(A_n(x-h, Y_n))}{h} \\ &= \lim_{m \to \infty} u_m(x), \end{aligned}$$

where

$$u_m(x) = \frac{g(A_n(x, Y_n)) - g(A_n(x-h_m, Y_n))}{h_m}$$

and $\{h_m\}$ is a decreasing sequence converging to 0 from above.

At the same time,

$$\begin{aligned} (T_n g)'_{(-)}(x) &= \lim_{h \to 0+} \frac{\mathsf{E} g(A_n(x, Y_n)) - \mathsf{E} g(A_n(x-h, Y_n))}{h} \\ &= \lim_{m \to \infty} \mathsf{E} u_m(x). \end{aligned}$$

Note that $u_1(x) \le u_2(x) \le \dots$, since $g(x) \in \mathbf{G}$ and condition **A2** imply that the functions $g(A_n(x, Y_n))$, $x \in \mathbb{R}^+$, $n = 0, 1, \dots, N$, are convex. Furthermore, for each m the function $u_m(x)$ is integrable, since $T_n g(x) < \infty$ for all $x \in \mathbb{R}^+$.

By Lebesgue's monotone convergence theorem,

$$\begin{aligned}(T_n g)'_{(-)}(x) &= \lim_{m\to\infty} \mathsf{E} u_m(x) = \mathsf{E} \lim_{m\to\infty} u_m(x) \\ &= \mathsf{E}[g(A_n(x, Y_n))]'_{(-)} \\ &= \mathsf{E} g'_{(-)}(A_n(x, Y_n)) A'_{n,(-)}(x, Y_n). \end{aligned} \qquad \square$$

The general solution of the maximization problem (2)–(3) can be described with the use of the so-called reward functions. Let us define the reward functions recursively beginning with

$$w_0(x) = g_N(x), \tag{6}$$

and for the moments $n = N-1, \ldots, 0$,

$$w_{N-n}(x) = \max\left\{g_n(x), e^{-r_{n+1}} T_{n+1} w_{N-n-1}(x)\right\}. \tag{7}$$

By the definition, the reward functions are defined on $\mathbb{R}^+$.

The next lemma follows from Lemma 1 and the definition of the reward functions.

Lemma 3. *If conditions* ***A2*** *and* ***G1*** *hold, then for every $n = N, \ldots, 0$ the reward functions $w_{N-n}(x) \in \mathbf{G}$.*

It has been shown in Chow, Robbins, and Siegmund (1971) and Shiryaev (1978) that the optimal stopping moment of the problem (2)–(3) exists and has a first hitting-time structure, i.e.,

$$\tau_{\text{opt}} = \min\{0 \le n \le N \colon S_n \in \Gamma_n\}, \tag{8}$$

where the optimal stopping domains are given by

$$\Gamma_n = \{x \colon w_{N-n}(x) = g_n(x)\} \tag{9}$$

for $n = 0, 1, 2, \ldots, N$, and

$$\Phi_g(\tau_{\text{opt}}) = w_N(S_0). \tag{10}$$

Note that

$$\Gamma_N = [0, \infty). \tag{11}$$

For $n = 0, \ldots, N-1$ using equation (7), we can rewrite (9) in an equivalent form:

$$\Gamma_n = \left\{x \colon g_n(x) \ge e^{-r_{n+1}} T_{n+1} w_{N-n-1}(x)\right\}. \tag{12}$$

The representations (9) and (12) are true for arbitrary payoff functions $g_n(x)$, not necessarily belonging to the class $\mathbf{G}$. Unfortunately, both formulas (9) and (12) give a too inexplicit description of optimal stopping domains.

3. Threshold structure of optimal stopping domains for put options

In this section we study conditions on the payoff function and the price process such that the optimal stopping domains have the following structure:

$$\Gamma_n = [0, d_n] \quad \text{with } d_n < \infty,\ n = 0, \ldots, N-1. \tag{13}$$

Let us assume that the following condition holds:

E1: For every $n = 0, \ldots, N-1$, there exists x_n such that

$$g_n(x_n) < e^{-r_{n+1}} T_{n+1} g_{n+1}(x_n).$$

We will show that condition **E1** guarantees that the optimal stopping domains

$$\Gamma_n \neq \mathbb{R}^+$$

for $n = 0, \dots, N-1$.

The following condition plays a key role:

B1: For every $n = 0, 1, \dots, N-1$ and $x \in \mathbb{R}^+$,

$$g'_{n,(-)}(x) \leq e^{-r_{n+1}} \mathsf{E} g'_{n+1,(-)}(A_{n+1}(x, Y_{n+1})) A'_{n+1,(-)}(x, Y_{n+1}).$$

Theorem 1. *Under conditions* ***A1****,* ***A2****,* ***G1****,* ***E1****, and* ***B1*** *for every $n = 0, 1, \dots, N-1$, the optimal stopping domain has the one-threshold structure, i.e.,*

$$\Gamma_n = [0, d_n]$$

with $d_n < \infty$.

Proof. The structure given in (13) corresponds to the following three properties of the optimal stopping domains: **(a)** $\Gamma_n \neq \mathbb{R}^+$; **(b)** Γ_n is a closed nonempty set; and **(c)** if $x' \in \Gamma_n$, then $x'' \in \Gamma_n$ for any $0 \leq x'' \leq x'$.

Consider moment $N-1$. It follows from condition **E1** and the identity $g_N(x) \equiv w_0(x)$ that

$$g_{N-1}(x_{N-1}) < e^{-r_N} T_N g_N(x_{N-1}) = e^{-r_N} T_N w_0(x_{N-1}). \tag{14}$$

Relation (14) implies that the point $x_{N-1} \in \Gamma^c_{N-1}$, where $\Gamma^c_{N-1} = \mathbb{R}^+ \setminus \Gamma_{N-1}$, is the continuation domain. Thus, **(a)** holds for the stopping domain Γ_{N-1}.

Property **(a)** for the optimal stopping domain Γ_{N-1} implies that

$$d_{N-1} = \inf \left\{ x \colon x \in \Gamma^c_{N-1} \right\} < \infty. \tag{15}$$

The functions $g_{N-1}(x)$ and $e^{-r_N} T_N g_N(x)$ are continuous by condition **G1** and Lemma 1, respectively. It follows from the continuity of these functions and the defining formula (12) that the set Γ^c_{N-1} is open and, therefore, the point $d_{N-1} \in \Gamma_{N-1}$. Thus, **(b)** holds for the optimal stopping domain Γ_{N-1}.

Using the identity $g_N(x) = w_0(x)$, condition **B1**, and Lemma 2 we have for any $x \geq 0$,

$$g'_{N-1,(-)}(x) \leq e^{-r_N} (T_N w_0)'_{(-)}(x). \tag{16}$$

Assume that $x' \in \Gamma_{N-1}$. Using this assumption and (16) we get for any $0 \leq x'' \leq x'$,

$$\begin{aligned} g_{N-1}(x'') &= g_{N-1}(x') - \int_{x''}^{x'} g'_{N-1,(-)}(t)\, dt \\ &\geq e^{-r_N} T_N w_0(x') - \int_{x''}^{x'} e^{-r_N} (T_N w_0)'_{(-)}(t)\, dt \\ &= e^{-r_N} T_N w_0(x''). \end{aligned}$$

Thus, **(c)** holds for the optimal stopping domain Γ_{N-1}.

So we proved that the optimal stopping domain Γ_{N-1} has the threshold structure described in (13).

Let us make one induction step and consider moment $N-2$. First of all, let us show that the following inequality holds for the point x_{N-2} that appears in condition **E1**:

$$g_{N-2}(x_{N-2}) < e^{-r_{N-1}} T_{N-1} w_1(x_{N-2}). \tag{17}$$

From the definitions of the operator and the reward functions for any $x \geq 0$,

$$\begin{aligned} T_{N-1} w_1(x) &= \mathsf{E} w_1(A_{N-1}(x, Y_{N-1})) \\ &= \mathsf{E} g_{N-1}(A_{N-1}(x, Y_{N-1})) I(A_{N-1}(x, Y_{N-1}) \leq d_{N-1}) \\ &\quad + \mathsf{E} e^{-r_N} T_N w_0(A_{N-1}(x, Y_{N-1})) I(A_{N-1}(x, Y_{N-1}) > d_{N-1}). \end{aligned} \tag{18}$$

But, if $A_{N-1}(x, Y_{N-1}) > d_{N-1}$, then

$$e^{-r_N} T_N w_0(A_{N-1}(x, Y_{N-1})) > g_{N-1}(A_{N-1}(x, Y_{N-1})).$$

Indeed, in this case $A_{N-1}(x, Y_{N-1}) \in \Gamma^c_{N-1}$ since, as was already proved,

$$\Gamma_{N-1} = [0, d_{N-1}].$$

Therefore,

$$\begin{aligned} &\mathsf{E} e^{-r_N} T_N w_0(A_{N-1}(x, Y_{N-1})) I(A_{N-1}(x, Y_{N-1}) > d_{N-1}) \\ &\quad \geq \mathsf{E} g_{N-1}(A_{N-1}(x, Y_{N-1})) I(A_{N-1}(x, Y_{N-1}) > d_{N-1}). \end{aligned} \tag{19}$$

Combining (18) and (19) we get the following inequality that is satisfied for any $x \geq 0$:

$$T_{N-1} w_1(x) \geq \mathsf{E} g_{N-1}(A_{N-1}(x, Y_{N-1})). \tag{20}$$

Now, using relation (20) and condition **E1**, we have

$$\begin{aligned} e^{-r_{N-1}} T_{N-1} w_1(x_{N-2}) &\geq e^{-r_{N-1}} \mathsf{E} g_{N-1}(A_{N-1}(x_{N-2}, Y_{N-1})) \\ &= e^{-r_{N-1}} T_{N-1} g_{N-1}(x_{N-2}) > g_{N-2}(x_{N-2}). \end{aligned} \tag{21}$$

Relation (21) implies that point $x_{N-2} \in \Gamma^c_{N-2}$, i.e., **(a)** holds for the optimal stopping domain Γ_{N-2}.

Property **(a)** for the optimal stopping domain Γ_{N-2} implies that

$$d_{N-2} = \inf \left\{ x \colon x \in \Gamma^c_{N-2} \right\} < \infty. \tag{22}$$

The functions $g_{N-2}(x)$ and $e^{-r_{N-1}} T_{N-1} w_{N-1}(x)$ are continuous by condition **G1** and Lemmas 1 and 3, respectively. It follows from continuity of these function and the defining formula (12) that the set Γ^c_{N-2} is open and, therefore, the point $d_{N-2} \in \Gamma_{N-2}$. Thus, **(b)** holds for the optimal stopping domain Γ_{N-2}.

Using Lemmas 1–3 we get the following differentiation formula of (18) for any $x \geq 0$:

$$(T_{N-1} w_1)'_{(-)}(x) = \mathsf{E}[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)}. \tag{23}$$

We now have to consider two cases concerning the expression on the right-hand side.

First case: $A_{N-1}(x, Y_{N-1}) \leq d_{N-1}$. This is true if and only if $0 < x \leq x^*(Y_{N-1})$, where $x^*(Y_{N-1}) = \sup\{x > 0 \colon A_{N-1}(x, Y_{N-1}) \leq d_{N-1}\}$. Then, for $x \leq x^*(Y_{N-1})$,

$$\begin{aligned} [w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)} &= [g_{N-1}(A_{N-1}(x, Y_{N-1}))]'_{(-)} \\ &= g'_{N-1,(-)}(A_{N-1}(x, Y_{N-1})) A'_{N-1,(-)}(x, Y_{N-1}). \end{aligned} \tag{24}$$

Second case: $A_{N-1}(x, Y_{N-1}) > d_{N-1}$. This is true if and only if $x > x^*(Y_{N-1})$. Then, for $x > x^*(Y_{N-1})$, by Lemma 2 and relation (16),

$$\begin{aligned} &[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)} \\ &\quad = \left[e^{-r_N} T_N w_0(z) \right]'_{(-)} \Big|_{z = A_{N-1}(x, Y_{N-1})} A'_{N-1,(-)}(x, Y_{N-1}) \\ &\quad \geq g'_{N-1,(-)}(A_{N-1}(x, Y_{N-1})) A'_{N-1,(-)}(x, Y_{N-1}). \end{aligned} \tag{25}$$

Combining (24) and (25) we get for any $x \geq 0$,

$$[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)} \geq g'_{N-1,(-)}(A_{N-1}(x, Y_{N-1})) A'_{N-1,(-)}(x, Y_{N-1}). \tag{26}$$

Take the expectation of (26) and insert into (23) for any $x \geq 0$:

$$\begin{aligned} (T_{N-1} w_1)'_{(-)}(x) &= \mathsf{E}[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)} \\ &\geq \mathsf{E} g'_{N-1,(-)}(A_{N-1}(x, Y_{N-1})) A'_{N-1,(-)}(x, Y_{N-1}). \end{aligned} \tag{27}$$

Finally, by condition **B1** and (27) for any $x \geq 0$,

$$e^{-r_{N-1}} (T_{N-1} w_1)'_{(-)}(x) \geq g'_{N-2,(-)}(x). \tag{28}$$

Assume that $x' \in \Gamma_{N-2}$. Using this assumption and relation (28) we have for any $0 \le x'' \le x'$,

$$\begin{aligned} g_{N-2}(x'') &= g_{N-2}(x') - \int_{x''}^{x'} g'_{N-2,(-)}(t)\,dt \\ &\ge e^{-r_{N-1}} T_{N-1} w_1(x') - \int_{x''}^{x'} e^{-r_{N-1}} (T_{N-1} w_1)'_{(-)}(t)\,dt \\ &= e^{-r_{N-1}} T_{N-1} w_1(x''). \end{aligned}$$

Thus **(c)** holds for the optimal stopping domain Γ_{N-2}.

So we proved that the optimal stopping domain Γ_{N-2} also has the threshold structure described in (13).

It should be noted that despite the similarity of the proofs given above for the cases $N-1$ and $N-2$, the latter proof should also be given. As a matter of fact, the proof of the key relations (21) and (28) cannot be realized in the same way as the proof of relations (14) and (16), which was based on the direct use of the identity $g_N(x) \equiv w_0(x)$ and condition **B1**.

By repeating the induction reasoning used for the case $N-2$, etc., it is possible to prove that the optimal stopping domains Γ_n have the threshold structure described in (13) for all $n = N-3, N-2, \dots, 0$. □

Theorem 1 gives effective sufficient conditions for the optimal stopping domains to have a simple one-threshold structure for the case where a payoff function possesses the following property:

$$g'_{n,(-)}(x) < 0, \qquad x \ge 0,\ n = 0, 1, \dots, N. \tag{29}$$

For example, the functions $g_n(x) = a_n e^{-b_n x}$ and $g_n(x) = a_n (x + c_n)^{-b_n}$, where $a_n, b_n, c_n > 0$, possess property (29).

The key condition **B1** takes in these cases a simple readable form. For example, in the case of the homogeneous-in-time multiplicative model, where $A_n(x, y) = xy$, and the exponential payoff function with coefficients $a_n = a$, $b_n = b$, condition **B1** takes the form of a simple inequality: $e^{-r} \mathsf{E} e^{-bx(Y_1 - 1)} Y_1 \ge 1$.

However, property (29) does not cover the case with the standard payoff function $g_n(x) = a_n [K_n - x]_+$ and other payoff functions with the following property:

$$g_n(x) = C_n, \qquad x \ge K_n,\ n = 0, 1, \dots, N, \tag{30}$$

where $C_n \ge 0$, $n = 0, 1, \dots, N$, are constants.

In this case the inequality in condition **B1** can fail for $x \ge K_n$. Hence, Theorem 1 requires some modification.

We now consider payoff functions with the following additional property:

G2: Let $0 < K_n < \infty$ be fixed for all $n = 0, 1, \dots, N$. For $n = 1, 2, \dots, N$ and each $x \ge K_n$,

$$g_n(x) = 0.$$

We suppose that $g'_{n,(-)}(K_n) < 0$ for all $n = 0, 1, \dots, N$. Condition **B1** needs to be modified in the following way:

B2: For every $n = 0, 1, \dots, N$ and each $0 < x \le K_n$,

$$\begin{aligned} g'_{n,(-)}(x) \le e^{-r_{n+1}} \Big[& \mathsf{E} g'_{n+1,(-)}(A_{n+1}(x, Y_{n+1})) A'_{n+1,(-)}(x, Y_{n+1}) \\ & + g'_{n+1,(-)}(K_{n+1}) \mathsf{E} A'_{n+1,(-)}(x, Y_{n+1}) I_{(K_{n+1}, \infty)}(A_{n+1}(x, Y_{n+1})) \Big]. \end{aligned}$$

Theorem 2. *Under conditions* ***A1****,* ***A2****,* ***G1****,* ***G2****, and* ***B2*** *for each* $n = 0, 1, \ldots, N-1$, *there exists* $0 \le d_n \le K_n$ *such that the optimal stopping domain has the one-threshold structure*

$$\Gamma_n = [0, d_n].$$

Proof. The proof is based on the same idea as the proof of Theorem 1. That is, for each $n = 0, 1, \ldots, N-1$, we should establish the three properties: **(a)** $\Gamma_n \neq \mathbb{R}^+$; **(b)** Γ_n is a closed nonempty set; and **(c)** if $x' \in \Gamma_n$, then $x'' \in \Gamma_n$ for any $0 \le x'' \le x'$.

Consider moment $N-1$. By condition **G2** the domain Γ_{N-1} is bounded above by K_{N-1}. Thus, property **(a)** holds for moment $N-1$. Furthermore, **(a)** implies that

$$d_{N-1} = \inf\left\{x \colon x \in \Gamma^C_{N-1}\right\} \le K_{N-1},$$

where $\Gamma^C_{N-1} = \mathbb{R}^+ \setminus \Gamma_{N-1}$.

The functions $g_{N-1}(x)$ and $e^{-r_N} T_N g_N(x)$ are continuous by property **G1** and Lemma 1, respectively. It follows by the continuity of these functions and the definition of the stopping domain (12) that Γ^C_{N-1} is open and therefore the point $d_{N-1} \in \Gamma_{N-1}$. Thus, **(b)** holds for the stopping domain Γ_{N-1}.

By condition **B2** and Lemma 2 for all $0 < x \le K_{N-1}$,

$$\begin{aligned} g'_{N-1,(-)}(x) &\le e^{-r_N} \mathsf{E} g'_{N,(-)}(A_N(x, Y_N)) A'_{N,(-)}(x, Y_N) \\ &\quad + e^{-r_N} g'_{N,(-)}(K_N) \mathsf{E} A'_{N,(-)}(x, Y_N) I_{(K_N,\infty)}(A_N(x, Y_N)) \\ &\le e^{-r_N} \mathsf{E} g'_{N,(-)}(A_N(x, Y_N)) A'_{N,(-)}(x, Y_N) \\ &= (e^{-r_N} T_N w_0)'_{(-)}(x), \end{aligned} \tag{31}$$

where the last inequality holds since $g'_{N,(-)}(K_N) < 0$.

Assume that $x' \in \Gamma_{N-1}$. By this assumption and (31) for any $0 \le x'' \le x'$,

$$\begin{aligned} g_{N-1}(x'') &= g_{N-1}(x') - \int_{x''}^{x'} g'_{N-1,(-)}(t)\, dt \\ &\ge e^{-r_N} T_N w_0(x') - \int_{x''}^{x'} e^{-r_N} (T_N w_0)'_{(-)}(t)\, dt \\ &= e^{-r_N} T_N w_0(x''). \end{aligned}$$

Thus, **(c)** holds for Γ_{N-1}.

We have proved that Γ_{N-1} has the structure described by (13).

Consider now moment $N-2$. As before, by condition **G2**, domain Γ_{N-2} is bounded above. Thus, property **(a)** holds for Γ_{N-2} and this implies that

$$d_{N-2} = \inf\left\{x \colon x \in \Gamma^C_{N-2}\right\} \le K_{N-2}.$$

The functions $g_{N-2}(x)$ and $e^{-r_{N-1}} T_{N-1} w_1(x)$ are continuous by property **G1** and Lemmas 1 and 3, respectively. It follows by the continuity of these functions and the definition of the optimal stopping domain (12) that Γ^C_{N-2} is open and, therefore, the point $d_{N-1} \in \Gamma_{N-2}$. Thus, **(b)** holds for the optimal stopping domain Γ_{N-2}.

By Lemmas 1–3, we have the following differentiation formula:

$$(T_{N-1} w_1)'_{(-)}(x) = \mathsf{E}[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)}.$$

We have to consider two cases concerning the expression on the right-hand side.

First case: $0 < A_{N-1}(x, Y_{N-1}) \le d_{N-1}$. This is true if and only if $0 < x \le x^*(Y_{N-1})$, where $x^*(Y_{N-1}) = \sup\{x > 0 \colon A_{N-1}(x, Y_{N-1}) \le d_{N-1}\}$. Then, for $0 < x \le x^*(Y_{N-1})$,

$$[w_1(A_{N-1}(x, Y_{N-1}))]'_{(-)} = g'_{N-1,(-)}(A_{N-1}(x, Y_{N-1})) A'_{N-1,(-)}(x, Y_{N-1}). \tag{32}$$

Second case: $A_{N-1}(x,Y_{N-1}) > d_{N-1}$. This is true if and only if $x > x^*$. Then

$$w_1(A_{N-1}(x,Y_{N-1})) = e^{-r_N} T_N g_N(z) \big|_{z=A_{N-1}(x,Y_{N-1})} .$$

If $d_{N-1} < A_{N-1}(x,Y_{N-1}) \le K_{N-1}$, then by Lemma 2, the nonpositivity of $g'_{N,(-)}$ and condition **B2**,

$$\begin{aligned}[w_1(A_{N-1}(x,Y_{N-1}))]'_{(-)} &= e^{-r_N}\big(T_N g_N(z)\big)'_{(-)}\big|_{z=A_{N-1}(x,Y_{N-1})} \cdot A'_{N-1,(-)}(x,Y_{N-1})\\ &\ge g'_{N-1,(-)}(A_{N-1}(x,Y_{N-1}))\cdot A'_{N-1,(-)}(x,Y_{N-1}).\end{aligned}$$

If $A_{N-1}(x,Y_{N-1}) > K_{N-1}$, then, since $(T_N g_N)'_{(-)}$ is nonpositive and nondecreasing and by (31),

$$\begin{aligned}[w_1(A_{N-1}(x,Y_{N-1}))]'_{(-)} &= e^{-r_N}\big(T_N g_N(z)\big)'_{(-)}\big|_{z=A_{N-1}(x,Y_{N-1})} \cdot A'_{N-1,(-)}(x,Y_{N-1})\\ &\ge e^{-r_N}\big(T_N g_N\big)'_{(-)}(K_{N-1})\cdot A'_{N-1,(-)}(x,Y_{N-1})\\ &\ge g'_{N-1,(-)}(K_{N-1})\cdot A'_{N-1,(-)}(x,Y_{N-1}).\end{aligned}$$

Thus, for all $x > 0$,

$$\begin{aligned}[w_1(A_{N-1}(x,Y_{N-1}))]'_{(-)} &\ge g'_{N-1,(-)}(z)A'_{N-1,(-)}(x,Y_{N-1})\\ &\quad + g'_{N-1,(-)}(K_{N-1})A'_{N-1,(-)}(x,Y_{N-1})I_{(K_{N-1},\infty)}(z),\end{aligned} \tag{33}$$

where $z = A_{N-1}(x,Y_{N-1})$. Taking the expectation of expression (33) and applying condition **B2**, we get for all $0 < x \le K_{N-2}$,

$$e^{-r_{N-1}}\mathsf{E}[w_1(A_{N-1}(x,Y_{N-1}))]'_{(-)} \ge g'_{N-2,(-)}(x),$$

and hence

$$e^{-r_{N-1}}(T_{N-1}w_1)'_{(-)}(x) \ge g'_{N-2,(-)}(x). \tag{34}$$

Assume that $x' \in \Gamma_{N-2}$. By this assumption and relation (34) for any $0 \le x'' \le x'$,

$$\begin{aligned}g_{N-2}(x'') &= g_{N-2}(x') - \int_{x''}^{x'} g'_{N-2,(-)}(t)\,dt\\ &\ge e^{-r_{N-1}}T_{N-1}w_1(x') - \int_{x''}^{x'} e^{-r_{N-1}}(T_{N-1}w_1)'_{(-)}(t)\,dt\\ &= e^{-r_{N-1}}T_{N-1}w_1(x'').\end{aligned}$$

Thus, **(c)** holds for Γ_{N-2}. We have proved that Γ_{N-2} has the threshold property given in (13).

Continuing in a similar way we can prove that Γ_n for $n = N-3,\dots,0$ has the structure given in (13). □

Remark. The following condition guarantees that $d_n > 0$ for all $n = 0,\dots,N-1$:

D1: For every $n = 0,\dots,N-1$,

$$g_n(0) > \max_{n+1\le k\le N} e^{-R_{n+1,k}} g_k(0),$$

where $R_{n+1,k} = r_{n+1} + \dots + r_k$, $1 \le n+1 \le k \le N$.

Indeed, note that by formula (10),

$$\begin{aligned}w_{n+1}(x) &= \sup_{n+1\le\tau\le N} \mathsf{E}\left\{e^{-R_{n+1,\tau}} g_\tau(S_\tau) \mid S_{n+1} = x\right\}\\ &\le \max_{n+1\le k\le N} e^{-R_{n+1,k}} g_k(0).\end{aligned} \tag{35}$$

Thus, by condition **D1**,

$$g_n(0) > e^{-r_{n+1}} w_{N-(n+1)}(x)$$

for any $x \geq 0$. Applying the operator T_{n+1} to both sides of the inequality we get

$$g_n(0) > e^{-r_{n+1}} T_{n+1} w_{N-(n+1)}(x)$$

for any $x \geq 0$. Now, by continuity of the functions $g_n(x)$ and $T_{n+1} w_{N-(n+1)}(x)$, there exists a $\delta > 0$ such that for every $x \in [0, \delta)$,

$$g_n(x) > e^{-r_{n+1}} T_{n+1} w_{N-(n+1)}(x).$$

Hence, $d_n > 0$.

Bibliography

1. A. Basso, M. Nardon, and P. Planca, *An analysis of the effects of continuous dividends on the exercise of American options*, Rend. Studi Econ. Quant. **2001** (2002), 41–68. MR1940220 (2003i:91048)
2. P. Boyle, M. Broadie, and P. Detemple, *Monte Carlo methods in security pricing*, Option Pricing, Interest Rates and Risk Management (E. Jouini, J. Cvitanić, and M. Musiela, eds.), Cambridge University Press, 2001. MR1848553
3. M. Broadie and J. Detemple, *American options on dividend-paying assets*, Fields Institute Communications **22** (1999), 69–97. MR1664664
4. P. Carr, R. Jarrow, and R. Myneni, *Alternative characterizations of American put options*, Mathematical Finance **2** (1992), 87–106.
5. Y. S. Chow, H. Robbins, and D. Siegmund, *The Theory of Optimal Stopping*, Houghton Mifflin Comp., 1971. MR0331675 (48:10007)
6. J. Jacka, *Optimal stopping and the American put*, Mathematical Finance **1** (1991), 1–14.
7. H. Jönsson, *Monte Carlo studies of American type options with discrete time*, Theory Stoch. Proc. **7(23)** (2001), no. 1–2, 163–188.
8. H. Jönsson, A. G. Kukush, and D. S. Silvestrov, *Threshold structure of optimal stopping domains for American type options*, Theory Stoch. Proc. **8(24)** (2002), no. 1–2, 169–177. MR2028749
9. H. Jönsson, A. G. Kukush, and D. S. Silvestrov, *Threshold Structure of Optimal Stopping Strategies for American Type Options*, Research Report 2004-2, Department of Mathematics and Physics, Mälardalen University, 2004.
10. I. J. Kim, *The analytic valuation of American options*, The Review of Financial Studies **3** (1990), 547–572.
11. A. G. Kukush and D. S. Silvestrov, *Optimal pricing of American type options with discrete time*, Theory Stoch. Proces. **10(26)** (2004), no. 1–2, 72–96.
12. A. N. Shiryaev, *Optimal Stopping Rules*, Springer-Verlag, 1978. MR0468067 (57:7906)
13. A. N. Shiryaev, Yu. M. Kabanov, D. O. Kramkov, and A. V. Mel'nikov, *Toward a theory of pricing options of European and American types. I. Discrete time*, Theory Probab. Appl. **39** (1994), 14–60. MR1348190 (97e:90008)
14. D. S. Silvestrov, V. G. Galochkin, and V. G. Sibirtsev, *Algorithms and programs for optimal Monte Carlo pricing of American type options*, Theory of Stoch. Process. **5(21)** (1999), no. 1–2, 175–187.
15. D. S. Silvestrov, V. G. Galochkin, and A. A. Malyarenko, *OPTAN — a pilot program system for analysis of options*, Theory of Stoch. Process. **7(23)** (2001), no. 1–2, 291–300.
16. P. van Moerbeke, *On optimal stopping and free boundary problems*, Archive for Rational Mechanics and Analysis **60** (1976), 101–148. MR0413250 (54:1367)

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